Journal of Algebraic Combinatorics **9** (1999), 251–269

Homotopy of Non-Modular Partitions and the Whitehouse Module

SHEILA SUNDARAM sheila@claude.math.wesleyan.edu
Department of Mathematics, Wesleyan University, Middletown, CT 06459

Received October 8, 1996; Revised July 17, 1997

For Benjamin

Abstract. We present a class of subposets of the partition lattice Π_n with the following property: The order complex is homotopy equivalent to the order complex of Π_{n-1}, and the S_n-module structure of the homology coincides with a recently discovered lifting of the S_{n-1}-action on the homology of Π_{n-1}. This is the Whitehouse representation on Robinson's space of fully-grown trees, and has also appeared in work of Getzler and Kapranov, Mathieu, Hanlon and Stanley, and Babson et al.

One example is the subposet P_n^{n-1} of the lattice of set partitions Π_n, obtained by removing all elements with a unique nontrivial block. More generally, for $2 \leq k \leq n-1$, let Q_n^k denote the subposet of the partition lattice Π_n obtained by removing all elements with a unique nontrivial block of size equal to k, and let $P_n^k = \bigcap_{i=2}^k Q_n^i$. We show that P_n^k is Cohen-Macaulay, and that P_n^k and Q_n^k are both homotopy equivalent to a wedge of spheres of dimension $(n-4)$, with Betti number $(n-1)!\frac{n-k}{k}$. The posets Q_n^k are neither shellable nor Cohen-Macaulay. We show that the S_n-module structure of the homology generalises the Whitehouse module in a simple way.

We also present a short proof of the well-known result that rank-selection in a poset preserves the Cohen-Macaulay property.

Keywords: poset, homology, homotopy, set partition, group representation

1. Introduction

In this paper we consider subposets of the partition lattice Π_n obtained by removing various modular elements. Recall that Π_n is the lattice of set partitions of an n-element set, ordered by refinement. We say a block of a partition is nontrivial if it consists of more than one element. The modular elements of Π_n are precisely those partitions with a unique nontrivial block (for this and other basic definitions see [25]). For a bounded poset P we denote by $\hat{P}$ the proper part of P, i.e., the poset P with the greatest element $\hat{1}$ and the least element $\hat{0}$ removed. We write $\Delta(P)$ for the order complex of P; the simplices of $\Delta(P)$ are the chains of $\hat{P}$. By the ith (reduced) homology $\tilde{H}_i(P)$ of P we mean the ith (reduced) simplicial homology of its order complex $\Delta(P)$. All homology in this paper is taken with integer coefficients except for representation theoretic discussions, in which case we take coefficients over the complex field. All posets are bounded unless explicitly stated otherwise.

For $2 \leq k \leq n-1$, define P_n^k to be the subposet of Π_n obtained by removing all modular elements whose unique nontrivial block has size $2 \leq i \leq k$, and define Q_n^k to be the subposet

Current address: 240 Franklin St. Ext., Danbury, CT 06811.

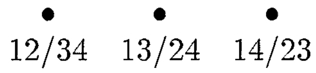

Figure 1. The poset $\hat{P}_4^3$.

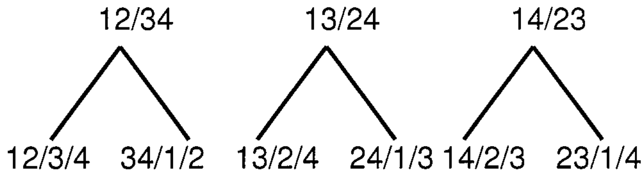

Figure 2. The poset $\hat{Q}_4^3$.

of Π_n obtained by removing all modular elements whose unique nontrivial block has size k. In particular, P_n^{n-1} consists of all partitions in Π_n with at least two nontrivial blocks, together with the greatest and least elements. It is not hard to see that the posets P_n^k are ranked, of rank $(n-2)$, one less than the rank of Π_n. On the other hand the subposets Q_n^k have full rank $n-1$ if $k \geq 3$.

Recall that a poset P is said to be Cohen-Macaulay if the reduced homology of the order complex of every interval $[x, y]$ of P, $\hat{0} \leq x \leq y \leq \hat{1}$, vanishes below the top dimension. The figures 1 and 2 show the (order complexes of) the posets P_4^3 and Q_4^3, respectively. Clearly Q_4^3 is not Cohen-Macaulay. Note that the zero-dimensional order complex of P_4^3 and the one-dimensional order complex of Q_4^3 both have the same homotopy type, and hence have the same homology.

We describe briefly the motivation for this work. In [26] some general techniques were developed for computing the homology representation of a poset for a finite group of automorphisms, and applied to Cohen-Macaulay subposets of the partition lattice. Note that the subposets P_n^k and Q_n^k are invariant under the action of the symmetric group S_n. In particular the Lefschetz module (i.e., the alternating sum (by degree) of the reduced homology modules), $\mathrm{Alt}(P_n^k)$, is a virtual S_n-module. By applying [26, Theorem 1.10 and Remark 1.10.1] to the subposets P_n^k, we can show that as (virtual) S_n-modules, $(-1)^{n-4}$ $\mathrm{Alt}(P_n^k)$ and $(-1)^{n-4}\mathrm{Alt}(Q_n^k)$ are both isomorphic to

$$\tilde{H}(\Pi_k)\uparrow_{S_k \times S_1 \times \cdots \times S_1}^{S_n} - \tilde{H}(\Pi_n). \tag{1.1}$$

(Here the up arrow indicates induction.) For $k = n-1$ the representation given by (1.1) is the complement of $\tilde{H}(\Pi_n)$ in the induction of $\tilde{H}(\Pi_{n-1})$ from S_{n-1} to S_n. This is precisely the representation of S_n on Robinson's space of fully grown trees, as computed by Sarah Whitehouse (see [13, 20, 21, 31]). The restriction of this representation to S_{n-1} is $\tilde{H}(\Pi_{n-1})$. Over the complex field, up to tensoring with the sign, this is also the lifting of the S_{n-1}-action on the multilinear component of the free Lie algebra Lie_{n-1} on $(n-1)$ generators up to S_n, described in [11]. There is an obvious surjective order-reversing map from the proper part of Hanlon's poset of homeomorphically irreducible trees with n labelled leaves (the poset T_{n-2}^1, in the notation of [13]), to the proper part of the poset P_n^{n-1}.

The paper [16] attempts to explain topologically the existence of this lifting, by studying the action on the cohomology of the complement of the braid arrangement. For two other contexts in which this lifting appears, see [1] and [14].

For arbitrary k it is not hard to see that (1.1) is in fact a true representation of S_n. Thus it is natural to ask whether the homology of the subposets P_n^k and Q_n^k is concentrated in a unique dimension. We answer this question affirmatively, showing that both posets have the same homotopy type, that of a wedge of $(n-4)$-spheres. We also show that P_n^k is Cohen-Macaulay over the integers. (It follows that the pure posets Q_n^k are *not* Cohen-Macaulay.)

Our main tool is Quillen's fibre lemma (see [8, 19])). In Section 2 we investigate the effect on homology of deleting an antichain from a poset (Theorem 2.1) and generalise this to an analogue for simplicial complexes (Theorem 2.5). As a consequence we obtain, using only the exact homology sequence of a pair, a simple proof of the well-known result that rank-selection in a poset preserves the Cohen-Macaulay property. In Section 3 we show that the subposets $\widehat{P_n^k}$ and $\widehat{Q_n^k}$ are homotopy equivalent (in fact S_n-homotopy equivalent), and determine the homotopy type. The representation theoretic aspects are addressed in Section 4, where we derive directly the formula (1.1), describing the S_n-module structure of the homology of Q_n^k (and hence of P_n^k) in terms of the homology of the partition lattices Π_k and Π_n. We conclude in Section 5 with a brief discussion of possible generalisations of this work.

