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Norms that Generate the Same Wijsman Topology on Convex Sets

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Dedicated to R. T. Rockafellar on his 60th Birthday

In a Banach space X, we introduce a criterion for comparing the Wijsman topologies that are induced by two equivalent norms of X on the hyperspace of closed convex sets $C(X)$. Thereafter, we study the duality map associated with the unit ball of a given norm of X in relation to its composition with the polarity map. This more geometrical description of the norm allows us to give a direct proof of a known theorem (see [3] and [1]): If X is reflexive and the duality map is n-to-n-usco, then the Wijsman topology coincides with the Mosco topology on $C(X)$.

1. Introduction

Let X be a Banach space and let $C(X)$ be the family of closed and convex subsets of X. The Wijsman topology on $C(X)$ is defined to be the weakest topology for which the distance functionals $d(x,.) : C(X) \longrightarrow \mathbb{R}$ are continous for all $x \in X$. This topology can also be viewed as the union of two other topologies: the lower Vietoris which is generated by all subsets of $C(X)$ of the form $V^- = \{A \in C(X) \mid A \cap V \neq \emptyset\}$, where V is open in X; and the upper Wijsman which is generated by all sets of the form $(B^c)^{++} = \{A \in C(X) \mid \inf_{a\in A\ b\in B} \|a - b\| = D(A,B) > 0\}$, where B is a homothetic translation of the unit ball of X. Clearly the lower Vietoris depends only on the open sets of X, hence only on the topology of X, while the upper Wijsman depends also on the norm chosen on X. Our purpose is to give a geometric criterion for deciding when two equivalent norms of X generate the same upper Wijsman topology on $C(X)$ and, in order to illustrate a possible application of this criterion, we will give another proof of a theorem of Borwein and Fitzpatrick (see [3]). This theorem says that, if X is reflexive and the norm p on X is sufficiently "smooth" (i.e. the duality map is n-to-n-usco), then the Wijsman topology associated to p is the maximum of all the Wijsman topologies induced by equivalent norms, i.e. is the Mosco topology (see [2] for this).

In the following, U and S will always denote, respectively, the closed unit ball and the unit sphere. We will add an upper-star to objects lying in the dual space.

In the next section we illustrate a geometrical criterion involving unit balls of X which allows us to compare the corresponding Wijsman topologies on $C(X)$. In the third section we study the relationship between the unit ball and the polar of its associated duality map under the hypothesis that the duality map is norm-to-norm-upper-semicontinous and

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norm-compact valued. Finally, in section 4 we give a different proof of the theorem of Borwein and Fitzpatrick.

2. A criterion for comparing Wijsman topologies

The criterion we propose in this section has been inspired by the one given in [4] in the case of the hyperspace of closed subsets of a metric space. Suppose that p_0 and p_1 are two equivalent norms on X. For any $x^* \in X^*$ with $\|x^*\|_1 = 1$ and any $\epsilon > 0$, consider the closed half-space $H(x^*, \epsilon) = \{x \in X \mid \langle x^*, x\rangle \leq 1 + \epsilon\}$. Note that $H(x^*, \epsilon)$ contains U_1, because $\sup_{x \in U_1} \langle x^*, x\rangle = \|x^*\|_1 = 1 < 1 + \epsilon$.

Definition 2.1. We say that p_0 covers p_1 if, for every $x^* \in S_1^*$ and for every $\epsilon > 0$, there are $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n > 0$ such that:

$$U_1 \subset \bigcup_{i=1}^{n} (x_i + \lambda_i U_0) \subset H(x^*; \epsilon)$$

In other words, p_0 covers p_1, if, whenever we choose a hyperplane with positive distance from the ball U_1, it is always possible to find a finite number of p_0–balls whose union contains U_1 but does not intersect that hyperplane. The next theorem will justify the choice of such a criterion, but first we need to state some preliminary well-known facts, which we will prove for the sake of completeness.

Lemma 2.2.
1. *If $D \subset X$, then for all $x \in X$ and $\lambda > 0$: $x + \lambda(D^c)^{++} = [(x + \lambda D)^c]^{++}$.*
2. *If $D_i \subset X$ for $i = 1, \ldots, n$, then: $[(\cup_{i=1}^n D_i)^c]^{++} = \cap_{i=1}^n (D_i^c)^{++}$.*
3. *If E is closed and $D \subset X$: $D \subset E \iff (E^c)^{++} \subset (D^c)^{++}$.*

