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Two Generalizations of Komlós' Theorem with Lower Closure-Type Applications

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Dedicated to R. T. Rockafellar on his 60th Birthday

Two generalizations of Komlós' theorem for functions and multifunctions with values in a Banach space are presented. The first generalization is partly new and the second one is a Komlós-type result of a completely new nature. It requires the Radon-Nikodym property for the Banach space and its dual. In both cases our approach relies on using Komlós' theorem by means of a diagonal extraction argument, as introduced in [5], [6], [7]. Two quite general lower closure-type results, which follow immediately from our main results, are shown to generalize or substantially extend a number of results in the literature.

1. Introduction

It is well-known that weak lower semicontinuity and lower closure results for integral functionals over a finite measure space $(\Omega, \mathcal{A}, \mu)$ can be obtained from relative compactness results for *K-convergence* [5], [6], [7]. In themselves, such relative compactness results can be used to develop new relative compactness results, e.g. for narrow convergence of Young measures [9] or to refine classical results for relative weak compactness in L_1-spaces. As a fine example of the latter we mention the recent work of Saadoune [44], where the well-known characterization of relative weak L_1-compactness of Diestel, Ruess and Schachermayer [28] is generalized. Recall that a sequence of functions or multifunctions (F_n), $F_n : \Omega \to Y$, is said to *K-converge* to another function or multifunction $F_0 : \Omega \to Y$ (written as $F_n \xrightarrow{K} F_0$) if for every subsequence (F_{n_j}) of (F_n) there exists a null set N such that for every $\omega \in \Omega \backslash N$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} F_{n_j}(\omega) = F_0(\omega). \tag{1.1}$$

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In other words, K-convergence (which is itself a nontopological notion) constitutes almost everywhere pointwise convergence of arithmetic averages (i.e., Cesaro-convergence) over any subsequence. Here the image space Y of the functions or multifunctions is a convex cone in the sense of e.g. [30] (i.e., $(Y,+)$ is an abelian semigroup satisfying the usual laws of multiplication with nonnegative scalars), equipped with a topology. Concretely, this paper uses for Y either a separable Banach space X or a subset of its hyperspace 2^X. As initiated in [5], [6], [7], relative compactness results for such K-convergence can be obtained by exploiting a deep and fundamental theorem of J. Komlós [40] for real-valued functions. In turn, this produces a powerful abstract version of Komlós' theorem, which is characterized by a convex version of *tightness* and the presence of a class of affine continuous integrands with a point-separating property (see [6], [7] and Appendix A).

This paper presents essentially two generalizations of Komlós' theorem for functions and multifunctions taking values in a separable Banach space X. The first generalization, Theorem 2.1, subsumes the two separate main results Theorems A, B of [5] (see Corollary 2.2). A result similar to Theorem 2.1 (a) can already be found in Theorem 2.2 of [6], and, actually, Theorem 2.1 (a) follows from the abstract version of Komlós' theorem in [6], [7]. This is demonstrated in Appendix A to this paper by proving what is actually an extension of Theorem 2.1 (a). Our Theorem 2.1 (b) is new, and we suspect that its proof contains elements that may be useful in other, related arguments as well.

Theorem 2.5, the second principal generalization of Komlós' theorem given in this paper, is completely new. It depends on the Banach space X and its dual X^* having the Radon-Nikodym property, and our proof depends on the Radon-Nikodym theorem for multimeasures. On the other hand, it is interesting to note that essential parts of the reasoning followed in proving Theorem 2.1 are also used (e.g., part 1 in section 3 serves in the proof of both theorems).

As mentioned in the first paragraph, there are natural by-products of these principal results in the form of lower closure-type results. In section 5 these results, Theorems 5.1 and 5.2 have been derived. In section 6 it is shown that together (or in conjunction with Appendix A) they generalize a number of results in the literature, and sometimes quite substantially. These include the Fatou-type lemmas in [18], which formed for us the starting point of the present paper.

The remainder of this introduction is used to establish some notation: Let $\|\cdot\|$ stand for the norm on the separable Banach space X; the associated dual space is denoted by X^* and the usual duality between X and X^* by $<\cdot,\cdot>$. Respectively by s and w we shall indicate the strong and the weak topology on X. A subset of X is said to be *w-ball-compact* if it has a w-compact intersection with every closed ball. Recall that the *sequential weak Kuratowski limes superior* w-Ls C_n of a sequence (C_n) of subsets of X is defined as the set of all w-limits of subsequences (x_{n_j}), with $x_{n_j} \in C_{n_j}$ for all indices n_j. Further, let $\mathcal{P}_{wkc}(X)$ be the collection of all nonempty w-compact convex subsets of X. For any C in $\mathcal{P}_{wkc}(X)$ we set

$$s(x^* \mid C) := \sup_{x\in C} < x, x^* >, \quad \|C\| := \sup_{x\in C} \|x\|,$$

to define for the set C respectively its *support function* $s(\cdot \mid C)$ and *radius* $\|C\|$. On $\mathcal{P}_{wkc}(X)$ we use the following topology: a sequence or generalized sequence (C_n) converges *scalarly* (alias *weakly* [26]) to C_0 in $\mathcal{P}_{wkc}(X)$ if

$$\lim_{n\to\infty} s(x^* \mid C_n) = s(x^* \mid C_0) \text{ for every } x^* \in X^*. \tag{1.2}$$

A function $F : \Omega \to \mathcal{P}_{wkc}(X)$, also called a *multifunction*, is said to be *Effros measurable* (or *measurable* for short) if the set $\{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$ belongs to $\mathcal{A}$ for any s-open subset U of X. As we shall see later, the Effros measurability of a multifunction taking values in $\mathcal{P}_{wkc}(X)$ is equivalent to its being scalarly measurable. By $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ we denote the set of all measurable multifunctions $F : \Omega \to \mathcal{P}_{wkc}(X)$ such that $\int_\Omega \|F(\omega)\|\mu(d\omega) < +\infty$ (the integrand $\|F(\cdot)\|$ is measurable by III.9 of [19]). Obviously, the singleton-valued multifunctions in $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ can be identified with elements of the space $\mathcal{L}^1_X$ of all Bochner-integrable functions from Ω into X. Finally, recall that for a set $A \in \mathcal{A}$ the integral of a multifunction $F : \Omega \to 2^X$ over A is the set given by

$$\int_A F \; d\mu := \{\int_A f \; d\mu : f \in \mathcal{L}^1_X, \;\; f(\omega) \in F(\omega) \text{ a.e.}\}.$$

As such, this general definition still allows the set in question to be empty; however, it is well-known (Theorem 3.6 (ii) of [39]) that for $F \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ the set $\int_A F \; d\mu$ is nonempty and belongs to $\mathcal{P}_{wkc}(X)$ for every $A \in \mathcal{A}$.

2. Main results

In this section we state our main results; sections 3 and 4 are devoted to the proofs. Our first principal result is an extension of Komlós' theorem in the spirit of [5], [6], [7], but with pointwise ball-compactness conditions in what can be seen as a multivalued elaboration of Remark 2.3 in [8]. A generalization of part (a) of this theorem will be derived from [6], [7] in Appendix A.