The study of partitions with forbidden block sizes has led to the discovery of two other classes of related subposets of Π_n. One has the same S_n-homotopy type as the poset P_n^{n-1}, and hence its homology affords the Whitehouse representation. The other has the same S_n-homotopy type as the poset P_n^k for arbitrary k, $3 \leq k \leq n-2$, and hence its homology affords the generalised Whitehouse representation. These ramifications are described in [27], and will be the subject of a future paper.

2. Deleting an antichain from a Cohen-Macaulay poset

Let P be any poset, and let A be an antichain in P. For our first result we use the exact sequence of a pair to obtain information on the homology of the subposet $P\backslash A$ of P, obtained by removing all elements of A, in the case when P is Cohen-Macaulay.

The hypotheses in the theorem below may be relaxed somewhat by considering the more general case of simplicial complexes; see Theorem 2.5 at the end of this section.

Theorem 2.1 *Let P be a Cohen-Macaulay poset of rank r over the integers. Let A be an antichain in $\hat{P}$. Let $P\backslash A$ denote the subposet of P obtained by deleting the elements of A. Then the reduced integral homology of $P\backslash A$ vanishes in all dimensions except possibly $r-2$ and $r-3$.*

Proof: Consider the long exact homology sequence of the pair $(\Delta(P), \Delta(P\backslash A))$ (see [17]). Since P is Cohen-Macaulay, the reduced homology of P vanishes for degrees not equal to $r-2$, and the long exact sequence reduces to the following two sequences:

$$0 \to \tilde{H}_{r-2}(P\backslash A) \to \tilde{H}_{r-2}(P) \to H_{r-2}(\Delta(P), \Delta(P\backslash A)) \to \tilde{H}_{r-3}(P\backslash A) \to 0 \tag{2.1}$$

and, for $i \le r - 3$,

$$0 \to H_i(\Delta(P), \Delta(P\backslash A)) \to \tilde{H}_{i-1}(P\backslash A) \to 0 \tag{2.2}$$

We must first compute the relative homology groups $H_i(\Delta(P), \Delta(P\backslash A))$. Clearly the ith quotient chain group $C_i(\Delta(P))/C_i(\Delta(P\backslash A))$ consists of classes of chains going through at least one element of A. Since A is an antichain, each such chain must go through exactly one element of A. Now consider the boundary $\tilde{\partial}$ map of this relative complex. By the preceding remarks it is clear that if $c = x_0 < \cdots < x_p = a < \cdots < x_i$ is a (representative of) a nonzero relative i-chain, where $x_p = a$ is the unique element of A in the chain, then

$$\tilde{\partial}_i(c) = \sum_{0 \le t \le i, t \ne p} (-1)^t (x_0 < \cdots < \hat{x}_t < \cdots < x_i),$$

where as usual the hat denotes suppression of an element.

Hence the complex of relative chains is isomorphic to the direct sum of tensor products (over the integers) of chain complexes

$$C_i(\Delta(P), \Delta(P\backslash A)) = \bigoplus_{\substack{a \in A \\ s+t=i-2}} \tilde{C}_s(\Delta(\hat{0}, a)_P) \otimes \tilde{C}_t(\Delta(a, \hat{1})_P). \tag{2.3}$$

By hypothesis, in each summand of (2.3) (at least one of) the intervals have free homology. Consequently, by the Künneth theorem, the relative homology is given by

$$H_i(\Delta(P), \Delta(P\backslash A)) = \bigoplus_{\substack{a \in A \\ s+t=i-2}} \tilde{H}_s(\hat{0}, a)_P \otimes \tilde{H}_t(a, \hat{1})_P. \tag{2.4}$$

Now use the fact that for the intervals $(\hat{0}, a)$ and $(a, \hat{1})$ in P, the reduced homology vanishes except in the top dimension. Hence in the above sum, the right-hand side vanishes unless $s = \text{rank}(a) - 2$ and $t = r - \text{rank}(a) - 2$, i.e., unless $i = r - 2$. The conclusion now follows from (2.2). $\square$

As a by-product of this general result, we obtain a simple proof of the fact that rank-selection preserves the Cohen-Macaulay property, a theorem due independently, and with different proofs, to Baclawski, Stanley and Munkres.

Corollary 2.2 ([2, Theorem 6.4; 23, Theorem 4.3; 18, Corollary 6.6]) *Let P be a Cohen-Macaulay poset over the integers, and let Q be a rank-selected subposet of P. Then Q is Cohen-Macaulay over the integers.*

Proof: Let $Q = P\backslash A$ where A is some subset of $\hat{P}$. It suffices to consider the case of removing one rank, so that A is an antichain. Then Q is ranked of rank $r - 1$, where r is the rank of P. Hence $\tilde{H}_{r-2}(Q) = 0$. Now use the preceding result.

The same argument applies to an interval in Q, which either coincides with the corresponding interval of P, or else is obtained from it by deleting one rank. Hence if Q is P minus one rank, then Q is Cohen-Macaulay. $\square$

If P is an arbitrary poset and A is an antichain of $\hat{P}$, then a special case of a well-known formula for the Möbius number $\mu(P)$ of P (see [3, Lemma 4.6]) says that

$$\mu(P\backslash A) = \mu(P) - \sum_{x\in A} \mu((\hat{0}, x)_P)\mu((x, \hat{1})_P).$$

Noting that $\mu(P)$ is simply the reduced Euler characteristic of the order complex of P, i.e., $\mu(P) = \sum_{i\geq -1}(-1)^i \dim \tilde{H}_i(P)$, we have the following formula (which also follows from the proof of Theorem 2.1):

Corollary 2.3 *Let P and A be as in Theorem* 2.1. *Then*

$$\begin{aligned}&\dim \tilde{H}_{r-3}(P\backslash A) - \dim \tilde{H}_{r-2}(P\backslash A)\\ &\quad = \sum_{x\in A} \dim \tilde{H}((\hat{0}, x)_P) \dim \tilde{H}((x, \hat{1})_P) - \dim \tilde{H}_{r-2}(P).\end{aligned}$$

We return now to the partition lattice Π_n. Recall that if λ is an integer partition of n, then a set partition x in Π_n is said to be of *type* λ if x has block sizes $\lambda_1, \lambda_2, \ldots$. For $2 \leq k \leq n-1$, let Q_n^k be the subposet obtained by deleting the antichain consisting of all elements of type $(k, 1^{n-k})$.

For $k \geq 3$, the poset Q_n^k is ranked of rank $(n-1)$. For let $a \in \Pi_n$ have a unique nontrivial block of size k, and suppose a covers x and is covered by y. Then all blocks of x are singletons except possibly for two blocks B_1, B_2 whose union is the k-block A of a. Assume first that B_1 has size less than or equal to $k-2$. Since y covers a, either y is a modular element with unique nontrivial block $A \cup \{p\}$ or else y has two nontrivial blocks A and $\{p_1, p_2\}$; here the p's are singletons of a. In either case there is a non-modular element z in Π_n in the interval (x, y): in the first case merge the block B_1 of x with the singleton $\{p\}$ to form z. In the second case merge the singletons p_1 and p_2.

