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# Extraction of a "Good" Subsequence From a Bounded Sequence of Integrable Functions

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**Dedicated to R. T. Rockafellar on his 60th Birthday**

For a uniformly integrable sequence, the Young measures allow to precise the Dunford-Pettis theorem: there exists a subsequence and two complementary subsets above which one has strong convergence and "pure" weak $L^1$-convergence. For a bounded sequence in $L^1$, the "biting lemma" permits the extraction of a subsequence presenting, besides the foregoing behaviors, concentration of mass. This structural result allows us to prove very quickly some known results.

*Keywords* : Bounded sequences in $L^1$, biting lemma, Young measures, uniform integrability

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## 1. Introduction

We consider functions belonging to $L^1(\Omega, \mu; \mathbb{R}^d)$, where $(\Omega, \mathcal{F}, \mu)$ is a measured space with $\mu$ a positive bounded measure.
Bounded sequences in $L^1$ may have weak limit points in the bidual $(L^1)''$ — identified to $(L^\infty)'$ — out of $L^1$. This appears when uniform integrability (briefly UI) does not hold: see J. Diestel [10, p.45]. The decomposition of such elements in the sum of a function of $L^1$ and of a non null *singular* linear form $\ell_s$ (see on this subject K. Yosida & E. Hewitt [33], [9, Ch.VIII]) has been very efficient in the study of convex integral functionals: at least four papers by R.T. Rockafellar [21–24], several Russian works, among them V.L. Levin [18], Ioffe-Tihomirov [15] (this last one has been resumed using the compact stonian spectrum of $L^\infty$ in [29]! For a strange singular linear form see [30]). Then, the "biting lemma" proved several times: by V.F. Gaposhkin [13], Brooks-Chacon [7], Acerbi-Fusco [1], M. Słaby [26], L. Egghe [12], Ball-Murat [6], C. Castaing [8], has showed its efficiency and P.-L. Lions [19] defined the concentration (of mass).

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Other interesting and useful weak limit points come from Young measures theory. This notion, introduced by L.C. Young [34] in 1937 to define generalized curves (and then generalized surfaces), presents a renewal of interest these last years, specially with the works of E.J. Balder [3–4] and J.M. Ball [5]. When the bounded sequence $(u_n)_{n\in\mathbb{N}}$ is UI, there exists a subsequence $(u'_n)_n$ whose associated Young measures converge and the limit Young measure, $\tau$, has a disintegration $(\tau_x)_{x\in\Omega}$ such that the function [1] $x \mapsto \mathrm{bar}(\tau_x)$ is the weak limit of $u'_n$ ([31, Th.19 p.169], [32, Th.9]). Moreover $\tau$ contains some precise information about the possible asymptotic oscillatory behavior of the $u'_n$.
When the sequence $(u_n)_n$ is not UI it may happen that all subsequences are not UI (think of $(n\,1_{[0,1/n]})_{n\geq 1}$ on $[0,1]$ equipped with the Lebesgue measure). If $\ell_s \neq 0$ is the singular part of a weak limit point $\ell \notin L^1$, there exists a sequence $(B_p)_p$ in $\mathcal{F}$ decreasing to $\emptyset$ such that $\forall p$, $B_p$ carries $\ell_s$ (see [9, VIII.9 p.239]). But, while Young measures allow to work with subsequences, here only a filter on $\mathbb{N}$ finer than the Fréchet filter[2] gives the $\sigma((L^\infty)', L^\infty)$ convergence $u_n \to \ell$. The biting lemma does not give a precise limit but allows the extraction of a subsequence presenting concentration of mass on sets $B_p$ decreasing to $\emptyset$.
Combining the biting lemma and Young measures techniques, we will prove a structural result (Theorem 4.5) which roughly speaking says: there exists a subsequence $(u'_n)_n$ and a partition of $\Omega$ in two measurable sets, one above which one has strong convergence, the other above which the functions oscillates in such a manner that[3] not any subsequence of $(u'_n)_n$ strongly converges on any non negligible subset and thirdly the possible non uniform integrability gives rise to concentration of mass on some $B_p$ sets.
The following example seems typical of the general case. Let

$$u_n(x,y) = n\,1_{[0,\frac{1}{n}]}(x) + 1_{[0,\frac{1}{2}]}(y)\,\sin(n\,x)$$

on $\Omega = [0,1]^2$ equipped with the Lebesgue measure. This example is studied in Section 4. In Section 5, Theorem 4.5 will be used to prove quickly some theorems by Hewitt-Stromberg, H.-A. Klei and Klei-Miyara.
One word about the infinite dimensional case. Surely our result does not extend to a bounded sequence in $L^1(\Omega,\mu;E)$. But one can notice some formulation analogy with the parametric versions of Rosenthal's theorem by M. Talagrand [27, Theorem 1] and Diestel-Ruess-Schachermayer [11, Theorem 2.4] : there are different behaviors on complementary subsets of $\Omega$.