Theorem 2.1. *Let (F_n) be a sequence in $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ satisfying the following hypotheses:*
(i) $\sup_n \int_\Omega \|F_n(\omega)\|\mu(d\omega) < +\infty$,
(ii) $\operatorname{clco} \cup_n F_n(\omega)$ *is w-ball-compact a.e.*

Then there exist a subsequence $(F_{n'})$ of (F_n) and $F_* \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ satisfying

(a) $F_{n'} \xrightarrow{K} F_*$,
(b) $F_*(\omega) \subset \operatorname{clco} w - \operatorname{Ls} F_{n'}(\omega)$ *a.e.*

In the single-valued case Theorem 2.1 takes the following form, in which it subsumes the two main results of [5]. [1]

Corollary 2.2. *Let (f_n) be a sequence of Bochner-integrable functions in $\mathcal{L}^1_X$ satisfying the following hypotheses:*
(i) $\sup_n \int_\Omega \|f_n(\omega)\|\mu(d\omega) < +\infty$,

[1] It is enough to observe that w-compactness always implies w-ball-compactness, and that w-closedness implies w-ball-compactness in case X is a reflexive Banach space.

(ii) $\mathrm{clco}\{f_n(\omega) : n \in \mathbf{N}\}$ *is w-ball-compact a.e.*
Then there exist a subsequence $(f_{n'})$ *of* (f_n) *and* $f_* \in \mathcal{L}^1_X$ *satisfying*
(a) $f_{n'} \stackrel{K}{\to} f_*$,
(b) $f_*(\omega) \subset \mathrm{clco}\,w\text{--}\mathrm{Ls}\ f_{n'}(\omega)$ *a.e.*

Remark 2.3. If one systematically replaces the w-topology by the s-topology, that is, one replaces $\mathcal{P}_{wkc}(X)$ by $\mathcal{P}_{skc}(X)$ (the set of all nonempty s-compact convex subsets of X) and one replaces w-ball-compactness in (ii) by s-ball-compactness, then the conclusions (a) and (b) can be improved as follows:

(α) $F_{n'} \stackrel{K_h}{\to} F_*$,
(β) $F_*(\omega) \subset \mathrm{clco}\,s\text{--}\mathrm{Ls}\ F_{n'}(\omega)$ a.e.,

where $F_{n'} \stackrel{K_h}{\to} F_*$ means now K-convergence as in (1.1) when $Y := \mathcal{P}_{skc}(X)$ is equipped with the Hausdorff metric h; i.e., for every subsequence $(F_{n'_j})$ of $(F_{n'})$ there exists a null set N such that for every $\omega \in \Omega \backslash N$

$$\lim_{m\to\infty} h\left(\frac{1}{m}\sum_{j=1}^{m} F_{n'_j}(\omega), F_*(\omega)\right) = 0.$$

This observation follows by adapting the proof of Theorem 2.1 given below, in particular by replacing part (a) of Lemma 3.2 by its part (b).

Remark 2.4. Both in Theorem 2.1 and in Remark 2.3 the convexity condition for the values of the multifunctions can be lifted.[2] In Theorem 2.1 this is evident by elementary properties of support functions and Krein's theorem. As for Remark 2.3, the removal of convexity is made possible by applying Lemma p. 307 of [2] or Lemma 2 of [37].
We now state our second principal result, a Komlós-type theorem of a completely new kind, characterized by the fact that X and its dual X^* now have the *Radon-Nikodym property* (RNP). Recall that X is said to have the RNP (with respect to $(\Omega, \mathcal{A}, \mu)$) if each μ-absolutely continuous vector measure $\nu : \mathcal{A} \to X$ with bounded variation has a density $f \in \mathcal{L}^1_X$ with respect to μ, i.e.,

$$\nu(A) = \int_A f \; d\mu \text{ for all } A \in \mathcal{A}.$$

Actually, following [17], [18] it can be observed that in this paper the full force of the RNP for X is not really needed: it is enough to require the above density property only for those ν which satisfy

$$\nu(A) \subset \mathrm{clco} \cup_n \int_A F_n \; d\mu \text{ for all } A \in \mathcal{A}.$$

Theorem 2.5. *Suppose that both the Banach space* X *and its dual* X^* *have the RNP. Let* (F_n) *be a sequence in* $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ *satisfying the following hypotheses:*

[2] We thank G. Krupa (Utrecht) for calling this point to our attention.

(i) $\sup_n \int_\Omega \|F_n(\omega)\|\mu(d\omega) < +\infty$,
(ii') $\mathrm{clco} \cup_n \int_A F_n \ d\mu$ *is* w*-compact for every* $A \in \mathcal{A}$,
Then there exist a subsequence $(F_{n'})$ *of* (F_n) *and* $F_* \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ *satisfying*

(a) $F_{n'} \overset{K}{\to} F_*$,
(b') $F_*(\omega) \subset \cap_m \mathrm{clco} \cup_{n' \geq m} F_{n'}(\omega)$ *a.e.*

Theorem 2.5 replaces the pointwise ball-compactness conditions (ii) by the w-compactness conditions (ii') for integrals, which is of an essentially different, more macroscopic nature. But it imposes additional RNP-conditions on X and its dual. For singleton-valued multifunctions we immediately have the following corollary, which is also new:

Corollary 2.6. *Suppose that both the Banach space* X *and its dual* X^* *have the RNP. Let* (f_n) *be a sequence in* $\mathcal{L}^1_X$ *satisfying the following hypotheses:*
i) $\sup_n \int_\Omega \|f_n(\omega)\|\mu(d\omega) < +\infty$,
ii') $\mathrm{clco}\{\int_A f_n \ d\mu : n \in \mathbf{N}\}$ *is* w*-compact for every* $A \in \mathcal{A}$.
Then there exist a subsequence $(f_{n'})$ *of* (f_n) *and* $f_* \in \mathcal{L}^1_X$ *satisfying*

a) $f_{n'} \overset{K}{\to} f_*$,
b') $f_*(\omega) \in \cap_m \mathrm{clco}\{f_{n'}(\omega) : n' \geq m\}$ *a.e.*

Remark 2.7. Evidently, (b') is a little weaker than (b); see p. 2.17 of [14] for examples, due to M. Valadier, which illustrate this, both in finite and infinite dimensions (see also Lemma 3.3 for a special situation where (b) and (b') are equivalent).

Remark 2.8. For a separable Banach space X it is well-known, by a theorem of Stegall (see p. 195 of [29]), that X^* has the RNP if and only if X^* is separable for the dual norm.

Example 2.9. In the presence of (i) and the following condition: for every $x^* \in X^*$

$$(s(x^* \mid F_n(\cdot))) \text{ is uniformly integrable over } \Omega,$$

hypothesis (ii) implies the validity of (ii'). This can be easily proven by means of Young measures, for instance, since (i)–(ii) constitute a tightness condition for the corresponding relaxations of any sequence of selectors. Conversely, hypothesis (ii') does not necessarily imply (ii), as the following examples show. For $\Omega := [0,1]$ with Lebesgue measure and $X := \ell^1$ we define $f_n(\omega) := 1_{A_n}(\omega)e_n$. Here (A_n) is a sequence of independent subsets of $[0,1]$ with Lebesgue measure $\mu(A_n) \to 0$ and $\sum_{n=1}^\infty \mu(A_n) = +\infty$ (e.g., see Example 4.14 of [11] for a concrete instance). By the second Borel-Cantelli theorem we have that for almost every ω the sequence $(f_n(\omega))$ contains an infinite number of $e_n's$, which implies that it cannot be compact for $\sigma(\ell^1, \ell^\infty)$. Therefore, (ii) does not hold. On the other hand, (ii') certainly holds, because $\| \int_A f_n \ d\mu\| = \mu(A \cap A_n) \to 0$ for every $A \in \mathcal{A}$. The previous example is not entirely convincing, because its dual space $X^* = \ell^\infty$ does not have the RNP. However, the following modification addresses this shortcoming. Let $X := J$, the space defined in Example 1.4.2 of [41], with canonical basis (e_n). Let (A_n) be as above, and define $f_n(\omega) := 1_{A_n}(\omega) \sum_{m=1}^n e_m$. Neither $(\sum_{m=1}^n e_m)$ nor any of its subsequences converges weakly [41], so the above argument can be repeated. This time, both X and X^* have the RNP. [3]

[3] We thank G. Godefroy, H. Klei and G. Pisier (Paris) for providing us with this information.

