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# Antihomogeneous Conjugacy Operators in Convex Analysis

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**Dedicated to R. T. Rockafellar on his 60th Birthday**

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An axiomatic description of antihomogeneous conjugacy operators is suggested and some properties of these operators are established. Moreover several examples are given.

*Keywords* : conic complete lattice, antihomogeneous conjugacy operator, calculus, operator of polarity.

## 1. Introduction

One of the main methods of Convex Analysis is the application of conjugacy operators which transfer a given convex object (set, function, multivalued mapping and so on ) to a conjugate object. Several conjugacy operators are used in Convex Analysis, for instance the operator of polarity which maps a convex closed set containing the origin to its polar set, the operator which transfers the same set to its Minkowski gauge, the operator which maps a convex lower semicontinuous (l.s.c.) function in its conjugate function (in the sense of Convex Analysis). Various kinds of conjugacy operators are described in the famous book by Rockafellar [14]. Conjugacy operators are used also in the theory of quasiconvex functions (see for instance [5, 11, 13, 17]).

It is most convenient to study conjugacy operators within framework of lattice theory as dual isomorphism between lattices. This approach was suggested the famous Russian mathematician G.P.Akilov in the 1970's. Unfortunately the important work of G.P.Akilov was not published. Varios classes of conjugacy operators and varios aspects of the theory of conjugacy operators are studied in framework of lattice theory by I. Singer, J.-E. Martinez-Legaz (see for example [20, 12]) and others. There is a survey of results in this direction in the paper [10] by J.-E. Martinez-Legaz. On the other hand a vector version of some conjugacy operators of Convex Analysis (with substitute functionals for operators and straight line for vector lattices) was studied by A.G.Kusraev and S.S.Kutateladze (see for example [6, 7]).

Conjugacy operators which are studied in Convex Analysis and related areas are defined as a rule on a lattice with multiplication by positive scalars, and these operators are

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antihomogeneous (i.e. homogeneous of degree (-1)). We are primarily concerned in this paper with conjugacy operators possessing this antihomogeneuos property.
The aim of this paper is to give an axiomatic description of antihomogeneous conjugacy operators and to study connections between these operators and algebraic operations (calculus). Our viewpoint sheds some new light on the calculus and allows us to obtain some results by the simplest way.

Examples are part and parcel of the paper.

In Section 2 we introduce so-called $c^2$-lattices which are natural domains of antihomogeneous conjugacy operators. In Section 3 antihomogeneous conjugacy operators are introduced. Sections 4 and 5 are devoted to the study of some algebraic operations and calculus. Finally in Section 6 the operator of polarity is studied.

## 2. $c^2$ -lattices.

The natural domain of an antihomogeneous conjugacy operator is a complete lattice with multiplication by a positive scalar.

**Definition 2.1.** A set $U$ is called conic complete lattice ($c^2$-lattice) if

i) $U$ is a conic set, i.e. multiplication by positive scalar is defined on $U$ with properties as follows:
$$1u = u; \lambda(\mu u) = (\lambda\mu)u \quad \forall u \in U, \quad \forall \lambda, \mu > 0.$$

ii) $U$ is a complete lattice i.e. $U$ is an ordered set and there exist supremum and infimum for an arbitrary subset of the set $U$.

Let us give some examples of $c^2$-lattice. We consider three situations.

**Situation 1. $c^2$-lattices of nonnegative functions.** Let $\overline{R}_+ = [0, +\infty]$ be the closed ray of all nonnegative numbers together with $+\infty$. Let us consider a set $U$ of functions defined on a set $X$ and acting into the ray $\overline{\mathbb{R}}_+$ with properties as follows

1) $0 \in U$;
2) If $u \in U$, $\lambda > 0$ then $\lambda u \in U$ where $(\lambda u)(x) = \lambda u(x) \quad \forall x \in X$;
3) If $A \subset U$ then $\overline{\sup} A \in U$ where $\overline{\sup} A$ is an upper envelope of the set $A$, i.e. $(\overline{\sup} A)(x) = \sup_{f \in A} f(x) \, \forall x \in X$.

We consider $U$ as an ordered set with natural order relation : $u_1 \geq u_2$ if and only if $u_1(x) \geq u_2(x) \quad \forall x \in X$. Let $A \subset U$. Clearly the function $\overline{\sup} A$ is the supremum of the set $A$ in the ordered set $U$.
If $f$ is an arbitrary function defined on $X$ and acting into $\overline{\mathbb{R}}_+$ then a function $\mathrm{co}_U = \overline{\sup}\{u \in U : u \leq f\}$ is called the $U$-hull of the function $f$.

Clearly a function $\mathrm{co}_U(\overline{\inf} A)$ is the infimum of the set $A$ in the ordered set $U$. Here and subsequently $\overline{\inf} A$ stands for the lower envelope of the set $A$. We will denote $\overline{\sup} A$ and $\mathrm{co}_U(\overline{\inf} A)$ as $\sup A$ and $\inf A$ accordingly. Thus $U$ is a complete lattice. Clearly $U$ is a $c^2$-lattice with regard to natural multiplication on a positive number and natural order relation.
Let us give some concrete examples of $c^2$-lattices which are sets of functions with properties 1) - 3) above.

1. A set $F(X)$ of all functions defined on a set $X$ and acting into $\overline{\mathbb{R}}_+$.

2. Sets $M_0(X)$ of all increasing functions and $M^0(X)$ of all decreasing functions defined on an ordered set $X$ and acting into $\overline{\mathbb{R}}_+$.

Below we will consider only functions defined on a locally convex Hausdorff space $X$ and acting into $\overline{\mathbb{R}}_+$. Recall that a function $f$ is called proper if there is $x \in X$ such that $f(x) \neq +\infty$. Let us continue the list of examples.

3. A set $PH(X)$ of all lower semicontinuous (l.s.c.) nonnegative positively homogeneous of degree one proper functions.
4. A set $SL(X)$ of all nonnegative proper sublinear l.s.c. functions.
5. A set $CF(X)$ of all nonnegative proper convex l.s.c. functions $f$ such that $f(0) = 0$.
6. A set $Q(X)$ of all nonnegative quasiconvex l.s.c. functions $f$ with property $f(0) = 0$.

Let us note that $SL(X) = PH(X) \cap CF(X)$. At the same time $SL(X) = PH(X) \cap Q(X)$. This was proved by J. P. Crouzeix (see, for example, [5]).

