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Topological Degree for Maximal Monotone Operators and Application to Parametric Optimization Problems[1]

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The generalized topological degree theory is based on the Brouwer and Leray-Schauder degrees. It can be defined for general classes of mappings. The purpose of this article is two-fold. One goal is to define the topological degree for maximal monotone operators. Particular attention is paid to the continuation methods for this kind of operators and real functions of convex type. This allows us to extend some recent results (see [5], [6]) by withdrawing the compactness assumptions.

Keywords : Topological degree, Pseudo-monotone operator, Operator of class (S_+), Maximal monotone operator, Generalized Yosida approximation, Graph-continuity.

1. Introduction

The question of stability in optimization deals with what happens to an optimization problem when the elements of the problem are in some way deformed. As being expressed by Felix E. Browder, the concept of degree of a mapping, in all its different forms, is one of the most effective tools for studying the properties of the existence and multiplicity of solutions of nonlinear equations. Historically, the well known topological degree is a useful tool in applied mathematics, for example to prove that some nonlinear equations have solutions and to investigate the stability by using the continuation method. The notion of the degree was first introduced explicitly by Brouwer in 1912 in the case of finite dimensional spaces. Leray and Schauder extended this theme in 1934 to the context of Banach spaces and mappings of the form $f = I - g$, with I the identity and g a compact mapping (we refer to [15], [27] and [39] for a wide bibliography on the subject.) Afterwards many authors defined and developed the topological degree theory for various classes of non-compact nonlinear mappings between Banach spaces. For references on these notions see [1], [2], [3], [15], [16], [22]–[24], [26], [27], [29], [31], [33]–[38], [41], [43], [47] and [50]. In a series of articles [15]–[18] in 1983 Browder has defined and extended this concept of the classical topological degree for operators of monotone type (class

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(S_+) and pseudomonotone operators). His method is based on Galerkin approximations for which the classical Brouwer degree is defined. Recently, Berkovits and Mustonen [11]–[13] introduced a new construction of the Browder degree which is based on the Leray-Schauder degree.

The degree theory obtained can be used by relaying on the continuation methods to deduce existence theorems for nonlinear inclusion differential equations (see the references mentioned above), fixed point theory ([14], [16], [25], [36], [44], [52], ...) and optimisation ([5], [6], [45]).

In this paper we show how Browder's degree, given for operators of class (S_+), can be naturally extended to the case of maximal monotone operators by relying on generalized Yosida approximates. Particular attention is paid to the normalization and invariance under homotopies for the topological degree we define. Homotopy methods are used to prove several theorems on the existence of solutions. This allows us to extend some recent results (see [5], [6]) of Attouch, Penot and Riahi about the continuation method for solutions of parametrized monotone nonlinear equations (Theorem 4.5). It is also possible, by relying on subdifferentials, that our definition could be used to define topological degrees for real convex functions and convex-concave saddle functions.

Here is the summary of the paper. In section 2, some basic properties of Browder's degree are set out. In section 3, we demonstrate an auxiliary continuation theorem (Theorem 3.3). It concerns to pass from graph continuity of maximal monotone operators family to class (S_+) property of the associated generalized Yosida approximation. Section 4 is devoted to define the topological degree of a maximal monotone operator and to give familiar properties (Theorem 4.2). Afterwards we give various results on parametrized nonlinear monotone equations (Theorem 4.5, Propositions 4.6, 4.7). Finally, in section 5 we apply the results of the previous sections to real functions of convex type (Propositions 5.2, 5.3).

2. Notation and preliminaries

Let be given a real reflexive Banach space X with the topological dual X^*. Without loss of generality we will always assume $(X, \|.\|)$ and $(X^*, \|.\|)$ to be locally unifomly convex, by virtue of the powerful renorming theorem of Asplund, Lindenstrauss, Trojanski and Zizler (see [27], p.185 or [49]). In particular this implies that the duality mapping J of X into X^* given by

$$J(x) = \{x^* \in X^*; \langle x^*, x\rangle = \|x\|.\|x^*\| = \|x\|^2\}$$

is a homeomorphism between X and X^*. The strong and the weak convergences in each of the spaces X and X^* are denoted by "$\xrightarrow{s}$" and "$\xrightarrow{w}$", respectively.

Let us recall some definitions and results that will be needed in the sequel.