## 2. Notations and backgrounds

For $u \in L^1(\Omega,\mu;\mathbb{R}^d)$ (to be correct one should write $u \in \mathcal{L}^1(\Omega,\mu;\mathbb{R}^d)$ ), *the Young measure associated to* $u$ is the measure on $\Omega\times\mathbb{R}^d$ which is the image of $\mu$ by $x \mapsto (x,u(x))$; it is denoted by $\nu$. If one considers functions $u_n$, their Young measures will be denoted by $\nu^n$. A general *Young measure* is a positive measure on $\Omega\times\mathbb{R}^d$, usually denoted by $\tau$,

[1] $\mathrm{bar}(\tau_x)$ denotes the barycenter of $\tau_x$.
[2] The extraction of a convergent subsequence is usually impossible.
[3] This will be called "pure" weak $L^1$-convergence. See section 4.

such that $\forall A \in \mathcal{F},\ \tau(A \times \mathbb{R}^d) = \mu(A)$ (i.e. the projection of $\tau$ on $\Omega$ is $\mu$). The set of all Young measures on $\Omega \times \mathbb{R}^d$ is denoted by $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$.
A *Carathéodory integrand* is a real function on $\Omega \times \mathbb{R}^d$ separately measurable on $\Omega$ and continuous on $\mathbb{R}^d$. The *narrow topology* on $\mathcal{Y}(\Omega, \mu; \mathbb{R}^d)$ is the weakest topology making continuous the maps $\tau \mapsto \int_{\Omega\times\mathbb{R}^d} \psi(x,\xi)\ d\tau(x,\xi)$ where $\psi$ runs through the set of all bounded Carathéodory integrands. If the Young measures $(\nu_n)_n$ are associated to the functions $u_n$, they converge narrowly to $\tau$ iff

$$\int_\Omega \psi(x, u_n(x))\ \mu(dx) =: \int_{\Omega\times\mathbb{R}^d} \psi\ d\nu^n \longrightarrow \int_{\Omega\times\mathbb{R}^d} \psi\ d\tau \tag{2.1}$$

for any bounded Carathéodory integrand $\psi$. The convergence in (2.1) still holds when $\psi$ is a Carathéodory integrand with linear growth (i.e. $|\psi(x,\xi)| \le \alpha(x) + K\|\xi\|$, where $\alpha \in L^1$) as soon as $(u_n)_n$ is UI. More generally there is a lower semi-continuity theorem ([3], [31, Th.16 p.166], [32, Th.6]): if the Young measures $\nu^n$ associated to $u_n$ converge to $\tau$, if $\psi\ \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$ is measurable in $(x,\xi)$ and l.s.c. in $\xi$ and if the sequence of negative parts $((\psi(., u_n(.))^-)_n$ is UI, then

$$\int_{\Omega\times\mathbb{R}^d} \psi\ d\tau \le \liminf_{n\to+\infty} \int_\Omega \psi(x, u_n(x))\ \mu(dx)\ . \tag{2.2}$$

For $\psi \ge 0$, this is an easy preliminary result ([31, Th.7 p.161], [32, Lemma 5]).
If $u_n$ ($n \in \mathbb{N} \cup \{+\infty\}$) are measurable functions, one has ([31, Prop.6 p.160], [32, Prop.1]):

$$\nu^n \to \nu^\infty \text{ narrowly } \iff u_n \to u_\infty \text{ in measure}\ . \tag{2.3}$$

If a sequence $(\tau^n)_n$ of Young measures satisfies

$$\sup_n \int_{\Omega\times\mathbb{R}^d} \|\xi\|\ d\tau^n(x,\xi) < +\infty\ ,$$

— this happens when the Young measures are associated to the functions of a bounded sequence in $L^1$ — there exists a narrowly convergent subsequence (more generally there is an extension of the Prokhorov theorem to Young measures: [3], [31, Th.11 p.162], [32, Th.7]).
The disintegration of a Young measure $\tau$ is a measurable family of probabilities on $\mathbb{R}^d$, $(\tau_x)_{x\in\Omega}$, characterized by: for any $\psi\ \Omega \times \mathbb{R}^d \to \overline{\mathbb{R}}$ measurable and $\ge 0$ or $\tau$-integrable,

$$\int_{\Omega\times\mathbb{R}^d} \psi\ d\tau = \int_\Omega [\int_{\mathbb{R}^d} \psi(x,\xi)\ \tau_x(d\xi)]\ \mu(dx)\ .$$

If $\nu$ is associated to $u$, $\nu_x = \delta_{u(x)}$, where $\delta_{u(x)}$ denotes the Dirac mass at $u(x)$. If $\int_{\Omega\times\mathbb{R}^d} \|\xi\|\ d\tau(x,\xi) < +\infty$, then for $\mu$-almost every $x$, $\tau_x$ is of first order, that is satisfies $\int_{\mathbb{R}^d} \|\xi\|\ \tau_x(d\xi) < +\infty$, hence has a barycenter: $\mathrm{bar}(\tau_x) := \int_{\mathbb{R}^d} \xi\ \ \tau_x(d\xi)$.