Remark 2.10. Notice that in the presence of (i), hypothesis (ii') is equivalent to
(ii$''$) $\mathrm{clco} \cup_n \int_A F_n \, d\mu$ is w-ball-compact for every $A \in \mathcal{A}$,
which resembles hypothesis (ii) a little more.

Remark 2.11. Hypothesis (ii$'$) is equivalent to the following two conditions:
(ii$'''$) the set $\mathrm{clco} \cup_n \int_\Omega F_n \, d\mu$ is w-compact,
(ii$''''$) there exists a sequence (f_n) in $\mathcal{L}^1_X$ such that f_n is a selector of F_n for each n and such that $C(A) := \mathrm{clco}\{\int_A f_n \, d\mu : n \in \mathbf{N}\}$ is w-compact for every $A \in \mathcal{A}$.
It is evident that (ii$'$) implies (ii$'''$)– (ii$''''$). For the converse, note that for any A in $\mathcal{A}$ the set $\int_A F_n \, d\mu + \int_{\Omega\setminus A} f_n \, d\mu$ is contained in $\int_\Omega F_n \, d\mu$. Hence, $\mathrm{clco} \int_A F_n \, d\mu$ is contained in $\mathrm{clco} \int_\Omega F_n \, d\mu - C(\Omega\setminus A)$; the latter set is w-compact by (ii$'''$) –(ii$''''$), so (ii$'$) follows.

Remark 2.12. Unlike Theorem 2.1, Theorem 2.5 does not have a variant when the s-topology systematically replaces the w-topology (cf. Remark 2.3). This is due to the fact that in this case the Radon-Nikodym theorem for multimeasures (which will play a crucial role in the proof of Theorem 2.5) only provides the existence of a derivative multifunction with w-compact values; see Example 1 in [22].

3. Proofs of the main results

We shall denote by τ the Mackey topology on the dual space X^*. Since X is supposed separable, the unit ball of X^* has a countable, τ-dense subset D with $D = -D$; also, the set H spanned by all rational linear combinations of vectors in D is τ-dense in X^* (III.32 of [19]). For any $C \in \mathcal{P}_{wkc}(X)$ one has

$$C = \cap_{x^* \in H} \{x \in X : \langle x, x^* \rangle \leq s(x^* \mid C)\}, \tag{3.1}$$

by an application of III.34 of [19]. On the space $\mathcal{P}_{wkc}(X)$ we also consider the H-scalar convergence topology. This is the topology of pointwise convergence of support functions on H, obtained by requiring (1.2) to hold only for all $x^* \in H$; evidently the H-scalar convergence topology is metrizable. For scalar convergence of sequences, the following consequences of (1.2) are immediate but useful:

$$\|C_0\| \leq \liminf_n \|C_n\|, \qquad \mathrm{clco}\,\mathrm{Ls}\ C_n \subset C_0 \subset \cap_p \ \mathrm{clco}\ \cup_{n \geq p} C_n, \tag{3.2}$$

where it can be remarked that the first inclusion becomes an identity in case $\cup_n C_n$ is relatively w-compact (by an obvious modification of Proposition 3.3 of [26]). For future use it is also important to note that equality (3.1) continues to hold for any closed convex and merely bounded subset C, provided that X^* is separable for the dual norm topology; in this case the separable dense subset H is the obvious one.
The set of all Bochner-integrable X-valued functions on $(\Omega, \mathcal{A}, \mu)$ is denoted by $\mathcal{L}^1_X$. For any multifunction $F \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ we denote

$$\|F\|_1 := \int_\Omega \|F\| d\mu.$$

It is well-known (Theorem 2.2 of [38]) that for Effros measurable $F : \Omega \to 2^X$, having closed values, the following identity holds:

$$s(x^* \mid \int_A F \; d\mu) = \int_A s(x^* \mid F) d\mu, \;\; x^* \in X^*, \;\; A \in \mathcal{A}, \tag{3.3}$$

provided that $\int_\Omega F \; d\mu$ is nonempty (as is always the case when $F \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$).

The following celebrated theorem, due to Komlós [40], will play a key role in the proofs of Theorems 2.1 and 2.5. A fairly short proof can be found in [20].

Theorem 3.1. *Suppose that* (ϕ_n) *is a sequence in* $\mathcal{L}^1_{\mathbf{R}}$ *such that*

$$\sup_n \int_\Omega |\phi_n| \; d\mu < +\infty.$$

Then there exist $\phi_* \in \mathcal{L}^1_{\mathbf{R}}$ *and a subsequence* $(\phi_{n'})$ *of* (ϕ_n) *such that for every further subsequence* $(\phi_{n'_j})$ *of* $(\phi_{n'})$ *there is a null set* N *with*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} \phi_{n'_j}(\omega) = \phi_*(\omega) \text{ for all } \omega \in \Omega \backslash N.$$

The next three lemmas will be used to prove Theorem 2.1. The first one is a Blaschke type compactness criterion for scalar convergence; it is Lemma 5.1 of [35] and can be seen as a consequence of e.g. Theorem 3.1 of [21]. The second one is recalled from Proposition 3.10 of [33]. For the sake of completeness, the proofs of both lemmas are given in section 4.

Lemma 3.2.

(a) *For every* K *in* $\mathcal{P}_{wkc}(X)$*, the subset* $\mathcal{K} := \{C \in \mathcal{P}_{wkc}(X) : C \subset K\}$ *of* $\mathcal{P}_{wkc}(X)$ *is metrizable and compact for the scalar convergence topology.*

(b) *For every* K *in* $\mathcal{P}_{skc}(X)$*, the subset* $\mathcal{K} := \{C \in \mathcal{P}_{skc}(X) : C \subset K\}$ *of* $\mathcal{P}_{skc}(X)$ *is compact for the Hausdorff metric* h*.*

Lemma 3.3. *Suppose* (C_n) *is a sequence of subsets of* X *such that there exists* $K \in \mathcal{P}_{wkc}(X)$ *for which*

$$\cup_n C_n \subset K. \tag{3.4}$$

Then

$$\text{clco}(w - \text{Ls } C_n) = \cap_k \text{ clco}(\cup_{n\geq k} C_n)$$

and

$$s(x^* \mid \text{clco}(w - \text{Ls } C_n)) = \limsup_{n\to\infty} s(x^* \mid C_n) \text{ for every } x^* \in X^*.$$

The third lemma is the well-known biting lemma, which goes back to V.F. Gaposhkin [31], and has been rediscovered by several authors (e.g., [13]).