Proof.
1. Note that $D(x + \lambda A, x + \lambda D) > 0 \Leftrightarrow \inf_{a \in A\ b \in D} \|(x + \lambda a) - (x + \lambda b)\| > 0 \Leftrightarrow \inf_{a \in A\ b \in D} \lambda \|a - b\| > 0 \Leftrightarrow D(A, D) > 0$. So $x + \lambda A \in [(x + \lambda D)^c]^{++} \Leftrightarrow A \in (D^c)^{++} \Leftrightarrow x + \lambda A \in x + \lambda(D^c)^{++}$.
2. If $A \in [(\cup_{i=1}^n D_i)^c]^{++}$, then $D(A, \cup_{i=1}^n D_i) > 0$. So $D(A, D_i) > 0$ for all $i = 1, \ldots, n$. Hence $A \in (D_i^c)^{++}$ for all i and $A \in \cap_{i=1}^n (D_i^c)^{++}$. Conversely, suppose $D(A, D_i) > 0$ for all $i = 1, \ldots, n$. Then $D(A, \cup_{i=1}^n D_i) = \min_{i=1 \ldots n} D(A, D_i) > 0$, so $A \in [(\cup_{i=1}^n D_i)^c]^{++}$.
3. Suppose $D \subset E$, then if $A \in (E^c)^{++}$, $D(A, D) \geq D(A, E) > 0$. So $A \in (D^c)^{++}$. Conversely, suppose $(E^c)^{++} \subset (D^c)^{++}$. If $D \not\subset E$, pick $x \in D \setminus E$. Then, since E is closed, $d(x, E) > 0$. So $\{x\} \in (E^c)^{++} \subset (D^c)^{++}$, hence $d(x, D) > 0$, but $x \in D$, and this is a contradiction. $\square$

Theorem 2.3. *For $i = 0, 1$, let $\tau^+_{W_{p_i}}$ be the upper Wijsman topology induced by the norm p_i on $C(X)$, the space of closed convex subsets of X. Then:*

$$p_0 \text{ covers } p_1 \iff \tau^+_{W_{p_0}} \geq \tau^+_{W_{p_1}}$$

Proof. First we assume that p_0 covers p_1. It is enough to show that the subbase of $\tau^+_{W_{p_1}}$, as defined in Section 1, is contained in $\tau^+_{W_{p_0}}$. Consider a homothetic translation of the unit ball for the norm p_1: $B = x + \lambda U_1$, with $x \in X$ and $\lambda > 0$. By Lemma 2.2 (1), we have $x + \lambda(U_1^c)^{++} = [(x + \lambda U_1)^c]^{++}$. Therefore, we only need to prove that $(U_1^c)^{++}$ is open in $\tau^+_{W_{p_0}}$, because the topology $\tau^+_{W_{p_0}}$ is invariant under translations and positive dilations (this is easily checked on its subbase). So, let $\hat{A}$ be a fixed closed convex set in $(U_1^c)^{++}$. Since $D(\hat{A}, U_1) > 0$, there is $\delta > 0$ such that $(1 + \delta)U_1 \cap \hat{A} = \emptyset$. Then, the Hahn–Banach theorem gives us $x^* \in S_1^*$ such that:

$$\sup_{z \in U_1} \langle x^*, z\rangle = 1 \ < \ 1 + \delta \ < \ \inf_{a \in \hat{A}} \langle x^*, a\rangle \tag{2.1}$$

We know that p_0 covers p_1, therefore we can cover U_1 with homothetic translations of U_0 without exceeding $H(x^*; \delta)$. That is to say, there are $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n > 0$ such that:

$$U_1 \subset \bigcup_{i=1}^{n} (x_i + \lambda_i U_0) \subset H(x^*; \delta) \tag{2.2}$$

Define $B_i = x_i + \lambda_i U_0$ and $B = \cup_{i=1}^n B_i$. We claim that $(B^c)^{++}$ is an open set in $\tau^+_{W_{p_0}}$ which contains $\hat{A}$ and is contained in $(U_1^c)^{++}$. From the second inclusion in (2.2) we have that $B \subset H(x^*; \delta)$. Moreover, since $H(x^*; \delta)$ is a half-space, for any $a \notin H(x^*; \delta)$, $d(a, H(x^*; \delta)) = \langle x^*, a\rangle - (1 + \delta)$. Hence, by the second inequality in (2.1), we have that: $D(\hat{A}, B) \geq D(\hat{A}, H(x^*, \delta)) = \inf_{a \in \hat{A}} \langle x^*, a\rangle - \ (1 + \delta) > 0$. Therefore $\hat{A} \in (B^c)^{++}$. Moreover, by the first inclusion in (2.2), $U_1 \subset B$, so by Lemma 2.2 (3), $(B^c)^{++} \subset (U_1^c)^{++}$. Thus, we proved that $\hat{A} \subset (B^c)^{++} \subset (U_1^c)^{++}$. Finally, by Lemma 2.2 (2), we have $(B^c)^{++} = [(\cup_{i=1}^n B_i)^c]^{++} = \cap_{i=1}^n (B_i^c)^{++}$, which ensures that $(B^c)^{++}$ is an open set in $\tau^+_{W_{p_0}}$. Since $\hat{A}$ was arbitrary, $(U_1^c)^{++}$ is also open in $\tau^+_{W_{p_0}}$. So $\tau^+_{W_{p_0}} \geq \tau^+_{W_{p_1}}$.