Definition 2.1. Let Ω be an open subset of the reflexive Banach space X and $\{f_t : \overline{\Omega} \longrightarrow X^*; t \in T\}$ be a family of demi-continuous operators from $\overline{\Omega}$ in X^*. Then the family $(f_t)_{t\in T}$ is called *pseudomonotone* (resp. of class (S_+)) if for any net $(t_i)_{i\in I}$ converging to t in T and $(x_i)_{i\in I}$ in $\overline{\Omega}$, the relations

$$x_i \xrightarrow{w} x \quad \text{and} \quad \limsup_{i\in I}\langle f_{t_i}(x_i), x_i - x\rangle \le 0$$

imply

$$\lim_{i\in I}\langle f_{t_i}(x_i), x_i - x\rangle = 0 \quad (\text{resp. } x_i \xrightarrow{s} x),$$

if moreover $x \in \overline{\Omega}$ the net $(f_{t_i}(x_i))_{i\in I}$ does weakly converge to $f_t(x)$.

Theorem 2.2. *(Browder's degree) Let X be a reflexive Banach space, Ω an open bounded subset of X, $\mathcal{F}(\Omega, y)$ be the family of all operators of class (S_+) such that $y \notin f(\partial\Omega)$, and $\mathcal{A}(\Omega, y)$ the family of homotopies in $\mathcal{F}(\Omega, y)$ of class (S_+).*
On these admissible triplets (f, Ω, y), i.e. $(f, \Omega, y) \in \mathcal{F}(\Omega, y)$, one can define a unique $\mathbf{Z}$ -valued function d that satisfies the three basic conditions corresponding to the ones of Brouwer's topological degree, namely :

($\mathbf{d}_1$) *$d(J, \Omega, y) = 1$ for $y \in \Omega$, and $d(f, \Omega, y) \neq 0$ implies $y \in f(\Omega)$;*

($\mathbf{d}_2$) *$d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ whenever Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \notin f(\overline{\Omega} \setminus \Omega_1 \cup \Omega_2)$;*

($\mathbf{d}_3$) *$d(f_t, \Omega, y(t))$ is independent of $t \in T$ whenever y is continuous on T, the homotopy $(f_t)_{t\in T}$ is of class (S_+) and $y(t) \notin f_t(\partial\Omega)$ on T.*

The reader is refered to Browder [15]–[20] for more details.

3. Generalized Yosida approximation

Let X be a reflexive Banach space. In the sequel we assume that X and X^* are locally uniformly convex, and we will identify a multi-valued mapping (or operator) $A : X \longmapsto X^*$ with its graph in $X \times X^*$ i.e. $A = \{(x, y) \in X \times X^*; y \in A(x)\}$. The domain of A is denoted by $\mathrm{dom}(A) = \{x \in X; A(x) \neq \emptyset\}$.
A multi-valued operator $A \subset X \times X^*$ is said to be monotone if for any $(x_i, y_i) \in A$, with i = 1,2, one has $\langle y_1 - y_2\, , \, x_1 - x_2\rangle \geq 0$.
A is said to be maximal monotone if it is maximal in the family of monotone operators in $X \times X^*$, ordered by inclusion.
Let now $A \subset X \times X^*$ be a maximal monotone operator. Then the resolvent $J_\lambda^A(x)$ for $\lambda > 0$ and A is defined as the unique solution of the equation

$$0 \in J(J_\lambda^A(x) - x) + \lambda A(J_\lambda^A(x)).$$

The Yosida approximation is given by $A_\lambda(x) = \frac{1}{\lambda}J(x - J_\lambda(x)) = (A^{-1} + \lambda J^{-1})^{-1}(x)$. For more details see [4], [10], [15], [27].
We will use an extension of the concept of Yosida approximation that is given by $A^\lambda = A_\lambda + \lambda J$. This notion will be called generalized Yosida approximation.

Definition 3.1. a) A net $\{A_i \subset X \times X^*; i \in I\}$ is said to be *graph convergent* to $A \subset X \times X^*$ if it converges in $X \times X^*$ in the Kuratowski-Painlevé sense. In other words $A = \mathrm{graph} - \lim A_i$ if the following inclusions hold :

$$\limsup_{i\in I} A_i \subset A \subset \liminf_{i\in I} A_i$$

with

$$\liminf_{i\in I} A_i = \{(x, y) \in X \times X^*;\ \exists (x_i, y_i) \in A_i \to (x, y)\}$$

and

$$\limsup_{i\in I} A_i = \{(x,y) \in X \times X^*; \exists K \subset I \quad \text{such that} \quad \forall\, j \in K\, \exists (x_j, y_j) \in A_j \to (x,y)\}.$$