## 3. Preliminary results

**Definition 3.1.** The *modulus of uniform integrability* of H.P. Rosenthal, $\eta((u_n))$, of the sequence $(u_n)_n$, is:

$$\eta((u_n)) := \lim_{\varepsilon \to 0,\ \varepsilon > 0} \left[ \sup\left\{ \int_A \|u_n(x)\| \,\mu(dx) : n \in \mathbb{N},\ \mu(A) \le \varepsilon \right\} \right] .$$

The sequence $(u_n)_n$ is UI iff $\eta((u_n)) = 0$.

**Lemma 3.2.** *Let $(u_n)_n$ be a bounded sequence in $L^1(\Omega, \mu; \mathbb{R}^d)$. Then*

$$\eta((u_n)) = \lim_{t \to +\infty} \left[ \sup_{n \in \mathbb{N}} \int_{\{\|u_n(.)\| > t\}} \|u_n(x)\| \,\mu(dx) \right] .$$

**Proof.** The two limits are decreasing limits, hence infima. Let us denote $\eta_1$ the modulus of the statement and $M := \sup_n \|u_n\|_{L^1}$.
By definition of $\eta_1$, $\forall \delta > 0$ one has $\forall t$, $\exists n$ such that

$$\int_{\{\|u_n(.)\| > t\}} \|u_n\| \,d\mu > \eta_1 - \delta .$$

Let $\varepsilon > 0$. There exists $t$ large enough such that $\dfrac{M}{t}$ is $\le \varepsilon$. Then if $n$ corresponds to such a $t$, $A := \{\|u_n(.)\| > t\}$ satisfies $\mu(A) \le \varepsilon$ and $\int_A \|u_n\| \,d\mu \ge \eta_1 - \delta$. Hence $\eta((u_n)) \ge \eta_1 - \delta$, and this holds for all $\delta > 0$, so $\eta((u_n)) \ge \eta_1$.
Let us prove the converse inequality. Let $t > 0$ and $\delta > 0$ be given. Take $\varepsilon > 0$ small enough such that $t\,\varepsilon < \delta$. By definition of $\eta((u_n))$ there exist $A$ and $n$ such that $\mu(A) \le \varepsilon$ and $\int_A \|u_n\| \,d\mu > \eta((u_n)) - \delta$. Then

$$\begin{aligned}
\int_{\{\|u_n\| > t\}} \|u_n\| \,d\mu &= \int_A \|u_n\| \,d\mu - \int_{A \cap \{\|u_n\| \le t\}} \|u_n\| \,d\mu + \int_{\{\|u_n\| > t\} \setminus A} \|u_n\| \,d\mu \\
&\ge \int_A \|u_n\| \,d\mu - \int_{A \cap \{\|u_n\| \le t\}} \|u_n\| \,d\mu \\
&\ge (\eta((u_n)) - \delta) - t\,\varepsilon \\
&\ge \eta((u_n)) - 2\,\delta .
\end{aligned}$$

So one has $\eta_1 \ge \eta((u_n)) - 2\,\delta$ for all $\delta > 0$, hence $\eta_1 \ge \eta((u_n))$. □

The expression of $\eta((u_n))$ given in Lemma 3.2 is used in Słaby's proof of the biting lemma (reproduced in L. Egghe [12, VIII.1.17 pp.303–305] and [31, Th.23 pp.173–174]). We reproduce this proof here in order to be able to refer to its notations.

**Theorem 3.3.** (Biting lemma) *Let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence in $L^1(\Omega, \mu; \mathbb{R}^d)$. There exist a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ and a sequence $(B_p)_{p \in \mathbb{N}}$ in $\mathcal{F}$ decreasing to $\emptyset$ such that the sequence $(1_{B_k^c}\, u_{n_k})_{k \in \mathbb{N}}$ is* UI.

**Proof.** Denote, for $i \in \mathbb{N}$,

$$g_i(t) := \sup_{n \geq i} \int_{\{\|u_n(.)\| > t\}} \|u_n(x)\| \, \mu(dx) \, .$$

The functions $g_i$ are decreasing with values in $[0, +\infty[$. Hence, $\lim_{t \to +\infty} g_i(t)$ exists. By Lemma 3.2, $\eta((u_n)) = \lim_{t \to +\infty} g_0(t)$. There exists a sequence increasing to $+\infty$, $(t_q)_{q \geq 1}$ such that $\forall q$, $g_0(t_q) \leq \eta((u_n)) + \frac{1}{q}$. Since $\forall i$, $\{u_0, \dots, u_{i-1}\}$ is UI, it is easy to check that $\lim_{t \to +\infty} g_i(t) = \eta((u_n))$. Hence $\forall i$, $\forall t$, $g_i(t) \geq \eta((u_n))$, so there exists a strictly increasing sequence $(m_q)_q$ such that

$$\int_{\{\|u_{m_q}(.)\| > t_q\}} \|u_{m_q}(x)\| \, \mu(dx) \geq \eta((u_n)) - \frac{1}{q} \, . \tag{3.1}$$