Lemma 3.4. [biting lemma] *Suppose* (ϕ_n) *is a sequence in* $\mathcal{L}^1_{\mathbf{R}}$ *such that*

$$\sup_n \int_\Omega |\phi_n| d\mu < +\infty.$$

Then there exists a nonincreasing sequence (B_p) *in* $\mathcal{A}$ *such that* $\mu(\cap_{p=1}^{\infty} B_p) = 0$ *and for every* p

$$(\phi_n) \text{ is uniformly integrable over } \Omega \backslash B_p.$$

Proof of Theorem 2.1. *Part 1.* [4] Using Komlós' theorem (Theorem 3.1) and the diagonal method, we obtain the existence of a subsequence $(F_{n'})$ of (F_n) and of functions ϕ_{x^*}, $x^* \in H$, and ψ in $\mathcal{L}^1_{\mathbf{R}}$ such that

$$\phi_{x^*}(\omega) = \lim_{m\to\infty} \frac{1}{m} \sum_{n'=1}^{m} s(x^* \mid F_{n'}(\omega)) \tag{3.5}$$

and

$$\psi(\omega) = \lim_{m\to\infty} \frac{1}{m} \sum_{n'=1}^{m} \|F_{n'}(\omega)\| \tag{3.6}$$

for every $x^* \in H$ and every ω outside some null set N_1. By the same result, this also holds for every further subsequence of $(F_{n'})$. (Note already that then the limit functions ϕ_{x^*}, $x^* \in H$, and ψ do not change, but that in general the null set N_1 will have to be replaced by another null set.) We define the partial sums $G_m : \Omega \to \mathcal{P}_{wkc}(X)$ by

$$G_m(\omega) := \frac{1}{m} \sum_{n'=1}^{m} F_{n'}(\omega) \tag{3.7}$$

for all $m \in \mathbf{N}$. Relationship (3.6) implies that for every $\omega \notin N_1$

$$\|G_m(\omega)\| \le \psi(\omega) + 1 \text{ for } m \text{ large enough (say } m \ge m_\omega). \tag{3.8}$$

Part 2. We apply the previous part 1. Denote $G(\omega) := \mathrm{clco} \cup_n F_n(\omega)$; by hypothesis (ii) this set is w-ball-compact whenever ω is outside some null set N_0. Hence, for any $\omega \notin N := N_0 \cup N_1$ and for every $m \ge m_\omega$ the set $G_m(\omega)$ is contained in the w-compact set $K(\omega) := G(\omega) \cap \{x \in X : \|x\| \le \psi(\omega) + 1\}$. Thus, Lemma 3.2 gives the existence of a subsequence $(G_{m_i}(\omega))$ of $(G_m(\omega))$ (possibly depending upon ω), which converges scalarly to some limit set $F_*(\omega)$ in $\mathcal{P}_{wkc}(X)$. But because of (3.1) and (3.5) it is easily seen that $F_*(\omega)$ acts as the *unique* cluster point of $(G_m(\omega))$, which therefore converges to it. Thus, we obtain that for all $\omega \notin N$

$$\lim_m s(x^* \mid G_m(\omega)) = s(x^* \mid F_*(\omega)) \text{ for all } x^* \in X^*. \tag{3.9}$$

On N we set $F_*(\omega)$ identically equal to some fixed singleton (say). Evidently, this implies the measurability of $s(x^* \mid F_*(\cdot))$ for every $x^* \in X^*$. Effros measurability of F_* then follows easily from

$$d(x, F_*(\omega)) = \sup_{x^*\in X^*, \|x^*\|^* \le 1} [< x, x^* > - s(x^* \mid F_*(\omega))] = \sup_{x^* \in D} [< x, x^* > - s(x^* \mid F_*(\omega))],$$

[4] This part of the proof will also serve in the proof of Theorem 2.5.

where the first identity is well-known and the second one follows by τ-denseness of D in the unit ball of X^* and τ-continuity of the support function (by $F_*(\omega) \in \mathcal{P}_{wkc}(X)$). Also, it follows from (3.9), by (3.2) and the classical Fatou lemma, that

$$\|F_*\|_1 \leq \int_\Omega \liminf_m \|G_m\| d\mu \leq \liminf_m \|G_m\|_1.$$

Here the limes inferior on the right is finite by hypothesis (i); hence, we conclude that F_* belongs to $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$. By part 1 it follows that the above argument can be repeated for the partial sums $H_m := \frac{1}{m}\sum_{j=1}^m F_{n'_j}$ corresponding to any subsequence $(F_{n'_j})$ of $(F_{n'})$. This gives the existence of a null set N' of Ω and a multifunction $F_{**} \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ such that for all $\omega \notin N'$

$$\lim_m s(x^* \mid H_m(\omega)) = s(x^* \mid F_{**}(\omega)) \text{ for all } x^* \in X^*. \tag{3.10}$$

From (3.5), (3.9) and (3.10) we conclude that for all $\omega \notin N \cup N'$

$$s(x^* \mid F_*(\omega)) = s(x^* \mid F_{**}(\omega)) = \phi_{x^*}(\omega) \text{ for all } x^* \in H.$$

Therefore, $F_{**} = F_*$ a.e., so (a) has been proven. We comment that, apart from the use of Komlós' theorem, the above technique is quite standard (e.g., see Lemma 5.1 of [35]).
Part 3. Next, we prove (b). We begin by observing that the multifunction w-Ls $F_{n'}$ is measurable by Theorem 4.4 of [33]. Suppose one did not have (b). Then there would exist $x^* \in D$ and a nonnull set A such that

$$s(x^*|F_*(\omega)) > s(x^*|w\text{-Ls } F_{n'}(\omega))$$

for all $\omega \in A$. For each n' there exists a measurable (and integrable) selector of $F_{n'}$ such that $s(x^*|F_{n'}(\omega)) =< f_{n'}(\omega), x^* >$ a.e. Without loss of generality we may suppose that the nonnull set A is such that on A the three sequences $(s(x^*|F_{n'}(\cdot)))$, $(\|f_{n'}\|)$ and $(< f_{n'}, x^* >)$ are uniformly integrable (apply the biting lemma three times: after three sufficiently small bites there is still a nonnull set left). Then $(f_{n'})$, restricted to A, is a uniformly integrable subset of $\mathcal{L}^1_X(A)$ which is $\mathcal{R}_w$-tight in the sense of [1]. Therefore, Theorem 6 of [1] implies that a subsequence $(f_{n''})$ of $(f_{n'})$ converges weakly to some $f_* \in \mathcal{L}^1_X(A)$. Further, by [1, Theorem 8] $f_*(\omega) \in \operatorname{clco} w - \operatorname{Ls} f_{n''}(\omega)$ a.e. in A (see Appendix B). On the one hand, elementary properties of K-convergence [7] imply

$$\lim_{n''} \int_A s(x^*|F_{n''}) d\mu = \int_A s(x^*|F_*) d\mu.$$

On the other hand, the stated properties of f_* imply

$$\lim_{n''} \int_A s(x^*|F_{n'}) = \lim_{n''} \int_A \langle f_{n''}, x^* \rangle = \int_A \langle f_*, x^* \rangle \leq \int_A s(x^*|w\text{-Ls } F_{n'}) < \int_A s(x^*|F_*).$$

This gives a contradiction. $\square$

We remark that the proof of part (b) can also be given by well-known results on Young measures. These results actually generalize the cited theorems from [1]; e.g., see Theorem 6.2 and Remark 6.5 in [9]. Here it should be observed (Remark 6.6 (a) of [9]) that L^1-boundedness and $\mathcal{R}_w$-tightness as in [1] imply tightness in the sense of [3], [6]. We also remark that for the partial sums (G_m) appearing in the proof of Theorem 2.1 the identity $F_*(\omega) = \text{clco}\, G_m(\omega)$ holds a.e.; this follows from the remark after (3.2).