For maximal monotone operators the graph-convergence of $(A_i)_{i\in I}$ to A is equivalent to $A \subset \liminf_{i\in I} A_i$.

b) A family $\{A_t \subset X \times X^*;\, t \in T\}$ is *graph-continuous* if whenever $t_i \to t$ in T one has

$$A_t = \text{graph} - \lim A_{t_i}.$$

Proposition 3.2. ([4], Prop. 3.60) *Let $(A_i)_{i\in I}$ and A be a family of maximal monotone operators in $X \times X^*$. The following statements are equivalent :*
(a) *$(A_i)_{i\in I}$ is graph-convergent to A ;*
(b) *$A_{i,\lambda}(x)$ converges strongly to $A_\lambda(x)$ for every $x \in X$ and every $\lambda > 0$.*

The following theorem summarizes some properties which will be needed in the sequel.

Theorem 3.3. *Let X be a reflexive Banach space such that both X and X^* are locally uniformly convex. Let $\{A_t \subset X \times X^*; t \in T\}$ be a graph-continuous family of maximal monotone operators. Then for any open bounded $\Omega \subset X$ the family $\{A_t^\lambda : \overline{\Omega} \longrightarrow X^*; t \in T, \lambda > 0\}$ is of class (S_+).*

Proof. Let Ω be an open bounded subset of X. Suppose that $t_i \to t$ in T, $\lambda_i \to \lambda > 0$ and $x_i \overset{w}{\to} x$ in X satisfy

$$\limsup_{i\in I} \langle A_{t_i}^{\lambda_i}(x_i), x_i - x\rangle \le 0.$$

First of all we obtain

$$\limsup_{i\in I} \langle A_{t_i,\lambda_i}(x_i), x_i - x\rangle \le -\lambda \liminf_{i\in I} \langle J(x_i), x_i - x\rangle \le \liminf_{i\in I} \langle J(x), x_i - x\rangle \le 0. \quad (1)$$

If we set $y_i = A_{t_i,\lambda_i}(x_i), y = A_{t,\lambda}(x)$ and $z_i = A_{t_i,\lambda}(x)$ for every $i \in I$ we obtain that the points $(x_i - \lambda_i J^{-1} y_i, y_i)$ and $(x - \lambda J^{-1} z_i, z_i)$ are in A_{t_i}. From the monotonicity of A_{t_i} and J^{-1} it follows that

$$0 \le \langle y_i - z_i, J^{-1}y_i - J^{-1}z_i\rangle \le \frac{1}{\lambda}\langle y_i - z_i, x_i - x\rangle + \frac{\lambda - \lambda_i}{\lambda}\langle y_i - z_i, J^{-1}y_i\rangle. \quad (2)$$

Because of Proposition 3.2, the net $(z_i)_{i\in I}$ converges strongly to y.
Now, let $(u_0, v_0) \in A_t$. Since A_{t_i} graph-converges to A_t , there exists $(u_i, v_i) \in A_{t_i}$ strongly convergent to (u_0, v_0). The monotonicity of A_{t_i} implies

$$0 \le \langle y_i - v_i\,,\, x_i - \lambda_i J^{-1} y_i - u_i\rangle \le -\lambda_i \|y_i\|^2 + (\|v_i\| + \frac{1}{\lambda_i}\|u_i - x_i\|)\|y_i\| + \langle v_i, u_i - x_i\rangle,$$

and $\|y_i\|^2 + \alpha\|y_i\| + \beta \le 0$ for some real numbers α and β. Hence $(J^{-1}y_i)_{i\in I}$ is bounded and by using (2) we obtain

$$\liminf_{i\in I} \langle y_i, x_i - x\rangle \ge \liminf_{i\in I} (\langle z_i, x_i - x\rangle + (\lambda_i - \lambda)\langle y_i - z_i, J^{-1}y_i\rangle) \ge 0.$$

Taking into account (1), we conclude that $\langle A_{t_i,\lambda_i}(x_i)\,,\,x_i - x\rangle$ converges to zero . Because of $\limsup_{i\in I}\langle A_{t_i}^{\lambda_i}(x_i), x_i - x\rangle \leq 0$ we deduce

$$\limsup_{i\in I}\langle J(x_i), x_i - x\rangle \leq -\frac{1}{\lambda}\limsup_{i\in I}\langle A_{t_i}^{\lambda_i}(x_i), x_i - x\rangle \leq 0.$$

As a consequence of the local uniform convexity of X and the weak convergence of $(x_i)_{i\in I}$ to x, we conclude that $(x_i)_{i\in I}$ converges strongly to x. □

4. Degree for maximal monotone operator

Now we are in a position to define the topological degree for maximal monotone operators. Let X be a reflexive Banach space, $A \subset X \times X^*$ a maximal monotone operator and Ω a bounded open subset of X.
We approximate the operator A, for $\lambda > 0$, by the generalized Yosida approximation $A^\lambda = A_\lambda + \lambda J$. Observe that A^λ is demicontinuous and of class (S_+).