Let $A_p = \{\|u_{m_p}(.)\| > t_p\}$. Then $t_p \, \mu(A_p) \leq \sup_{n \in \mathbb{N}} \|u_n\|_{L^1}$ implies $\mu(A_p) \to 0$. Now we show that $(1_{A_q^c} \, u_{m_q})_{q \in \mathbb{N}}$ is UI. Let

$$g(t) := \sup_{q \in \mathbb{N}} \int_{A_q^c \cap \{\|u_{m_q}(.)\| > t\}} \|u_{m_q}(x)\| \, \mu(dx) \, .$$

We have to prove that $g(t) \to 0$ when $t \to +\infty$. One has

$$\begin{aligned}
g(t_j) &= \sup_{q > j} \int_{\{t_j < \|u_{m_q}(.)\| \leq t_q\}} \|u_{m_q}(x)\| \, \mu(dx) \\
&= \sup_{q > j} \Big[ \int_{\{\|u_{m_q}(.)\| > t_j\}} \|u_{m_q}(x)\| \, \mu(dx) - \int_{\{\|u_{m_q}(.)\| > t_q\}} \|u_{m_q}(x)\| \, \mu(dx) \Big] \\
&\leq \sup_{q > j} \Big[ g_0(t_j) - \int_{\{\|u_{m_q}(.)\| > t_q\}} \|u_{m_q}(x)\| \, \mu(dx) \Big] \\
&\leq \sup_{q > j} [(\eta((u_n)) + \frac{1}{j}) - (\eta((u_n)) - \frac{1}{q})] \\
&\leq \frac{2}{j} \, .
\end{aligned}$$

This proves: $t \to +\infty \Rightarrow g(t) \to 0$. It remains to replace $(A_p)_p$ by a sequence decreasing to $\emptyset$ (or to a negligible set). There exists a strictly increasing sequence $(l_k)_k$ such that $\mu(A_{l_k}) \leq 2^{-k}$. Then $B_p := \cup_{k \geq p} A_{l_k}$ decreases to a negligible set. The subsequence $u_{n_k} := u_{m_{l_k}}$ (or $u_{m_{(l_k)}}$) has the required properties. The uniform integrability of $(1_{B_k^c} \, u_{n_k})_{k \in \mathbb{N}}$ follows from the inclusion $B_k^c \subset A_{l_k}^c$. □

## 4. Main results

**Definition 4.1.** Let $u_n$ ($n \in \mathbb{N} \cup \{+\infty\}$) be functions in $L^1(\Omega, \mu; \mathbb{R}^d)$.
1) One says that $u_n$ converges *strongly* to $u_\infty$ if $\|u_n - u_\infty\|_{L^1} \to 0$.

2) One says that $u_n$ converges *weakly* to $u_\infty$ if

$$\forall p \in L^\infty, \ \int_\Omega \langle p, u_n \rangle \, d\mu \longrightarrow \int_\Omega \langle p, u_\infty \rangle \, d\mu \ .$$

3) One says that $u_n$ converges *purely weakly* to $u_\infty$ *on* $W \in \mathcal{F}$ if the restrictions $u_n|_W$ converge weakly to $u_\infty|_W$ and, for any non negligible measurable subset $A$ of $W$, not any subsequence of $(u_n|_A)_n$ converges strongly to $u_\infty|_A$.

**Remark 4.2.** In 3) it is equivalent to say: not any subsequence of $(u_n|_A)_n$ converges in measure (to any function).
For clarity we give first the simplified version of the general result in the case when the sequence is UI. All assertions are consequence of the Prokhorov theorem, of (2.3) and of (2.1) extended to Carathéodory integrands with linear growth.

**Theorem 4.3.** *Let $(u_n)_n$ be a uniformly integrable sequence in $L^1(\Omega, \mu; \mathbb{R}^d)$. There exist a subsequence $(u'_n)_n$, a function $u_\infty \in L^1(\Omega, \mu; \mathbb{R}^d)$ and $M \in \mathcal{F}$ such that*[4]*:*
1b) *$u'_n$ converges weakly to $u_\infty$,*
1c) *the restrictions $u'_n|_M$ converge strongly to $u_\infty|_M$,*
1d) *the restrictions $u'_n|_{M^c}$ converge purely weakly to $u_\infty|_{M^c}$.*
2) *There exists a Young measure $\tau$, such that*

$$M^c = \{x \in \Omega : \int_{\mathbb{R}^d} \|\xi - u_\infty(x)\| \, \tau_x(d\xi) > 0\}$$

*and such that, for any Carathéodory integrand $\psi$ with linear growth* (i.e. $|\psi(x,\xi)| \le \alpha(x) + K\|\xi\|$, where $\alpha \in L^1$)

$$\int_\Omega \psi(x, u'_n(x)) \, \mu(dx) \longrightarrow \int_{\Omega \times \mathbb{R}^d} \psi \, d\tau \ . \tag{4.1}$$

**Remark 4.4.** We will not give the proof since the statement, except (4.1), is a particular case of Theorem 4.5. First versions of this statement appear in [31, Th.19 p.169] and [32, Th.9]. The novelty is the introduction of the partition $(M, M^c)$.