**Situation 2. $c^2$-lattices of $H$-convex functions.** Let $Y$ be a cone in a locally convex Hausdorff space $X$ and $H$ be a cone of functions defined on the $Y$ and acting into $\mathbb{R} = (-\infty, +\infty)$ . A function $f$ defined on $Y$ and acting into $\overline{\mathbb{R}} = (-\infty, +\infty]$ is called $H$-convex (see [8]) if there is a set $A \subset H$ such that $f(y) = \sup_{h \in A} h(y)\ \forall y \in Y$ . We will denote by $C^H(Y)$ the set of all $H$-convex functions. It easy to check (see [8]) that $C^H(Y)$ is a convex cone. The set $C^H_*(Y) = C^H(Y) \cup \{-\infty\}$, where $(-\infty)(x) = -\infty \quad \forall x$ is a complete lattice as regards the natural order relation: $f_1 \geq f_2$ if and only if $f_1(y) \geq f_2(y)\ \forall y$ . Really if $A \subset C^H(Y)$ then $\sup A = \overline{\sup} A$ ; if $A$ bounded from below by a function $h \in H$, i.e. $h \leq f \quad \forall f \in A$ then $\inf A = \mathrm{co}_H \overline{\inf} A$ where $\mathrm{co}_H g = \sup\{h \in H : h \leq g\}$ ; otherwise $\inf A = -\infty$. There are various ways in order to introduce multiplication by a positive scalar in the lattice $C^H_*(Y)$. We need the following two methods:

$$(\lambda \cdot f)(x) = \lambda f(x) \tag{1}$$

$$(\lambda \otimes f)(x) = \frac{1}{\lambda} f(\lambda x). \tag{2}$$

Accordingly we consider two $c^2$-lattices in this case. We will write $C^H_1(Y)$ for $C^H_*(Y)$ with multiplication $\cdot$ defined by formula (1), and $C^H_2(Y)$ for $C^H_*(Y)$ with multiplication $\otimes$ defined by formula (2).

Let us give some examples.

1. If $Y = X$ and $H$ consist of all affine continuous functions defined on the space $X$ then $C^H_*(X)$ is the set of all convex l.s.c. functions defined on this space.
2. Let $Y$ be a closed cone, $Y - Y = X$ and $H$ be the set of all nonnegative linear functions $h$ defined on $Y$ with the next property: there is a continuous linear function $\hat{h}$ on $X$ such that $h$ is restriction of $\hat{h}$. A sublinear l.s.c. function $\hat{p}$ defined on $X$ is called a monotonic extension of a sublinear l.s.c. function $p$ defined on $Y$ if $\hat{p}(y) = p(y) \quad \forall y \in Y$ and $\hat{p}(x_1) \geq \hat{p}(x_2)$ if $x_1, x_2 \in X$ and $x_1 - x_2 \in Y$. It can be shown (see [8]) that $C^H(Y)$ is the set of all sublinear l.s.c. functions defined on $Y$ which have a monotonic extension.
3. Let $X$ be an Euclidean space with inner product $[\cdot,\cdot]$ and $Y = X$. Let us consider the cone $H$ of all quadrics $h$ which have a form

$$h(x) = \alpha \parallel x \parallel^2 + [l, x] + \beta,$$

where $\alpha \leq 0$, $l \in X$, $\beta$ is a scalar. Applying results of the papers [8, 2] it is easy to check that $C^H(X)$ is the set of all l.s.c. functions defined on $X$ and bounded from below by a function $h \in H$.

**Situation 3. $c^2$- lattices of sets.** Let $U$ be a totality of subsets of a vector space $X$ such that

1) $0 \in \Omega$ for all $\Omega \in U$ ; $X \in U$;
2) If $\Omega \in U$ then $\lambda \cdot \Omega = \{\lambda x : x \in \Omega\} \in U$ for all $\lambda > 0$ ;
3) If $A \subset U$ then $\bigcap\{\Omega : \Omega \in A\} \in U$ .

If $\Omega$ is a subset of the space $X$ and $0 \in \Omega$ then $U$- hull of $\Omega$, denoted by $\mathrm{co}_U\Omega$ is the set $\bigcap\{\Omega' : \Omega' \in U,\ \Omega' \supset \Omega\}$.

Let us consider $U$ as an ordered space: $\Omega_1 \geq \Omega_2$ if and only if $\Omega_1 \supset \Omega_2$ . It is easy to check that for arbitrary subset $A$ of the set $U$ there exist supremum and infimum of $A$ in the ordered set $U$ and the next formulae hold

$$\inf A = \bigcap\{\Omega : \Omega \in A\}; \quad \sup A = \mathrm{co}_A \bigcup\{\Omega : \Omega \in A\}.$$

Clearly $U$ is a $c^2$-lattice as regards the natural multiplication of a positive number and natural order relation. Let us give some concrete examples of $c^2$-lattices of sets with properties 1) - 3) above.

1. A set $ST(X)$ of all star with regard to zero closed subsets of a locally convex space $X$. Recall that a subset $\Omega$ of the space $X$ is called star with regard to zero if relation $x \in \Omega$ implies $\lambda x \in \Omega$ for all $\lambda \in [0, 1]$ .
2. A set $CS(X)$ of all convex closed subsets of the space $X$ such that $0 \in X$.
3. A set $\mathrm{Cone}(X)$ of all closed convex cones $K \subset X$.
4. A set $\mathrm{Lin}(X)$ of all closed linear subspaces of the space $X$.

## 3. Antihomogeneous conjugacy operators

**Definition 3.1.** Let $U$ and $V$ be $c^2$-lattices. A mapping $P : U \to V$ is called antihomogeneous conjugacy operator if

1. The mapping $P$ is a dual isomorphism of lattices $U$ and $V$ i.e. $P$ is one-to-one correspondence and inequality $Pu \geq Pv$ is true if and only if $u \leq v$ .
2. $P(\lambda u) = \frac{1}{\lambda}P(u)$ for all $u \in U$ and $\lambda > 0$ .

Clearly if $P : U \to V$ is an antihomogeneous conjugacy operator then there exist the inverse operator $P^{-1} : V \to U$ and $P^{-1}$ is an antihomogeneous conjugacy operator too.

Let us give some examples of antihomogeneous conjugacy operators. We use the denotations of $c^2$-lattices which describe above without reminding.

**Example 3.2.** Mappings $P : F(X) \to F(X)$ , $P : M_0(X) \to M^0(X)$ , $P : M^0(X) \to M_0(X)$ where $P(f)(x) = \frac{1}{f(x)}$ are antihomogeneous conjugacy operators.

**Example 3.3.** Let $\Omega$ be a star with regard to zero closed set and $\mu_\Omega(x) = \inf\{\lambda > 0 : x \in \lambda\Omega\}$ is a Minkowski gauge of $\Omega$. (We assume that $\inf \emptyset = +\infty$). The operator $P : ST(X) \to PH(X)$ where $P(\Omega) = \mu_\Omega$ is an antihomogeneous conjugacy operator.

In examples below we will consider dual vector spaces $X$ and $L$ with a bilinear form $[\cdot,\cdot]$.

**Example 3.4.** 3. Mappings $P : CS(X) \to CS(L)$ , $P : \mathrm{Cone}(X) \to \mathrm{Cone}(L)$ and $P : \mathrm{Lin}(X) \to \mathrm{Lin}(L)$ defined by the formula $P(\Omega) = \Omega^\circ$ are antihomogeneous conjugacy operators. Here $\Omega^\circ$ is the polar set of $\Omega$.