Definition 4.1. We define the *topological degree* of A over Ω at 0 by the formula :

$$\deg(A, \Omega, 0) = \lim_{\lambda\searrow 0} d(A^\lambda, \Omega, 0). \tag{3}$$

We first verify that in the definition above, the degree function deg is independent of $\lambda > 0$ for λ sufficiently small. On any closed subinterval of $[0, \lambda]$, the family $\{A^\lambda; \lambda \in I\}$ is of class (S_+). Therefore, by invariance of the Browder's degree function under homotopies of class (S_+), the function $d(A^\lambda, \Omega, 0)$ will be independent of $\lambda \in I$, provided $0 \notin A^\lambda(\partial\Omega)$ for every $\lambda \in I$. Otherwise, one can find a decreasing sequence (λ_n) converging to zero such that $0 \in A^{\lambda_n}(\partial\Omega)$. Since $\partial\Omega$ is bounded and the Banach space X is reflexive, one can find a weakly convergent sequence $x_n \in \partial\Omega$ to x in X such that

$$0 = A^{\lambda_n}(x_n) = A_{\lambda_n}(x_n) + \lambda_n J x_n.$$

Thus

$$\limsup_{n\to\infty}\langle A^{\lambda_n}(x_n)\,,\,x_n - x\rangle \leq 0.$$

By Theorem 3.3 it follows that the sequence (x_n) converges strongly to x and so $x \in \partial\Omega$. On the other hand $0 = A^{\lambda_n}(x_n)$ is equivalent to $v_n \in Au_n$ with $v_n = -\lambda_n J x_n$ and $u_n = x_n - {\lambda_n}^2 J^*(-Jx_n)$. Since A is maximal we deduce that $0 \in Ax$, for $v_n \xrightarrow{s} 0$ and $u_n \xrightarrow{w} x$. Thus we reach a contradiction with the assumption that $0 \notin A(\partial\Omega)$.
This common value permits us to define the extended degree function (3).
With the above definition one can state the familiar properties of degree theory for maximal monotone mappings.

Theorem 4.2. *Let Ω be an open bounded subset of a reflexive Banach space X which is, with its dual X^*, locally uniformly convex . Let $A \subset X \times X^*$ be a maximal monotone operator. Then we have*

(i) $\deg(J,\Omega,0)=1$, *provided* $0\in\Omega$;

(ii) $\deg(A,\Omega,0)=0$ *whenever* $0\notin A(\overline{\Omega})$;

(iii) *if the homotopy of the maximal monotone operators* $\{A_t\subset X\times X^*; t\in T\}$ *is graph continuous and satisfies* $0\notin\bigcup\{A_t(\partial\Omega); t\in T\}$, *then* $\deg(A_t,\Omega,0)$ *is independent of* t *in* T;

(iv) *if* Ω_1 *and* Ω_2 *are two disjoint open subsets of* Ω *such that* $0\notin A(\overline{\Omega}\setminus\Omega_1\bigcup\Omega_2)$, *then*

$$\deg(A,\Omega,0)=\deg(A,\Omega_1,0)+\deg(A,\Omega_2,0).$$

Proof. Part (i) is obvious, since $J^\lambda=J_\lambda+\lambda J=\dfrac{1+\lambda+\lambda^2}{1+\lambda}J$ and