Now suppose x is obtained from a by splitting the unique nontrivial block A into the block B_1 and a singleton p'. (Thus x is itself modular.) If y is modular with nontrivial block $A \cup \{p\}$, merge the singletons p and p'. If y has a second nontrivial block $\{p_1, p_2\}$ then merge the singletons p_1 and p_2. In each case this produces a non-modular partition z in the interval (x, y).

Note that $Q_n^2 = P_n^2$ is the rank-selected subposet obtained by deleting the atoms. For $n \geq 5$ Q_n^k is not a lattice. The smallest interesting example is Q_4^3, whose order complex is disconnected and one-dimensional, and is homotopy equivalent to a wedge of two 0-spheres (see figure 2 of Section 1). In particular, Q_4^3 is not Cohen-Macaulay. In the next section we shall see that this is true in general.

Finally, we note that Theorem 2.1 gives the following fact, which will play a crucial role in the next section.

Proposition 2.4 *The reduced integral homology of Q_n^k vanishes in all dimensions different from $n-3$ and $n-4$.*

In the next section we shall show that the homology of Q_n^k is concentrated in a unique degree. It is not difficult to construct examples of a Cohen-Macaulay poset P and an antichain A which show that $P\backslash A$ can have homology in both degrees.

We can relax the hypotheses of Theorem 2.1 by considering the appropriate analogue for simplicial complexes. Recall that the link $\ell k(v)$ of a vertex v of a simplicial complex Δ is the subcomplex whose simplices are the faces F of Δ such that $v \notin F$ and $F \cup \{v\}$ is (a simplex) in Δ.

Theorem 2.5 *Let Δ be a finite simplicial complex, and let A be a subset of the vertices of Δ such that every facet* (*i.e.*, *maximal face*) *of Δ has at most one vertex in A. Assume that there is an integer d such that*
(i) *the ith reduced homology of Δ vanishes for all degrees $i \neq d$, and*
(ii) *for every vertex $a \in A$, the ith reduced homology of the link of a in Δ vanishes for all degrees $i \neq d-1$.*

Let Δ' be the subcomplex of Δ obtained by removing all faces having a vertex in the set A. Then $\tilde{H}_i(\Delta') = 0$ for all $i \neq d-1$ and $i \neq d$.

Proof: The following observations are sufficient, since the essential ideas are as in the proof of Theorem 2.1. The key point now is that the relative chain complex $C(\Delta)/C(\Delta')$ is isomorphic to the direct sum, over $a \in A$, of the chain complex of the suspension of the link $\ell k(a)$ of a in Δ.

Hence the relative homology is given by the formula

$$\tilde{H}_i(\Delta, \Delta') = \bigoplus_{\substack{a \in A \\ j=i-1}} \tilde{H}_j(\ell k(a)).$$

But by hypothesis, the link $\ell k(a)$ has zero homology in degrees $j \neq d-1$. That is, the relative homology is zero for degrees $\neq d$. Now the conclusion follows exactly as in Theorem 2.1. □

In the particular case when Δ is a pure d-dimensional Cohen-Macaulay simplicial complex, conditions (i) and (ii) of the above theorem are automatically satisfied. The conclusion of Theorem 2.1 may thus be obtained by taking Δ to be the order complex of a Cohen-Macaulay poset of rank $d+2$.

The full result of [18, Corollary 6.6] also follows from the above. In addition, just as we obtained Corollary 2.2, we recover Stanley's result on subcomplexes of completely balanced Cohen-Macaulay complexes (see [23, Theorem 4.3]) from Theorem 2.5. The details are identical to the above proof and the proof of Corollary 2.2.

3. A homotopy equivalence

We begin by stating a powerful theorem of Quillen, which we shall use repeatedly throughout this paper. For a survey of the variations on this useful principle see [8].

Theorem 3.1 (Quillen's fibre lemma) [19, Proposition 1.6] *Let P and Q be bounded posets and let $f : \hat{P} \mapsto \hat{Q}$ be an order-preserving map. Assume that for all $a \in \hat{Q}$, the fibre $F_a = \{z \in \hat{P} : f(z) \geq a\}$ is contractible. Then f induces a homotopy equivalence of the order complexes $\Delta(P)$ and $\Delta(Q)$. (The same conclusion holds if the fibre $F^a = \{z \in \hat{P} : f(z) \leq a\}$ is contractible for all $a \in \hat{Q}$.)*

Recall that P_n^k is the subposet of Π_n obtained by deleting all modular elements of type $(i, 1^{n-i})$, for $2 \leq i \leq k$. Thus $P_n^k = \bigcap_{i=2}^k Q_n^i$. It follows from the remarks about Q_n^k that P_n^k is also ranked, but of rank $n-2$ (since the atoms have been deleted). The aim of this section is to show that the $(n-4)$-dimensional complex $\Delta(P_n^k)$ and the $(n-3)$-dimensional complex $\Delta(Q_n^k)$ have the same homology. In fact the following stronger result holds.

Theorem 3.2 *The order complexes of P_n^k and Q_n^k are homotopy equivalent. More generally, for any subset I of $\{2, \dots, k-1\}$, the inclusion $\widehat{P_n^k} \hookrightarrow \widehat{Q_n^k \cap (\bigcap_{i\in I} Q_n^i)}$ induces a homotopy equivalence of the corresponding order complexes.*

Proof: We shall only prove the first statement, since the second follows by the identical argument.

Consider the inclusion map $\iota : \widehat{P_n^k} \to \widehat{Q_n^k}$. By Quillen's fibre lemma we need only show that the fibres $F_a = \{z \in \widehat{P_n^k} : z \geq a\}$ are contractible. This is clearly true if $a \in P_n^k$, so assume $a \in Q_n^k \backslash P_n^k$. Then a is a modular element with a unique nontrivial block B of size i, $2 \leq i \leq k-1$. For notational convenience assume a is the partition (with $n-i+1$ blocks) in which the elements $1, 2, \dots, n-i$ are the singletons. We may view a as a partition of $n-i+1$ elements with one distinguished element consisting of the block B. The fibre F_a is thus poset isomorphic to the poset $R_{n-i+1}(S(k))$ obtained from $\hat{\Pi}_{n-i+1}$ by removing a set $S(k)$. This set $S(k)$ consists of all modular elements whose unique nontrivial block is of cardinality s, $2 \leq s \leq k+1-i$, and contains the distinguished element B.

The fact that these posets are contractible follows from the next lemma. □

Lemma 3.3 *Let $k \geq 2$, and let S be the subset of modular elements of Π_n of type $(j, 1^{n-j})$, $2 \leq j \leq k$, such that n is in the unique nontrivial block of every element of S.*

Let $R_n(S)$ be the subposet of Π_n obtained by removing all elements of S. Then (the order complex of) $R_n(S)$ is contractible.

Proof: Let α_n denote the partition in Π_n consisting of exactly two blocks, one of which is the singleton block $\{n\}$. Note that $\alpha_n \in R_n(S)$. Define a map $f : \widehat{R_n(S)} \mapsto \Pi_n$ by

$$f(x) = x \wedge \alpha_n.$$

Here $\wedge$ denotes the meet operation in the lattice Π_n. Note that the effect of taking the meet of x with α_n is to fix x if n is a singleton of x, or else to produce a new partition x', where x' is obtained from x by splitting the block B containing n into two blocks so that n is a singleton. Now observe that

(a) f is order-preserving;
(b) the image of f is contained in $\widehat{R_n(S)}$ (for this it suffices to note that $\hat{0}$ is not in the image of f, and this is ensured by the fact that S contains all the atoms whose unique nontrivial block contains n);
(c) $f(x) \leq x$ and $f(f(x)) = f(x)$ for all x.