**Example 3.5.** Let $P : SL(X) \to SL(L)$ be the operator defined by formula $P(f) = f^\circ$ , where $f^\circ$ is the polar function of $f$. Then $P$ is an antihomogeneous conjugacy operator. Recall (see [14]) that

$$f^\circ(l) = \inf\{\mu > 0 : [l,x] \le \mu f(x) \quad \forall x\}.$$

**Example 3.6.** A mapping $P : CF(X) \to CF(X)$ where

$$P(f)(l) = \inf\{\mu > 0 : [l,x] \le 1 + \mu f(x) \quad \forall x\}$$

is an antihomogeneous conjugacy operator. This operator was introduced and studied by R. T. Rockafellar [14].

**Example 3.7.** A mapping $P : Q(X) \to Q(X)$ where

$$P(f)(l) = \sup_{[l,x]>1} \frac{1}{f(x)}$$

is an antihomogeneous conjugacy operator. This operator was introduced and studied in [17].

**Example 3.8.** Let $X_1$ and $X_2$ be locally convex Hausdorff spaces, $Y \subset X_1$ and $H \subset X_2$ be closed convex cones. Let us consider a function $(y,h) \to \langle y,h\rangle$ defined on the Cartesian product $Y \times H$ with properties as follows:

$$\langle \lambda y, h\rangle = \langle y, \lambda h\rangle = \lambda\langle y,h\rangle \quad \forall \lambda > 0, y \in Y, h \in H;$$

$$\forall y_1, y_2 \in Y \quad \exists h \in H \quad \text{such that} \quad \langle y_1, h\rangle \ne \langle y_2, h\rangle;$$

$$\forall h_1, h_2 \in H \quad \exists y \in Y \quad \text{such that} \quad \langle y, h_1\rangle \ne \langle y, h_2\rangle.$$

This function allows us to consider an element $y \in Y$ as a function defined on $H$ and an element $h \in H$ as a function defined on $Y$. We will write either $y(h)$ or $h(y)$ instead of $\langle y,h\rangle$ in these cases.

Let $\overline{Y}$ be the set of all functions $f$ defined on $H$ which have a form $f(h) = y(h) + c$ and let $\overline{H}$ be the set of all functions $g$ defined on $Y$ such that $g(y) = h(y) + c$ where $y \in Y$, $h \in H$, $c \in R$ . Now we consider $c^2$-lattices $C_1^{\overline{H}}(Y)$ and $C_2^{\overline{Y}}(H)$ which were defined in the Situation 2 of the Section 2. Set

$$(Pf)(h) = \sup_{y\in Y}(\langle y,h\rangle - f(y)). \tag{3}$$

It is well known (see for example [8, 2]) that $P$ is a dual isomorphism of the complete lattices $C_1^{\overline{H}}(Y)$ and $C_2^{\overline{Y}}(H)$. At the same time $P(\lambda \cdot f) = \frac{1}{\lambda} \otimes P(f)$ for all $\lambda > 0$. So $P$ is an antihomogeneous conjugacy operator.

Let us return to the dual vector spaces $X$ and $L$. Let $X_1 = X$, $X_2 = L$ , $Y = X$ and $H = L$. Clearly $f \in C^{\overline{H}}(X)$ if and only if $f$ is a l.s.c. convex function defined on the space $X$. In this case the function $P(f)$ coincides with the conjugate function $f^*$ in the sense of Convex Analysis.

**Example 3.9.** Here we will mention antihomogeneous conjugacy operators defined on $c^2$-lattices of convex processes. Recall that a multivalued mapping $a : X_1 \to X_2$ is called a convex process if $a(x+y) \supset a(x)+a(y)$ and $a(\lambda x) = \lambda a(x) \quad \forall \lambda > 0$. Conjugacy operators for finite dimensional convex processes were introduced and studied by R. T. Rockafellar [14, 15]. There is a complete theory of conjugate convex processes in the paper [4] by J. M. Borwein. (Some results from [4] were made more precisely in [21]. We consider two pairs $X_1, L_1$ and $X_2, L_2$ of dual vector spaces and a convex process $a : X_1 \to X_2$. The convex process $a^* : L_2 \to L_1$ is called conjugate with regards to $a$ if we have for all $g \in L_2$

$$a^*(g) = \{f \in L_1 : [f, x] \geq [g, y] \quad \forall x \in X,\ y \in a(x)\}.$$

It is convenient for applications to Mathematical Economics to consider only convex processes with the following properties:

i) dom $a = \{x \in X_1 : a(x) \neq \emptyset\}$ coincides with a given cone $K_1$.

ii) If $x \in K_1$ then $a(x)$ is part of a given cone $K_2$ and $a(x)$ is a normal set: $a(x) = \mathrm{cl}((a(x) - K_2) \cap K_2)$.

The conjugate mapping $a^*$ is defined in this case only on a cone $K_2^*$ by the formula

$$a^*(g) = \{f \in K_1^* : [f, x] \geq [g, y] \quad \forall x \in K_1,\ y \in a(x)\}.$$

Here $K^*$ is the conjugate cone for the cone $K$. A conjugate mapping in this form and its generalization are studied in detail by author [9, 16]. We must define various $c^2$-lattices of convex processes in order to study various kinds of conjugacy operators.

## 4. Calculus

The connection between an antihomogeneous conjugacy operator and multiplication by a positive number is given in the definition of an antihomogeneous conjugacy operator. It is easy to verify using the simplest properties of lattices (see for example [3]) that the following is true.

**Proposition 4.1.** *Let $P : U \to V$ be an antihomogeneous conjugacy operator and $A$ a subset of a $c^2$-lattice $U$. Then*

$$P(\sup_{u \in A} u) = \inf_{u \in A} P(u); \qquad P(\inf_{u \in A} u) = \sup_{u \in A} P(u).$$

Summation is not present in the definitions both of a $c^2$-lattice and an antihogeneous conjugacy operator, but there is a lot of $c^2$-lattices with summation and it is very interesting to establish connections between summation and the antihomogeneous conjugacy operator.

At first we define multiplication by zero and $+\infty$ . Let $U$ be a $c^2$-lattice and $u \in U$ . By definition

$$0 \cdot u = \inf_{0<\lambda<+\infty} \lambda u, \qquad +\infty \cdot u = \sup_{0<\lambda<+\infty} \lambda u.$$

Now we introduce two algebraic operations defined on the $c^2$-lattice $U$. If $u, v \in U$ then we set

$$u \hat{+} v = \inf_{0 \le \alpha \le 1} \sup \left( \frac{1}{\alpha} u, \frac{1}{1-\alpha} v \right), \tag{4}$$

$$u \oplus v = \sup_{0 \le \alpha \le 1} \inf (\alpha u, (1-\alpha) v). \tag{5}$$

Let us note then these operations were introduced and studied for $c^2$-lattice $Q(X)$ of all nonegative quasiconvex l.s.c.functuons $q$ such that $q(0) = 0$ in [17].

