Journal of Convex Analysis
Volume 2 (1995), No.1/2, 19–40

Partial Differential Inclusions Governing Feedback Controls

Jean-Pierre Aubin, Hélène Frankowska
CEREMADE, Université de Paris-Dauphine,
75775 Paris Cedex 16, France.

Received 10 June 1994
Revised manuscript received 7 December 1994

Dedicated to R. T. Rockafellar on his 60th Birthday

We derive partial differential inclusions of hyperbolic type solutions of which are feedbacks governing viable (controlled invariant) solutions of a control system. We also show that the *tracking property* leads to such partial differential inclusions. We state a variational principle and an existence theorem of a (single-valued contingent) solution to such an inclusion.

1. Introduction

Let X, Y, Z denote finite dimensional vector-spaces. We studied in [14] the existence of dynamical closed-loop controls regulating state-control solutions of a control system:

$$\begin{cases} x'(t) &= f(x(t), u(t)) \\ u(t) &\in U(x(t)) \end{cases} \tag{1.1}$$

where $U : X \rightsquigarrow Y$ is a closed set-valued map and $f : \mathrm{Graph}(U) \mapsto X$ a continuous (single-valued) map with linear growth (in the sense that $\|f(x,u)\| \leq c(\|x\| + \|u\| + 1)$). Consider a nonnegative continuous function $\varphi : \mathrm{Graph}(U) \mapsto \mathbb{R}_+$ with linear growth and set $K := \mathrm{Dom}(U)$. We address in this paper the problem of finding feedback (or closed-loop) controls $r : K \mapsto Y$ satisfying the constraint

$$\forall\, x \in K, \quad r(x) \in U(x)$$

and the regulation property: *for any $x_0 \in K$, there exists a solution to the differential equation*

$$x'(t) = f(x(t), r(x(t))) \ \ \& \ \ x(0) = x_0$$

such that $u(t) := r(x(t)) \in U(x(t))$ is absolutely continuous and fulfils the growth condition

$$\|u'(t)\| \leq \varphi(x(t), u(t))$$

ISSN 0944-6532 / $ 2.50

We observe that *the graphs of such feedback controls are viability domains (whose definition is recalled in the appendix) of the system of differential inclusions*

$$\begin{cases} x'(t) & = & f(x(t), u(t)) \\ u'(t) & \in & \varphi(x(t), u(t))B \end{cases} \tag{1.2}$$

contained in the graph of U. By the Viability Theorem such feedback controls are *solutions to the following contingent differential inclusion*

$$\forall\, x \in K, \quad 0 \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B$$

satisfying the constraints

$$\forall\, x \in K, \quad r(x) \ \in \ U(x)$$

where $Dr(x)$ is the contingent derivative[1] of r at x. We shall study this partial differential inclusion, provide a variational principle and an existence theorem.

We observe that the existence of a dynamical closed loop is a particular case of the *tracking problem,* which is studied under several names in many fields, and specially, arises in engineering (see for instance [18]). Indeed, consider two set-valued maps $F : X \times Y \leadsto X$, $G : X \times Y \leadsto Y$ and the *system of differential inclusions*

$$\begin{cases} x'(t) & \in & F(x(t), y(t)) \\ y'(t) & \in & G(x(t), y(t)) \end{cases}$$

We would like to characterize a set-valued map $H : X \leadsto Y$, regarded as an *observation map* satisfying the following *tracking property*: for every $x_0 \in \mathrm{Dom}(H)$ and every $y_0 \in H(x_0)$, there exists a solution $(x(\cdot), y(\cdot))$ to this system starting at (x_0, y_0) and satisfying

$$\forall\, t \geq 0, \quad y(t) \ \in \ H(x(t))$$

The answer to this question is again a solution to a *viability problem*, since we actually look for $(x(\cdot), y(\cdot))$ which remains viable in the graph of H. By the Viability Theorem the tracking property is equivalent to the fact that H is a solution to the *contingent differential inclusion*

$$\forall\, (x, y) \in \mathrm{Graph}(H), \quad 0 \ \in \ DH(x, y)(F(x, y)) - G(x, y)$$

When F and G are single-valued maps f and g and H is a differentiable single-valued map h, the contingent differential inclusion boils down to the *quasi-linear hyperbolic system of*

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[1] The contingent derivative $DH(x, y)$ of a set-valued map $H : X \leadsto Y$ at $(x, y) \in \mathrm{Graph}(H)$ is defined by

$$\mathrm{Graph}(DH(x, y)) \ := \ T_{\mathrm{Graph}(H)}(x, y)$$

where $T_{\mathrm{Graph}(H)}(x, y)$ denotes the contingent cone to $\mathrm{Graph}(H)$ at (x, y) (whose definition is recalled in the appendix). When $H = h$ is single-valued, we set $Dh(x) := Dh(x, h(x))$. See [10, Chapter 5] for more details on differential calculus of set-valued maps.

first-order partial differential equations[2]:

$$\forall\, j = 1,\ldots,m, \quad \sum_{i=1}^{n} \frac{\partial h_j}{\partial x_i} f_i(x, h(x)) - g_j(x, h(x)) \;=\; 0$$

Knowing F and G, we have to find observation maps H satisfying the tracking property, i.e., to solve the above contingent differential inclusion. Furthermore, we can address other questions such as : Find single-valued solutions h to the contingent differential inclusion which then becomes

$$\forall\, x \in K, \;\; 0 \in Dh(x)(F(x, h(x))) - G(x, h(x)) \tag{1.3}$$

In this case, the tracking property states that there exists a solution to the *"reduced" differential inclusion*

$$x'(t) \;\in\; F(x(t), h(x(t)))$$

so that $(x(\cdot), y(\cdot) := h(x(\cdot)))$ is a solution to the initial system of differential inclusions starting at $(x_0, h(x_0))$. Knowing h allows to divide the system by half, so to speak.
Let us mention right now that looking for "weak" solutions to this contingent differential inclusion in Sobolev spaces or other spaces of distributions does not help since we require *solutions h to be defined through their graph, and thus, solutions which are defined everywhere.*
It may seem strange to accept set-valued maps as solutions to an hyperbolic system of partial differential inclusions. But this may offer a way to describe shocks by the set-valued character of the solution. Derivatives in the sense of distributions do not provide the unique way to describe weak or generalized solutions. Contingent derivatives offer another possibility to weaken the required properties of a derivative, losing the linear character of the differential operator, but allowing a pointwise definition. It may be useful for tackling hyperbolic problems. This has been already noticed in [16, 17, 24, 25] for conservation laws.
Indeed, let us consider a single-valued map $f : X \mapsto Y$ and its differential quotients $\nabla_h f(x)(v) := \dfrac{f(x+hv) - f(x)}{h}$. If f is Gâteaux differentiable, then these differential quotients converge for the pointwise convergence topology (when $h \to 0$). This strong requirement can be weakened in (at least) two ways, each way sacrificing different groups of properties of the usual derivatives.

- The distributional derivative is the limit of the difference-quotients $x \mapsto \nabla_h f(x)(v) := \dfrac{f(x+hv) - f(x)}{h}$ (when $h \to 0$) **in the space of distributions**, and the limit is a **vectorial distribution** $D_v f \in \mathcal{D}'(X;Y)$ (i.e. no longer necessarily a single-valued function). Furthermore, one can define differential quotients of any **vectorial distribution** $T \in \mathcal{D}'(X;Y)$ and the derivative of a distribution as their limit (when $h \to 0$) **in the space of distributions**.

[2] For several special types of systems of differential equations, the graph of such a map h (satisfying some additional properties) is called a *center manifold*. Existence of local center manifolds have been widely used for the study of stability near an equilibrium and in control theory. See [7, 8, 20, 23] for instance.

- The contingent derivative is the *upper graphical limit* of the difference-quotients $v \mapsto \nabla_h f(x)(v) := \dfrac{f(x+hv)-f(x)}{h}$ (when $h \to 0+$), and the limit is a **set-valued map** $Df(x) : X \leadsto Y$ (and no longer necessarily a single-valued function). Furthermore, one can define differential quotients for any **set-valued map** $F : X \leadsto Y$ at $(x,y) \in \mathrm{Graph}(F)$ by $v \leadsto \frac{F(x+hv)-y}{h}$, $h > 0$ and the contingent derivative of a set-valued map as their *upper graphical limit* (when $h \to 0+$).