Conditions (b) and (c) together imply that the fibres $F_a = \{y : f(y) \geq a\}$ of f are contractible for all a in the image of f. Hence, by Quillen's fibre lemma again, f is a homotopy equivalence between $\widehat{R_n(S)}$ and the image of f. But the image of f clearly consists of all partitions in Π_n in which n is a singleton, except for the least element of Π_n. That is, the image of f is poset-isomorphic to $\hat{\Pi}_{n-1} \cup \hat{1}$, where the $\hat{1}$ is provided by the two-block partition α_n. Hence the image of f is contractible.

This completes the proof of Theorem 3.2. □

Remark 3.3.1 The conclusion of Lemma 3.3 is valid for more general subsets S of modular elements, as long as S contains all the modular elements of type $(2, 1^{n-2})$, (i.e., atoms) and that n is in the nontrivial block of all elements of S. The special case of Lemma 3.3, when S consists only of atoms, follows from [29, Theorem 6.1]; here S is the set of complements of the two-block partition α_n in which n is a singleton (for elaborations of this principle see the references in [8]).

Theorem 3.4 *Let $2 \leq k \leq n-1$. The reduced integral homology of the posets P_n^k and Q_n^k is free everywhere and vanishes except in dimension $(n-4)$. This holds more generally for the posets $\widehat{Q_n^k \cap (\bigcap_{i\in I} Q_n^i)}$, $I \subseteq \{2, \ldots k-1\}$.*

In particular for $n \geq 4$ and $k \geq 3$, the (pure) posets Q_n^k, $\widehat{Q_n^k \cap (\bigcap_{i\in I} Q_n^i)}$, $2 \notin I$, are not Cohen-Macaulay.

Proof: From Theorem 3.2 it follows that the two posets have the same homology. Since P_n^k has rank $(n-2)$, its order homology vanishes for all degrees greater than $n-4$, and is free in the top degree. On the other hand, Proposition 2.4 says that Q_n^k can have nonvanishing homology only in degrees $n-3$ and $n-4$. The result follows. □

As one more application of these arguments, we also obtain the following.

Theorem 3.5 *The poset $\widehat{P_n^{n-1}}$, and hence also $\widehat{Q_n^{n-1}}$ and $\widehat{Q_n^{n-1} \cap (\bigcap_{i\in I} Q_n^i)}$, $I \subseteq \{2, \ldots, n-2\}$, is homotopy-equivalent to $\widehat{\Pi}_{n-1}$. Hence the order complexes of P_n^{n-1}, Q_n^{n-1} and $\widehat{Q_n^{n-1} \cap (\bigcap_{i\in I} Q_n^i)}$, $I \subseteq \{2, \ldots, n-2\}$, have the homotopy type of a wedge of $(n-2)!$ spheres of dimension $(n-4)$.*

Proof: Consider the map $f : \widehat{P_n^{n-1}} \mapsto \widehat{\Pi}_n$ as defined in Lemma 3.3. The image of this map consists of all partitions in $\widehat{\Pi}_n$ such that n is a singleton, except for the two-block partition α_n of Lemma 3.3; it is therefore isomorphic to $\widehat{\Pi}_{n-1}$. The fibres (with respect to the image!) are contractible by the same argument as in the proof of Theorem 3.2. More precisely, we consider only fibres $F_a = \{z \in \widehat{P_n^{n-1}} : f(z) \geq a\}$ for a in the image of f. Note that the fibre of the two-block partition α_n of Lemma 3.3 is empty and hence not contractible. The result now follows by Lemma 3.3 and Quillen's fibre lemma.

The final statement follows from the well-known fact that the order complex of the partition lattice Π_n is shellable ([5, Example 2.9]), and hence (see [6, Theorem 1.3], [9, Theorem 4.1]) has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $(n-3)$ (see [24] for the Möbius (Betti) number computation). $\square$

From Corollary 2.3 we now have

Corollary 3.6 *For* $2 \leq k \leq n-1$, *let* β_n^k *denote the common dimension of the unique nonvanishing homology of the posets* P_n^k *and* $\widehat{Q_n^k \cap (\bigcap_{i \in I} Q_n^i)}$, $I \subseteq \{2, \ldots k-1\}$. *Then*

$$\beta_n^k = (-1)^{n-4} \mu(P_n^k) = (-1)^{n-4} \mu(Q_n^k) = (n-1)! \frac{n-k}{k}.$$

In order to investigate whether or not P_n^k is Cohen-Macaulay, we need to look at proper intervals in the poset. Note that the obvious analogue of Theorem 3.2 is false for arbitrary intervals of P_n^k. For example, in Q_6^5 the interval $J' = (\hat{0}, 12|3456)$ is homotopy equivalent to a wedge of six spheres S^2 (it coincides with the same interval in Π_6), whereas in P_6^5 the interval $J = (\hat{0}, 12|3456)$ has rank 3. It is not hard to see that J has the homotopy type of a wedge of 7 spheres of dimension 1.

To obtain information on intervals $(\hat{0}, y)$ in P_n^k, we need the following generalisation of Lemma 3.3.

Lemma 3.7 *Let S be the subset of the modular partitions in* Π_n *as in Lemma* 3.3 *and let* $y \in \widehat{\Pi}_n$, *such that* $y \notin S$ *and n is in a nontrivial block of y. Then* (*the order complex of*) *the subposet* $[\hat{0}, y] \backslash S$ *of the interval* $[\hat{0}, y]$ *is contractible.*

Proof: Note that $I = [\hat{0}, y] \backslash S$ is simply the interval $[\hat{0}, y]$ in the poset $R_n(S)$ of Lemma 3.3. Restrict the map f of Lemma 3.3 to the interval $\hat{I} = (\hat{0}, y) \cap R_n(S)$. Clearly $f(\hat{I}) \subseteq \hat{I}$. The image of f consists of all partitions in I such that n is a singleton, except for the $\hat{0}$. Also $f(y) \in \hat{I}$: this is because n is not a singleton in y, and hence $f(y) \neq y$. Clearly $f(y)$ is the (unique) greatest element of $f(I)$, and hence $f(I)$ is contractible. Now by the arguments of Lemma 3.3, I is contractible. $\square$

Proposition 3.8 *Let* $y \in P_n^k$. *Let J denote the interval* $(\hat{0}, y)$ *in* P_n^k, *and let J' denote the subset of the interval* $(\hat{0}, y)$ *in* Q_n^k *obtained by removing the set* $M_{y,k}$ *of all modular elements whose unique nontrivial block coincides with a block of y, and has size* $\leq k$. *Then the inclusion* $J \hookrightarrow J'$ *induces a homotopy equivalence of order complexes.*

Proof: This follows by checking that the fibres are contractible, as in Theorem 3.2, except that now we make use of Lemma 3.7. Note that removal of the elements in the set $M_{y,k}$ is necessary in order to apply the lemma. □

Proposition 3.9 *Let y, J, J' be as in Proposition* 3.8*. Then the homology of J (and J') vanishes in all degrees different from* $\mathrm{rank}_{\Pi_n}(y) - 3$*, the top dimension of the interval J of P_n^k (here* rank_{Π_n} *denotes the rank function of* Π_n*).*

Proof: Proposition 3.8 implies that J and J' have the same homology. There are two key observations. First, J' is obtained from the interval $(\hat{0}, y)$ in Π_n by deleting an antichain. Hence by Theorem 2.1, J' can have nonzero homology only in degrees $\mathrm{rank}_{\Pi_n}(y) - 2$ and $\mathrm{rank}_{\Pi_n}(y) - 3$. Second, the dimension of the order complex of J is the smaller of these two degrees. The result follows. □

Let $J = [x, y]$, $x \neq \hat{0}$, $y < \hat{1}$ be an interval in the poset P_n^k. First assume there are two nontrivial blocks of x which are contained in distinct blocks of y. In this case it is clear that the interval $[x, y]$ of P_n^k coincides with the interval between x and y in Π_n, and is therefore Cohen-Macaulay.