**Proposition 4.2.** *Let $P : U \to V$ be an antihomogeneous conjugacy operator. Then* $P(u \hat{+} v) = Pu \oplus Pv, \quad P(u \oplus v) = Pu \hat{+} Pv$ .

**Proof.** It is clear.
Let $u, v \in \overline{\mathbb{R}}_+$. It is easy to check (see for example [18]) that

$$\inf_{0 \le \alpha \le 1} \sup \left( \frac{1}{\alpha} u, \frac{1}{1-\alpha} v \right) = u + v. \tag{6}$$

Straightforward calculation shows also that

$$\sup_{0 \le \alpha \le 1} \inf (\alpha u, (1-\alpha) v) = \frac{1}{\frac{1}{u} + \frac{1}{v}}. \tag{7}$$

We now apply formulae (6) and (7) in order to describe the operations $\hat{+}$ and $\oplus$ defined on special $c^2$-lattices of functions. We will concider a $c^2$-lattice $U$ of functions defined on a set $X$ such that

i) the order relation defined on $U$ determined by natural way: $u_1 \ge u_2$ if and only if $u_1(x) \ge u_2(x) \; \forall x \in X$;

ii) if $A \subset U$ then $\sup A = \overline{\sup} A$ where $\overline{\sup} A$ is an upper envelope of the set $A$.

In the sequel $\mathbf{U}$ denotes the totality of all $c^2$-lattices with properties i) and ii). Clearly if $U \in \mathbf{U}$ and $A \subset U$ then $\inf A \le \overline{\inf} A$ where $\overline{\inf} A$ is the lower envelope of the set $A$. Let us note that $c^2$-lattices of functions which were indicated in the Section 2 (see situation 1 and situation 2 in this section) are members of the totality $U$.

**Proposition 4.3.** *Let $U \in \mathbf{U}$.*

1) *Suppose that the pointwise sum $u_1 + u_2 \in U$ whenever $u_1, u_2 \in U$. Then $u_1 \hat{+} u_2 = u_1 + u_2$ .*

2) *Suppose that the pointwise infimum $\overline{\inf}(u_1, u_2) \in U$ whenever $u_1, u_2 \in U$. Then $u_1 \oplus u_2$ coinsides with the inverse sum $u_1 \dot{+} u_2$ where*

$$u_1 \dot{+} u_2 = \frac{1}{\frac{1}{u_1} + \frac{1}{u_2}}.$$

*Let us note that the function $u_1 \dot{+} u_2$ belongs to $U$ in this case.*

**Proof.** 1) Let

$$w_\alpha = \overline{\sup} \left( \frac{1}{\alpha} u_1, \frac{1}{1-\alpha} u_2 \right) = \sup \left( \frac{1}{\alpha} u_1, \frac{1}{1-\alpha} u_2 \right) \quad (0 \le \alpha \le 1).$$

Formula (6) shows that $u_1 + u_2 = \overline{\inf}_{0\le\alpha\le 1} w_\alpha$. Since $u_1 + u_2 \in U$ it follows that $\overline{\inf}_{0\le\alpha\le 1} w_\alpha \in U$. Therefore

$$u_1 \hat{+} u_2 = \inf_{0\le\alpha\le 1} w_\alpha = \overline{\inf}_{0\le\alpha\le 1} w_\alpha = u_1 + u_2.$$

2) Applying (7) we see that

$$u_1 \oplus u_2 = \sup_{0\le\alpha\le 1} \inf(\alpha u_1, (1-\alpha)u_2) = \overline{\sup}_{0\le\alpha\le 1}\overline{\inf}(\alpha u_1, (1-\alpha)u_2) = \frac{1}{\frac{1}{u_1}+\frac{1}{u_2}}.$$

**Corollary 4.4.** *Let $U$ and $V$ be $c^2$-lattices and $P : U \to V$ be an antihomogeneous conjugacy operator. Then*

1. *If $V \in \mathbf{U}$ and $V$ contains $\overline{\inf}(v_1, v_2)$ whenever $v_1, v_2 \in V$ (for example $V$ is one of the following $c^2$-lattices : $F(X)$, $M_0(X)$, $M^0(X)$, $PH(X)$ then the operator $P$ maps a "sum" $u_1 \hat{+} u_2$ to the inverse sum of the functions $P(u_1)$ and $P(u_2)$.*
2. *If $V \in \mathbf{U}$ and $V$ contains the pointwise sum $v_1 + v_2$ whenever $v_1, v_2 \in V$ (for example $V$ is any one of the following $c^2$-lattices : $F(X)$, $M_0(X)$, $M^0(X)$, $PH(X)$, $SL(X)$, $CF(X)$ or $c^2$-lattice $C_1^H(Y)$ of $H$-convex functions) then the operator $P$ maps the "inverse sum" $u_1 \oplus u_2$ to the pointwise sum of the functions $P(u_1)$ and $P(u_2)$.*
3. *If $U \in \mathbf{U}$ and $U$ contains the pointwise sum $u_1 + u_2$ whenever $u_1, u_2 \in U$ then the operator $P$ maps a pointwise sum $u_1 + u_2$ to the "inverse sum" $P(u_1) \oplus P(u_2)$ .*

Let us give some examples. At first we consider $c^2$-lattices of sets. Let $A, B$ be subsets of a locally convex Hausdorff space $X$. We will use the following notation:

$$A + B = \{a + b : a \in A, b \in B\}, \qquad A \hat{\oplus} B = \bigcup_{0\le\alpha\le 1} [(\alpha A) \cap ((1-\alpha)B)].$$

It is easily seen that if $A$, $B$ are closed star with regards to zero sets then $A \hat{\oplus} B$ also is closed and star with regards to zero, convexity of $A$, $B$ implies convexity of $A \hat{\oplus} B$ . Let us note that operation $\hat{\oplus}$ is well known in Convex Analysis (see for example [1]). This operation was studied in [18] for special class of star with regards to zero sets. Below we consider dual vector spaces $X$ and $L$.

**Example 4.5.** Let $ST(X)$ be a $c^2$-lattice of all closed star with regards to zero subsets of the space $X$. Clearly if $A, B \in ST(X)$ then

$$A \hat{+} B = \bigcap_{0\le\alpha\le 1} [(\frac{1}{\alpha} A) \cup (\frac{1}{1-\alpha} B)], \qquad A \oplus B = A \hat{\oplus} B.$$

**Example 4.6.** Let $CS(X)$ be a $c^2$-lattice of all closed convex containing zero subsets of the space $X$. We have by definition

$$A \hat{+} B = \bigcap_{0\le\alpha\le 1} \mathrm{conv}[(\frac{1}{\alpha} A) \cup (\frac{1}{1-\alpha} B)], \quad A \oplus B = A \hat{\oplus} B.$$

where conv$A$ is a convex hull of the set $A$. On the other hand we can show that the next assertion is true.