**Theorem 4.5.** *Let $(u_n)_n$ be a bounded sequence in $L^1(\Omega, \mu; \mathbb{R}^d)$. There exist a subsequence $(u'_n)_n$, a function $u_\infty \in L^1(\Omega, \mu; \mathbb{R}^d)$, a sequence $(B_p)_p$ decreasing to $\emptyset$* (if $(u_n)_n$ is UI, one may take $\forall p$, $B_p = \emptyset$) *and $M \in \mathcal{F}$ such that:*
1a) $(1_{B_n^c} \, u'_n)_n$ *is* UI
1b) *$\forall p$, the restrictions $u'_n|_{B_p^c}$ converge weakly to $u_\infty|_{B_p^c}$,*
1c) *the restrictions $u'_n|_M$ converge in measure to $u_\infty|_M$ and $\forall p$, the restrictions $u'_n|_{(M \setminus B_p)}$ converge strongly to $u_\infty|_{(M \setminus B_p)}$,*
1d) *for any non negligible $A \in \mathcal{F}$ contained in $W := \Omega \setminus M$, not any subsequence of $(u'_n|_A)_n$ converges in measure to $u_\infty|_A$ (nor to another function).*

[4] Here 1a) of Theorem 4.5 is obviously satisfied.

1e) *for any subsequence* $(u''_n)_n$ *of* $(u'_n)_n$, $\eta((u''_n)) = \eta((u_n))$.

2) *There exists a Young measure* $\tau$ *whose disintegration is a measurable family of first order probabilities on* $\mathbb{R}^d$, $(\tau_x)_{x\in\Omega}$, *such that* $\tau_x$ *is carried by the set* $\mathrm{Ls}(u_n(x))$ *of limit points of the sequence* $(u_n(x))_n$, *such that for* $x \in W$, $\tau_x$ *is not a Dirac mass and* $\mathrm{bar}(\tau_x) = u_\infty(x)$, *such that for* $x \in M$, $\tau_x = \delta_{u_\infty(x)}$ *and such that, for any* $u \in L^1(\Omega, \mu; \mathbb{R}^d)$, $\|u'_n - u\|_{L^1}$ *converges to*

$$\eta((u_n))+\int_{\Omega\times\mathbb{R}^d} \|\xi - u(x)\|\, d\tau(x,\xi)$$
$$= \eta((u_n)) + \int_{W\times\mathbb{R}^d} \|\xi - u(x)\|\, d\tau(x,\xi) + \|1_M\,(u_\infty - u)\|_{L^1}\ .$$

*Moreover for any bounded Carathéodory integrand* $\psi$

$$\int_\Omega \psi(x, u_n(x))\, \mu(dx) \longrightarrow \int_{\Omega\times\mathbb{R}^d} \psi\, d\tau\ .$$

**Remark 4.6.**

1) Assertion 1e) means that, for the subsequence $(u'_n)_n$, the concentration of mass (non null iff $\eta((u_n))$ is $> 0$) equals the maximum of mass concentrations among all the subsequences of $(u_n)_n$. Maybe the following idea is a good motivation: often to prove an implication where there is a regularity assumption, it is useful to be able to extract an irregular subsequence. In this line the second author has already given the following result ([31, Th.20 p.169], [32, Th.9]): if $(u_n)_n$ converges weakly (hence is UI) and does not converge strongly, there exists a subsequence whose associated Young measures converge to a Young measure $\tau$ not associated to a function.

2) The norm $\|u'_n - u\|_{L^1}$ equals $\int_{\Omega\times\mathbb{R}^d} \psi\, d\nu^n$ where $\psi$ is the integrand $\psi(x,\xi) := \|\xi - u(x)\|$. This is a Carathéodory integrand with linear growth (for some application of these integrands see [20]).

**Example 4.7.** Here is an example where $B_p$ necessarily bites both $M$ and $W$. Moreover this example seems to be a rather general one. Let $\Omega = [0,1]^2$ with the Lebesgue measure, $d = 1$ and

$$u_n(x,y) = n\ 1_{[0,\frac{1}{n}]}(x) + 1_{[0,\frac{1}{2}]}(y)\ \sin(n\ x)\ .$$

Then $u_\infty = 0$, $W = [0,1]\times[0,\frac{1}{2}]$ and [5] $M = [0,1]\times\left]\frac{1}{2},1\right]$. Roughly speaking, mass concentration appears on $\{0\}\times[0,1]$. In any way the $B_p$ are chosen, they meet $M$ and $W$: since $(1_{B_k^c}\, u_{n_k})_k$ is UI, $\|1_{[0,1/n_k]\times[0,1]}\ 1_{B_k^c}\ u_{n_k}\|_{L^1}$ tends to 0 when $k \to +\infty$, hence

$$\iint_{[0,1/n_k]\times[0,1]} 1_{B_k^c}(x,y)\ [n_k + 1_{[0,\frac{1}{2}]}(y)\ \sin(n_k\ x)]\ dx\ dy \longrightarrow 0\ .$$

[5] One can check (see [32, Th.4]), that for $(x,y)\in W$, $\tau_{(x,y)}$ is the probability on $\mathbb{R}$ with the density $\xi \mapsto (\pi\ \sqrt{1-\xi^2}\ )^{-1}$ on $]-1,1[$.