$$\deg(J,\Omega,0)=\lim_{\lambda\to 0^+}d(J^\lambda,\Omega,0)=d(J,\Omega,0).$$

To prove (ii), suppose $\deg(A,\Omega,0)\neq 0$. Let $\lambda_0>0$ be such that $d(A^\lambda,\Omega,0)\neq 0$ for each $0<\lambda\le\lambda_0$. By Browder's degree theorem $(\mathbf{d}_1)$ there exists $x_\lambda\in\overline{\Omega}$ such that $0=A^\lambda(x_\lambda)$ for each $\lambda\in]0,\lambda_0]$. Let $(\lambda_i)_{i\in I}$ be a net of positive constants which converge to zero. $\overline{\Omega}$ is bounded, therefore we can find a subnet, also denoted by $(x_i=x_{\lambda_i})_{i\in I}$ and elements of $\overline{\Omega}$, such that $(x_i)_{i\in I}$ converges weakly to x in X and $((1+\lambda_i^2)x_i,-\lambda_i Jx_i)\in A$ for every $i\in I$. Since A is maximal monotone we conclude that $0\in A(x)$.
On the other hand $\{A^\lambda;\lambda>0\}$ is of class (S_+), as pointed out in Theorem 3.3. We deduce that $(x_i)_{i\in I}$ converges strongly to x and $x\in\overline{\Omega}$. Thus $0\in A(\overline{\Omega})$, as desired.

(iii) By relying on Browder's degree theorem $(\mathbf{d}_3)$ and Theorem 3.3 it suffices to show that for each $t\in T$ there exists $\lambda(t)>0$ such that for every $\lambda\in]0,\lambda(t)]$ one has $0\notin A_t^\lambda(\partial\Omega)$ and $\inf_{t\in T}\lambda(t)>0$. Assuming the contrary, since $\overline{\Omega}$ is bounded, we can find t_0 in T and nets $(\lambda_i)_{i\in I}$ in $]0,+\infty[$ and $(x_i)_{i\in I}$ in $\partial\Omega$ such that $\lambda_i\to 0$, $x_i\overset{w}{\to}x$ in X and $0\in A_{t_0}^{\lambda_i}(x_i)$. As in (ii) one deduces $0\in A_{t_0}(\overline{x})$. From this and the fact that A_{t_0} is monotone, we infer that

$$\langle\lambda_i Jx_i\,,\,\overline{x}-(1+\lambda_i^2)x_i\rangle\le 0$$

and

$$\limsup_{i\in I}\|x_i\|\le\limsup_{i\in I}\frac{1}{1+\lambda_i^2}\|\overline{x}\|\le\|\overline{x}\|.$$

Hence $(x_i)_{i\in I}$ converges strongly to $\overline{x}$ since X is locally uniformly convex. As $\partial\Omega$ is closed we conclude that $x\in\partial\Omega$, a contradiction to $0\notin A_{t_0}(\partial\Omega)$.
For (iv), let $\lambda_0>0$ be such that for any $0<\lambda\le\lambda_0$ we have $0\notin A^\lambda(\overline{\Omega}\setminus(\Omega_1\cup\Omega_2))$. Otherwise there are $\lambda_i\to 0$ and $x_i\overset{w}{\to}\overline{x}\in X$ such that $(x_i)_{i\in I}\subset(\overline{\Omega}\backslash(\Omega_1\cup\Omega_2))$. From this and the maximality of A we infer, by virtue of (iii), that $\overline{x}\in(\overline{\Omega}\backslash(\Omega_1\cup\Omega_2)\cap A^{-1}(0)$, which is a contradiction. According to the property (d_2) of Browder's degree theorem we obtain

$$d(A^\lambda,\Omega,0)=d(A^\lambda,\Omega_1,0)+d(A^\lambda,\Omega_2,0),$$

and so, by letting $\lambda \to 0$,

$$\deg(A, \Omega, 0) = \deg(A, \Omega_1, 0) + \deg(A, \Omega_2, 0).$$

□

Theorem 4.3. *On the family of maximal monotone operators there exists one and only one degree function with invariance under graph-continuous homotopies.*

Proof. Let A be a maximal monotone operator and Ω a bounded open subset of X. Suppose that $0 \notin A(\partial\Omega)$. As stated in Definition 4.1, for $\lambda > 0$ sufficiently small we have