In both cases, the approaches are similar: they use (different) convergencies **weaker than the pointwise convergence** for increasing the possibility for the difference-quotients to converge, at the price of losing some properties by passing to these weaker limits (the pointwise character for distributional derivatives, the linearity of the differential operator for graphical derivatives).

The use of contingent derivatives (for instance of the value function for optimal control problems) is by no means new (see for instance [1], [6, Chapter 6], [28], [29]). It has been shown in [28], [29] that "contingent solutions" are related by duality to the "viscosity solutions" introduced in the context of Hamilton-Jacobi equations by Crandall & Lions in [22] (see also [21] and the literature following these papers). In the context of this paper (quasi-linear but set-valued hyperbolic differential inclusions), Proposition 4.4 makes explicit the duality relations between contingent solutions and solutions very closed in spirit to the *viscosity solutions* in the case when $Y = \mathbf{R}$. The variational principle we prove below (Theorem 4.1) states that for systems of partial differential equations or inclusions, the contingent solutions are adaptations to the vector-valued case of viscosity solutions.

Solutions to the contingent differential inclusion are defined in section 2. We then devote section 3 to the study of the transpose of contingent derivatives, and in particular, to a series of new convergence results, which play here as well as elsewhere a crucial role. For instance Proposition 3.5 below implies that a functional involving such transposes of contingent derivatives is lower semicontinuous for the ... pointwise convergence topology. The *variational principle* is the topic of section 4 and the existence of solutions the object of section 5. These results are applied to characterize and find feedback controls regulating solutions in section 6.

2. Contingent Differential Inclusion

Consider two finite dimensional vector-spaces X and Y, two set-valued maps $F : X \times Y \leadsto X$, $G : X \times Y \leadsto Y$ and a set-valued map $H : X \leadsto Y$. Throughout the whole paper we assume that F, G are upper semicontinuous, have nonempty convex compact images and the linear growth :

$$\exists\, c > 0,\ \forall\, x \in X,\ y \in Y,\ \ \|F(x,y)\| + \|G(x,y)\| \leq c(\|x\| + \|y\| + 1)$$

where $\|F(x,y)\| := \sup_{v \in F(x,y)} \|v\|$.

We associate with these data the *contingent differential inclusion*

$$\forall\, (x,y) \in \mathrm{Graph}(H),\ \ 0 \in DH(x,y)(F(x,y)) - G(x,y) \tag{2.1}$$

Definition 2.1. A set-valued map $H : X \leadsto Y$ satisfying (2.1) is a solution to the above *contingent differential inclusion* if its graph is a *closed* subset of $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$.

When $H = h : \mathrm{Dom}(h) \mapsto Y$ is a single-valued map with closed graph contained in $\mathrm{Dom}(F) \cap \mathrm{Dom}(G)$, the partial contingent differential inclusion (2.1) becomes

$$\forall\, x \in \mathrm{Dom}(h),\ \ 0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) \tag{2.2}$$

Actually solutions to (2.1) enjoy some stability. Recall that the graph of the *graphical upper limit* $H^\sharp$ of a sequence of set-valued maps $H_n : X \rightsquigarrow Y$ is, by definition, the graph of the upper limit of the graphs of the maps H_n. (See [10, Chapter 7].) Then results from [5] imply

Theorem 2.2. [Stability]
Let us consider a sequence of upper semicontinuous set-valued maps $F_n : X \times Y \rightsquigarrow X$, $G_n : X \times Y \rightsquigarrow Y$ with nonempty convex compact images and uniform linear growth in the sense that there exists a constant $c > 0$ such that

$$\sup_{n \geq 0} \left(\|F_n(x,y)\| + \|G_n(x,y)\|\right) \ \leq\ c(\|x\| + \|y\| + 1)$$

and their graphical upper limit $F^\sharp$ and $G^\sharp$. Consider also a sequence $H_n : X \rightsquigarrow Y$ of solutions to the contingent differential inclusions

$$\forall\, (x,y) \ \in\ \mathrm{Graph}(H_n),\ \ 0 \in DH_n(x,y)(F_n(x,y)) - G_n(x,y) \tag{2.3}$$

Then the graphical upper limit $H^\sharp$ of the solutions H_n is a solution to

$$\forall\, (x,y) \ \in\ \mathrm{Graph}(H^\sharp),\ \ 0 \in DH^\sharp(x,y)(\overline{co}F^\sharp(x,y)) - \overline{co}(G^\sharp(x,y)) \tag{2.4}$$

In particular, if the set-valued maps F_n and G_n converge graphically to F and G respectively, then the graphical upper limit $H^\sharp$ of the solutions H_n is a solution to (2.1).

Since the graphical convergence of single-valued maps is weaker than pointwise convergence, the graphical limits of single-valued maps which are converging pointwise may well be set-valued.
In the next section we provide some convergence results implying a dual characterization of solutions to partial differential inclusions.

3. Codifferentials

A set-valued map whose graph is a closed cone is called a *closed process*. It is a *closed convex process* if its graph is furthermore convex. They were introduced by Rockafellar in [32, 33]. Closed convex processes enjoy most of the properties of continuous linear operators, as it was shown in [32, 33] and [10, Chapter 2]. The transpose of a closed process $A : X \rightsquigarrow Y$ is the closed convex process $A^* : Y^* \rightsquigarrow X^*$ defined by Rockafellar in [32] by

$$p \ \in A^*(q) \ \text{ if and only if } \ \forall\, (x,y) \in \mathrm{Graph}(A),\ \ \langle p, x\rangle \ \leq\ \langle q, y\rangle$$

We define in a symmetric way the *bitranspose* $A^{**} : X \rightsquigarrow Y$ of A, the graph of which is the closed convex cone spanned by the graph of A:

$$\mathrm{Graph}(A^{**}) \ = \ (\mathrm{Graph}(A))^{-\,-}$$

Definition 3.1. Let $H : X \rightsquigarrow Y$ be a set-valued map and (x, y) belong to its graph. The transpose $DH(x, y)^\star : Y^\star \rightsquigarrow X^\star$ of the contingent derivative $DH(x, y)$ is called the *codifferential of* H *at* (x, y). When $H := h$ is single-valued, we set $Dh(x)^\star := Dh(x, h(x))^\star$.

Recall that whenever h is Lipschitz around x, $Dh(x)(u) \neq \emptyset$ for every $u \in X$ (see [10, Proposition 5.1.4]).

Lemma 3.2. *Let $K \subset X$ and $h : K \mapsto Z$ be a single-valued map Lipschitz around $x \in K$. Then $p \in Dh(x)^\star(q)$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\forall\, y \in B(x, \delta) \cap K, \;\; \langle p, y - x\rangle - \langle q, h(y) - h(x)\rangle \;\leq\; \varepsilon\|x - y\| \tag{3.1}$$

Proof. The sufficient condition being straightforward, let us prove the necessary one. Assume the contrary: there exists $\varepsilon > 0$ and a sequence of elements $x_n \in K$ converging to x such that

$$\langle p, x_n - x\rangle - \langle q, h(x_n) - h(x)\rangle \;>\; \varepsilon\|x - x_n\|$$

We set $\varepsilon_n := \|x_n - x\|$ which converges to 0 and $u_n := (x_n - x)/\varepsilon_n$, a subsequence of which converges to some u of the unit sphere. Since h is Lipschitz around x, there exists a cluster point $v \in Dh(x)(u)$ of the sequence

$$(h(x + \varepsilon_n u_n) - h(x))/\varepsilon_n$$

We thus derived the following contradiction:

$$\langle p, u\rangle - \langle q, v\rangle \;\leq\; 0 \;\;\&\;\; \langle p, u\rangle - \langle q, v\rangle \;\geq\; \varepsilon$$

□

The *contingent epiderivative* of an extended function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ at a point x of its domain is defined by

$$D_\uparrow V(x)(u) \;:=\; \liminf_{h \to 0+,\; u' \to u} \frac{V(x + hu') - V(x)}{h}$$

See [10, Chapter 6] for more details on this topic.