To prove Theorem 2.5, we need a version of the Radon-Nikodym theorem for multivalued measures taking values in $\mathcal{P}_{wkc}(X)$. Recall that a map M from $\mathcal{A}$ into $\mathcal{P}_{wkc}(X)$ is called *additive* if

$$M(A \cup B) = M(A) + M(B),$$

whenever $A, B \in \mathcal{A}$ are disjoint sets. If in addition the function $s(x^* \mid M(\cdot))$ is a finite measure for every $x^* \in X^*$, then M is said to be a *weak multimeasure* (or multimeasure for short); this notion was introduced by Costé and Pallu de La Barrière in [24], [25]. Interestingly, by a result of Costé (p. III.4 of [23]) this definition of a multivalued measure with values in $\mathcal{P}_{wkc}(X)$ is equivalent to the following one: for every sequence (A_n) of pairwise disjoint sets in $\mathcal{A}$ one has

$$\lim_{m\to\infty} h(M(\cup_n A_n), \sum_{n=1}^{m} M(A_n)) = 0,$$

where h stands for the Hausdorff distance on $\mathcal{P}_{wkc}(X)$. The multimeasure M is said to be *μ-absolutely continuous* if $\mu(A) = 0$ implies $M(A) = \{0\}$ for every $A \in \mathcal{A}$. The *total variation* V_M of M is given by

$$V_M(\Omega) := \sup \sum_{i=1}^{p} \|M(A_i)\|,$$

the supremum being taken over all finite measurable partitions $\{A_1, A_2, \ldots, A_p\}$ of Ω. If $V_M(\Omega)$ is finite, the multimeasure M is said to be of *bounded variation*. The following result, due to Costé Théorème 3 of [22] or Théorème 8, p. III.31 of [23], is a multivalued analogue of the classical Radon-Nikodym theorem:

Theorem 3.5. *Suppose that X and X^* both have the RNP. Let M be a weak multimeasure of bounded variation with values in $\mathcal{P}_{wkc}(X)$. If M is μ-absolutely continuous, there exists a multifunction $\Gamma \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$, (a version of) the Radon-Nikodym derivative of M with respect to μ, such that*

$$M(A) = \int_A \Gamma \; d\mu \textit{ for every } A \in \mathcal{A}.$$

Actually, in [22], [23] the measure space $(\Omega, \mathcal{A}, \mu)$ is assumed to be complete. However, by reasoning as in Remark, p. 163 of [38] this condition is seen to be removable. Klei has given a very interesting generalization of the above theorem (Theorem 5.3 of [39]); his result also demonstrates the basic necessity of the RNP for both X and X^*.
In the absence of (ii), Lemma 3.3 can no longer be used, and an additional tool is required to obtain the pointwise inclusion in (b'):

Lemma 3.6. *Let (C_n) be a sequence in $\mathcal{P}_{wkc}(X)$. Then*

$$\cap_p \operatorname{clco} (\cup_{m\geq p} \frac{1}{m} \sum_{n=1}^{m} C_n) \subset \cap_p \operatorname{clco} \cup_{n\geq p} C_n.$$

Proof of Theorem 2.5. By Remark 2.8 we can take H to be a countable, strongly dense subset of X^*, closed for taking rational linear combinations. Let $(F_{n'})$ be as obtained in part 1 of the proof of Theorem 2.1. By (3.6) the sets $G_m(\omega)$, $m \in \mathbf{N}$, of (3.7) are of course (pointwise) uniformly bounded; hence $(s(\cdot \mid G_m(\omega)))$ is equicontinuous on X^* (recall that the strong topology is used on X^* in the current proof). Hence, it follows from (3.5) that

$$\phi_{x^*}(\omega) := \lim_{m\to\infty} \frac{1}{m} \sum_{n'=1}^{m} s(x^* \mid F_{n'}(\omega)) \text{ exists }, \tag{3.11}$$

not only for $x^* \in H$, but for all $x^* \in X^*$. As is clear from (3.8), $\phi_{\cdot}(\omega)$ is Lipschitz-continuous on X^* with Lipschitz-constant $\psi(\omega) + 1$ for every $\omega \notin N_1$. By (3.11) the function ψ_A, defined on X^* by $\psi_A(x^*) := \int_A \phi_{x^*}\ d\mu$, is subadditive on X^* and it satisfies

$$\psi_A(x^*) \leq s(x^* \mid K(A)) \text{ for all } x^* \in X^*, \tag{3.12}$$

where $K(A) := \operatorname{clco} \cup_n \int_A F_n\ d\mu$ is w-compact by hypothesis (ii'). Hence, by classical properties of sublinear functions, ψ_A is τ-continuous on X^*. Therefore, it is also w^*-lower semicontinuous, which by Theorem II.16 of [19] implies the existence of a nonempty closed convex subset $M(A)$ of X such that

$$\psi_A(\cdot) = s(\cdot \mid M(A)). \tag{3.13}$$

By (3.12) it follows that $M(A)$ is contained in the w-compact set $K(A)$, so $M(A)$ itself is also w-compact. We now show that $M : \mathcal{A} \to \mathcal{P}_{wkc}(X)$ has the following properties:

(P_1) M is additive,
(P_2) M is absolutely continuous with respect to μ,
(P_3) M has bounded variation.
(P_4) $s(x^* \mid M(\cdot))$ is σ-additive for every $x^* \in X^*$,

As for (P_1), note that this property follows from

$$s(x^* \mid M(A \cup B)) = \int_A \phi_{x^*}\ d\mu + \int_B \phi_{x^*}\ d\mu = s(x^* \mid M(A) + M(B))$$

for every $x^* \in H$ by the definition of ϕ_{x^*} and (3.13). So by (3.1) it follows that $M(A\cup B) = M(A) + M(B)$. Also, (P_2) follows easily, for if $A \in \mathcal{A}$ satisfies $\mu(A) = 0$, then $\psi_A(x^*) = 0$ for all $x^* \in H$. To prove (P_3) we observe that for all $x^* \in D$

$$s(x^* \mid M(A)) = \int_A \phi_{x^*}\ d\mu \leq \int_A \psi\ d\mu \tag{3.14}$$

by (3.5)–(3.6). Therefore, as in the case of scalar-valued measures (p. 101 of [29]),

$$\|M(A)\| = \sup_{x^* \in D} s(x^* \mid M(A)) \leq \int_A \psi \; d\mu.$$

Of course, this implies $V_M(\Omega) \leq \int_\Omega \psi \; d\mu < +\infty$, which proves (P$_3$). By sublinearity of support functions, (3.14) gives

$$|s(x^* \mid M(A))| \leq \int_A \psi \; d\mu$$

for all $x^* \in D$ (by $D = -D$). Hence, the total variation V_{x^*} of the additive set function $s(x^* \mid M(\cdot))$ satisfies $V_{x^*}(A) \leq \int_A \psi \; d\mu$ for all $A \in \mathcal{A}$. Thus, (P$_4$) follows.
In view of (P$_1$)–(P$_4$), we may now invoke Theorem 3.5, the multivalued Radon-Nikodym theorem. This yields the existence of a multifunction $F_* \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ which satisfies

$$M(A) = \int_A F_* \; d\mu \text{ for all } A \in \mathcal{A}. \tag{3.15}$$

Moreover, F_* is unique up to null sets. The proof of (a) is easily finished by substituting (3.3) in (3.14) and using the Lipschitz-continuity of $\phi.(\omega)$ on X^* for $\omega \notin N_1$.
To prove (b'), note first that by Lemma 3.6 it is enough to prove that for every p

$$F_*(\omega) \subset L_p(\omega) := \operatorname{clco} \cup_{m \geq p} G_m(\omega) \text{ a.e.} \tag{3.16}$$