Conversely, suppose that $\tau^+_{W_{p_0}} \geq \tau^+_{W_{p_1}}$. Fix $x^* \in S_1^*$ and $\epsilon > 0$. Then the closed half-space $A = \{x \in X \mid \langle x^*, x\rangle \geq 1 + \epsilon\}$ belongs to $(U_1^c)^{++} \cap C(X)$, because $D(A, U_1) = \inf_{a \in A \ z \in U_1} \|a - z\| \geq \inf_{a \in A \ z \in U_1} \langle x^*, a - z\rangle = \inf_{a \in A} \langle x^*, a\rangle - \sup_{z \in U_1} \langle x^*, z\rangle = (1 + \epsilon) - 1 = \epsilon > 0$. So there is an open set $\mathcal{B}$ in $\tau^+_{W_{p_0}}$ that contains A and is itself contained in $(U_1^c)^{++}$. Now, $\mathcal{B}$ can be chosen to be in the base of $\tau^+_{W_{p_0}}$, i.e. $\mathcal{B} = \cap_{i=1}^n (B_i^c)^{++}$, where B_i are homothetic translations of U_0. By Lemma 2.2, $\cap_{i=1}^n (B_i^c)^{++} = [(\cup_{i=1}^n B_i)^c]^{++} \subset (U_1^c)^{++}$ implies $U_1 \subset \cup_{i=1}^n B_i$; while, $A \in [(\cup_{i=1}^n B_i)^c]^{++}$ implies $\cup_{i=1}^n B_i \subset A^c \subset cl(A^c) = H(x^*; \epsilon)$. So $U_1 \subset \cup_{i=1}^n B_i \subset H(x^*; \epsilon)$. □

Accordingly, if a norm p covers all other equivalent norms, then the Wijsman topology induced by p will be the maximum of all the Wijsman topologies induced by equivalent norms. This observation will be our main tool for proving the theorem of Borwein and Fitzpatrick. The intuition is that the unit ball U of a norm that covers any equivalent norm cannot have "sharp corners", i.e. must satisfy some rotundity property, in order to finitely cover any other equivalent ball without exceeding any given hyperplane. In this context the right object to consider for studying the corners of U is the polar of the duality map.

3. The unit ball and the polar of its duality map

3.1. The smallest cones containing the unit ball

Let p be a given norm on X. The duality map associated with p is a multifunction $J : S \longrightarrow S^*$ defined as follows:

$$\text{For all } x \in S, \qquad J(x) = \{x^* \in S^* \mid \langle x^*, x\rangle = 1\}.$$

Note that J has always non-empty, weak*-compact and convex values (see [6]). Recall that for an arbitrary set $A \subset X^*$ we can define its polar set in X to be:

$$A^\circ = \{x \in X \mid \sup_{a\in A}\langle a, x\rangle \leq 1\} = \bigcap_{a\in A}\{x \in X \mid \langle a, x\rangle \leq 1\}$$

Clearly, A° is always closed and convex. In what follows our attention will focus on the polar set of the set $J(\bar{x})$ for a fixed $\bar{x} \in S$. We will show that $J(\bar{x})^\circ$ is the smallest closed convex cone with vertex at $\bar{x}$ containing the unit ball U. But first we need to better describe the latter.
So, fix $\bar{x} \in S$. Define $B_\lambda = \bar{x} + \lambda(U - \bar{x})$, for all $\lambda > 0$, and let $B = \cup_{\lambda>0} B_\lambda$.

Lemma 3.1. *The B_λ form an increasing family of balls; $\bar{x}$ is in every $\partial B_\lambda = \bar{x}+\lambda(S-\bar{x})$, for all $\lambda > 0$; also $B = \cup_{\lambda>0}\partial B_\lambda = \bar{x} + \cup_{\lambda>0}\lambda(S - \bar{x})$; and finally, $cl(B)$ is the smallest closed convex cone with vertex at $\bar{x}$ containing U.*