Next suppose all the nontrivial blocks of x are contained in a single block of y. Let a_i be the size of the nontrivial block A_i of x, $1 \leq i \leq r$, and let s be the size of the nontrivial block B of y which contains them. Note that $r \geq 2$. Let x' be the partition of the set B induced by x (x' has r nontrivial blocks A_i and $s - r$ singletons). Then the interval $[x, y]$ of P_n^k is isomorphic to a product of the interval $[x', \hat{1}]$ in P_s^k, together with a collection of partition lattices.

These observations and the preceding results show that P_n^k is Cohen-Macaulay if and only if all intervals of the form $[x, \hat{1}]$ have homology which vanishes in all dimensions less than the highest. Although the analogue of Theorem 3.2 does hold for such intervals, this fact is not as helpful in this case. The difficulty occurs because there is no longer a shift in the dimensions of the order complexes of the intervals J and J'.

Proposition 3.10 *Let $J = [x, \hat{1}]$, $x \neq \hat{0}$, be an interval in P_n^k. Let J' be the interval $[x, \hat{1}]$ in the poset Q_n^k. Then the inclusion map $J \hookrightarrow J'$ is a homotopy equivalence, and hence J and J' can have nonvanishing reduced homology only in dimension* $n - 3 - \mathrm{rank}_{\Pi_n}(x)$ *or* $n - 4 - \mathrm{rank}_{\Pi_n}(x)$.

Proof: The statements of the theorem are immediate if J' (and hence J) coincides with the interval $[x, \hat{1}]$ of Π_n, i.e., if x is not smaller than a modular element of type $(k, 1^{n-k})$. Hence we consider the other case.

We use the same argument as in Theorem 3.2. We need to show that the fibres $F_a = \{z \in J : z \geq a\}$ for $a \in J' \backslash J$ of type $(j, 1^{n-j})$, $2 \leq j \leq k - 1$, are contractible. Let B be the unique nontrivial block of a.

The fibre F_a is isomorphic to a poset $R_m(S)$ as in Lemma 3.3, where m is the number of blocks of a, and S is as described in the proof of Theorem 3.2. Hence it is contractible by Lemma 3.3.

The conclusion now follows from Theorem 2.1. □

Let $2 \leq k \leq n-1$. Fix an integer a between 2 and k. Define $T_{n,a}^{\leq k}$ to be the subposet obtained from Π_n by deleting all modular elements x of type $(j, 1^{n-j})$, $a \leq j \leq k$, such that the unique nontrivial block of x contains the a largest integers $n-a+1, \ldots, n$. Similarly define $T_{n,a}^{=k}$ to be the subposet obtained from Π_n by deleting all modular elements x of type $(k, 1^{n-k})$, such that the unique nontrivial block of x contains the elements $n-a+1, \ldots, n$. Let $x \in P_n^k$ be of rank $\leq k-1$, and assume x has at least one singleton block. Then it is easy to see that $[x, \hat{1}]_{P_n^k}$ is poset isomorphic to $T_{m,a}^{\leq k}$, while $[x, \hat{1}]_{Q_n^k}$ is poset isomorphic to $T_{m,a}^{=k}$, where m is the number of blocks of x, and a is the number of nontrivial blocks of x.

Hence Proposition 3.10 may be rephrased as follows:

Let $2 \leq a \leq k \leq n-1$. The inclusion $T_{n,a}^{\leq k} \hookrightarrow T_{n,a}^{=k}$ is a homotopy equivalence.

Note that the order complexes of $T_{n,a}^{\leq k}$ and $T_{n,a}^{=k}$ both have the same dimension $(n-3)$, and hence, by Theorem 2.1, we can only conclude that they both have nonvanishing homology only in degrees $n-3$ and $n-4$. Moreover from Corollary 2.3 we have

$$\dim \tilde{H}_{n-3}\big(T_{n,a}^{\leq k}\big) - \dim \tilde{H}_{n-4}\big(T_{n,a}^{=k}\big) = (n-a)!\left(\frac{(n-1)!}{(n-a)!} - \frac{(k-1)!}{(k-a)!}\right).$$

In particular, since the right-hand side is clearly positive, we are forced to conclude that homology is nonzero in degree $(n-3)$.

Fortunately it is not hard to show that

Proposition 3.11 *The posets $T_{n,a}^{=k}$ are (pure) shellable. Hence the posets $T_{n,a}^{\leq k}$ and $T_{n,a}^{=k}$ are both homotopy equivalent to a wedge of $(n-a)!(\frac{(n-1)!}{(n-a)!} - \frac{(k-1)!}{(k-a)!})$ spheres of dimension $(n-3)$. Hence (the order complexes of) all intervals of the form $[x, \hat{1}]$ and $[x, y]$, $x \neq \hat{0}$, in P_n^k and Q_n^k have the homotopy type of a wedge of spheres.*

Proof: We shall use the following simple EL-labelling of the partition lattice due to Wachs [28]. If $u \to v$ is a covering relation in Π_n, so that v is obtained from u by merging two blocks B_1 and B_2, define the label of the edge $(u < v)$ to be $\max(B_1 \cup B_2)$. We shall show that this EL-labelling restricts to an EL-labelling of $T_{n,a}^{=k}$.

With respect to this labelling, there is a unique strictly increasing chain $c_{(x,y)}$ in every interval (x, y) of Π_n. By [5, Proposition 2.8], it suffices to show that for every $x < y$ in $T_{n,a}^{=k}$, the chain $c_{(x,y)}$ is a chain of $T_{n,a}^{=k}$.

We need only consider those elements $x < y$ of $T_{n,a}^{=k}$ for which the interval $(x, y)_{\Pi_n}$ contains elements forbidden in $T_{n,a}^{=k}$. Such an element z must have a unique nontrivial block B of size k containing the a largest integers $n-a+1, \ldots, n$. Suppose the unique strictly increasing chain $c_{(x,y)} = (x = z_0 < z_1 < \cdots < z_i = y)$ contains the element z; since $x \neq z$, it must therefore have the label n on one of its edges. This edge can only be the last edge of the chain, which implies that $z = z_i = y$, contradicting the fact that $y \in T_{n,a}^{=k}$.

The remaining statements follow from the remarks preceding the proposition. $\square$

Putting together the work of this section, we have shown

Theorem 3.12 *The poset P_n^k is Cohen-Macaulay over the integers.*

For $k = 2$, P_n^2 is simply a rank-selected subposet of Π_n, hence its order complex is shellable by [5, Theorem 4.1]). It follows from the general theory of shellability (see [6, Theorem 1.3] and [9, Theorem 4.1]) that the order complex has the homotopy type of a wedge of spheres. The subposet P_n^3 (in fact the intersection lattice of a codimension 2 orbit arrangement, and denoted $\Pi_{(2,2,1,\ldots,1)}$ in this context [7]), was shown to be CL-shellable by this author and V. Welker (1993, unpublished), and independently in recent far-reaching work of Kozlov ([15]). However this argument seems to break down at a key point for P_n^4. For $k \geq 5$ it can be seen that upper intervals $(x, \hat{1})$ in P_n^k are not totally semimodular, making it difficult to show CL-shellability.