Thus $n_k\ \mu(B_k^c \cap ([0,1/n_k] \times [0,1])) \to 0$ and

$$\frac{\mu(B_k \cap ([0,1/n_k] \times [0,1]))}{\mu([0,1/n_k] \times [0,1])} \longrightarrow 1\ .$$

Consequently

$$\frac{\mu(B_k \cap W)}{\mu([0,1/n_k] \times [0,1/2])} \longrightarrow 1 \quad \text{and} \quad \frac{\mu(B_k \cap M)}{\mu([0,1/n_k] \times ]1/2,1])} \longrightarrow 1\ .$$

Hence $B_k$ covers a big part of $W \cap ([0,1/n_k] \times [0,1])$ and a big part of $M \cap ([0,1/n_k] \times [0,1])$. Note that $\sin(n\ y)$ in place of $\sin(n\ x)$ would give the same $W$ and $\tau$ despite the change of directions of the “waves.” When $\Omega$ is an open subset of $\mathbb{R}^N$, there exist more powerful tools than Young measures: see G. Allaire [2] and L. Tartar [28].

**Proof of Theorem 4.5**

1) Let $(B_p)_p$ denote the sequence of the proof of the biting lemma and $(u'_n)_n$ the subsequence denoted by $(u_{n_k})_k$ in that proof. Let $(u''_n)_n$ be a further subsequence. A priori $\eta((u''_n)) \le \eta((u_n))$. Using the notations of the proof of Theorem 3.3, $u''_n$ is some $u_{m_q}$. For all $t$, there exists $q$ large enough such that $t_q \ge t$ and such that the function $u_{m_q}$ is some $u''_n$. By (3.1) $\eta((u_n))$ is approximated with a gap less than $1/q$ by

$$\int_{\{\|u''_n(\cdot)\| > t_q\}} \|u''_n(x)\|\ \mu(dx)$$

hence a fortiori by

$$\int_{\{\|u''_n(\cdot)\| > t\}} \|u''_n(x)\|\ \mu(dx)\ .$$

This proves $\eta((u''_n)) = \eta((u_n))$.

2) Since $(u_n)_n$ is bounded, thanks to the Prokhorov theorem extended to Young measures, one may assume that the Young measures associated to the $u'_n$ converge to a Young measure $\tau$. Thanks to the lower semi-continuity result (2.2),

$$\int_{\Omega \times \mathbb{R}^d} \|\xi\|\ d\tau(x,\xi) \le \sup_n \|u_n\|_{L^1}\ . \tag{4.2}$$

Then $\mu$-a.e. the disintegration $\tau_x$ has a barycenter $\mathrm{bar}(\tau_x)$ and, with $u_\infty(x) := \mathrm{bar}(\tau_x)$, $u_\infty$ is integrable. Moreover $\tau_x$ is carried by $\mathrm{Ls}(u_n(x))$ ([31, Prop.5 p.159]).

For a fixed $p$, since $u'_n|_{B_p^c}$ is UI, $u'_n|_{B_p^c} \to u_\infty|_{B_p^c}$ weakly (this is again a consequence of (2.1) extended to Carathéodory integrands with linear growth. For a reference see [31, Th.19 p.169]; see also the proof of Th.9 in [32]). The set $W$ of all $x$ where $\tau_x$ is not a Dirac mass, is $\{x \in \Omega : \int_{\mathbb{R}^d} \|\xi - u_\infty(x)\|\ \tau_x(d\xi) > 0\}$. By the Fubini theorem it is measurable. Let $M := \Omega \setminus W$. Note that $\mu$-almost everywhere on $M$, $\tau_x = \delta_{u_\infty(x)}$. From (2.3), $u'_n|_M \to u_\infty|_M$ in measure and, for any non negligible $A$ contained in $W$, not any subsequence of $(u'_n)_n$ converges in measure on $A$. Thanks to UI and to the Lebesgue-Vitali theorem, strong convergence holds on $M \setminus B_p$.

3) Let $u \in L^1(\Omega, \mu; \mathbb{R}^d)$ and $\varepsilon > 0$. Thanks to (4.2), for $p$ large enough one has:

$$\begin{aligned}\int_{B_p \times \mathbb{R}^d} \|\xi - u(x)\| \, d\tau(x,\xi) &\leq \int_{B_p \times \mathbb{R}^d} \|\xi\| \, d\tau(x,\xi) + \int_{B_p \times \mathbb{R}^d} \|u(x)\| \, d\tau(x,\xi) \\ &= \int_{B_p \times \mathbb{R}^d} \|\xi\| \, d\tau(x,\xi) + \int_{B_p} \|u(x)\| \, \mu(dx) \\ &\leq \frac{\varepsilon}{4} \, .\end{aligned}$$