$$\deg(A, \Omega, 0) = d(A^\lambda, \Omega, 0)$$

and

$$0 \notin A^\lambda(\partial\Omega).$$

Let us consider now a degree function d_0 satisfying the properties (i), (ii), (iii) and (iv) of Theorem 4.2, and let $\mathcal{M}_\lambda$ be the family of the maximal monotone operators $A_t = tA^\lambda + (1-t)A$, where $\lambda > 0$ is sufficiently small and $t \in T = [0, 1]$. It is clear that $\mathcal{M}_\lambda$ is graph-continuous. So that, if we suppose that $0 \notin \bigcup\{A_t(\partial\Omega);\, t \in T\}$, we deduce by Theorem 4.2, (iii) that $d_0(A, \Omega, 0) = d_0(A^\lambda, \Omega, 0)$; it suffices to take $t = 0, 1$. Using the unicity of the Browder's degree for operators of class (S_+), one has $d_0(A^\lambda, \Omega, 0) = d(A^\lambda, \Omega, 0)$. As the degree deg is independent of $\lambda > 0$ sufficiently small, it follows that $d_0(A, \Omega, 0) = \deg(A, \Omega, 0)$. Hence, if we suppose that the assertion of the degree function were false, there would exist $u_n \in \partial\Omega$ such that $u_n \xrightarrow{w} u$ in X, $\lambda_n \to 0$ and $t_n \to t$ with $0 \in A_{t_n}(u_n)$. By taking some $v_n \in A(u_n)$ and $w_n = A_{\lambda_n}(u_n)$ one has $-t_n\lambda_n J u_n = t_n w_n + (1-t_n)v_n$. Now for (x, y) in A, we have by monotonicity that

$$\langle w_n - y \,,\, u_n - \lambda_n J^{-1} w_n - x\rangle \geq 0$$

and

$$\langle v_n - y \,,\, u_n - x\rangle \geq 0.$$

Thus, by multiplying these two relations respectively by t_n and $(1-t_n)$, then adding, we obtain

$$\langle t_n w_n + (1-t_n) y \,,\, u_n - x\rangle = \langle t_n\lambda_n J u_n - y \,,\, x - u_n\rangle \leq t_n\lambda_n\langle w_n - y \,,\, J^{-1} w_n\rangle. \qquad (4)$$

A straightforward calculation shows that

$$t_n\lambda_n\|w_n\|^2 \leq (t_n\lambda_n)^{1/2}\|y\| + (\|t_n\lambda_n J u_n - y\|.\|u_n - u\|)^{1/2}.$$

Thus the sequence $(t_n\lambda_n\|w_n\|^2)$ is bounded. This permits us to deduce that $(t_n\lambda_n w_n)$ converges strongly to zero. Returning to (4), we have

$$\langle y \,,\, u_n - x\rangle \leq t_n\lambda_n\|w_n\|.\|y\| + t_n\lambda_n\|u_n\|.\|u_n - x\|.$$

Letting $n \to \infty$, it follows that $\langle y, x - u \rangle \geq 0$ for all $(x, y) \in A$. Hence $0 \in A(u)$ by maximility. Let us take now $x = u$ and $y = 0$ in (4), then we have

$$\limsup_{n \to \infty} \langle J u_n\,,\, u_n - u \rangle \leq 0.$$

Since J is of class (S_+) it follows that u_n converges strongly to u. This implies that $0 \in A(\partial\Omega)$ and gives a contradiction. □

Remark 4.4. Other properties of the topological degree given by Definition 4.1 can be proved by using the arguments of the preceding proposition. For example :

a) Up to a translation one can define the degree for maximal operators relative to Ω at a point $y \in X$, i.e. $\deg(A, \Omega, y) = \deg(A', \Omega', 0)$ where $A'(x) = A(x) - y$ and $\Omega' = \Omega - y$.

b) *(Homotopy)* Suppose that all the conditions of Theorem 4.2 (iii) hold except that condition $0 \notin \bigcup\{A_t(\partial\Omega)\ ;\ t \in T\}$ is replaced by: $t \longmapsto y(t)$ is continuous and $y(t) \notin \bigcup\{A_t(\partial\Omega)\ ;\ t \in T\}$ for each t in T. Then $\deg(A_t, \Omega, y(t))$ is independent of t. In fact, it is not hard to show that $\{A_t = A_t - y(t)\,;\, t \in T\}$ is a graph-continuous homotopy of maximal operators.

Comment. Let us now indicate how the above concepts and results for maximal monotone operators can be extended to m-accretive (hyperaccretive) mappings.
An operator $A \subset X \times X$ is called *accretive* if for each $(x_i, y_i) \in A$, $i = 1, 2$, one has

$$\langle J(x_1 - x_2),\, y_1 - y_2 \rangle \geq 0.$$

The operator A is *m-accretive* iff $A + \lambda I$ is onto X for each $\lambda > 0$.
To construct the topological degree for A it is sufficient to replace in the generalized Yosida approximation the duality mapping J by the identity I of X :

$$A^{\lambda} = (A^{-1} + \lambda I^{-1})^{-1} + \lambda I.$$

Note. Recently, Attouch, Penot and Riahi (see [5], Thm. 2.7) have investigated via the continuation methods the existence of solutions for parametrized nonlinear monotone problems. The approch in [5], [6] was based upon a connectedness argument. As a consequence of Theorem 4.2 we shall sharpen and extend these results from Hilbert to reflexive spaces, and also rule out the compactness conditions.