The following result characterizes the *transpose of the contingent derivative* of a set-valued map H in terms of the contingent epiderivatives of its support function:

Proposition 3.3. *Assume that $H : X \rightsquigarrow Y$ has compact convex values. We associate with any $q \in Y^\star$ the functions $H_q^\flat : X \mapsto \mathbb{R}$ and $H_q^\sharp : X \mapsto \mathbb{R}$ defined by*

$$\forall\, x \in X, \;\; H_q^\flat(x) \;:=\; \inf_{y \in H(x)} \langle q, y\rangle \;\;\&\;\; H_q^\sharp(x) \;:=\; \sup_{y \in H(x)} \langle q, y\rangle$$

Let $y_q^\flat,\, y_q^\sharp \in H(x)$ satisfy $\left\langle q, y_q^\flat\right\rangle = H_q^\flat(x)$ and $\left\langle q, y_q^\sharp\right\rangle = H_q^\sharp(x)$. Then

$$\left\{ p \in X^\star \mid \forall\, u \in X, \; \langle p, u\rangle \;\leq\; D_\uparrow H_q^\flat(x)(u) \right\} \;\subset\; DH(x, y_q^\flat)^\star(q)$$

If H is Lipschitz at x (in the sense that there exists $l > 0$ such that $H(x) \subset H(y) + l\|x - y\|B$ for every y in a neighborhood of x), then

$$DH(x, y_q^\sharp)^*(q) \subset \left\{ p \in X^* \mid \forall\, u \in X,\ \langle p, u\rangle \leq D_\uparrow H_q^\sharp(x)(u) \right\}$$

Consequently, when $H = h$ is single-valued and Lipschitz at x, setting $h_q^*(x) := \langle q, h(x)\rangle = h_q^\sharp(x) = h_q^\flat(x)$ we obtain the equality

$$Dh(x)^*(q) \;=\; \{ p \in X^* \mid \forall\, u \in X,\ \langle p, u\rangle \;\leq\; D_\uparrow h_q^*(x)(u) \}$$

Proof. Assume first that $p \in X^*$ satisfies

$$\forall\, u \in X,\ \langle p, u\rangle \;\leq\; D_\uparrow H_q^\flat(x)(u)$$

We prove that for every $v \in DH\left(x, y_q^\flat\right)(u)$,

$$D_\uparrow H_q^\flat(x)(u) \;\leq\; \langle q, v\rangle$$

Indeed, there exist sequences $t_n > 0$, $u_n \in X$ and $v_n \in Y$ converging to 0, u and v respectively such that $y_q^\flat + t_n v_n \in H(x + t_n u_n)$. Therefore,

$$D_\uparrow H_q^\flat(x)(u) \;\leq\; \liminf_{n\to\infty} \frac{H_q^\flat(x + t_n u_n) - H_q^\flat(x)}{t_n} \;\leq\; \liminf_{n\to\infty} \langle q, v_n\rangle$$

Consequently, $\langle p, u\rangle \leq \langle q, v\rangle$ for every $(u, v) \in \mathrm{Graph}(DH(x, y_q^\flat))$, so that $p \in DH(x, y_q^\flat)^*(q)$.

Conversely, assume that H is Lipschitz at x, $p \in DH\left(x, y_q^\sharp\right)^*(q)$ and fix $u \in X$. Then there exist sequences $t_n > 0$ and u_n converging to 0 and u such that

$$D_\uparrow H_q^\sharp(x)(u) \;=\; \lim_{n\to\infty} \frac{H_q^\sharp(x + t_n u_n) - H_q^\sharp(x)}{t_n}$$

Since H is Lipschitz at x, there exists $l > 0$ such that, for n large enough, $y_q^\sharp$ belongs to $H(x + t_n u_n) + l t_n \|u_n\| B$, so that it can be written $y_q^\sharp = y_n - t_n v_n$ where $y_n \in H(x + t_n u_n)$ and $\|v_n\| \leq l\|u_n\|$. Therefore a subsequence (again denoted by) v_n converges to some v, which belongs to $DH(x, y_q^\sharp)(u)$. Since $\langle q, y_n\rangle \leq H_q^\sharp(x + t_n u_n)$ and $\left\langle q, y_q^\sharp\right\rangle = H_q^\sharp(x)$, we infer that

$$D_\uparrow H_q^\sharp(x)(u) \;\geq\; \langle q, v\rangle \;\geq\; \langle p, u\rangle$$

□

When h is real-valued, we need only to know the values of $Dh(x)^*$ at the points 0, $+1$ and -1 to reconstruct the whole set-valued map $Dh(x)^*$.

We observe that for $q = +1$, $D_\uparrow h_q^*(x)(u) \;=\; D_\uparrow h(x)(u)$ and that for $q = -1$, $D_\uparrow h_q^*(x)(u)$ $= D_\uparrow(-h)(x)(u)$ and that for $q = 0$, $Dh(x)^*(0) = (\mathrm{Dom}(Dh(x)))^-$.
Recall (see for instance [10, Definition 6.4.7 and Proposition 6.4.8] or [21]) that:

$$\{\, p \in X^* \mid \forall\, u \in X,\ \langle p, u\rangle \;\leq\; D_\uparrow h(x)(u) \,\} \;=\; \partial_- h(x)$$

is the *local subdifferential* and

$$\{\, p \in X^* \mid \forall\, u \in X,\ \langle p, u\rangle \;\leq\; D_\uparrow(-h)(x)(u) \,\} \;=\; -\partial_+ h(x)$$

is the *local superdifferential* of h at x. The above characterization then becomes

Proposition 3.4. *Let $h : X \mapsto \mathbf{R}$ be a continuous at x function. Then*

$$Dh(x)^*(+1) \;=\; \partial_- h(x) \;\;\&\;\; Dh(x)^*(-1) \;=\; -\partial_+ h(x)$$

Proof. We already know that

$$\partial_- h(x) \;\subset\; Dh(x)^*(+1)$$

Assume now that $p \in Dh(x)^*(+1)$. We have to show that for all u, $\langle p, u\rangle \leq D_\uparrow h(x)(u)$. There is nothing to prove if $D_\uparrow h(x)(u) = +\infty$. If $D_\uparrow h(x)(u)$ is finite, then $v := D_\uparrow h(x)(u)$ belongs to $Dh(x)(u)$, so that $\langle p, u\rangle \leq D_\uparrow h(x)(u)$.
We finally claim that the continuity of h at x implies that $D_\uparrow h(x)(u) > -\infty$ for any $u \in X$, which is equivalent to $D_\uparrow h(x)(0) = 0$ thanks to [10, Propositions 6.1.3].
Indeed if not, by [10, Propositions 6.1.4 and Lemma 6.1.1], the pair $(0, -1)$ belongs to the contingent cone to the epigraph of h at $(x, h(x))$. Then there exist sequences $t_n > 0$ converging to 0, u_n converging to 0 and a sequence of $v_n > 0$ going to 1 such that $h(x + t_n u_n) \leq h(x) - t_n v_n$. On the other hand, h being continuous at x, the continuous function φ defined by $\varphi(t) := h(x + t u_n)$ satisfies for all large n

$$\varphi(t_n) \;\leq\; h(x) - t_n v_n \;\leq\; \varphi(0)$$

and therefore, there exist $s_n \in [0, t_n]$ such that $\varphi(s_n) = h(x) - t_n v_n$. Setting $\tilde{u}_n := \frac{s_n}{t_n} u_n$, which also converges to 0, we observe that $h(x + t_n \tilde{u}_n) = h(x) - t_n v_n$. This means that $-1 \in Dh(x)(0)$. But $p \in Dh(x)^*(1)$, and thus, we derived the contradiction:

$$0 \;=\; \langle p, 0\rangle \;\leq\; \langle 1, -1\rangle \;=\; -1$$

□

Remark. The above proposition allows to reformulate the notion of viscosity solution of a scalar Hamilton-Jacobi equation $\Psi(x, h'(x)) = 0$ in the following way: h is a viscosity solution if and only if

$$\left\{ \begin{array}{ll} i) & \forall\, p \in Dh(x)^*(+1),\;\; \Psi(x, p) \;\geq\; 0 \\ ii) & \forall\, p \in Dh(x)^*(-1),\;\; \Psi(x, -p) \;\leq\; 0 \end{array} \right. \tag{3.2}$$

□

The variational principle given in section 4 is based on the following convergence result:

Proposition 3.5. *Let $K \subset X$ be a closed subset. Assume that h is the pointwise limit of an equicontinuous family of maps $h_n : K \mapsto Y$. Let $x \in K$ and $p \in Dh(x)^*(q)$ be fixed. Then there exist subsequences of elements $x_{n_k} \in K$ converging to x, q_{n_k} converging to q and $p_{n_k} \in Dh_{n_k}(x_{n_k})^*(q_{n_k})$ converging to p.*
If the functions h_n are differentiable, we deduce that there exist subsequences of elements $x_{n_k} \in K$ converging to x and q_{n_k} converging to q such that $h'_{n_k}(x_{n_k})^*(q_{n_k})$ converges to p.