**Proposition 4.7.** *If $A, B \in CS(X)$ then $A \hat{+} B = cl(A+B)$ .*

**Proof.** We apply the isomorphism $\varphi$ of $c^2$-lattices $CS(X)$ and $SL(L)$ where $\varphi(A)$ is the support function $p_A$ of the set $A$. By definition $p_A(l) = \sup_{x \in A}[l, x]$. Since $\varphi$ is isomorphism it follows that

$$\varphi(\inf_\alpha A_\alpha) = \inf_\alpha \varphi(A_\alpha), \qquad \varphi(\sup_\alpha A_\alpha) = \sup_\alpha \varphi(A_\alpha), \qquad \varphi(\lambda A) = \lambda\varphi(A) \quad \forall \lambda > 0.$$

We have

$$\begin{aligned} p_{A\hat{+}B} &= \varphi(A\hat{+}B) \\ &= \varphi(\inf_\alpha \sup\left(\frac{1}{\alpha}A, \frac{1}{1-\alpha}B\right)) \\ &= \inf_\alpha \sup\left(\frac{1}{\alpha}\varphi(A), \frac{1}{1-\alpha}\varphi(B)\right) \\ &= \inf_\alpha \sup\left(\frac{1}{\alpha}p_A, \frac{1}{1-\alpha}p_B\right) \\ &= p_A \hat{+} p_B = p_A + p_B. \end{aligned}$$

It is well known that $p_A + p_B = p_{\text{cl}(A+B)}$ . Therefore $A\hat{+}B = \text{cl}(A+B)$ .

**Corollary 4.8.** *If $A,\ B \in CS(X)$ and if $P : CS(X) \to CS(L)$ is an antihomogeneous conjugacy operator then $P(cl(A+B)) = P(A)\hat{\oplus}P(B)$ and $P(A\hat{\oplus}B) = cl(P(A)+P(B))$ .*

**Example 4.9.** Let us consider the $c^2$-lattice $SL(X)$. Proposition 4.3 shows that $p+q = p\hat{+}q$ for all $p, q \in SL(X)$.

**Proposition 4.10.** *If $A, B \in SL(X)$ then*

$$p \oplus q = \sup_{0 \le \alpha \le 1} (\alpha p)\tilde{\oplus}((1-\alpha)q)$$

*where $f\tilde{\oplus}g$ is the inf-convolution of the functions $f$ and $g$.*

**Proof.** Let $\partial p = \{x \in X : [l, x] \le p(l)\ \forall l \in L\}$, the subdifferential of a sublinear function $p$. Clearly $\partial p = \varphi^{-1}(p)$ where the mapping $\varphi$ was defined in the proof of the proposition 4.7. We have

$$\partial(p \oplus q) = \varphi^{-1}(p \oplus q) = \varphi^{-1}(\sup_\alpha \inf(\alpha p, (1-\alpha)q)) =$$

$$\sup_\alpha \inf(\alpha\varphi^{-1}(p), (1-\alpha)\varphi^{-1}(q)) = \sup_\alpha \inf(\alpha\partial p, (1-\alpha)\partial q) = \partial p \oplus \partial q = \partial p \hat{\oplus} \partial q.$$

It is well known ( see for instance [1]) that $\partial p \hat{\oplus} \partial q = \partial s$ where $s = \sup_{0 \le \alpha \le 1}(\alpha p)\tilde{\oplus}(1-\alpha)q$. Therefore $p \oplus q = s$.

**Example 4.11.** Let us compute a set whose gauge coincides with the sum of gauges of given sets.

**Proposition 4.12.** *Let $A, B \in ST(X)$ and $\mu_A$ be the gauge of $A$, and $\mu_B$ be the gauge of $B$. Then $\mu_A + \mu_B$ is a gauge of the set $A \hat{\oplus} B$.*

**Proof.** Let us consider the antihomogeneous conjugacy operator $P : PH(X) \to ST(X)$ where $P(\mu) = \{x : \mu(x) \le 1\}$. Clearly $P(\mu_A) = A,\ P(\mu_B) = B$. We have

$$P(\mu_A + \mu_B) = P(\mu_A) \oplus P(\mu_B) = A \oplus B = A \hat{\oplus} B$$

therefore the gauge of $A \hat{\oplus} B$ is equal to the sum $\mu_A + \mu_B$.

**Example 4.13.** Now let us consider the $c^2$-lattice $C_1^{\overline{H}}(Y)$ of $\overline{H}$-convex functions defined on the cone $Y$ and the $c^2$-lattice $C_2^{\overline{Y}}(H)$ of $\overline{Y}$-convex functions defined on the cone $H$ (see Example 3.8 of Section 3) and the antihomogeneous conjugacy operator $P : C_1^{\overline{H}}(Y) \to C_2^{\overline{Y}}(H)$ defined by formula (3). We have

$$P(f_1 + f_2)(x) = (P(f_1) \oplus P(f_2))(x) = \sup_{0 \le \alpha \le 1} \inf(\alpha \otimes P(f_1), (1-\alpha) \otimes P(f_2))(x)$$

$$= \sup_{0 \le \alpha \le 1} \mathrm{co}_H \overline{\inf}\, [\frac{1}{\alpha} P(f_1)(\alpha x), \frac{1}{1-\alpha} P(f_2)((1-\alpha)x)].$$

In particular if $f_1, f_2$ are l.s.c convex functions and $P(f)$ coincides with the conjugate function (in the sense of Convex Analysis) $f^*$ then

$$(f_1 + f_2)^* = \sup_{0 \le \alpha \le 1} \mathrm{conv}\, \overline{\inf}\, [\frac{1}{\alpha} f_1^*(\alpha x), \frac{1}{1-\alpha} f_2^*(1-\alpha)x)]$$

where conv$g$ is the convex regularization of the function $g$ i.e. $\mathrm{conv} g(x) = \sup\{a(x) : a \le g,\ a$ is an affine function$\}$. By the famous Moreau -Rockafellar theorem the equality $(f_1 + f_2)^* = \mathrm{conv}(f_1^* \tilde{\oplus} f_2^*)$ holds. Therefore the following is true.

**Proposition 4.14.** *If $f_1, f_2$ are convex l.s.c. functions then*

$$conv(f_1 \tilde{\oplus} f_2)(x) = \sup_{0 \le \alpha \le 1} conv\, \overline{\inf}[\frac{1}{\alpha} f_1(\alpha x), \frac{1}{1-\alpha} f_2((1-\alpha)x)].$$

Let us discuss Proposition 4.14. Clearly the inf-convolution is a global operation. In other words we must know quantities $f_1(y)$ and $f_2(y)$ for all $y \in X$ in order to compute the function $f_1 \tilde{\oplus} f_2$ at the given point $x \in X$. Proposition 4.14 shows that the main reason for the global nature of inf-convolution is the use of the convex hull of a nonconvex function. If $\mathrm{conv}(f_1 \tilde{\oplus} f_2)(x) = (f_1 \tilde{\oplus} f_2)(x)$ and if we can apply the convex hull for computing the function $f_1 \tilde{\oplus} f_2$ at the point $x$ then we have to know only the quantities $f_1(\alpha x)$ and $f_2(\alpha x)$ for $0 \le \alpha \le 1$.