Let $\nu^n$ denotes the Young measure associated to $u'_n$. On $B_p^c \times \mathbb{R}^d$ the integrand $\psi$ defined by $\psi(x,\xi) := \|\xi - u(x)\|$ is Carathéodory with linear growth and the $u'_n$ are UI on $B_p^c$, so by (2.1)

$$\|1_{B_p^c} \, (u'_n - u)\|_{L^1} = \int_{B_p^c \times \mathbb{R}^d} \|\xi - u(x)\| \, d\nu^n(x,\xi)$$

converges to

$$\int_{B_p^c \times \mathbb{R}^d} \|\xi - u(x)\| \, d\tau(x,\xi) \, .$$

Hence, $p$ being fixed, for $n$ large enough,

$$\left| \|1_{B_p^c} \, (u'_n - u)\|_{L^1} - \int_{\Omega \times \mathbb{R}^d} \|\xi - u(x)\| \, d\tau(x,\xi) \right| \leq \frac{\varepsilon}{2} \, .$$

As for $\|1_{B_p} \, (u'_n - u)\|_{L^1}$, firstly, if $p$ is large enough,

$$\big| \|1_{B_p} \, (u'_n - u)\|_{L^1} - \|1_{B_p} \, u'_n\|_{L^1} \big| \leq \|1_{B_p} \, u\|_{L^1} \leq \frac{\varepsilon}{4} \, .$$

It remains to show that, for $n$ large enough, one has

$$\big| \|1_{B_p} \, u'_n\|_{L^1} - \eta((u_n)) \big| \leq \frac{\varepsilon}{4} \, .$$

Recall (notations of the proof of Theorem 3.3) that $B_p = A_{l_p} \cup A_{l_{p+1}} \cup \ldots$ and that, from (3.1),

$$\int_{A_{l_k}} \|u_{m_{l_k}}(x)\| \, \mu(dx) \geq \eta((u_n)) - \frac{1}{l_k} \, ,$$

hence

$$\forall k \geq p, \quad \int_{B_p} \|u_{m_{l_k}}(x)\| \, \mu(dx) \geq \eta((u_n)) - \frac{1}{l_p} \, .$$

If one has chosen $p$ such that $l_p \geq 4/\varepsilon$, then, for $n$ large enough, $\|1_{B_p} \, u'_n\|_{L^1} \geq \eta((u_n)) - \frac{\varepsilon}{4}$. Finally, setting

$$\eta((u_n);\delta) := \sup\left\{ \int_A \|u_n(x)\| \, \mu(dx) : n \in \mathbb{N}, \ \mu(A) \leq \delta \right\} ,$$

one has, as soon as $\mu(B_p) \leq \delta$,

$$\sup_n \int_{B_p} \|u'_n\| \, d\mu \leq \eta((u_n);\delta) \, .$$

Consequently, if one has chosen $\delta$ small enough such that $\eta((u_n);\delta) \leq \eta((u_n)) + \frac{\varepsilon}{4}$ and then $p$ large enough such that $\mu(B_p) \leq \delta$, one has

$$\forall n, \quad \int_{B_p} \|u'_n\| \, d\mu \leq \eta((u_n)) + \frac{\varepsilon}{4} \, .$$

Finally, for a given $\varepsilon$, a good choice of $p$ is possible and one gets, for $n$ large enough,

$$\left| \|u'_n - u\|_{L^1} - \left[ \eta((u_n)) + \int_{\Omega \times \mathbb{R}^d} \|\xi - u(x)\| \, d\tau(x,\xi) \right] \right| \leq \varepsilon \, .$$

□

## 5. Applications

**Theorem 5.1.** (Hewitt & Stromberg [14, Th.13.47 p.208]) *Let $u_n$ ($n \in \mathbb{N} \cup \{+\infty\}$) be functions of $L^1(\Omega,\mu;\mathbb{R}^d)$. Suppose that $u_n$ converge in measure to $u_\infty$ and that*

$$\limsup_{n \to +\infty} \|u_n\|_{L^1} \leq \|u_\infty\|_{L^1} \, . \tag{5.1}$$

*Then $u_n$ converge strongly to $u_\infty$.*

**Proof.** If $u_n$ does not converge strongly to $u_\infty$, one may, extracting a further subsequence, assume $\forall n, \|u_n - u_\infty\|_{L^1} \geq \varepsilon > 0$. Then there exists a subsequence satisfying part 1) of Theorem 4.5. Necessarily $W$ is negligible and the function $u_\infty$ is the same. From part 2) of Theorem 4.5 applied with $u = 0$,

$$\|u'_n\|_{L^1} \longrightarrow \eta((u_n)) + \|u_\infty\|_{L^1} \, .$$

By (5.1), $\eta((u_n)) = 0$, hence UI holds and, thanks to Lebesgue-Vitali, $u'_n$ converges strongly to $u_\infty$, which gives a contradiction. □