However, by using a topological result and a technical lemma due to Bouc, we can show that

Theorem 3.13 *Let $2 \leq k \leq n-1$. The order complex of the posets $Q_n^k \cap (\bigcap_{i \in I} Q_n^i)$, $I \subseteq \{2, \ldots, k-1\}$ is homotopy equivalent to a wedge of (β_n^k) spheres of dimension $n-4$.*

Proof: The case $k = n-1$ was settled by Theorem 3.5, while the case $n = 2$ follows from Theorem 3.1 and the fact that the order complex of Q_n^2 is shellable. Assume $3 \leq k \leq n-2$.

The cases $n \leq 5$ follow easily by inspection and using Theorem 3.4. Thus we assume $n \geq 6$.

It suffices by Theorem 3.2 to consider the poset Q_n^k. We have shown that the order complex has the same integral homology as that of a wedge of β_n^k spheres of dimension $n-4$. In order to show that the homotopy type is also the same, we invoke a result from homotopy theory: By [8, 9.15], it suffices to show that the order complex of Q_n^k is simply connected.

Lemma 3.14 below provides a technical tool for obtaining information about the fundamental group of the order complex of a poset.

Consider the inclusion map $\iota : \widehat{P_n^{n-1}} \to \widehat{Q_n^k}$. We claim that, for every maximal element a in $\widehat{Q_n^k}$, the fibres $\iota^a = \{x \in \widehat{P_n^{n-1}} : x \leq a\}$ are nonempty and connected. This is obvious if $a \in \widehat{P_n^{n-1}}$. The maximal elements in $Q_n^k \backslash P_n^{n-1}$ clearly all have two blocks, one of which has size $n-1$. If a is such an element, the (order complex of the) fibre ι^a is clearly homotopy equivalent to the order complex of P_{n-1}^{n-2}, and hence has the homotopy type of a wedge of $(n-5)$-spheres. Since $n \geq 6$, the claim follows.

Note that when $n \geq 6$, the order complex of P_n^{n-1} is connected and simply connected by Theorem 3.5. Hence Lemma 3.14 applies, showing that the fundamental group of $\widehat{Q_n^k}$ is trivial. □

Lemma 3.14 ([10, Section 2.2.2, Lemme 6]) *Let $f : X \to Y$ be an order-preserving map of posets X and Y, and assume that the order complex of X is connected. If for every maximal element y in $\hat{Y}$, the order complex of the fibre $f^y := \{x \in \hat{X} : x \leq y\}$ is nonempty and connected, then the order complex of Y is connected and the induced homomorphism of fundamental groups $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ is surjective.*

4. The representation of the symmetric group S_n on the homology

In this section all homology is taken over the field of complex numbers. We shall first compute the S_n-module structure of the unique nonvanishing homology of the poset Q_n^k. For this we need to recall some of the results of [26]. For a finite poset Q and a finite group G of automorphisms of Q, we denote by $\mathrm{Alt}(Q)$ the Lefschetz (G-)module of Q, i.e., $\mathrm{Alt}(Q) = \sum_i (-1)^i \tilde{H}_i(Q)$.

Theorem 4.1 (See [26, Theorem 1.10 and Remark 1.10.1]) *Let P be a Cohen-Macaulay poset of rank r, G a finite group of automorphisms of P, and Q a G-invariant subposet of P.*

Then as G-modules:

$$\begin{aligned}&(-1)^r \mathrm{Alt}(Q) - \tilde{H}(P)\\ &\quad= \bigoplus_{\substack{c=(\hat{0}<x_1<\cdots<x_k<\hat{1})\\ x_i\notin Q}} (-1)^k (\tilde{H}(\hat{0}, x_1)_P \otimes \tilde{H}(x_1, x_2)_P \otimes \cdots \otimes \tilde{H}(x_k, \hat{1})_P)\uparrow_{G_c}^{G};\end{aligned}$$

where the sum runs over all representatives of G-orbits of chains c of elements not in Q, and G_c is the stabiliser of the chain c in P.

In the special case when $P\setminus Q$ is an antichain, this result simplifies, giving

Theorem 4.2 *Let P be a Cohen-Macaulay poset of rank r and G a finite group of automorphisms of P. Let Q be a G-invariant subposet of P such that $P\setminus Q$ is an antichain. Then, as a G-module, the Lefschetz module* $\mathrm{Alt}(Q)$ *of Q is determined by*

$$(-1)^{r-1}\mathrm{Alt}(Q) + \tilde{H}(P) = \bigoplus_{\substack{\hat{0}<x<\hat{1}\\ x\in P/G, x\notin Q}} (\tilde{H}(\hat{0}, x)_P \otimes \tilde{H}(x, \hat{1})_P)\uparrow_{G_x}^{G}.$$

Another way to obtain Theorem 4.2 is to observe that all the maps in the exact homology sequence of the pair (P, Q) are G-equivariant; consequently the proof of Theorem 2.1 can be made G-equivariant to yield Theorem 4.2.

The hypotheses of the next theorem arise frequently in the study of subposets of the partition lattice. The theorem is a general result on the homology representation of upper intervals in posets of partitions, and was used extensively in [26]. The details of the proof are identical to the proof of [26, Theorem 1.4].

Theorem 4.3 [26] *Let $A_n \subseteq \Pi_n$ be a family of posets of set partitions and let $x \in A_n$ be of type λ where λ is an integer partition of n with m_i blocks of size i. Assume that $(x, \hat{1})_{A_n}$ is poset isomorphic to a poset B_r, where r is the number of blocks of x. There is an action of the symmetric group S_r on the poset B_r, by permuting the blocks of x. Let α_r denote the* (*possibly virtual*) *representation of S_r on the Lefschetz module* $\mathrm{Alt}(B_r)$. *Note that there is a copy of the Young subgroup $\times_i S_{m_i}$ in S_r. Let G_λ denote the stabiliser of x; thus G_λ is* *the direct product of wreath product groups $\times_i S_{m_i}[S_i]$, where $S_a[S_b]$ is the wreath product group obtained by letting S_a act on a copies of S_b.*

Finally assume that the restriction of the representation α_r to $\times_i S_{m_i}$ can be written (uniquely) as the following sum of irreducible modules:

$$\alpha_r\downarrow_{\times_i S_{m_i}} = \sum_{\bar{\nu}} c_{\bar{\nu}} \otimes_i V_{\nu^{(i)}},$$

where $\bar{\nu}$ denotes the ordered tuple of partitions $\nu^{(i)}$ of m_i, and $V_{\nu^{(i)}}$ denotes the irreducible S_{m_i}-module indexed by the integer partition $\nu^{(i)}$.

Then the (possibly virtual) representation of G_λ on the Lefschetz module of $(x, \hat{1})_{A_n}$, $\mathrm{Alt}((x, \hat{1})_{A_n})$ is given by

$$\sum_{\bar{\nu}} c_{\bar{\nu}} \otimes_i V_{\nu^{(i)}}\big[1_{S_i}\big],$$

where $V_{\nu^{(i)}}[1_{S_i}]$ denotes the wreath product $S_{m_i}[S_i]$-module of the irreducible $V_{\nu^{(i)}}$ with the trivial S_i-module 1_{S_i}.

The formula in the preceding theorem is more compactly expressed in terms of the plethysm operation and symmetric functions; see [26] for details.