## 5. Properties of the operations $\hat{+}$ and $\oplus$

Let $U$ be a $c^2$-lattice with the following properties:

i) If $0 < \lambda < +\infty$ and $\Omega$ is a subset of $U$ then

$$\sup\{\lambda u : u \in \Omega\} = \lambda \sup\{u : u \in \Omega\}, \qquad \inf\{\lambda u : u \in \Omega\} = \lambda \inf\{u : u \in \Omega\}.$$

ii) If $0 < \lambda, \mu < +\infty$ and $u \in U$ then

$$\sup(\lambda u, \mu u) = \sup(\lambda, \mu)u, \qquad \inf(\lambda u, \mu u) = \inf(\lambda, \mu)u.$$

Let us study some algebraic properties of the operations $\hat{+}$ and $\oplus$ defined by formulae (4) and (5) accordingly. Clearly these operations are commutative:

$$u \hat{+} v = v \hat{+} u, \qquad u \oplus v = v \oplus u.$$

Now let us clear up the connections between multiplication by a positive scalar and the operations $\hat{+}$ and $\oplus$. We have with $0 < \lambda < +\infty$

$$\lambda u \hat{+} \lambda v = \inf_{0 \le \alpha \le 1} \sup\left(\frac{\lambda}{\alpha}u, \frac{\lambda}{1-\alpha}v\right) = \lambda(u \hat{+} v),$$

$$\lambda u \oplus \lambda v = \sup_{0 \le \alpha \le 1} \inf(\lambda \alpha u, \lambda(1-\alpha)v) = \lambda(u \oplus v).$$

Applying (6) we have also

$$\lambda u \hat{+} \mu u = \inf_{0 \le \alpha \le 1} \sup\left(\frac{\lambda}{\alpha}u, \frac{\mu}{1-\alpha}u\right) = \inf_{0 \le \alpha \le 1} \sup\left(\frac{\lambda}{\alpha}, \frac{\mu}{1-\alpha}\right) u = (\lambda + \mu)u. \tag{8}$$

In the same manner we can see applying (7) that

$$\lambda u \oplus \mu u = \frac{1}{\frac{1}{\lambda} + \frac{1}{\mu}} u. \tag{9}$$

We see that the $c^2$-lattice $U$ generates two algebraical systems. The first of them is $U$ with operations $\hat{+}$ and $\cdot$ , the second is $U$ with operations $\oplus$ and $\odot$ where $\lambda \odot u = \frac{1}{\lambda} \cdot u$. We showed that commutative and distributive laws hold in these systems. Clearly a conjugacy operator $P : U \to V$ is an isomorphism between $(U, \hat{+}, \cdot)$ and $(V, \oplus, \odot)$.

Unfortunately associativity does not hold without additional assumptions in the algebraical systems under consideration. Let us give an example.

**Example 5.1.** Let us consider the $c^2$-lattice $Q(\mathbb{R}^2)$ of all nonnegative l.s.c. vanishing at zero and quasiconvex functions defined on the plane $\mathbb{R}^2$ (see Situation 1 of Section 2). We set for $x = (x_1, x_2)$

$$q_1(x) = \begin{cases} 1 & \text{if } x_1 \ge 1 \\ 0 & \text{if } x_1 < 1 \end{cases}$$

$$q_2(x) = \begin{cases} 1 & \text{if } x_2 \ge 1 \\ 0 & \text{if } x_2 < 1 \end{cases}$$

$$q_3(x) = \begin{cases} +\infty & x \neq \lambda(1,1) \quad (\lambda \geq 0) \\ \lambda & x = \lambda(1,1) \quad (\lambda \geq 0) \end{cases}$$

Clearly $q_i \in Q(\mathbb{R}^2)$, $i = 1,2,3$. Let $E = \{x = (x_1, x_2) : x_1 \leq 1, x_2 \leq 1\}$. Since $(q_1 \hat{+} q_2) = \mathrm{co}_{Q(\mathbb{R}^2)}(q_1 + q_2)$ we can compute that

$$(q_1 \hat{+} q_2)(x) = \begin{cases} 1 & x \in R^2 \setminus E \\ 0 & x \in E \end{cases}$$

Therefore $q_1 \hat{+} q_2 \neq q_1 + q_2$. On the other hand $(q_1 \hat{+} q_2) \hat{+} q_3 = (q_1 \hat{+} q_2) + q_3$ and $q_1 \hat{+} (q_2 \hat{+} q_3)$ $= q_1 + (q_2 + q_3)$. Applying these equalities we can show that

$$(q_1 \hat{+} q_2) \hat{+} q_3 \neq q_1 \hat{+} (q_2 \hat{+} q_3).$$

There are many $c^2$-lattices where the operation $\hat{+}$ and $\oplus$ are associative. By proposition 3 we have: if $U \in \mathbf{U}$ and $u_1 + u_2 \in U$ whenever $u_1, u_2 \in U$ then associativity holds for operation $\hat{+}$; if $U \in \mathbf{U}$ and $\overline{\inf}(u_1, u_2) \in U$ whenever $u_1, u_2 \in U$ then associativity holds for operation $\oplus$. Let us give one more condition which guarantees associativity.

**Proposition 5.2.** *If $U$ is a $c^2$-lattice with distributive laws: for all $u \in U$ and for an arbitrary family $(v_\xi)_\xi \in U$ the following equalities hold*

$$\inf(u, \sup_\xi v_\xi) = \sup_\xi \inf(u, v_\xi) \tag{10}$$

$$\sup(u, \inf v_\xi) = \inf_\xi \sup(u, v_\xi) \tag{11}$$

*then operations $\hat{+}$ and $\oplus$ are associative.*

**Proof.** We first compute $(u \oplus v) \oplus w$. Let us take the formula (10) for this purpose. We have

$$(u \oplus v) \oplus w = \sup_{0 \leq \gamma \leq 1} \inf(\gamma(u \oplus v), (1-\gamma)w) =$$

$$\sup_{0 \leq \gamma \leq 1} \inf[\sup_{0 \leq \alpha \leq 1} \inf(\gamma\alpha u, \gamma(1-\alpha)v), (1-\gamma)w] =$$

$$\sup_{0 \leq \gamma \leq 1} \sup_{0 \leq \alpha \leq 1} \inf(\gamma\alpha u, \gamma(1-\alpha)v, (1-\gamma)w) =$$

$$\sup_{\lambda,\mu,\nu \geq 0; \lambda+\mu+\nu=1} \inf(\lambda u, \mu v, \nu w).$$

In the same manner we can see that

$$u \oplus (v \oplus w) = \sup_{\lambda,\mu,\nu \geq 0; \lambda+\mu+\nu=1} \inf(\lambda u, \mu v, \nu w).$$

Therefore associativity holds for the operation $\oplus$. Applying (11) we obtain associativity for the operation $\hat{+}$.