We observe that for every $p \in \mathbf{N}$ and for a.e. $\omega \in \Omega$ the set $L_p(\omega)$ is bounded. Indeed, one has $\|G_m(\cdot)\| \leq \frac{1}{m}\sum_{n'=1}^m \|F_{n'}(\cdot)\|$, which, by (3.6), implies that there is a null set N_1 such that for every $\omega \notin N_1$ and every $p \in \mathbf{N}$ $\|L_p(\omega)\| \leq \sup_m \frac{1}{m}\sum_{n'=1}^m \|F_{n'}(\omega)\| < +\infty$. On the other hand, (3.15) gives $\int_A \phi_{x^*} \; d\mu = \int_A s(x^* \mid F_*) d\mu$ for every $A \in \mathcal{A}$ and $x^* \in H$. Hence, for every ω outside a suitable null set N_2, $\phi_{x^*}(\omega) = s(x^* \mid F_*(\omega))$ for all $x^* \in H$. Fix $\omega \notin N_1 \cup N_2$; then (3.5) gives $s(x^* \mid F_*(\omega)) = \lim_m s(x^* \mid G_m(\omega))$, which entails $s(x^* \mid F_*(\omega)) \leq s(x^* \mid L_p(\omega))$ for every p and $x^* \in H$. By boundedness of the $L_p(\omega)$ this gives $F_*(\omega) \subset L_p(\omega)$ for all p, in view of (3.1) and the separability of X^*. This proves (3.16), so the proof of Theorem 2.5 is finished. □

4. Proofs of auxiliary results

Proof of Lemma 3.2 (a). (See also the proof of Lemma 5.1 of [35].) It is obvious that $\mathcal{K}$ is closed for scalar convergence; so let us show that it is also relatively compact. Just as was done in the proof of Theorem 2.5, one shows that the collection $\{s(\cdot \mid C) : C \in \mathcal{K}\}$ is τ-equicontinuous (see also (4.1) below). Consequently, on $\mathcal{K}$, the scalar convergence topology coincides with the metrizable H-scalar convergence topology. Thus, it remains to show that an arbitrary sequence (C_n) in $\mathcal{K}$ has a convergent subsequence. By definition of $\mathcal{K}$ one has for every $n \in \mathbf{N}$ and every $x^* \in X^*$ that

$$-s(-x^* \mid K) \leq s(x^* \mid C_n) \leq s(x^* \mid K). \tag{4.1}$$

By the diagonal method this gives the existence of a subsequence $(C_{n'})$ of (C_n) such that the limit $\pi(x^*) := \lim_{n'\to\infty} s(x^* \mid C_{n'})$ exists for every $x^* \in H$. From (4.1) we deduce that the sublinear function π satisfies

$$|\pi(x^*)| \leq \max[s(x^* \mid K), s(-x^* \mid K)] \tag{4.2}$$

for every $x^* \in H$. Hence, π is τ-continuous on H; therefore, it admits a τ-continuous extension to X^*. This extension, denoted by $\tilde{\pi}$, still satisfies (4.2), but now for all $x^* \in X^*$. As in the proof of Theorem 2.5, by invoking II.16 of [19] we can see that this yields the existence of $C_* \in \mathcal{P}_{wkc}(X)$ such that $\tilde{\pi}(x^*) = s(x^* \mid C_*)$ for every $x^* \in X^*$. In view of (4.1), C_* is a subset of K (observe that $\tilde{\pi}(x^*) \leq s(x^* \mid K)$ for all $x^* \in X^*$). It is now elementary to conclude that the subsequence $(C_{n'})$ converges scalarly to C_*.
b. Repeating similar arguments as in the proof of part *a*, but replacing the topology τ on X^* by the topology τ_c of uniform convergence on s-compact subsets of X, we can prove the existence of a subsequence $(C_{n'})$ such that $\pi(x^*) := \lim_{n'\to\infty} s(x^* \mid C_{n'})$ for every $x^* \in H$. Likewise, we obtain existence and τ_c-continuity of $\tilde{\pi}$, the extension of π to X^*, and this implies the existence of an s-compact subset C_* of X such that $\tilde{\pi} = s(\cdot \mid C_*)$. By Ascoli's theorem, the τ_c-equicontinuity of $(s(\cdot \mid C_{n'}))$ causes the D-scalar convergence of $(C_{n'})$ to C_* to coincide with uniform convergence of $(s(\cdot \mid C_{n'}))$ to $s(\cdot \mid C_*)$ on any τ_c-compact subset of X^*. By the Banach-Dieudonné theorem (Théorème 1, p. IV.24 of [12]) the τ_c- and w^*-topologies coincide on the unit ball of X^*. So it follows that $(s(\cdot \mid C_{n'}))$ converges uniformly to $s(\cdot \mid C_*)$ on the dual unit ball; thus, $(C_{n'})$ converges in the Hausdorff metric to C_*.
An alternative proof can be given, based upon the remark following Theorem II.4 in [19]. □

Proof of Lemma 3.3. Let C be the right-hand side of the first identity that we have to prove. From the definition of w-Ls C_n it is clear that C contains the left-hand side of that same identity. It remains to prove the opposite inclusion. By definition of C it easily follows that for any $x^* \in X^*$

$$s(x^* \mid C) \leq \limsup_{n\to\infty} s(x^* \mid C_n). \tag{4.3}$$

Further, it is possible to find a subsequence (C_{n_j}) of (C_n) such that $\lim_j s(x^* \mid C_{n_j}) = \limsup_n s(x^* \mid C_n)$. Now, for any j, there exists $x_j \in C_{n_j}$ with $< x_j, x^* >\geq s(x^* \mid C_{n_j}) - 1/j$; then clearly $\lim_j < x_j, x^* >= \limsup_n s(x^* \mid C_n)$. By condition (3.4), the sequence (x_j), contained in K, has a subsequence which w-converges to some $\bar{x} \in K$. Clearly, such $\bar{x}$ belongs to w-Ls C_n, so

$$\limsup_{n\to\infty} s(x^* \mid C_n) =< \bar{x}, x^* >\leq s(x^* \mid \mathrm{clco}(w - \mathrm{Ls}\ C_n)).$$

Hence, the desired inclusion is obtained, by the arbitrariness of x^*. Finally, the second identity to be proven is an obvious consequence of (4.3) and our last inequality. □

Lemma 4.1. *Let (C_n) consist of w-compact and (D_n) of w-closed subsets of X. Assume that (C_n) and (D_n) are both nonincreasing. Then $\cap_n(C_n + D_n) = \cap_n C_n + \cap_n D_n$.*

Proof. Evidently, the right-hand side is contained in the left side. The proof of the converse inclusion is straightforward: let x be an element of the intersection on the left, i.e., for every n there exist $c_n \in C_n$ and $d_n \in D_n$ with $x = c_n + d_n$. Now by w-compactness of C_1 and the Eberlein-Šmulian theorem, a subsequence (c_{n_j}) of (c_n) will converge to some $\bar{c} \in X$. Of course, then (d_{n_j}) converges to $x - \bar{c}$. It follows immediately from the monotone inclusions that $\bar{c} \in \cap_n C_n$ and $x - \bar{c} \in \cap_n D_n$. □