Proof. We have to prove that there exist subsequences $x_{n_k} \in K$ and

$$(p_{n_k}, -q_{n_k}) \in \left(T_{\mathrm{Graph}(h_{n_k})}(x_{n_k}, h_{n_k}(x_{n_k}))\right)^-$$

converging to x and $(p, -q)$ respectively. So our proposition follows from

Theorem 3.6. *Let us consider a sequence of closed subsets K_n and an element $x \in \mathrm{Liminf}_{n\to\infty} K_n$. Set $K^\sharp := \mathrm{Limsup}_{n\to\infty} K_n$.*
Then, for any $p \in (T_{K^\sharp}(x))^-$, there exist subsequences of elements $x_{n_k} \in K_{n_k}$ and $p_{n_k} \in \left(T_{K_{n_k}}(x_{n_k})\right)^-$ converging to p and x respectively, i.e.

$$(T_{K^\sharp}(x))^- \subset \mathrm{Limsup}_{n\to\infty,\ x_n \to_{K_n} x} (T_{K_n}(x_n))^-$$

Proof. It is sufficient to consider the case when $x \in \bigcap_{n=1}^\infty K_n$. If not, we set $\widehat{K_n} := K_n + x - u_n$ where $u_n \in K_n$ converges to x and observe that $x \in \bigcap_{n=1}^\infty \widehat{K}_n$ and $T_{\widehat{K}_n}(x_n) = T_{K_n}(x_n - x + u_n)$.
Let $p \in (T_{K^\sharp}(x))^-$ be given with norm 1. We associate with any positive λ the projection x_n^λ of $x + \lambda p$ onto K_n:

$$\left\|x + \lambda p - x_n^\lambda\right\| = \min_{x_n \in K_n} \|x + \lambda p - x_n\| \tag{3.3}$$

and set

$$v_n^\lambda := \frac{x_n^\lambda - x}{\lambda} \quad \& \quad p_n^\lambda := p - v_n^\lambda \in \left(T_{K_n}(x_n^\lambda)\right)^-$$

Let us fix for the time $\lambda > 0$. By taking $x_n = x \in K_n$ in (3.3), we infer that $\|v_n^\lambda\| \leq 2$. Therefore some subsequences $x_{n'}^\lambda$ and $v_{n'}^\lambda$ converge to elements $x^\lambda \in K^\sharp$ and $v^\lambda = \frac{x^\lambda - x}{\lambda}$ respectively. Furthermore, there exists a sequence $\lambda_i \to 0+$ such that v^{λ_i} converge to some $v \in T_{K^\sharp}(x)$ because $\|v^\lambda\| \leq 2$ and $x^\lambda = x + \lambda v^\lambda \in K^\sharp$. Therefore $\langle p, v\rangle \leq 0$.
On the other hand, we deduce from (3.3) the inequalities

$$\left\|p - v_n^\lambda\right\|^2 = \|p\|^2 + \|v_n^\lambda\|^2 - 2\langle p, v_n^\lambda\rangle \leq \|p\|^2$$

which imply, by passing to the limit, that $\|v\|^2 \leq 2\langle p, v\rangle \leq 0$. Thus $v = 0$. We deduce that a subsequence $v_{n_k}^{\lambda_k} = p - p_{n_k}^{\lambda_k}$ converges also to 0. The lemma ensues. □

We shall need stronger convergence results, where in the conclusion of Proposition 3.5 we require that q_n and/or x_n remain constant. We have to pay some price for that: stronger convergence assumptions and the use of graphical derivatives $D_\delta h(x)$ contained in the graph of $Dh(x)$ *which are closed convex processes.*

For instance, the *circatangent derivative* $Ch(x)$, defined in the following way from the Clarke tangent cone:

$$\mathrm{Graph}(Ch(x)) := C_{\mathrm{Graph}(h)}(x, h(x))$$

is a closed convex process contained in the contingent derivative $Dh(x)$. They coincide whenever h is sleek at x. We can also use the *asymptotic derivative* $D_\infty h(x)$, whose graph is the *asymptotic cone* to the graph of h at $(x, h(x))$. (See [10, Chapters 4,5] for further details.)

We prove for instance the following

Proposition 3.7. *Let $K \subset X$ be a closed subset. Assume that h is Lipschitz around x on K and consider a sequence of continuous maps h_n converging to h uniformly on compact subsets of K. Let $x \in K$ and $p \in Dh(x)^*(q)$ be fixed. Then there exist a sequence $x_n \in K$ converging to x and a sequence $p_n \in D_\delta h_n(x_n)^*(q)$ converging to p.*
If the functions h_n are differentiable, we infer that there exists a sequence $x_n \in K$ converging to x such that $h'_n(x_n)^*(q)$ converges to p.

Proof. Let $\mu > 0$, $L := K \cap B(x, \mu)$ be a compact neighborhood of x on which the maps h_n converge uniformly to h. We apply Ekeland's Theorem to the functions $y \mapsto \langle q, h_n(y)\rangle - \langle p, y\rangle$ defined on this subset. Fix $\varepsilon \in]0, \mu[$. Then there exists $x_n \in L$ satisfying

$$\begin{cases} \langle q, h_n(x_n)\rangle - \langle p, x_n\rangle + \varepsilon\|x_n - x\| \ \leq\ \langle q, h_n(x)\rangle - \langle p, x\rangle \\ \\ \quad \forall\, y \in L,\ \ \langle q, h_n(x_n)\rangle - \langle p, x_n\rangle \leq\ \langle q, h_n(y)\rangle - \langle p, y\rangle + \varepsilon\|y - x_n\| \end{cases}$$

The first inequality implies that

$$\varepsilon\|x - x_n\| \ \leq\ \langle q, h_n(x) - h(x)\rangle + \langle q, h(x_n) - h_n(x_n)\rangle + \langle p, x_n - x\rangle$$

$$-\langle q, h(x_n) - h(x)\rangle \leq\ 2\|q\|\|h_n - h\| + \langle p, x_n - x\rangle - \langle q, h(x_n) - h(x)\rangle$$

By Lemma 3.2, there exists $0 < \delta \leq \mu$ such that

$$\forall\, y \in B(x, \delta) \cap K,\ \ \langle p, x_n - x\rangle - \langle q, h(x_n) - h(x)\rangle \ \leq\ \varepsilon\|x_n - x\|/2$$

Hence, $\|x_n - x\| \leq 4\|q\|\|h_n - h\|/\varepsilon < \mu$ for n large enough.

On the other hand, consider any $v \in Dh_n(x_n)(u)$: There exist $\varepsilon_p > 0$ converging to 0, u_p converging to u and v_p converging to v such that $h_n(x_n + \varepsilon_p u_p) = h_n(x_n) + \varepsilon_p v_p$ for all p. Taking $y = x_n + \varepsilon_p u_p$ for p large enough in the second inequality, we infer that

$$0 \leq \langle q, v_p\rangle - \langle p, u_p\rangle + \varepsilon\|u_p\|$$

and thus, by passing to the limit,

$$\forall\, (u, v) \in \mathrm{Graph}(Dh_n(x)),\ \ 0 \leq \langle q, v\rangle - \langle p, u\rangle + \varepsilon\|u\|$$

In particular, taking the restriction to $\mathrm{Graph}(D_\delta h_n(x_n))$ and noticing that $\|u\| = \sup_{e\in B_*}\langle u, e\rangle$, this inequality can be written in the form:

$$0 \ \leq \inf_{(u,v)\in \mathrm{Graph}(D_\delta h_n(x))} \sup_{e\in B_*} (\langle q, v\rangle - \langle p, u\rangle + \varepsilon\langle e, u\rangle)$$

Since B_* is convex compact and since the graph of $D_\delta h_n(x)$ is convex, the lop-sided minimax theorem (see for instance [9]) implies the existence of $e_0 \in B_*$ such that

$$0 \ \leq \inf_{(u,v)\in \mathrm{Graph}(D_\delta h_n(x))} (\langle q, v\rangle - \langle p, u\rangle + \varepsilon\langle e_0, u\rangle)$$