**Proposition 5.3.** *If $U, V$ are $c^2$-lattices and if there is an antihomogeneous conjugacy operator $P : U \to V$ then the following are equivalent:*

i) *associativity holds for operation* $\hat{+}$ *in the* $U$,
ii) *associativity holds for operation* $\oplus$ *in the* $V$.

**Proof.** This is clear.

Let us give some examples.

**Example 5.4.** Operation $\oplus$ is associative in the $c^2$-lattices $SL(X), CF(X), CS(X)$, and $ST(X)$.

**Example 5.5.** Since there is an antihomogeneous conjugacy operator $P : Q(X) \to Q(X)$ operation $\oplus$ is not associative in the $Q(X)$.

**Example 5.6.** Let $U$ be a $c^2$-lattice such that at least one of the operations $\hat{+}$ and $\oplus$ is associative. Then there is no antihomogeneous conjugacy operator $P : Q(X) \to U$.

## 6. Operator of polarity

In this section we consider the operator of polarity $\pi : CS(X) \to CS(X')$ where $X$ is a locally convex Hausdorff space with conjugate space $X'$. We consider $X$ and $X'$ as dual vector spaces, in particular we assume that weak* topology is introduced in the space $X'$. By definition $\pi(A) = A^\circ$ where $A^\circ$ is the polar set of a set $A$. Let $T : X \to X$ be a weakly continuous linear operator. It is well known that

$$\pi \circ T = (T^*)^{-1} \circ \pi$$

where $T^*$ is the conjugate operator with regard to $T$ . So we are interested in operators $P : CS(X) \to CS(X')$ such that

$$P \circ T = (T^*)^{-1} \circ P. \tag{12}$$

As it turns out an antihomogeneous conjugacy operator $P : CS(X) \to CS(X')$ is the operator of polarity if equality (12) holds only for the class of simplest operators $T$ and besides there is the unique set $A$ such that $P(A) = \pi(A)$.

Let $l \in X'$ and $u \in X$ such that $[l, u] = 1$. Set $T_{l,u}(x) = x - [l, x]u$ for all $x \in X$. Clearly $T_{l,u}$ is a projection from $X$ on hyperplane $H = l^{-1}(0)$. On the other hand if $H = l^{-1}(0)$ is a hyperplane and $T$ is a projection from $X$ on $H$ then there is a vector $u \in X$ such that $[l, u] = 1$ and $T = T_{l,u}$. Let $\tau$ be the set of all projections from $X$ on closed hyperplanes. We have the following:

**Proposition 6.1.** *Let* $P : CS(X) \to CS(X')$ *be an antihomogeneous conjugacy operator. Assume that (12) is true for all* $T \in \tau \cup \{-Id\}$ *where* $Id$ *is the identity and there is a nonzero vector* $v \in X$ *such that* $P(s[0, v]) = \pi(s[0, v])$ *where* $s[0, v]$ *is the segment* $\{\lambda v : 0 \leq \lambda \leq 1\}$. *Then* $P = \pi$.

**Proof.** Let us consider $w \in X$, $w \neq \lambda v$ for all real $\lambda$ and an operator $T \in \tau$ such that $T(v) = w$. For example $T(x) = x - [l, x](v - w)$ with $L \in X'$, $[l, v] = 1$, $[l, w] = 0$. We have

$$P(s[0, w]) = P(T(s[0, v]) = (T^*)^{-1}P(s[0, v]) = (T^*)^{-1}\pi(s[0, v]) = \pi(s[0, w]).$$

In the same manner we can see that $P(s[0,-v]) = \pi(s[0,-v])$. Since $P$ and $\pi$ are antihomogeneous conjugacy operators we have $P(s[0,\lambda v]) = \pi(s[0,\lambda v])$ for all real $\lambda$. Now let $A \in CS(X)$. We have $A = \cup_{x\in A}[0,x]$ and therefore

$$P(A) = \bigcap_{x\in A=} P(s[0,x]) = \bigcap_{x\in A} \pi(s[0,x]) = \pi(A).$$

Sometimes we can consider a more interesting set instead of a segment $s[0,v]$. Let $H$ be a Hilbert space. It is well known that the unit ball $B$ is the unique fixed point of the polarity operator $\pi : CS(H) \to CS(H)$. An antihomogeneous conjugacy operator $P : CS(H) \to CS(H)$ with the property $P(B) = \pi(B)$ has been considered in the paper [19] and conditions which guarantee either equality $P = \pi$ or equality $P = -\pi$ were given in that paper. Unfortunately there is a minor defect in the proof of the theorem in [19]. Below we will consider a set $B$ in a locally convex Hausdorff space $X$ with the following properties:

i) $B$ is a symmetrical closed convex set and $\mathrm{int} B \neq \emptyset$;

ii) there is a linear function $l \in X'$ and projection $T_0$ from $X$ on $H = l^{-1}(0)$ such that $T_0(B) \subset B$;

iii) there is an element $y \in B$ such that $[l,y] = \sup_{x\in B}[l,x]$.

Clearly the unit ball of a Hilbert space possesses properties i) - iii); there are many sets with properties i) - iii) in a finite - dimensional space. A strip $\{x : -1 \le [l,x] \le 1\}$ in the locally convex space $X$ where $l \in X'$ has these properties also.

We will denote by $\tau_1$ the set of all linear projections from $X$ on a straight line. Clearly $T \in \tau_1$ if and only if there are $l \in X'$ and $u \in X$ such that $T(x) = [l,x]u$.

**Theorem 6.2.** *Let $P : CS(X) \to CS(X')$ be an antihomogeneous conjugacy operator. Assume that the formula (12) is true for all $T \in \tau \cup \tau_1 \cup \{-Id\}$ and there is a subset $B$ of the space $X$ such that $P(B) = \pi(B)$ and properties* i) - iii) *above hold. Then either $P = \pi$ or $P = -\pi$, where $-\pi(A) = -A^\circ$*

**Proof.** Let the projector $T_0$ (see ii) above) have a form $T_0(x) = x - [l,x]u$ with $[l,u] = 1$. We will prove that either $P(s[0,u]) = \pi(s[0,u])$ or $P(s[0,u]) = -\pi(s[0,u])$ and then apply Proposition 6.1. The proof will be divided into four steps.