Proof of Lemma 3.6. For every $p \in \mathbf{N}$

$$\operatorname{clco} \cup_{m \geq p} \left(\frac{1}{m} C_1 + \frac{1}{m} \sum_{n=2}^{m} C_n\right) \subset \operatorname{clco} \cup_{m \geq p} \frac{1}{m} C_1 + \operatorname{clco} \cup_{m \geq p} \frac{1}{m} \sum_{n=2}^{m} C_n.$$

The right-hand side is closed, because $\operatorname{clco} \cup_{m \geq p} \frac{1}{m} C_1$ is w-compact. Upon taking the intersection over all p one thus obtains

$$\cap_p \operatorname{clco} \cup_{m \geq p} \frac{1}{m} \sum_{n=1}^{m} C_n \subset \cap_p \left(\operatorname{clco} \cup_{m \geq p} \frac{1}{m} C_1 + \operatorname{clco} \cup_{m \geq p} \frac{1}{m} \sum_{n=2}^{m} C_n\right).$$

By Lemma 4.1 this gives

$$\cap_p \operatorname{clco} \cup_{m \geq p} \frac{1}{m} \sum_{n=1}^{m} C_n \subset \cap_p \operatorname{clco} \cup_{m \geq p} \frac{1}{m} C_1 + \cap_p \operatorname{clco} \cup_{m \geq p} \frac{1}{m} \sum_{n=2}^{m} C_n. \tag{4.4}$$

It is easy to check that $\operatorname{clco} \cup_{m \geq p} \frac{1}{m} C_1$ is actually identical to $\operatorname{co}(\{0\} \cup \frac{1}{p} C_1)$ (note that $\{0\} \cup \frac{1}{p} C_1$, being w-compact, has a convex hull that is also w-compact). Therefore, it follows immediately that $\cap_p \operatorname{clco} \cup_{m \geq p} \frac{1}{m} C_1$ is equal to $\{0\}$, in view of the boundedness of C_1. Substitution in (4.4) gives

$$\cap_p \operatorname{clco} \cup_{m \geq p} \frac{1}{m} \sum_{n=1}^{m} C_n \subset \cap_p \operatorname{clco} \cup_{m \geq p} \frac{1}{m} \sum_{n=2}^{m} C_n \subset \cap_p \operatorname{clco}(\cup_{m \geq p} \operatorname{clco} \cup_{n=2}^{m} C_n),$$

which leads to

$$\cap_p \operatorname{clco} \cup_{m \geq p} \frac{1}{m} \sum_{n=1}^{m} C_n \subset \cap_p \operatorname{clco} \cup_{n \geq 2} C_n \subset \operatorname{cl} \operatorname{co} \ \cup_{n \geq 2} C_n.$$

The proof is now easily completed by induction. □

5. Applications: lower closure results

This section gives immediate lower closure-type applications of Theorems 2.1 and 2.5.

Theorem 5.1. *Let (F_n) be a sequence in $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ satisfying the following hypotheses:*

(i) $\sup_n \int_\Omega \|F_n(\omega)\| \mu(d\omega) < +\infty$,

(ii) $\mathrm{clco} \cup_n F_n(\omega)$ *is w-ball-compact a.e.*
Also, let (v_n) *be a sequence of scalarly measurable functions* $v_n : \Omega \to X^*$ *such that*
(iii) $v_n(\omega) \xrightarrow{\tau} v_0(\omega)$ *a.e.,*
where τ *again denotes the Mackey topology on* X^*. *Then the subsequence* $(F_{n'})$ *of* (F_n) *and* $F_* \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ *of* Theorem 2.1 *satisfy, next to (a)–(b),*
c) $\liminf_{n'} \int_\Omega s(v_{n'}(\cdot) \mid F_{n'}(\cdot))d\mu \geq \int_\Omega s(v_0(\cdot) \mid F_*(\cdot))d\mu,$
provided that $\min(0, s(v_{n'}(\cdot) \mid F_{n'}(\cdot)))$ *is uniformly integrable.*

Theorem 5.2. *Suppose that both the Banach space* X *and its dual* X^* *(equipped with the dual norm) have the RNP. Let* (F_n) *be a sequence in* $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ *satisfying the following hypotheses:*
(i) $\sup_n \int_\Omega \|F_n(\omega)\|\mu(d\omega) < +\infty,$
(ii') $\mathrm{clco} \cup_n \int_A F_n \; d\mu$ *is w-compact for every* $A \in \mathcal{A}$.
Also, let (v_n) *be a sequence of strongly measurable functions* $v_n : \Omega \to X^*$ *such that*
(iii')$\|v_n(\omega) - v_0(\omega)\|^* \to 0$ *a.e.*
Then the subsequence $(F_{n'})$ *of* (F_n) *and* $F_* \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ *of Theorem 2.5 satisfy, next to (a)-(b′),*
(c) $\liminf_{n'} \int_\Omega s(v_{n'}(\cdot) \mid F_{n'}(\cdot))d\mu \geq \int_\Omega s(v_0(\cdot) \mid F_*(\cdot))d\mu,$
provided that $\min(0, s(v_{n'}(\cdot) \mid F_{n'}(\cdot)))$ *is uniformly integrable.*

Proof of Theorem 5.1. Observe that in the proof of Theorem 2.1 the sequence $(s(\cdot \mid G_m(\omega))$ was a.e. equicontinuous for the τ-topology thanks to the intervention of the compact sets $K(\omega)$. Now there exists a subsequence $(F_{n'_j})$ of $(F_{n'})$ such that the left hand side of (c) is equal to $\lim_{n'_j} \int_\Omega s(v_{n'_j}(\cdot) \mid F_{n'_j}(\cdot))d\mu$, and without loss of generality we may suppose that this limit is finite. By the equicontinuity observed above and the definition of K-convergence as in (a), we have

$$\lim_m s(v_m(\omega) \mid G_m(\omega)) = s(v_0(\omega) \mid F_*(\omega)) \text{ a.e.}$$

So by the Fatou-Vitali lemma and the uniform integrability hypothesis for the negative parts (which obviously transfers to the negative parts of the $s(v_m \mid G_m)$), we get

$$\lim_{n'_j} \int_\Omega s(v_{n'_j} \mid F_{n'_j})d\mu = \lim_m \int_\Omega s(v_m \mid G_m)d\mu \geq \int_\Omega s(v_0 \mid F_*),$$

which amounts to (c). □

The proof of Theorem 5.2 is completely similar and will be omitted (by pointwise uniform boundedness of the $G_m(\omega)$ and the strength of the dual norm topology we again have equicontinuity).

6. Comparisons with other work

In this section we compare the results of this paper with those in the literature. Most of the literature can only be compared with our Theorems 5.1 and 5.2, and throughout such comparisons it is instructive to keep in mind that those lower closure type theorems are of a less general nature than our principal Komlós-type results of section 2.

1. Theorem 2.1 subsumes the two Komlós-type Theorems A and B of [5]. Theorem 2.5 generalizes Theorem A of [5] (which has X separable and reflexive). It also implies the Komlós-type Théorème 3.2 of [15] (its condition (2), which amounts to our (ii), is not needed – see also the remarks on p. 3.9). Two similar extensions of Komlós' theorem were also given in section 3 of [7]. However, Theorem 3.2 of [7] is only correct under an additional separability condition for the dual Banach space.[5]

2. Théorème 2.8 of [16] follows from Theorem 5.2, if we combine it with the biting Lemma 3.4. The remark following Théorème 2.7 of [16] (but not that theorem itself) follows from Theorem 5.2. Théorème 2.9 in [16] follows from the singleton-valued version of Theorem 7.1 by setting $h(\omega, x) := \phi(\omega, x) + \|x\|$, with ϕ as in [16], and by invoking Lemma 3.4 again.