Proof. Suppose $x \in B_\lambda$; then $x = \bar{x} + \lambda(y - \bar{x})$ for some $y \in U$. Let $\mu > \lambda$, then we also have $x = \mu[\frac{\lambda}{\mu}y + (1 - \frac{\lambda}{\mu})\bar{x}] + (1 - \mu)\bar{x}$. But $0 < \frac{\lambda}{\mu} < 1$, so by convexity $\frac{\lambda}{\mu}y + (1 - \frac{\lambda}{\mu})\bar{x} \in U$, hence $x \in B_\mu$. Therefore B_λ is increasing in λ.
Next, since $\bar{x} \in S$, $\bar{x} = \bar{x} + \lambda(\bar{x} - \bar{x}) \in \partial B_\lambda$ for all $\lambda > 0$.
To see that $B = \bar{x} + \cup_{\lambda>0}\lambda(U - \bar{x}) = \bar{x} + \cup_{\lambda>0}\lambda(S - \bar{x})$, first note that $S \subset U$. So $\bar{x}+\cup_{\lambda>0}\lambda(S-\bar{x}) \subset B$. Hence, it is enough to show that $B \subset \bar{x}+\cup_{\lambda>0}\lambda(S-\bar{x})$. Let $x \in B$, then $x \in B_\lambda$ for some $\lambda > 0$. Let $\lambda_0 = \inf\{\lambda > 0 \mid x \in B_\lambda\} = \inf\{\lambda > 0 \mid \frac{x-\bar{x}}{\lambda} \in U - \bar{x}\}$. We can suppose that $x \neq \bar{x}$, so, for $\lambda < \frac{\|x-\bar{x}\|}{2}$, we have $\|\frac{x-\bar{x}}{\lambda}\| > 2$, i.e. $\frac{x-\bar{x}}{\lambda} \notin U - \bar{x}$, thus $\lambda_0 > 0$. For all $\lambda > \lambda_0$, $\frac{x-\bar{x}}{\lambda} \in U - \bar{x}$, but $U - \bar{x}$ is closed, so $\frac{x-\bar{x}}{\lambda_0} \in U - \bar{x}$. Moreover, for $\lambda < \lambda_0$, $\frac{x-\bar{x}}{\lambda} \notin U - \bar{x}$. Therefore, $\frac{x-\bar{x}}{\lambda_0} \in (U - \bar{x}) \cap cl((U - \bar{x})^c) = S - \bar{x}$. Hence $x \in \bar{x} + \lambda_0(S - \bar{x})$. Thus $B = \bar{x} + \cup_{\lambda>0}\lambda(S - \bar{x})$.
Now let us prove that B is a convex cone with vertex in $\bar{x}$. For this we show that $C = \cup_{\lambda>0}\lambda(U - \bar{x})$ is a convex cone with vertex at the origin. Consider $x_1, x_2 \in C$, then there are $\lambda_1, \lambda_2 > 0$ and $y_1, y_2 \in U$ such that: $x_i = \lambda_i(y_i - \bar{x})$ for $i = 1, 2$. Therefore:

$$\begin{aligned} x_1 + x_2 &= \lambda_1 y_1 + \lambda_2 y_2 - (\lambda_1 + \lambda_2)\bar{x} \\ &= (\lambda_1 + \lambda_2)\frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2} - (\lambda_1 + \lambda_2)\bar{x} \end{aligned}$$

Since U is convex $x_1 + x_2 \in (\lambda_1 + \lambda_2)(U - \bar{x}) \subset C$. Hence, $C + C \subset C$. Thus, C is a convex cone. Therefore, $cl(B) = cl(\bar{x} + C)$ is a closed convex cone with vertex at $\bar{x}$. Moreover, all convex cones containing $U - \bar{x}$ must contain $\lambda(U - \bar{x})$ for all $\lambda > 0$, hence must contain C. Therefore $cl(B)$ is the smallest convex cone containing U with vertex at $\bar{x}$. □

Proposition 3.2. *Let $\bar{x} \in S$ and B be defined as above. Then $J(\bar{x})^\circ = cl(B)$.*

Proof. Consider $x \in B_\lambda$; then $x = \bar{x} + \lambda(y - \bar{x})$ with $y \in U$. Thus, for all $x^* \in J(\bar{x})$, $\langle x^*, x\rangle = 1 + \lambda(\langle x^*, y\rangle - 1) \leq 1$. So $x \in J(\bar{x})^\circ$. Therefore, $B \subset J(\bar{x})^\circ$, hence $cl(B) \subset J(\bar{x})^\circ$, since, by its definition, $J(\bar{x})^\circ$ is closed. Conversely, suppose that $x \in J(\bar{x})^\circ \setminus cl(B)$. Since $cl(B)$ is closed and convex, by the Hahn–Banach theorem we can find $d > 0$ and $x^* \in S^*$ such that: $\langle x^*, x\rangle > d > \sup_{b \in cl(B)} \langle x^*, b\rangle$. Thus $\sup_{b\in cl(B)}\langle x^*, b\rangle < \infty$ and since $cl(B)$ is a cone with vertex at $\bar{x}$, this implies that $\sup_{b\in cl(B)}\langle x^*, b\rangle = \langle x^*, \bar{x}\rangle$. Moreover, $U \subset cl(B)$, therefore we have:

$$1 \geq \langle x^*, \bar{x}\rangle = \sup_{b\in cl(B)} \langle x^*, b\rangle \geq \sup_{b\in U} \langle x^*, b\rangle = \|x^*\| = 1$$