**Theorem 5.2.** (Klei [16, Th.6]) *Let $(u_n)_n$ be a bounded sequence in $L^1(\Omega,\mu;\mathbb{R}^d)$ and $u \in L^1(\Omega,\mu;\mathbb{R}^d)$. Suppose* $\limsup \|u_n - u\|_{L^1} \leq \eta((u_n))$. *Then there exists a subsequence $(u'_n)_n$ with the following properties: $u'_n$ converges in measure to $u$, $\eta((u_n)) = \lim \|u'_n - u\|_{L^1}$ and, for any subsequence $(u''_n)_n$ of $(u'_n)_n$, $\eta((u''_n)) = \eta((u_n))$.*

**Proof.** Part 1) of Theorem 4.5 gives a subsequence $(u'_n)_n$, $u_\infty$ et $W$. And for any subsequence $(u''_n)_n$ of $(u'_n)_n$, $\eta((u''_n)) = \eta((u_n))$. From part 2) of Theorem 4.5,

$$\|u'_n - u\|_{L^1} \longrightarrow \eta((u_n)) + \int_{W \times \mathbb{R}^d} \|\xi - u(x)\| \, d\tau(x,\xi) + \|1_M \, (u_\infty - u)\|_{L^1} \, ,$$

hence

$$\eta((u_n)) + \int_{W\times\mathbb{R}^d} \|\xi - u(x)\| \, d\tau(x,\xi) + \|1_M \, (u_\infty - u)\|_{L^1} \leq \eta((u_n)) \, .$$

So $\|u'_n - u\|_{L^1} \to \eta((u_n))$ and

$$\int_{W\times\mathbb{R}^d} \|\xi - u(x)\| \, d\tau(x,\xi) + \|1_M \, (u_\infty - u)\|_{L^1} = 0 \, . \tag{5.2}$$

For $x \in W$, $\tau_x$ is not a Dirac mass, so $\int_{\mathbb{R}^d} \|\xi - u(x)\| \, \tau_x(d\xi)$ is $> 0$. By (5.2) and the formula

$$\int_{W\times\mathbb{R}^d} \|\xi - u(x)\| \, d\tau(x,\xi) = \int_W [\int_{\mathbb{R}^d} \|\xi - u(x)\| \, \tau_x(d\xi)] \, \mu(dx) \, ,$$

the set $W$ is $\mu$-negligible, which implies $1_M = 1_\Omega$ $\mu$-a.e. By (5.2) we have also $\|1_M \, (u_\infty - u)\|_{L^1} = 0$. Hence $u = u_\infty$ $\mu$-a.e. and $u'_n \to u$ in measure. □

**Theorem 5.3.** (Klei-Miyara [17]) *Let $(u_n)_n$ be a bounded sequence in $L^1_+(\Omega, \mu; \mathbb{R})$ such that $\int_\Omega u_n \, d\mu \to \ell$. Then*

$$\int_\Omega (\liminf u_n) \, d\mu \leq \ell - \eta((u_n)) \, .$$

**Remark 5.4.** This means that in the Fatou lemma, after extraction of a subsequence whose integrals converge[6], the gap is at least $\eta((u_n))$, the modulus of the subsequence. One can observe that the negative parts of the $u_n$ are UI, so the possible concentration of mass concerns only positive mass.

**Proof.** Let $(u'_n)_n$, $u_\infty$, $M$, $W$ and $\tau$ given by Theorem 4.5. Here $\tau_x$ is carried by $[0, +\infty[$ and $\tau$ is carried by $\Omega \times [0, +\infty[$, so in the following formulas $\xi$ replaces $|\xi|$. Part 2) of Theorem 4.5 applied with $u = 0$ gives

$$\begin{aligned} \ell = \lim \int_\Omega u'_n \, d\mu &= \eta((u_n)) + \int_{W\times\mathbb{R}} \xi \;\; d\tau(x,\xi) + \int_M u_\infty \, d\mu \\ &= \eta((u_n)) + \int_W [\int_{\mathbb{R}} \xi \;\; \tau_x(d\xi)] \, \mu(dx) + \int_M u_\infty \, d\mu \, . \end{aligned}$$

On $M$, $u_n \to u_\infty$ in measure, hence $\liminf u_n(x) \leq u_\infty(x)$. For $x \in W$, $\mu$-a.e. $\tau_x$ is carried by $\mathrm{Ls}(u_n(x))$, hence by $[\liminf u_n(x), +\infty[$. Consequently[7] $\liminf u_n(x) \leq \int_{\mathbb{R}} \xi \;\; \tau_x(d\xi)$ and

$$\int_\Omega (\liminf u_n) \, d\mu \leq \int_W [\int_{\mathbb{R}} \xi \;\; \tau_x(d\xi)] \, \mu(dx) + \int_M u_\infty \, d\mu = \ell - \eta((u_n)) \, .$$

This proves the expected inequality. □

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6 Without extraction the result does not hold: let $\Omega = [0,1]$, $u_n = 0$ if $n$ is even, $u_n = n \, 1_{[0,1/n]}$ if $n$ is odd. Then $\liminf(\int u_n) - \int(\liminf u_n) = 0$ but $\eta((u_n)) = 1$.

7 This holds for all $x \in \Omega$, but we give a direct argument in the case $x \in M$, because it is so easy and $W = \emptyset$ is possible !

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