3. Theorem 5.1 generalizes Théorème 4.2 of [18]. A similar comment applies to Théorème 3.4 of [16]. Here it is important to note that the RNP-conditions in [18] cause the sequential and topological Kuratowski limes superior to coincide Corollary 3.9(i) of [33]. Incidentally, by the same comment we also conclude that Theorem 2.1 immediately implies an improvement of Théorème 3.5 of [15] Theorem 5.2 applies as well to the context of Théorème 4.2 of [18], but it only generalizes partially: the first part of Théorème 4.2 follows directly, but this time the second part seems a little stronger than our (b').

4. Theorem 5.2 improves upon Théorème 3.4 of [16], which takes, as in the case of Théorème 4.2 of [18] discussed above, the sets in (ii) to be locally compact, not containing any lines and which supposes the v_n to take their values in the unit ball of X^*. Here again the comment involving Corollary 3.9 of [33] should be kept in mind.

5. Théorème 3.5 of [16] presents us with an interesting open problem. In Remark 6.6 (a) of [9] it has been shown that the $\mathcal{R}$-tightness notion of [1], also used in [16] and related papers, is for all practical purposes – it is invariably accompanied by uniform L^1-boundedness – a tightness condition in the sense of [3]. Therefore, Théorème 3.5 of [16] follows from well-known lower semicontinuity results for Young measures. However, we cannot use the function h, constructed in Remark 6.6 (a) of [9], in Theorem 7.1 (applied to singleton-valued multifunctions), because it lacks convexity. We suspect that the solution to this problem lies in adapting ideas from [44].

6. Of course, several other, more traditional Fatou-type results can be found in the literature, for ordinary functions as well as multifunctions. Recently, the present authors gave comprehensive results of this type, including cases where the F_n have unbounded values [10]. In such Fatou type lemmas the prime interest lies in obtaining inclusions for

[5] The first author is indebted to S.I. Suslov (Novosibirsk), M. Talagrand (Paris), and S. Diaz (Sevilla), who independently brought this omission to his attention.

the integrals of the multifunctions F_n. Roughly, such inclusions are of the type

$$w - \mathrm{Ls}_{\ n} \int_\Omega F_n \ d\mu \subset \mathrm{cl} \int_\Omega F_0 \ d\mu,$$

where $F_0(\omega) := w\text{-Ls } F_n(\omega)$. While this gives a greater precision per point (because pointwise F_0 is a subset of the "upper bound" for the multifunction F_* – see Theorems 2.1 (b) and 2.5 (b')), precision about $\int_\Omega F_0 \ d\mu$ is reduced by the presence of the additional closure operator. Objects like our F_* sometimes also figure in the statements of such *approximate Fatou lemmas* (e.g., see Theorems 2.1 and 2.2 of [4]), and as a rule they can at least be found in the *proofs* of such results, although they do not involve K-convergence explicitly.

7. The above comments apply to approximate Fatou lemmas corresponding to situations similar to the one addressed in Theorem 2.1. Whether a true approximate Fatou lemma is also associated to Theorem 2.5 remains to be studied. As yet, we know of no such result, but Theorem 2.5 could certainly be considered as a first step in this direction. The novel method of proof employed here, which combines K-convergence methods with the Radon-Nikodym theorem for multivalued measures, could well turn out to be significant in this regard.

7. An Extension of Theorem 2.1 (a)

As we mentioned before, Theorem 2.1 (a) (and of course Corollary 2.2 (a)) can also be derived from the abstract Komlós-type theorem in [7]. We now show this by actually deriving the following extension of Theorem 2.1 from [7]. It can be seen as a multivalued version of Theorem 2.1, Remark 2.3 of [8], which itself implies Corollary 2.2. Its formulation uses outer integration over Ω (indicated by $\int_\Omega^*$), so as to avoid unnecessary measurability considerations.

Theorem 7.1. *Let (F_n) be a sequence in $\mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ satisfying the following hypothesis: there exists a function $h : \Omega \times \mathcal{P}_{wkc}(X) \to [0, +\infty]$ such that*

(i') $\sup_n \int_\Omega^* h(\omega, F_n(\omega))\mu(d\omega) < +\infty$,

(i") *$h(\omega, \cdot)$ is convex and inf-compact*[6] *on $\mathcal{P}_{wkc}(X)$ for almost every ω,*

(i"') $\sup_n \int_\Omega |s(x^* \mid F_n(\omega))|\mu(d\omega) < +\infty$ *for every $x^* \in X^*$.*

Then there exist a subsequence $(F_{n'})$ of (F_n) and $F_* \in \mathcal{L}^1_{\mathcal{P}_{wkc}(X)}$ satisfying

(a) $F_{n'} \xrightarrow{K} F_*$,

(b) $F_*(\omega) \subset \mathrm{clco}\, w - \mathrm{Ls}\ F_{n'}(\omega)$ *a.e.*

This result implies Theorem 2.1, because under the hypotheses (i) and (ii) the function h, given by

$$h(\omega, C) := \begin{cases} \|C\| & \text{if } C \subset \mathrm{clco} \cup_n F_n(\omega), \\ +\infty & \text{otherwise,} \end{cases}$$

meets the conditions of Theorem 7.1 by the definition of ball-compactness and by Lemma 3.2.

[6] I.e., for every $\beta \in \mathbf{R}$ the set $\{C \in \mathcal{P}_{wkc}(X) : h(\omega, C) \leq \beta\}$ is compact.

Proof of Theorem 7.1. Part (a) of this result follows directly from Theorem 2.1 of [7] by setting its collection (a_j) equal to the set of all $C \mapsto s(x^* \mid C)$, $x^* \in D$. Observe that this family satisfies the conditions of that result, since the $s(x^* \mid \cdot)$ are affine and continuous on $\mathcal{P}_{wkc}(X)$ and also separate its points. Part (b) now follows as in the proof of Theorem 2.1 (b). □

8. Addendum on sequential limits

Let X be a Hausdorff topological space (e.g., a Banach space with the weak topology). The *sequential closure* of a subset Y of X is defined by

$$\text{seq cl } Y := \{x \in X : x = \lim_n y_n, \ \ y_n \in Y\}.$$

Let (x_n) be a given sequence in X. In part 3 of the proof of Theorem 2.1 we need the following reformulation of the sequential closure in terms of a concept used in [1]:

$$\text{Ls } (x_n) = \cap_{k=1}^{\infty} \text{seq cl } Y_k, \tag{8.1}$$

where $Y_k := \{x_n : n \geq k\}$. We prove this identity as follows: For any $k \in \mathbf{N}$, we let (x_n^k) be the sequence given by $x_n^k := x_{n+k}$, $n \in \mathbf{N}$. Then it is not difficult to check that

$$\text{seq cl } Y_k = Y_k \cup \text{Ls }_n(x_n^k) = Y_k \cup \text{Ls } (x_n)$$

for every $k \in \mathbf{N}$. Consequently, we have

$$\cap_k \text{seq cl } Y_k = (\cap_k Y_k) \cup \text{Ls } (x_n).$$

The proof of (8.1) is finished by noting that $\cap_k Y_k$ is actually the *repetition set* of (x_n), defined to consist of all $x \in X$ for which $x = x_n$ for infinitely many indices n. This repetition set already belongs to the left side of (8.1).

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