So $\langle x^*, \bar{x}\rangle = 1$, hence $x^* \in J(\bar{x})$. But then $\langle x^*, x\rangle > d > 1$ contradicts the fact that $x \in J(\bar{x})^\circ$. □

This proposition says that $J(\bar{x})^\circ$ is generated by positive linear combinations of U and $\bar{x}$, so $J(\bar{x})^\circ$ is the smallest closed convex cone with vertex at $\bar{x}$ containing U. In our "down-to-earth" discussion we will say that $J(\bar{x})^\circ$ is the *corner* of U at $\bar{x}$. Then, Proposition 3.2 can be restated by saying that the dilations of U spreading from $\bar{x}$ form an exhaustion for the corner of U at $\bar{x}$.
Since in the sequel we will assume that J is norm-compact valued we need to study more carefully the polar of a compact subset of S^*. The next section is devoted to this purpose.

3.2. The polar of a compact subset of S^*

We defined the polar of a set A to be $A^\circ = \bigcap_{x^*\in A}\{x \in X \mid \langle x^*, x\rangle \leq 1\}$. In the next proposition we will look at the consequences of A being a compact subset of S^*. It is easy to show that $int(A^\circ)$ is weakly open in X if and only if A is bounded and finite dimensional in S^*. Here we will show that, if A is norm compact in S^*, then the norm interior of A° is *bounded-weakly* open in X, that is to say, its intersection with any bounded set E in X is relatively weakly open in E (see [5] p.48).

Proposition 3.3. *If A is a norm-compact subset of S^*, then:*
1. *$int(A^\circ) = \bigcap_{x^*\in A}\{x \in X \mid \langle x^*, x\rangle < 1\}$.*
2. *$int(A^\circ)$ is bounded-weakly open in X.*

Proof. First suppose $x \in int(A^\circ)$, then there is $r > 0$ such that $x + rU \subset int(A^\circ) \subset A^\circ$. So for all $x^* \in A$, $\sup_{y\in U}\langle x^*, x + ry\rangle \leq 1$, hence $\langle x^*, x\rangle + r\|x^*\| \leq 1$, i.e. $\langle x^*, x\rangle \leq 1 - r < 1$. Conversely, suppose that for all $x^* \in A$, $\langle x^*, x\rangle < 1$. Then, since A is norm-compact and $f(x^*) = \langle x^*, x\rangle$ is norm-continous on X^*, we have $\max_{x^*\in A}\langle x^*, x\rangle = \delta < 1$. Let $r = 1 - \delta$ and consider $z \in x + rU$, then $z = x + ry$ with $y \in U$, so, for all $x^* \in A$, $\langle x^*, z\rangle = \langle x^*, x\rangle + r\langle x^*, y\rangle \leq (1 - r) + r = 1$. Therefore $x + rU \subset A^\circ$, i.e. $x \in int(A^\circ)$.

To show that $int(A^\circ) \cap E$ is relatively weakly open in E for any bounded set E, we will show that the complement is relatively weakly closed. Suppose x_α is a net in E weakly convergent to $x_0 \in E$ such that, for all α, $x_\alpha \notin int(A^\circ)$. By part (1), this means that for all α we can find $x^*_\alpha \in A$ such that $\langle x^*_\alpha, x_\alpha \rangle \geq 1$. Since x_α is bounded, and converges weakly to x_0, this implies that $\langle x^*_0, x_0 \rangle = \lim \langle x^*_\alpha, x_\alpha \rangle \geq 1$, for some $x^*_0 \in A$ so $x_0 \notin int(A^\circ)$. Hence $E \setminus int(A^\circ)$ is weakly closed in E, so $int(A^\circ) \cap E$ is weakly open in E. □

This Proposition applied to our study of the duality map tells us that, when J is compact-valued, $int(J(\bar{x})^\circ)$ is bounded-weakly open in X for any $\bar{x} \in S$. That is to say, the corner of U at $\bar{x}$ is sufficiently "flat", since weakly open sets are in a sense "bigger" than norm-open sets.

3.3. A stronger continuity property for the duality map

A multivalued map F between two topological spaces T and S is said to be uppersemicontinous if for each $x \in T$ and any neighborhood V of $F(x)$ in S there is a neighborhood W of x in T such that $F(W) \subset V$. Note that if S is a metric space, e.g. a normed linear space, and $F(x)$ is compact, then the neighborhoods V of $F(x)$ can be assumed to be ϵ-expansions of $F(x)$: $S_\epsilon[F(x)] = \cup_{y \in F(x)} \{z \in S \mid d(y, z) < \epsilon\})$.