Consequently, $(p - \varepsilon e_0, -q)$ belongs to the polar cone to $\mathrm{Graph}(D_\delta h_n(x_n))$, so that $p_n := p - \varepsilon e_0 \in D_\delta h_n(x_n)^*(q)$. Summarizing, for any $\varepsilon > 0$ and for any n such that $\|h_n - h\| \leq \varepsilon^2/4\|q\|$, we have proved the existence of $x_n \in K$ and $p_n \in D_\delta h_n(x_n)^*(q)$ such that $\|x_n - x\| \leq \varepsilon$, $\|p_n - p\| \leq \varepsilon$. □

Let $K \subset X$ be a closed subset and $\mathcal{C}_\Lambda(K, Z)$ denote the space of Lipschitz (single-valued) bounded maps from K to a finite dimensional vector-space Z,

$$\|h\|_\Lambda := \sup_{x\neq y\in K} \frac{\|h(x) - h(y)\|}{\|x - y\|} \ \ \& \ \ \|h\|_\infty := \sup_{x\in K} \|h(x)\|$$

denote the Lipschitz semi-norm and the sup-norm. Define the norm of the Banach space $\mathcal{C}_\Lambda(K, Z)$ by $\|h\| = \|h\|_\Lambda + \|h\|_\infty$.
We observe the following continuity properties of the contingent derivative:

Lemma 3.8. *Let $x \in K$ be fixed. Then the map*

$$(h, u) \in \mathcal{C}_\Lambda(K, Y) \times X \ \rightsquigarrow Dh(x)(u)$$

is Lipschitz:

$$\forall\, h,\ g\ \in \mathcal{C}_\Lambda(K, Y),\ \ Dh(x)(u) \subset Dg(x)(v) + \|h - g\|_\Lambda\|u\| + \|g\|_\Lambda\|u - v\|$$

Proof. The proof is straightforward from the inequality

$$\left\|\frac{h(x + tu) - h(x)}{t} - \frac{g(x + tv) - g(x)}{t}\right\| \ \leq \ \|h - g\|_\Lambda\|u\| + \|g\|_\Lambda\|u - v\|$$

□

We shall need the following stronger statement than Proposition 3.7 :

Proposition 3.9. *Let $K \subset X$ be a closed subset. Assume that h is Lipschitz and consider a sequence of Lipschitz maps h_n converging to h in $\mathcal{C}_\Lambda(K, Y)$. Let $x \in K$ and $p \in Dh(x)^*(q)$ be fixed. Then there exists a sequence $p_n \in D_\delta h_n(x)^*(q)$ converging to p. In particular, if the maps h_n are differentiable, $h_n'(x)^*q$ converges to p.*

Proof. Set $\varepsilon_n := 2\|q\|\|h_n - h\|_\Lambda$. By Lemma 3.2 , there exists $\mu > 0$ such that

$$\langle p, y - x\rangle - \langle q, h(y) - h(x)\rangle \leq \varepsilon_n\|y - x\|/2$$

whenever $y \in K \cap B(x,\mu)$. Therefore

$$\langle p, y-x\rangle - \langle q, h_n(y) - h_n(x)\rangle$$

$$\leq \varepsilon_n \|y-x\|/2 + \|q\| \|(h_n - h)(y) - (h_n - h)(x)\|$$

$$\leq (\varepsilon_n/2 + \|q\| \|h_n - h\|_\Lambda)\|y-x\| \ \leq \ \varepsilon_n \|y-x\|$$

On the other hand, consider any $v \in Dh_n(x)(u)$ and $t_p \to 0+$, $u_p \to u$ and $v_p \to v$ be such that $h_n(x+t_p u_p) = h_n(x) + t_p v_p$ for all p. Taking $y = x + t_p u_p \in K \cap B(x,\mu)$ for n large enough, we infer that

$$0 \ \leq \ \langle q, v_p\rangle - \langle p, u_p\rangle + \varepsilon_n \|u_p\|$$

and thus, by letting u_p and v_p converge to u and v,

$$\forall\, (u,v) \in \mathrm{Graph}(Dh_n(x)), \ \ 0 \ \leq \ \langle q, v\rangle - \langle p, u\rangle + \varepsilon_n \|u\|$$

In particular, taking the restriction to $\mathrm{Graph}(D_\delta h_n(x))$ which is convex,

$$0 \leq \inf_{(u,v)\in \mathrm{Graph}(D_\delta h_n(x))} \sup_{e \in B_*} (\langle q, v\rangle - \langle p, u\rangle + \varepsilon_n \langle e, u\rangle)$$

The lop-sided minimax theorem implies the existence of $e_n \in B_*$ such that $(p - \varepsilon_n e_n, -q)$ belongs to the negative polar cone to $\mathrm{Graph}(D_\delta h_n(x))$, i.e., $p_n := p - \varepsilon_n e_n \ \in \ D_\delta h_n(x)^*(q)$ □

4. The Variational Principle

We characterize in this section solutions to the contingent differential inclusion (2.2) through a *variational principle*. For that purpose, we denote by

$$\sigma(M,p) \ := \ \sup_{z\in M} \langle p, z\rangle \ \ \& \ \ \sigma^\flat(M,p) \ := \ \inf_{z\in M} \langle p, z\rangle$$

the support functions of $M \subset X$ and by B_* the unit ball of Y^*.

Consider a closed subset $K \subset X$. We introduce the nonnegative functional Φ defined on the space $\mathcal{C}(K,Y)$ of continuous maps by

$$\Phi(h) = \inf\{c \geq 0 \,|\, \forall\, x, q, \ \sup_{p \in Dh(x)^*(q)} (\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q)) \leq c\,\|q\|\}$$

Observe that c is finite if and only if

$$\sup_{p \in Dh(x)^*(0)} \sigma^\flat(F(x,h(x)),p) \leq 0$$

and

$$\Phi(h) = \sup_{q \in B_*} \sup_{x \in K} \sup_{p \in Dh(x)^*(q)} \Big(\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q)\Big)$$

Theorem 4.1. [Variational Principle] *Let $c \geq 0$. Then a single-valued map $h : K \mapsto Y$ is a solution to the contingent differential inclusion*

$$\forall\, x \in K, \;\; 0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) + cB$$

if and only if $\Phi(h) \leq c$.
Consequently, h is a solution to the contingent differential inclusion (2.2) if and only if $\Phi(h) = 0$.

Proof. Let $u \in F(x,h(x))$, $v \in G(x,h(x))$ and $e \in cB$ be such that $v - e \in Dh(x)(u)$. Then, for any $q \in Y^*$ and $p \in Dh(x)^*(q)$,

$$\langle p, u\rangle - \langle q, v - e\rangle \;\leq\; 0$$

so that

$$\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),q) \leq \langle p,u\rangle - \langle q,v\rangle \leq \langle q,e\rangle \leq c\,\|q\|$$

Taking the supremum with respect to $p \in Dh(x)^*(q)$, we infer that $\Phi(h) \leq c$.
Conversely, we can write inequality $\Phi(h) \leq c$ in the form of the minimax inequality: for any $x \in K,\; q \in Y^*$,

$$\sup_{p \in Dh(x)^*(q)} \; \inf_{u \in F(x,h(x))} \; \inf_{v \in G(x,h(x))} (\langle p,u\rangle - \langle q,v\rangle) \leq c\|q\|$$

Setting

$$\beta(p,q;u,v,e) \;:=\; \langle p,u\rangle - \langle q, v-e\rangle$$

this inequality can be written in the form: for every $x \in K$,

$$\sup_{(p,-q)\in \mathrm{Graph}(Dh(x))^-} \; \inf_{(u,v,e)\in F(x,h(x))\times G(x,h(x))\times cB} \beta(p,q;u,v,e) \leq 0$$

The lop-sided minimax theorem (see for instance [9]) implies the existence of $u_0 \in F(x,h(x))$, $v_0 \in G(x,h(x))$ and $e_0 \in cB$ such that

$$\sup_{(p,-q)\in \mathrm{Graph}(Dh(x))^-} (\langle p,u_0\rangle - \langle q, v_0 - e_0\rangle) \;=$$

$$\sup_{(p,-q)\in \mathrm{Graph}(Dh(x))^-} \; \inf_{(u,v,e)\in F(x,h(x))\times G(x,h(x))\times cB} \beta(p,q;u,v,e) \;\leq\; 0$$

This means that $(u_0, v_0 - e_0)$ belongs to $\overline{co}(\mathrm{Graph}(Dh(x)))$. In other words, we have proved that

$$(F(x,h(x)) \times (G(x,h(x)) + cB)) \cap \overline{co}\left(T_{\mathrm{Graph}(h)}(x,h(x))\right) \;\neq\; \emptyset$$

But by Proposition 7.1 of the Appendix, this is equivalent to the condition

$$(F(x,h(x)) \times (G(x,h(x)) + cB)) \cap T_{\mathrm{Graph}(h)}(x,h(x)) \;\neq\; \emptyset$$

i.e., h is a solution to our contingent differential inclusion. □

Remark. Since the bipolar cone of $\mathrm{Graph}(Dh(x))$ is the graph of the bitranspose $Dh(x)^{**}$, we have actually proved that h is a solution to the contingent differential inclusion if and only if it is a solution to the "relaxed" contingent differential inclusion

$$0 \in Dh(x)^{**}(F(x,h(x))) - G(x,h(x)) + cB$$

□

Theorem 4.2. *Let $\mathcal{H} \subset \mathcal{C}(K,Y)$ be a compact subset for the compact convergence topology. Assume that $c := \inf_{h\in\mathcal{H}} \Phi(h) < +\infty$. Then there exists a solution $h \in \mathcal{H}$ to the contingent differential inclusion*

$$0 \in Dh(x)(F(x,h(x))) - G(x,h(x)) + cB$$

Thanks to Theorem 4.1 it is sufficient to prove that the functional Φ is lower semicontinuous on the space $\mathcal{C}(K,Y)$ for the compact convergence topology.