For the purposes of this paper we shall only need to apply Theorem 4.3 to the upper interval $(x, \hat{1})$ of the partition lattice Π_n, when x is an element of type $(k, 1^{n-k})$. In this case all the posets involved are Cohen-Macaulay. We write π_n for the representation of S_n on the top homology of Π_n. The interval $(x, \hat{1})$ is isomorphic to the partition lattice Π_{n-k+1}, and hence in applying Theorem 4.3 we need to compute the restriction of π_{n-k+1} to the stabiliser of x, which is conjugate to the Young subgroup $S_{n-k} \times S_1$. But, by [24], this is just the regular representation of S_{n-k}. Hence we have the following result, which was also worked out in [26].

Corollary 4.4 (See [26, Example 2.11]) *Let x be an element of type $(k, 1^{n-k})$ in Π_n. The representation of the Young subgroup $S_{n-k} \times S_k$ on the top homology of the interval $(x, \hat{1})$ is*

$$\rho_{n-k} \otimes 1_{S_k},$$

where ρ_{n-k} denotes the regular representation of S_{n-k}.

It is now easy to compute the homology representation of Q_n^k:

Theorem 4.5 *Let $2 \le k \le n-1$. The representation of the symmetric group S_n on the unique nonvanishing homology $\tilde{H}_{n-4}(Q_n^k)$ is given by the quotient module*

$$(\rho_{n-k} \otimes \pi_k)\uparrow_{S_{n-k}\times S_k}^{S_n} \big/ \pi_n.$$

Proof: Let x_0 denote a partition of type $(k, 1^{n-k})$ whose stabiliser is the Young subgroup $S_k \times S_{n-k}$. Theorem 4.2 gives the following equality of S_n-modules:

$$\tilde{H}_{n-4}\big(Q_n^k\big) \oplus \tilde{H}_{n-3}(\Pi_n) = (\tilde{H}(\hat{0}, x_0) \otimes \tilde{H}(x_0, \hat{1}))\big\uparrow_{S_{n-k}\times S_k}^{S_n}.$$

Now use Corollary 4.4 and the fact that $(\hat{0}, x_0)$ is isomorphic to Π_k. □

Our next goal is to compute the homology representation of P_n^k. We indicate two approaches. The first is a straightforward application of Theorem 4.2, and uses the same arguments as in the proof of Theorem 4.5.

Theorem 4.6 *Let $2 \le k \le n-1$. As an S_n-module the unique nonvanishing homology $\tilde{H}_{n-4}(P_n^k)$ of P_n^k is given by the quotient module*

$$\pi_{n,k} = (\rho_{n-k} \otimes \pi_k)\big\uparrow_{S_{n-k}\times S_k}^{S_n} \big/ \pi_n; \tag{4.1}$$

here ρ_{n-k} denotes the regular representation of S_{n-k}.

Proof: We proceed by induction on k. The result holds for $k = 2$ by [26, Theorem 2.10 and Example 2.11]. Assume it holds for all parameters $2 \le k' \le k-1$. Now P_n^k is the subposet of P_n^{k-1} obtained by deleting the elements of type $(k, 1^{n-k})$. Hence, if x_0 is a partition of type $(k, 1^{n-k})$ whose stabiliser is the Young subgroup $S_k \times S_{n-k}$, then using Theorem 4.2 (with $P = P_n^{k-1}$ and $Q = P_n^k$) we have the equality of S_n-modules

$$\tilde{H}_{n-4}\big(P_n^{k-1}\big) - \tilde{H}_{n-4}\big(P_n^k\big) = \big(\tilde{H}\big((\hat{0}, x_0)_{P_n^{k-1}}\big) \otimes \tilde{H}\big((x_0, \hat{1})_{P_n^{k-1}}\big)\big)\big\uparrow_{S_{n-k}\times S_k}^{S_n}.$$

The interval $(x_0, \hat{1})_{P_n^{k-1}}$ in P_n^{k-1} is isomorphic to a partition lattice, and the $(S_{n-k} \times S_k)$-module structure of its homology follows from Corollary 4.4. The interval $(\hat{0}, x_0)_{P_n^{k-1}}$ in P_n^{k-1} is clearly isomorphic to P_k^{k-1}. By induction hypothesis the structure of the homology of P_k^{k-1} as an S_k-module is given by the representation $\pi_{k,k-1}$. It follows that as an $(S_{n-k} \times S_k)$-module, the homology of $(\hat{0}, x_0)_{P_n^{k-1}}$ is given by $1_{S_{n-k}} \otimes \pi_{k,k-1}$. Now by routine manipulations the result follows. □

Corollary 4.7 *Let $2 \le k \le n-1$. The character values of the representation of the symmetric group S_n on the unique nonvanishing homology of P_n^k and of Q_n^k, for an element in S_n of cycle-type σ, are*

$$\begin{cases} (-1)^{k-\frac{k}{d}} \dfrac{\mu(d)}{k} d^{\frac{k}{d}} \left(\dfrac{k}{d}\right)! (n-k)!, & \text{if } \sigma = \big(d^{\frac{k}{d}}, 1^{n-k}\big), d \mid k \\ -(-1)^{n-\frac{n}{d}} \dfrac{\mu(d)}{n} d^{\frac{n}{d}} \left(\dfrac{n}{d}\right)!, & \text{if } \sigma = \big(d^{\frac{n}{d}}\big), d \mid n \\ 0, & \text{otherwise}. \end{cases}$$

Proof: By a well-known result of Hanlon (see [12, Theorem 4.1], [24, Lemma 7.1]), the character values of the representation π_n on an element of cycle-type σ in S_n are given by

$$\begin{cases} (-1)^{n-\frac{n}{d}} \dfrac{\mu(d)}{n} d^{\frac{n}{d}} \left(\dfrac{n}{d}\right)!, & \text{if } \sigma = \left(d^{\frac{n}{d}}\right), d \mid n \\ 0, & \text{otherwise.} \end{cases}$$

Now the result follows from formula (4.1). □

By Theorems 4.5 and 4.6, the posets P_n^k and Q_n^k have S_n-isomorphic homology. In fact we can show that the homotopy equivalence of Theorem 3.2 is an S_n-homotopy, thereby establishing the result in another way. First we state the group-equivariant version of Quillen's fibre lemma.

Theorem 4.8 (See, e.g., [4, Chapter 6]) *Let P and Q be bounded posets, let G be a finite group of automorphisms of P and Q, and let $f : \hat{P} \mapsto \hat{Q}$ be an order-preserving G-map of posets. For $a \in \hat{Q}$ let G_a denote the stabiliser of a. Assume that for all $a \in \hat{Q}$, the fibre $F_a = \{z \in \hat{P} : f(z) \geq a\}$ is G_a-contractible (i.e., the fixed-point subposet $F_a^{G_a}$ of points in F_a fixed by G_a, is contractible). Then f induces a G-homotopy equivalence of the order complexes $\Delta(P)$ and $\Delta(Q)$. (The same conclusion holds if the fibre $F^a = \{z \in \hat{P} : f(z) \leq a\}$ is G_a-contractible for all $a \in \hat{Q}$.)*

In order to show that the homotopy equivalence of Theorem 3.2 is group equivariant, we need to show that the fibres F_a in the proof of the theorem are G_a-contractible, where G_a is the stabiliser of the element a of type $(j, 1^{n-j})$. (Thus G_a is isomorphic to the Young subgroup $S_{n-j} \times S_j$.) This in turn will follow from the group-equivariant version of Lemma 3.3.