Given a norm on a Banach space X, the duality map J associated with it is a multivalued map from the sphere S to the dual sphere S^* which is uppersemicontinous if we endow S with the norm topology and S^* with the weak*-topology (see [6]). In the following we require a stronger condition, i.e. we want J to be norm to norm uppersemicontinous (S^* endowed with the dual norm topology). Moreover, the values of J are always weak*compact by definition, but here we ask that J be norm-compact valued. In short we say that J is n-to-n-usco. The next lemma is well-known.

Lemma 3.4. *Suppose that J is n-to-n-usco. If $\{x_k\}$, $\bar{x} \in S$, $z^*_k \in J(x_k)$, and $x_k \longrightarrow \bar{x}$ in norm, then there is $x^*_0 \in J(\bar{x})$ such that, "up to subsequences", $z^*_k \longrightarrow x^*_0$ in norm.*

Proof. Since $J(\bar{x})$ is convex and weak*-compact, there is $x^*_k \in J(\bar{x})$ such that $\|z^*_k - x^*_k\| = d(z^*_k, J(\bar{x}))$. By assumption, $J(\bar{x})$ is norm-compact, so, up to subsequences, x^*_k converges in norm to $x^*_0 \in J(\bar{x})$, i.e. given $\epsilon > 0$ there is k_0 such that $\|x^*_k - x^*_0\| \leq \epsilon$ for $k \geq k_0$. Moreover, J is norm-to-norm uppersemicontinous in $\bar{x}$, hence there is k_1 such that for all $k \geq k_1$: $J(x_k) \subset S_\epsilon[J(\bar{x})] = \{x^* \in X^* \mid d(x^*, J(\bar{x})) < \epsilon\}$. Therefore, for all $k \geq k_1$, $z^*_k \in S_\epsilon[J(\bar{x})]$, i.e. $\|z^*_k - x^*_k\| = d(z^*_k, J(\bar{x})) < \epsilon$, so $\|z^*_k - x^*_0\| \leq \|z^*_k - x^*_k\| + \|x^*_k - x^*_0\| \leq \epsilon + \epsilon = 2\epsilon$, for $k \geq \max\{k_0, k_1\}$. □

The map J being n-to-n-usco has an interesting consequence for the polar of $J(\bar{x})$, for a fixed $\bar{x} \in S$. In Section 3.1 we showed that $B_\lambda = \bar{x} + \lambda(U - \bar{x})$ is an increasing family of balls such that $int(J(\bar{x})^\circ) \subset \cup_{\lambda > 0} B_\lambda \subset J(\bar{x})^\circ$. The same is true if we take the balls B_λ to be open, i.e. $int(B_\lambda)$ instead of B_λ. So, given any norm-compact subset K of $int(J(\bar{x})^\circ)$, by compactness, there is λ_0 such that K is contained in $int(B_\lambda)$ for all $\lambda \geq \lambda_0 > 0$. In a similar way, when J is n-to-n-usco, given any weakly compact subset K of $int(J(\bar{x})^\circ)$, we can find λ_0 so that K is contained in all the B_λ's for $\lambda \geq \lambda_0 > 0$.

Proposition 3.5. *Suppose J is n-to-n-usco at $\bar{x}$. If $K \subset int(J(\bar{x})^\circ)$ is a weakly compact set, then there is $\lambda_0 > 0$ such that $K \subset B_{\lambda_0} = \bar{x} + \lambda_0(U - \bar{x})$.*

Proof. Since X is a Banach space, by the Eberlein-Smulian Theorem (see [5] Theorem 1. p. 58), K is weakly sequentially compact. Suppose that for every natural number $n > 0$ there is $x_n \in K \setminus B_n$. Then we can extract a subsequence y_k which converges weakly to $x_0 \in K$, where $y_k = x_{n_k} \in K \setminus B_{n_k}$. By Lemma 3.1, we have $int(J(\bar{x})^\circ) = int(cl(B)) = int(B) \subset \bar{x} + \cup_{\lambda>0}\lambda(S - \bar{x})$, since $B = \cup_{\lambda>0} B_\lambda$ is convex and has non-empty interior. For each k, $y_k \in K \subset int(J(\bar{x})^\circ)$, so we can find $\lambda_k > 0$ such that $y_k \in \bar{x} + \lambda_k(S - \bar{x}) \subset B_{\lambda_k}$. Note that $y_k \notin B_{n_k}$, and since the sets B_λ are increasing with λ, we must have $\lambda_k > n_k$, so λ_k converges to infinity. Moreover, we can write $y_k = \bar{x} + \lambda_k(z_k - \bar{x})$ for some $z_k \in S$. Now recall that K is weakly compact in X and that X is a Banach space so K is norm-bounded by a constant $M > 0$, so that z_k converges in norm to $\bar{x}$, because $\|z_k - \bar{x}\| = \frac{\|y_k - \bar{x}\|}{\lambda_k} \leq \frac{M + \|\bar{x}\|}{\lambda_k} \longrightarrow 0$, as $k \longrightarrow \infty$. For each k pick $z_k^* \in J(z_k) \neq \emptyset$. Then by Lemma 3.4, there is $x_0^* \in J(\bar{x})$ so that, up to a subsequence, $z_k^* \longrightarrow x_0^*$ in norm. Now, we compute $\langle x_0^*, x_0\rangle$. Given $\epsilon > 0$, eventually