Proposition 4.3. *The functional Φ is lower semicontinuous on equicontinuous subsets of the space $\mathcal{C}(K,Y)$ for the compact convergence topology.*

Proof. We may assume that Φ is proper. Let h_n be a sequence of equicontinuous maps satisfying for any n, $\Phi(h_n) \leq c$ and converging uniformly to some h. We have to show that $\Phi(h) \leq c$. Indeed, fix $x \in K$, $q \in B_*$ and $p \in Dh(x)^*(q)$. By Proposition 3.5, there exist subsequences (again denoted by) $x_n \in K$ converging to x, q_n converging to q and $p_n \in Dh_n(x_n)^*(q_n)$ converging to p such that $h_n(x_n)$ converges to $h(x)$.
Since F and G are upper semicontinuous with compact values, we know that for any (p,q) and $\varepsilon > 0$, we have

$$\begin{cases} \sigma^{\flat}(F(x,h(x)),p) - \sigma(G(x,h(x)),q) \leq \\ \sigma^{\flat}(F(x_n,h_n(x_n)),p_n) - \sigma(G(x_n,h_n(x_n)),q_n) + \varepsilon \leq (\Phi(h_n)+\varepsilon)\,\|q_n\| + \varepsilon \end{cases}$$

for n large enough. Hence, by letting n go to ∞, we infer that for any $\varepsilon > 0$,

$$\sigma^{\flat}(F(x,h(x)),p) - \sigma(G(x,h(x)),q) \leq (c+\varepsilon)\,\|q\| + \varepsilon$$

Letting ε converge to 0 and taking the supremum on $p \in Dh(x)^*(q)$, we infer that $\Phi(h) \leq c$. □

In the case when $Y = \mathbf{R}$, the contingent solutions are very closed in spirit to the *viscosity solutions*:

Proposition 4.4. *Assume that $Y = \mathbf{R}$. Then a continuous function $h : X \mapsto R$ is a solution to (2.2) on a closed set $K \subset X$ if and only if for every $x \in K$,*

$$\begin{cases} \text{i)} \ \sup_{p\in\partial_- h(x)} \left(\sigma^{\flat}(F(x,h(x)),p) - \sup(G(x,h(x)))\right) \ \leq \ 0 \\[2ex] \text{ii)} \ \inf_{p\in\partial_+ h(x)} (\sigma(F(x,h(x)),p) - \inf(G(x,h(x)))) \ \geq \ 0 \end{cases}$$

Proof. Set

$$\begin{cases} \Phi_+(h,x) = \sup_{p\in Dh(x)^\star(+1)} \Big(\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),+1)\Big) \\ \\ \Phi_-(h,x) = \sup_{p\in Dh(x)^\star(-1)} \Big(\sigma^\flat(F(x,h(x)),p) - \sigma(G(x,h(x)),-1)\Big) \\ \\ \Phi_0(h,x) = \sup_{p\in Dh(x)^\star(0)} \Big(\sigma^\flat(F(x,h(x)),p)\Big) \end{cases}$$

If h is a solution, then properties $i)$, $ii)$ follow from Proposition 3.4 and Theorem 4.1 with $c=0$.

Conversely if $i)$, $ii)$ hold true, then $\sup_{x\in K}\max\left(\Phi_+(h,x),\Phi_-(h,x)\right)\le 0$. It remains to show that $\Phi_0(h,x)\le 0$. Fix any $p\in Dh(x)^\star(0)$. Then $(p,0)\in\left(T_{\mathcal{E}p(h)}(x,h(x))\right)^-$. By Rockafellar's result from [35] there exist $x_n\to x$, $(p_n,q_n)\to(p,0)$ such that $q_n\neq 0$, $\frac{p_n}{|q_n|}\in\partial_-h(x_n)$. Hence, by $i)$,

$$\sigma^\flat(F(x_n,h(x_n)),p_n) - \|q_n\|\,\sigma(G(x_n,h(x_n)),+1)\le 0$$

Taking the limit we get $\sigma^\flat(F(x,h(x)),p)\le 0$. □

We can relate solutions to the contingent differential inclusion (2.2) to viscosity solutions when the set-valued map $F: X \rightsquigarrow X$ does not depend on y and when G is equal to 0. The above proposition implies that both h and $-h$ are viscosity subsolutions to the Hamilton-Jacobi equation

$$-\sigma(F(x),h'(x)) \;=\; 0$$

The apparent discrepancy comes from the fact that solutions h of the contingent partial differential inclusion are energy functions and not the value function of an optimal control problem.

5. Single-Valued Solutions to Partial Differential Inclusions

We shall look for solutions in a compact convex subset $\mathcal{H}$ of the space $\mathcal{C}_\Lambda(K,Y)$ of Lipschitz maps from K to Y.

Theorem 5.1. *Let K be a closed subset of X. Consider a compact convex subset $\mathcal{H}$ of $\mathcal{C}_\Lambda(K,Y)$. When $h\in\mathcal{H}$, we denote by $T_{\mathcal{H}}(h(\cdot))\subset\mathcal{C}(K,Y)$ the tangent cone to $\mathcal{H}$ at h for the pointwise convergence topology. Let $D_\delta h(x)$ be a family of closed convex processes satisfying $D_\delta h(x)\subset Dh(x)$ and assume that for every $h\in\mathcal{H}$, there exist v, $w\in\mathcal{C}(K,Y)$ such that $\forall\, x\in K$,*

$$w(x)\in D_\delta h(x)(F(x,h(x))),\ v(x)\in G(x,h(x))\ \&\ w(\cdot)-v(\cdot)\in T_{\mathcal{H}}(h(\cdot))$$

Then there exists a solution $h\in\mathcal{H}$ to the contingent differential inclusion

$$\forall\, x\in K,\ \ 0\in Dh(x)(F(x,h(x))) - G(x,h(x))$$

Proof. We assume that there is no solution to the contingent differential inclusion and we shall derive a contradiction.
Indeed, thanks to Proposition 7.1, this amounts to assume that for any $h \in \mathcal{H}$, there exists $x \in K$ such that

$$0 \notin \overline{co}\left(T_{\mathrm{Graph}(h)}(x, h(x))\right) - F(x, h(x)) \times G(x, h(x))$$

Since the images of F and G are compact and convex, the separation theorem implies that there exists also $(p, -q) \in X^\star \times Y^\star$ such that

$$0 < \sigma^\flat(F(x, h(x)), p) - \sigma(G(x, h(x)), q) \ \& \ p \in Dh(x)^\star(q)$$

Set

$$a(h; x, q) := \sup_{p \in D_\delta h(x)^\star(q)} \sigma^\flat(F(x, h(x)), p) - \sigma(G(x, h(x)), q)$$

Since $Dh(x)^\star(q) \subset D_\delta h(x)^\star(q)$, we observe that

$$0 < a(h; x, q)$$

On the other hand, the function $(y, p) \mapsto \sigma^\flat(F(x, y), p) - \sigma(G(x, y), q)$ being lower semicontinuous (because F and G are upper semicontinuous with compact values), there exist neighborhoods $N_1(h(x))$ and $N_2(p)$ such that

$$\forall\, y \in N_1(h(x)),\ \forall\, p' \in N_2(p),\ \ 0 < \sigma^\flat(F(x, y), p') - \sigma(G(x, y), q)$$