Theorem 4.5. *Suppose that assumptions* (iii) *of* Theorem 4.2 *hold and the following condition is satisfied: for some* $t_0 \in T$ *one has* $\deg(A_{t_0}, \Omega, 0) \neq 0$.
Then for each $t \in T$, $A_t^{-1}(0)$ *is nonempty and contained in* Ω.

The proof relies only on the independence of $d(A_t, \Omega, 0)$ of t in T and the convexity (connectedness) of $A_t^{-1}(0)$ since A_t is maximal monotone.

Proposition 4.6. *Let A and B be two maximal monotone operators such that* $\mathrm{dom}(A) \cap \mathrm{int}(\mathrm{dom}(B)) \neq \emptyset$ *and* $A(x) = B(x)$ *for each* $x \in \partial\Omega$, *and moreover* $0 \notin A(\partial\Omega)$.
Then $\deg(A, \Omega, 0) = \deg(B, \Omega, 0)$.

Proof. The homotopy $\{A_t = tA + (1-t)B\,;\, t \in T = [0,1]\}$ is a graph-continuous family of maximal monotone operators and satisfies all assumptions of Theorem 4.2 (iii). Then $\deg(A_t, \Omega, 0)$ is independent of t in T. Take $t = 0$ and $t = 1$ and the claim follows. □

Proposition 4.7. *Suppose that A is maximal monotone and $-\mu Ju \notin A(u)$ for each u in $\partial\Omega$ and $\mu > 1$ (resp. $\mu \geq 0$). Then $\deg(A+J,\Omega,0) = 1$ if $0 \in \Omega$ and $\deg(A+J,\Omega,0) = 0$ if $0 \notin \Omega$ (resp. $\deg(A,\Omega,0) = 1$ if $0 \in \Omega$ and $\deg(A,\Omega,0) = 0$ if $0 \notin \Omega$).*

Proof. Considering the homotopy of maximal monotone operators $A_t = tA + J$ (resp. $A_t = tA + (1-t)J$), we have $0 \notin A_t(\partial\Omega)$ for each $t \in T = [0,1]$. Thus by Theorem 4.2 (iii) the statement follows, i.e. $\deg(A+J,\Omega,0) = \deg(J,\Omega,0)$ (resp. $\deg(A,\Omega,0) = \deg(J,\Omega,0)$). □

Corollary 4.8. *Let A be a maximal monotone operator. Suppose that for some $r > 0$ and for each $\lambda \geq 0$ $\|(A^{-1})_\lambda\| \neq r$. Then $A^{-1}(0)$ is nonempty and contained in B_r, the ball with radius r.*

Proof. We apply Theorem 4.5 to $\{A_t = tA + (1-t)J;\ t \in [0,1]\}$. Since $\|(A^{-1})_\lambda\| \neq r$ for each $\lambda \geq 0$, one has $0 \notin \bigcup\{A_t(\partial B_r);\ t \in [0,1]\}$. We obtain the desired conclusion.

5. Application to real functions of convex type

In this section we apply the results of section 4 to convex functions and convex-concave saddle bifunctions.

A - Before we state further consequences, let us first introduce some definitions. For further details see [4], [10], [27] and [52].

Definition 5.1. 1) Let X be a reflexive space. A function $f : X \longmapsto \mathbb{R} \cup \{+\infty\}$ is said to be *convex lower semicontinuous* (*lsc*) whenever its epigraph, $\mathrm{epi}(f) = \{(x,t) \in X \times \mathbb{R}\,; f(x) \leq t\}$, is convex and closed. Here $X \times \mathbb{R}$ is endowed with the product topology.
2) Let $\mathcal{F} = \{f_t : X \longmapsto \mathbb{R} \cup \{+\infty\}\,; t \in T\}$ be a family of convex lsc functions. $\mathcal{F}$ is said to be *Mosco-epicontinuous* provided the homotopy $\{\mathrm{epi}(f_t) \subset X \times \mathbb{R}\,; t \in T\}$ is graph-continuous with respect to the strong topology and the weak convergence of X.
3) Let f be a proper convex lsc function. Then the *subdifferential* of f at x in X and the *minimum set* are given by

$$\partial f(x) = \{x^* \in X^*;\ f(x) \leq f(u) + \langle x^*,\ x - u\rangle\ \forall u \in X\}$$

and

$$\mathcal{M}(f) = \{x \in X;\ f(x) \leq f(u)\ \forall u \in X\}.$$

Since the subdifferential of f is a maximal monotone operator, one can define the degree of a proper convex lsc function at y relatively to an open bounded subset Ω of X by:

$$\deg(f,\Omega,y) = \deg(\partial f,\Omega,y). \tag{5}$$

Remark 5.2. This definition permits us to extend the next result from Hilbert (see [45]) to reflexive spaces.