$$\begin{aligned}
\langle x_0^*, x_0\rangle &\geq \langle x_0^*, y_k\rangle - \epsilon \qquad (y_k \to x_0 \text{ weakly}) \\
&= \langle z_k^*, y_k\rangle + \langle x_0^* - z_k^*, y_k\rangle - \epsilon \\
&\geq \langle z_k^*, \bar{x} + \lambda_k(z_k - \bar{x})\rangle - \|x_0^* - z_k^*\|\|y_k\| - \epsilon \\
&\geq \lambda_k\langle z_k^*, z_k\rangle + (1 - \lambda_k)\langle z_k^*, \bar{x}\rangle - \epsilon M - \epsilon \\
&\geq \lambda_k + (1 - \lambda_k) - \epsilon(M+1) = 1 - \epsilon(M+1)
\end{aligned}$$

The last line follows from the fact that $\langle z_k^*, \bar{x}\rangle \leq 1$ and $1 - \lambda_k < 0$. Now, since ϵ is arbitrary, we get $\langle x_0^*, x_0\rangle \geq 1$. But $x_0 \in K \subset int(J(\bar{x})^\circ)$, so, by Proposition 3.3, this is a contradiction. Therefore there is $n_0 \in \mathbb{N}$ such that $K \subset B_{n_0}$, hence $K \subset B_\lambda$ for all $\lambda \geq n_0$. □

The Proposition we just proved tells us that, when J is n-to-n-usco, the corners of U are "nicely" exhausted by the dilations of U itself.
With this more thorough description of the duality map at hand we are now able to prove the theorem of Borwein and Fitzpatrick in one direction.

4. Another proof of the B-F theorem

The idea of the proof is based on the fact that, when J is n-to-n-usco, the corners of the unit ball U are sufficiently flat (i.e. bounded-weakly open) and are nicely exhausted by the dilations of U (i.e. any weakly compact subset of $int(J(\bar{x})^\circ)$ is in reality contained in one dilated ball B_λ), together with the fact that in a reflexive space any equivalent ball is weakly compact. But first recall the following fact.

Lemma 4.1. *If Y is a compact Hausdorff space and $V_1, \ldots, V_n$ are open sets that cover Y, then there is a finite refinement $\{W_j\}$ of $\{V_i\}$ with $W_1, \ldots, W_j$ closed in Y.*

Proof. For each $y \in Y$, we have $y \in V_i$ for some i. Since Y is compact, $Y \setminus V_i$ is compact, and since Y is regular, there is an open neighborhood G_y of y such that $y \in G_y \subset cl(G_y) \subset V_i$. By compactness, $Y = \cup_{j=1}^m G_{y_j}$ for some $y_1, \ldots, y_m$ in Y. Set $W_j = cl(G_{y_j})$. Then $\{W_j\}$ is a closed finite refinement of $\{V_i\}$. □

Theorem 4.2. *If X is reflexive, with norm p and J its duality map, let π be the set of norms that are equivalent to p and $C(X)$ be the set of closed and convex subsets of X; then:*

$$J \text{ n-to-n-usco} \Longrightarrow \tau_{W_p} = \sup_{p' \in \pi} \tau_{W_{p'}} \text{ on } C(X)$$