By Proposition 3.9, there exists $\eta(x) > 0$ such that whenever $\|l - h\|_\Lambda \le \eta(x)$, there exists $p' \in D_\delta l(x)^\star(q)$ satisfying $l(x) \in N_1(h(x))$ and $p' \in N_2(p)$. Hence

$$0 < \sigma^\flat(F(x, l(x)), p') - \sigma(G(x, l(x)), q) \le a(l; x, q)$$

Consequently, h belongs to the subset $N(x, q)$ defined by

$$N(x, q) := \{l \in \mathcal{C}_\Lambda(K, Y) \mid 0 < a(l; x, q)\}$$

which is *open in* $\mathcal{C}_\Lambda(K, Y)$ by Proposition 3.9.
Summing up, we just have proved that if there is no solution to the contingent differential inclusion, then $\mathcal{H}$ can be covered by the open subsets $N(x, q)$. Being compact, it can be covered by a finite number m of such neighborhoods $N(x_i, q_i)$. Let $\alpha_i(\cdot)$ be a continuous partition of unity associated with this covering.
We introduce now the function $\varphi : \mathcal{H} \times \mathcal{H} \mapsto \mathbf{R}$ defined by

$$\varphi(h, l) := \sum_{i=1}^{m} \alpha_i(h) \langle q_i, l(x_i) - h(x_i) \rangle$$

It is continuous with respect to h on $\mathcal{C}_\Lambda(K, Y)$ (because the α_i's are so and $h \mapsto \langle q_i, h(x_i) \rangle$ are continuous for the pointwise topology), affine with respect to l and satisfies $\varphi(l, l) = 0$. Hence, $\mathcal{H}$ being convex and compact, Ky Fan Inequality (see [10, Theorem 3.1.1]) implies the existence of $\bar{h} \in \mathcal{H}$ such that for every $l \in \mathcal{H}$, $\varphi(\bar{h}, l) \leq 0$. This means that the discrete measure $\sum_{i=1}^m \alpha_i(\bar{h}) q_i \otimes \delta(x_i)$ belongs to the normal cone to $\mathcal{H}$ at $\bar{h}$ since for every $l \in \mathcal{H}$,

$$\langle \sum_{i=1}^m \alpha_i(\bar{h}) q_i \otimes \delta(x_i), l - \bar{h} \rangle \ = \ \sum_{i=1}^m \alpha_i(\bar{h}) \langle q_i, l(x_i) - \bar{h}(x_i) \rangle \ \leq \ 0$$

We then deduce from the assumption of theorem that $\sum_{i=1}^m \alpha_i(\bar{h}) a(\bar{h}; x_i, q_i) \leq 0$. Indeed, there exist continuous functions $v(\cdot)$ and $w(\cdot)$ such that

$$\forall\, x \in K, \ \ v(x) \in G(x, \bar{h}(x)) \ \ \& \ \ w(x) \in D_\delta \bar{h}(x)(F(x, \bar{h}(x)))$$

and $w(\cdot) - v(\cdot) \in T_{\mathcal{H}}(\bar{h}(\cdot))$. Therefore, for any $p_i \in D_\delta \bar{h}(x_i)^\star(q_i)$, there exists $u_i \in F(x_i, \bar{h}(x_i))$ such that

$$\begin{cases} \quad \sigma^\flat(F(x_i, \bar{h}(x_i)), p_i) - \sigma(G(x_i, \bar{h}(x_i)), q_i) \\ \\ \leq \ \langle p_i, u_i \rangle - \langle q_i, v(x_i) \rangle \ \leq \ \langle q_i, w(x_i) - v(x_i) \rangle \end{cases}$$

So that, by taking the supremum on $p_i \in D_\delta \bar{h}(x_i)^\star(q_i)$, we obtain

$$a(\bar{h}; x_i, q_i) \ \leq \ \langle q_i, w(x_i) - v(x_i) \rangle$$

Multiplying by $\alpha_i(\bar{h}) \geq 0$ and summing from $i = 1$ to m, we obtain

$$\sum_{i=1}^m \alpha_i(\bar{h}) a(\bar{h}; x_i, q_i) \ \leq \ \left\langle \sum_{i=1}^m \alpha_i(\bar{h}) q_i \otimes \delta(x_i), w(\cdot) - v(\cdot) \right\rangle \ \leq \ 0$$

because $w(\cdot) - v(\cdot) \in T_{\mathcal{H}}(\bar{h}(\cdot))$ and $\sum_{i=1}^m \alpha_i(\bar{h}) q_i \otimes \delta(x_i)$ belongs to the normal cone to $\mathcal{H}$ at $\bar{h}$. We claim that in the same time $\sum_{i=1}^m \alpha_i(\bar{h}) a(\bar{h}; x_i, q_i) > 0$. Indeed, whenever $\alpha_i(\bar{h}) > 0$, then $\bar{h}$ belongs to $N(x_i, q_i)$, which implies that $0 < a(\bar{h}, x_i, q_i)$. We have therefore obtained a contradiction. □

Lemma 5.2. *Let $H : K \rightsquigarrow Y$ be a set-valued map and let $\mathcal{H}$ be a subset of continuous selections of H. Then*

$$T_{\mathcal{H}}(h(\cdot)) \ \subset \ \{v \in \mathcal{C}(K, Y) \ | \ \forall\, x \in K, \ \ v(x) \in T_{H(x)}(h(x))\}$$

If in addition for any finite sequence $(x_i, y_i) \in \mathrm{Graph}(H)$ $(i = 1, \ldots, m)$ such that $x_i \neq x_j$ when $i \neq j$, there exists a selection $s \in \mathcal{H}$ interpolating it:

$$\forall\, i = 1, \ldots, m, \ \ s(x_i) = y_i$$

then equality holds true:

$$T_{\mathcal{H}}(h(\cdot)) \ = \ \{v \in \mathcal{C}(K, Y) \ | \ \forall\, x \in K, \ \ v(x) \in T_{H(x)}(h(x))\}$$

Proof. Indeed, let $v \in \mathcal{C}(K, Y)$ such that $v(x) \in T_{H(x)}(h(x))$ for all $x \in K$. Then there exists $\varepsilon_\lambda(\cdot)$ converging to 0 with λ for the pointwise convergence topology such that

$h(x) + \lambda v(x) + \lambda \varepsilon_\lambda(x) \in H(x)$. Let us consider any neighborhood of 0 for the pointwise topology

$$\mathcal{V} := \{l \in \mathcal{C}(K,Y) \mid \sup_{i=1,\ldots,n} \|l(x_i)\| \leq \varepsilon\}$$

associated with a finite subset $\{x_1, \ldots, x_n\}$ and λ small enough for $\varepsilon_\lambda(\cdot)$ to belong to it. By the interpolation assumption there exists $l_\lambda \in \mathcal{H}$ such that

$$\forall\, x_i, \;\; l_\lambda(x_i) = h(x_i) + \lambda v(x_i) + \lambda \varepsilon_\lambda(x_i) \in H(x_i)$$

Thus the continuous function $u_\lambda := (l_\lambda - h)/\lambda$ is such that $h + \lambda u_\lambda \in \mathcal{H}$ and belongs to the neighborhood $v + \mathcal{V}$ of v for the topology of the pointwise convergence. In other words, v belongs to the tangent cone to $\mathcal{H}$ at h for the pointwise topology. □

6. Feedback Controls Regulating Smooth Evolutions

Consider a control system (U, f):

$$\begin{cases} x'(t) &= f(x(t), u(t)) \\ u(t) &\in U(x(t)) \end{cases} \tag{6.1}$$