Proposition 5.3.
a) *For a proper convex lsc function f such that $\mathcal{M}(f) \cap \partial\Omega = \emptyset$, $\deg(f,\Omega,0) \neq 0$ implies $\emptyset \neq \mathcal{M}(f) \subset \Omega$.*

b) *Let* $\{f_t;\, t \in T\}$ *be a Mosco-epicontinuous family of proper convex lsc functions such that* $\mathcal{M}(f_t) \cap \partial\Omega = \emptyset \quad \forall t \in T$ *and* $\deg(f_{t_0}, \Omega, 0) \neq 0$ *for some* $t_0 \in T$.

Then for each t *in* T *one has* $\emptyset \neq \mathcal{M}(f_t) \subset \Omega$.

Proof. a) This follows immediately from Theorem 4.2 (ii) and the relation (5).
b) Since the Mosco-epicontinuity of $\{f_t;\, t \in T\}$ implies that the family $\{\partial f_t\,;\, t \in T\}$ is graph-continuous and $\mathcal{M}(f_t) = (\partial f_t)^{-1}(0)$, the conclusion follows from Theorem 4.5. □

B - For bivariate functions (bi-functions), let X and Y be two reflexive Banach spaces which are in separate duality with X^* and Y^* via pairings denoted by $\langle , \rangle$.

Let us consider a closed convex-concave bifunction $F : X \times Y \longmapsto \overline{\mathbb{R}}$, which is convex lower semicontinuous (resp. concave upper semicontinuous) with respect to the variable x in X (resp. y in Y). In [46] R. T. Rockafellar introduced the operator $A = \partial_1 F \times -\partial_2 F$, where $\partial_1 F$ and $\partial_2 F$ denote the partial subdifferentials of the convex functions $F(.,y)$ and $-F(x,.)$. and proved that A is maximal monotone in $X \times Y$.
It is well known that the set $\mathcal{S}(F) = \{(x,y) \in X \times Y;\ \inf_{u \in X} F(u,y) = \sup_{v \in Y} F(x,v)\}$ of saddle points of F is exactly $A^{-1}(0,0)$.
Let us consider a family $\{F_t\,;\, t \in T\}$ of closed convex-concave bifunctions, which is Mosco-epi/hypocontinuous whenever the homotopy

$$\{C_t(x,y^*) = \sup_{y \in Y}(F_t(x,y) + \langle y^*, y\rangle)\,;\, t \in T\}$$

of convex parents is Mosco-epicontinuous. Then the homotopy $\{\partial C_t\,;\, t \in T\}$ is graph-continuous. Hence $\{A_t = \partial_1 F_t \times \partial_2 F_t\,;\, t \in T\}$ is graph-continuous. Further properties of these notions can be found in [7]–[9], [30], [46] and [52].
Let $F : X \times Y \longmapsto \overline{\mathbb{R}}$ be a closed convex concave bifunction, Ω an open bounded subset of $X^* \times Y^*$ and $(x,y) \in X \times Y$. The degree of F at (x,y) is defined by

$$\deg(F, \Omega, (x,y)) = \deg(A, \Omega, (x,y)).$$

With the above definition and properties one can easily state the analogue of Proposition 5.2 for bifunctions :

Proposition 5.4.
a) $\deg(F, \Omega, 0) \neq 0$ *and* $S(F) \cap \partial\Omega = \emptyset$ *imply that there exists a saddle point of* F *at* Ω.
b) *Let* $\{F_t; t \in T\}$ *be a Mosco-epi/hypocontinuous homotopy such that* $\deg(F_{t_0}, \Omega, 0) \neq 0$ *for some* $t_0 \in T$ *and* $S(F_t) \cap \partial\Omega = \emptyset$ *for each* t *in* T. *Then for every* $t \in T$ *one has*

$$\emptyset \neq S(F_t) \subset \Omega.$$

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