Proof. We must show that for every norm p' equivalent to p we have $\tau^+_{W_p} \geq \tau^+_{W_{p'}}$. Therefore, by Theorem 2.3, it's enough to show that p covers every equivalent norm p'. So, let U' be the unit ball of the norm p'. Fix $x^* \in S'^*$ and $\epsilon > 0$. Then $U' \subset H(x^*, \epsilon) = \{x \in X \mid \langle x^*, x\rangle \leq 1 + \epsilon\}$. We must show that U' can be covered by a finite number of p-balls which remain contained in $H(x^*, \epsilon)$. As a matter of notation, let N be the hyperplane $\{x \in X \mid \langle x^*, x\rangle = 1 + \epsilon\}$, that is the boundary of $H(x^*, \epsilon)$. Since the Wijsman topology is invariant for homothetic norms we can suppose, by dilating U if necessary, that the unit ball of the norm p is such that $\sup_{x \in U}\langle x^*, x\rangle = 1 + \epsilon$, so that U is also contained in $H(x^*, \epsilon)$. Also, since X is reflexive, there is $\bar{x} \in S$ such that $\langle x^*, \bar{x}\rangle = 1 + \epsilon$, so that $\bar{x} \in N$.
By Proposition 3.2, the corner of U at $\bar{x}$, $J(\bar{x})^\circ$, is the smallest closed convex cone with vertex at $\bar{x}$ that contains U, therefore $J(\bar{x})^\circ \subset H(x^*, \epsilon)$, because $H(x^*, \epsilon)$ is a closed convex cone containing U and has vertex at $\bar{x}$, since $\bar{x} \in N$. Then, for each $x \in N$, consider the open cone $V_x = int(J(\bar{x})^\circ) + (x - \bar{x})$, i.e. the corner of U at $\bar{x}$ translated so that its vertex is still in N. Since $J(\bar{x})^\circ \subset H(x^*, \epsilon)$, every V_x is also contained in $H(x^*, \epsilon)$. The fact that, when J is n-to-n-usco, the corner of U at $\bar{x}$ is sufficiently flat comes to use in the next Claim.

Claim: $U' \subset \cup_{i=1}^n V_{x_i}$, *for some* $x_1, \ldots, x_n \in N$.

Consider $y \in U'$. Then $\langle x^*, y\rangle = k \leq 1$, because $x^* \in S'^*$. Let $x = y + (1 - \frac{k}{1+\epsilon})\bar{x}$. Then $\langle x^*, x\rangle = \langle x^*, y\rangle + (1 - \frac{k}{1+\epsilon})\langle x^*, \bar{x}\rangle = k + (1 - \frac{k}{1+\epsilon})(1 + \epsilon) = 1 + \epsilon$. So $x \in N$. Note that we can write $y = \frac{k}{1+\epsilon}\bar{x} + (x - \bar{x})$. Moreover, for all $y^* \in J(\bar{x}) \subset S^*$, $\langle y^*, \frac{k}{1+\epsilon}\bar{x}\rangle = \frac{k}{1+\epsilon} < 1$, and since $J(\bar{x})$ is norm-compact in X^*, Proposition 3.3 (1) applied to $J(\bar{x})^\circ$, tells us that $\frac{k}{1+\epsilon}\bar{x} \in int(J(\bar{x})^\circ)$. Therefore $y \in V_x$. So $U' \subset \cup_{x \in N} V_x$. Now, since U' is bounded (in every equivalent norm), Proposition 3.3 (2) says that, for each $x \in N$, $V_x \cap U'$ is weakly open in U'. So $\bigcup_{x \in N}(V_x \cap U')$ is a relatively weakly open cover of U'. But X is reflexive, so U' is weakly-compact. Therefore, we can find $x_1, \ldots, x_n \in N$ such that $U' \subset \cup_{i=1}^n V_{x_i}$. Now, by Lemma 4.1, we can find a finite refinement $\{W_j\}_{j=1}^m$ of the cover $\{V_{x_i}\}$ where the W_j's are relatively weakly closed in U'. Hence, for each $j = 1, \ldots, m$, $W_j \subset V_{x_{i_j}}$, for some $1 \leq i_j \leq n$, and W_j is weakly compact. Proposition 3.5 says that each W_j is in fact contained in a p-ball, i.e. there is $\lambda_j > 0$ such that $W_j \subset B_{\lambda_j} + (x_{i_j} - \bar{x}) = \lambda_j U + (1 - \lambda_j)\bar{x} + (x_{i_j} - \bar{x}) = (x_{i_j} - \lambda_j \bar{x}) + \lambda_j U$. Call these p-balls $D_j = (x_{i_j} - \lambda_j \bar{x}) + \lambda_j U$. Then, $U' \subset \cup_{j=1}^m W_j \subset \cup_{j=1}^m D_j \subset \cup_{j=1}^m V_{x_{i_j}} \subset H(x^*, \epsilon)$. Hence $U' \subset \cup_{j=1}^m D_j \subset H(x^*; \epsilon)$. Since $x^* \in S'^*$ and $\epsilon > 0$ were arbitrarily fixed, the hypothesis of Theorem 2.3 are satisfied, thus $\tau^+_{W_p} \geq \tau^+_{W_{p'}}$. So $\tau_{W_p} = \sup_{p' \in \pi} \tau_{W_{p'}}$. □

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