Let $(x, u) \to \varphi(x, u)$ be a non negative continuous function with linear growth.
We have proved in [14] that there exists a closed regulation map $R^\varphi \subset U$ larger than any closed regulation map $R : K \leadsto Z$ contained in U and enjoying the following viability property: *For any initial state $x_0 \in \mathrm{Dom}(R)$ and any initial control $u_0 \in R(x_0)$, there exists a solution $(x(\cdot), u(\cdot))$ to the control system (6.1) starting at (x_0, u_0) such that*

$$\forall\, t \geq 0, \;\; u(t) \;\in\; R(x(t))$$

and

$$\textit{for almost all } t \geq 0, \; \|u'(t)\| \;\leq\; \varphi(x(t), u(t))$$

Let $K \subset \mathrm{Dom}(U)$ be a closed subset. We recall that *a closed set-valued map $R : K \leadsto Z$ is a feedback control regulating viable solutions to the control problem satisfying the above growth condition if and only if R is a solution to the contingent differential inclusion*

$$\forall\, x \in K, \;\; 0 \in DR(x,u)(f(x,u)) - \varphi(x,u)B$$

satisfying the constraint

$$\forall\, x \in K, \;\; R(x) \;\subset\; U(x)$$

In particular, a *closed graph single-valued regulation map $r : K \mapsto Z$* is a solution to the *contingent differential inclusion*

$$\forall\, x \in K, \;\; 0 \in Dr(x)(f(x, r(x))) - \varphi(x, r(x))B \tag{6.2}$$

satisfying the constraint

$$\forall\, x \in K, \;\; r(x) \in U(x)$$

Such solution can be obtained by a variational principle. We introduce the functional Φ defined by

$$\Phi(r) := \inf\{c \geq 0 \,|\, \forall\, x,\, q, \sup_{p \in Dr(x)^\star(q)} (\langle p, f(x, r(x))\rangle \leq (\varphi(x, r(x)) + c)\|q\|)\}$$

Theorem 6.1. *Let $\mathcal{R} \subset \mathcal{C}(K,Y)$ be a nonempty compact subset of selections of the set-valued map U (for the compact convergence topology). Suppose that the functions f and φ are continuous and that $c := \inf_{r\in\mathcal{R}} \Phi(r) < +\infty$. Then there exists a solution $r(\cdot) \in \mathcal{R}$ to the contingent differential inclusion*

$$\forall\, x \in K,\ \ 0 \in Dr(x)(f(x,r(x))) - (\varphi(x,r(x)) + c)B$$

As for the existence of such a feedback, we deduce from Theorem 5.1 the following consequence:

Theorem 6.2. *Consider a nonempty convex subset $\mathcal{R} \subset \mathcal{C}_\Lambda(K,Z)$ of selections of the set-valued map U which is compact in $\mathcal{C}_\Lambda(K,Z)$. Suppose that the functions f and φ are continuous and fix any family of closed convex processes $D_\delta r(x) \subset Dr(x)$. If for every $r \in \mathcal{R}$, there exist v, $w \in \mathcal{C}(K,Y)$ such that $\forall\, x \in K$,*

$$w(x) \in D_\delta r(x)(f(x,r(x))),\ v(x) \in \varphi(x,r(x))B \ \&\ w(\cdot) - v(\cdot) \in T_{\mathcal{R}}(r(\cdot))$$

then there exists a solution $r \in \mathcal{R}$ to the contingent differential inclusion (6.2).

7. Appendix: Dual Characterization of the Viability Domain

Let $F : X \rightsquigarrow X$ be a set-valued map and $K \subset \mathrm{Dom}(F)$ be a nonempty subset. We denote by

$$T_K(x) \ := \ \left\{ v \in X \mid \liminf_{h\to 0+} \frac{d(x+hv;K)}{h} \ = \ 0 \right\}$$

the contingent cone to K at $x \in K$. The set K is called a *viability domain* of F if

$$\forall\, x \in K,\ F(x) \cap T_K(x) \ \neq \ \emptyset$$

For every $x \in K$ denote by

$$N_K^P(x) \ := \ \{y - x \mid \ \ \mathrm{dist}(y,K) = \|y - x\|\}$$

The following result provides a very useful *duality* characterization of viability domains:

Proposition 7.1. *Assume that the set-valued map $F : K \rightsquigarrow X$ is upper semicontinuous with convex compact values. Then the four following properties are equivalent:*

$$\begin{cases} \text{i) } \forall\, x \in K,\ \ F(x) \cap T_K(x) \ \neq \ \emptyset \\ \\ \text{ii) } \forall\, x \in K,\ \ F(x) \cap \overline{co}\,(T_K(x)) \ \neq \ \emptyset \\ \\ \text{iii) } \forall\, x \in K,\ \ \forall\, p \in (T_K(x))^-,\ \ \sigma(F(x), -p) \ \geq \ 0 \\ \\ \text{iv) } \forall\, x \in K,\ \forall\, p \in N_K^P(x),\ \ \sigma(F(x), -p) \geq 0 \end{cases}$$

Remark. The equivalence between i) and ii) was first noticed in a different context in [26]. A more simple proof of this fact was given in [13]. The equivalence of i) and iv) was proved in [36] and [19]. However the proof from [13] also implied that i) $\Longleftrightarrow$ iv). We provide it below for the reader convenience. □

Proof. Clearly *i)* yields *ii)*. The equivalence between *ii)* and *iii)* follows obviously from the Separation Theorem. Since $y - x \in N_K^P(x)$ implies that $y - x \in (T_K(x))^-$ we have iii) $\Longrightarrow$ iv).
Assume that iv) holds true and fix $x \in K$. Let $u \in F(x)$ and $v \in T_K(x)$ achieve the distance between $F(x)$ and $T_K(x)$:

$$\|u - v\| \; = \inf_{y \in F(x),\, z \in T_K(x)} \|y - z\|$$

and set $w := \frac{u+v}{2}$. We have to prove that $u = v$. Assume the contrary.
Since v is contingent to K at x, there exist sequences $h_n > 0$ converging to 0 and v_n converging to v such that $x + h_n v_n$ belongs to K for every $n \geq 0$. We also introduce a projection of best approximation

$$x_n \; \in \; \Pi_K(x + h_n w) \;\text{ of } x + h_n w \text{ onto } K \text{ and we set } \; z_n \; := \; \frac{x_n - x}{h_n}$$

so that $h_n(w - z_n) \in N_K^P(x_n)$. By assumption *iv)*,

$$\exists\, y_n \in F(x_n), \;\text{ such that }\; \langle w - z_n, y_n \rangle \; \leq \; 0 \tag{7.1}$$

Since x_n converges to x, the upper semicontinuity of F implies that for any $\varepsilon > 0$, there exists N_ε such that for $n \geq N_\varepsilon$, y_n belongs to the neighborhood $F(x) + \varepsilon B$, which is compact. Thus a subsequence (again denoted by) y_n converges to some element $y \in F(x)$.
We shall now prove that z_n converges to v. Indeed, inequality

$$\|w - z_n\| \; = \; \frac{1}{h_n}\|x + h_n w - x_n\| \; \leq \; \frac{1}{h_n}\|x + h_n w - x - h_n v_n\| \; = \; \|w - v_n\|$$

implies that the sequence z_n has a cluster point and that every cluster point z of the sequence z_n belongs to $T_K(x)$. Furthermore, every such z satisfies $\|w - z\| \leq \|w - v\|$.
We now observe that v *is the unique best approximation of* w *by elements of* $T_K(x)$. If not, there would exist $p \in T_K(x)$ satisfying either $\|w - p\| < \|w - v\|$ or $p \neq v$ and $\|w-p\| = \|w-v\| = \|w-u\|$. In the latter case, we have $\langle u-w, w-p\rangle < \|u-w\|\|w-p\|$, since the equality holds true only for $p = v$. Each of these conditions together with the estimates

$$\|u - p\|^2 = \|u - w\|^2 + \|w - p\|^2 + 2\langle u - w, w - p\rangle \leq \|u - v\|^2$$

imply the strict inequality $\|u-p\| < \|u-v\|$, which is impossible since v is the projection of u onto $T_K(x)$. Hence $z = v$.
Consequently, all the cluster points being equal to v, we conclude that z_n converges to v. Therefore, we can pass to the limit in inequalities (7.1) and obtain, observing that $w - v = (u - v)/2$,

$$\langle u - v, y \rangle \; = \; 2\langle w - v, y \rangle \; \leq \; 0 \;\text{ where }\; y \in F(x) \tag{7.2}$$

Since $F(x)$ is closed and convex and since $u \in F(x)$ is the projection of v onto $F(x)$, we infer that

$$\langle u - v, u - y \rangle \leq 0 \tag{7.3}$$

Finally, $T_K(x)$ being a cone and $v \in T_K(x)$ being the projection of u onto this cone, and in particular, onto the half-line $v\mathbf{R}_+$, we deduce that

$$\langle u - v, v \rangle = 0 \tag{7.4}$$

Therefore, properties (7.2, 7.3, 7.4) imply that

$$\|u - v\|^2 = \langle u - v, -v \rangle + \langle u - v, u - y \rangle + \langle u - v, y \rangle \leq 0$$

and thus, that $u = v$. □

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