1. Let us consider the sets $C = \{x \in B : [l,x] = 0\} = B \cap H$ and $C' = \{f \in B^\circ : [f,u] = 0\} = B^\circ \cap H'$. Here $H' = u^{-1}(0)$. We first show that the equality $P(C) = C' + \mathbb{R}l$ holds where $\mathbb{R}l$ is a line $(\lambda l)_{-\infty<\lambda<+\infty}$. It is easy to check that the conjugate with regard to $T_0$, operator$T_0^*$, is a projection from $X'$ on $H'$ which has the form $T_0^*(f) = f - [f,u]l$ . It is easily seen that $(T_0^*)^{-1}(f) = f + \mathbb{R}l$. Let us check the equality $T_0(B) = C$. Applying inclusions $T_0(B) \subset B$ and $T_0(B) \subset H$ we obtain $T_0(B) \subset C$. On the other hand inclusion $B \supset C$ implies $T_0(B) \supset T_0(C) = C$. Since $T_0(B) = C$ and $T_0 \in \tau$ we have

$$P(C) = (T_0^*)^{-1}P(B) = (T_0^*)^{-1}(B^\circ) = (T_0^*)^{-1}(C') = C' + \mathbb{R}l.$$

2. Let $W = \{x \in B : [l,x] \ge 0\}$ and

$$Z_+ = B^\circ + \mathbb{R}_+ l, \qquad Z_- = B^\circ - \mathbb{R}_+ l. \tag{13}$$

where $\mathbb{R}_+ l$ is the ray $(\lambda l)_{\lambda\ge 0}$. Since $B$ is a symmetrical set it follows that $B^\circ$ is a symmetrical set too. Therefore $Z_- = -Z_+$. Now we prove that either $P(W) = Z_+$

or $P(W) = Z_-$. We will denote $P(W)$ by $Z$. Since $B$ is a symmetrical set we have $-W = \{x \in B : [l, x] \leq 0\}$. Therefore

$$W \cup (-W) = B \qquad W \cap (-W) = C.$$

On the other hand $P(-W) = P((-Id)W) = -Z$. As $P$ is an antihomogeneous conjugacy operator and $P(C) = C' + \mathbb{R}l$ we have the next system of two equations with regard to an unknown convex $w^*$-closed set $Z$:

$$Z \cap (-Z) = B^\circ \tag{14}$$

$$\text{cl conv}(Z \cup (-Z)) = C' + \mathbb{R}l. \tag{15}$$

We will show that there are only two solutions of the system (14) - (15): one of them is $Z_+$ and the other is $Z_- = -Z_+$, where $Z_+$, $Z_-$ are defined by formula (13).
First of all let us note that assumption $\text{int} B \neq \emptyset$ implies $w^*$ -compactness of the set $B^\circ$ and therefore the set $C'$ is $w^*$ -compact. Let $K$ be the recession cone of the set $Z$. Applying (15) and weak* - compactness of $C'$ we conclude that $K \subset \mathbb{R}l$. Clearly $-K$ is a recession cone of $(-Z)$ so (14) shows that $K \neq \mathbb{R}l$. Our next goal is to show that $K \neq \{0\}$.
Let $q$ denote the function defined on the set $C'$ by formula

$$q(f) = \sup\{\lambda : f + \lambda l \in Z\}$$

Since $Z$ is a convex $w^*$ -closed set it follows that $q$ is a concave and $w^*$ -upper semicontinuous function. Assume that $K = \{0\}$. We have $q(f) < +\infty$ for all $f \in C'$ in this case. The $w^*$ -compactness of the set $C'$ implies that the functuon $q$ is bounded from above on this set. Thus

$$\sup_{f \in C'} \sup\{\lambda : f + \lambda l \in Z\} < +\infty.$$

In the same manner we can conclude that

$$\sup_{f \in C'} \sup\{\lambda : f + \lambda l \in (-Z)\} < +\infty.$$

From (15) we obtain that $\text{cl conv}(Z \cup (-Z)) \subset C' + \mathbb{R}l$. Since $[f, u] = 0 \ \forall f \in C'$ and $[l, u] = 1$ we have

$$\begin{aligned}
&\sup\{[g, u] : g \in \text{cl conv}(Z \cup (-Z)\} \\
&= \sup\{[g, u] : g \in (Z \cup (-Z)\} \\
&= \sup\{[f + \lambda l, u] : f \in C', f + \lambda l \in Z \cup (-Z)\} \\
&= \sup\{\lambda : f \in C', f + \lambda l \in Z \cup (-Z)\} < +\infty
\end{aligned} \tag{16}$$

On the other hand applying (15) we can assert that

$$\sup\{[g, u] : g \in \text{cl conv}(Z \cup (-Z))\} = \sup\{[g, u] : g \in C' + \mathbb{R}l\} = +\infty$$

which contradict the formula (16).

Thus $K \subset \mathbb{R}l$, $K \neq \mathbb{R}l$ and $K \neq \{0\}$. Therefore $K$ is either $\mathbb{R}_+l$ or $-\mathbb{R}_+l$. Assume that $K = \mathbb{R}_+l$. It follows from (14) that $B^\circ \subset Z$. By the definition of recession cone we have

$$Z_+ = B^\circ + \mathbb{R}l \subset Z + \mathbb{R}l = Z$$

Let us show that $Z_+ = Z$ in this case. Let $f \in Z$. Applying (15) we can find $g \in C'$ and a real $\mu$ such that $f = g + \mu l$. If $\mu \geq 0$ then $f \in C' + \mathbb{R}_+l \subset B^\circ + \mathbb{R}_+l = Z_+$. Now let us assume $\mu < 0$. We have in this case

$$-f = -g + (-\mu)l \subset C' + \mathbb{R}_+l \subset Z_+ \subset Z.$$

Therefore $f \in -Z$ and $f \in (-Z) \cap Z$. Applying (14) we have

$$f \in B^\circ \subset B^\circ + \mathbb{R}_+l = Z_+.$$

The equality $Z = Z_+$ is proved. In the same manner we can see that $Z = Z_-$ if $K = -\mathbb{R}_+$
3. Let us compute $\pi(W)$. We will denote by $H_+$ the half-space $\{x : [l, x] \geq 0\}$ of the space $X$. Clearly $\pi(H_+) = -\mathbb{R}_+l$ . Since $W = B \cap H_+$ we have

$$\pi(W) = \operatorname{conv}(B^\circ \cup (-\mathbb{R}_+l)) = B^\circ - \mathbb{R}_+l = Z_-.$$

It follows that either $P(W) = \pi(W)$ or $P(W) = -\pi(W)$.
4. Let $T_*$ be the projection operator from $X$ on the line $\mathbf{R}u$ which is defined by formula $T_*(x) = [l, x]u$. If $P(W) = \pi(W)$ then

$$P(s[0, \lambda u]) = P(T_*(W)) = T_*^{-1}P(W) = T_*^{-1}\pi(W) = \pi(T_*(W)) = \pi(s[0, \lambda u]).$$

In the same manner we can see that $P(s[0, \lambda u]) = -\pi(s[0, \lambda u])$ if $P(W) = -\pi(W)$.
Now we can use either Proposition 6.1 itself or a version of this proposition for the operator $-\pi$ in order to complete the proof.

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