It is not hard to see that the homotopy equivalence of Lemma 3.3 is an S_{n-1}-homotopy, where we identify S_{n-1} with the subgroup of S_n which fixes n. For any subgroup H of S_{n-1}, it is easy to check that the map f restricts to a homotopy equivalence on the fixed point subposet $R_n(S)^H$ consisting of points fixed by H, and that the image remains contractible. Hence the posets $R_n(S)$ are in fact S_{n-1}-contractible.

Proposition 4.9 *The inclusion $\widehat{P_n^k} \hookrightarrow \widehat{Q_n^k}$ and more generally, for any subset $I \subseteq \{2, \ldots, k-1\}$, the inclusion*

$$\widehat{P_n^k} \hookrightarrow \widehat{Q_n^k \cap \left(\bigcap_{i \in I} Q_n^i\right)}$$

induces an S_n-homotopy equivalence of the corresponding order complexes.

These observations also imply that the homotopy equivalence between $\widehat{P_n^{n-1}}$ and $\hat{\Pi}_{n-1}$ in Theorem 3.5 is an S_{n-1}-homotopy. Because the case $k = n-1$ is of particular interest, we state it separately:

Corollary 4.10 *The posets P_n^{n-1} and Q_n^{n-1}, and more generally, the posets*

$$\widehat{Q_n^{n-1} \cap \left(\bigcap_{i \in I} Q_n^i \right)}, \quad I \subseteq \{2, \ldots, n-2\},$$

are S_{n-1}-homotopy equivalent to $\hat{\Pi}_{n-1}$ and have homology modules that are S_n-isomorphic to the representation

$$(\pi_{n-1})\uparrow_{S_{n-1}}^{S_n} / \pi_n. \tag{4.2}$$

This is the representation of S_n computed by Sarah Whitehouse [31] on the tree complex of Alan Robinson (see also [13, 20, 21]). It follows from Corollary 4.10 (or by inspecting character values in Corollary 4.7) that the restriction to S_{n-1} is the representation π_{n-1}.

Denote by $\bar{\pi}_n$ the lifting of π_{n-1} given by the representation (4.2). Let $V_{(n-1,1)}$ denote the irreducible S_n-module indexed by the integer partition $(n-1, 1)$. By basic manipulations one sees that

$$\bar{\pi}_n \otimes V_{(n-1,1)} \simeq \pi_n, \tag{4.3}$$

a formula which appears in [11].

5. Conclusion

In this final section we discuss some questions raised by the phenomena exhibited in this paper for the partition lattice Π_n.

Let M_n denote the subposet of Π_n consisting of the modular partitions in $\hat{\Pi}_n$, together with the elements $\hat{0}$ and $\hat{1}$. Clearly M_n is just the truncated Boolean lattice of subsets of an n-set, with the subsets of size 1 (i.e., the rank one elements) deleted. It follows from Stanley's theory of R-labellings ([25]) that the Möbius number is $\mu(M_n) = (-1)^{n-1}(n-1)$.

On the other hand, by Theorem 3.4, we know that $P_n^{n-1} = \Pi_n \backslash M_n$ has Möbius number $(-1)^{n-4}(n-2)!$

Hence we have, at the level of Möbius numbers, the equation

$$|\mu(\Pi_n \backslash M_n)\mu(M_n)| = |\mu(\Pi_n)|. \tag{5.1}$$

We also have the topological result that

$$\Delta(\Pi_n \backslash \hat{M}_n) \simeq \Delta(\Pi_{n-1}). \tag{5.2}$$

The formula (4.3) of the preceding section further suggests that the factorisation (5.1) carries over to the homology, at the level of S_n-modules, with the introduction of a sign twist. By a result of Solomon ([22], see also [24]), the representation of S_n on the homology of M_n is precisely the irreducible indexed by the integer partition $(2, 1^{n-2})$. Hence (4.3) says that as modules over the integers,

$$\tilde{H}_{n-4}(\Pi_n \backslash \hat{M}_n) \otimes \tilde{H}_{n-1}(M_n) = \tilde{H}_{n-3}(\Pi_n), \tag{5.3}$$

and as S_n-modules,

$$\tilde{H}_{n-4}(\Pi_n \backslash \hat{M}_n) \otimes \tilde{H}_{n-1}(M_n) = \tilde{H}_{n-3}(\Pi_n) \otimes \mathrm{sgn}_{S_n}. \tag{5.4}$$

It would be interesting to see if these phenomena, e.g., (5.1) and (5.3), occur for other instances of removing modular elements from a supersolvable geometric lattice. For example, the analogues of (5.1) and (5.3) hold trivially for the Boolean lattice, where every element is modular. The analogue of (5.2) however is clearly false.

Acknowledgment

Research supported by National Science Foundation Grant No. DMS9400875.

References

1. E. Babson, A. Björner, S. Linusson, J. Shareshian, and V. Welker, "Complexes of not i-connected graphs, extended abstract," *Proceedings of the 9th Formal Power Series and Algebraic Combinatorics Conference*, Vienna, July 1997.
2. K. Baclawski, "Cohen-Macaulay ordered sets," *J. Algebra* **63** (1980), 226–258.
3. K. Baclawski, "Cohen-Macaulay connectivity and geometric lattices," *Europ. J. Combinatorics* **3** (1982), 293–305.
4. D.J. Benson, *Representations and Cohomology*, Cambridge University Press, 1992, Vol. 2.
5. A. Björner, "Shellable and Cohen-Macaulay partially ordered sets," *Trans. Amer. Math. Soc.* **260** (1980), 159–183.
6. A. Björner, "Some combinatorial and algebraic properties of coxeter complexes and tits buildings," *Adv. Math.* **52** (1984), 173–212.
7. A. Björner, "Subspace arrangements," *Proc. First European Congress of Mathematics*, Paris 1992, Birkhaüser, Basel, pp. 321–370, 1994.
8. A. Björner, "Topological methods," in *Handbook of Combinatorics*, R. Graham, M. Grötschel and L. Lóvasz (Eds.), North-Holland, 1995, pp. 1819–1872.
9. A. Björner and M.L. Wachs, "Shellable nonpure complexes and posets I," *Trans. Amer. Math. Soc.* **348**(4) (1996), 1299–1327.
10. S. Bouc, "Homologie de certains ensembles de 2-sous-groupes du groupe symétrique," *J. Algebra* **150** (1992), 158–186.
11. E. Getzler and M. Kapranov, "Cyclic operads and cyclic homology," *Geometry, Topology and Physics*, International Press, Cambridge, MA, 1995, pp. 167–201.
12. P. Hanlon, "The fixed-point partition lattices," *Pacific J. Math.* **96** (1981), 319–341.
13. P. Hanlon, "Otter's method and the homology of homeomorphically irreducible k-trees," *J. Comb. Theory (A)* **74**(2) (1996), 301–320.
14. P. Hanlon and R.P. Stanley, "A q-deformation of a trivial symmetric group action," *Trans. Amer. Math. Soc.* **350** (1998), 4445–4459.
15. D. Kozlov, "General lexicographic shellability and orbit arrangements," preprint (1996), annals of combinatorics, to appear.
16. O. Mathieu, "Hidden Σ_{n+1}-actions," *Commun. Math. Phys.* **176** (1996), 467–474.
17. J. R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, 1984.
18. J.R. Munkres, "Topological results in combinatorics," *Mich. Math. J.* **31** (1984), 113–128.
19. D. Quillen, "Homotopy properties of the poset of nontrivial p-subgroups of a group," *Adv. Math.* **28** (1978), 101–128.
20. Alan Robinson, "The space of fully grown trees," Sonderforschungsbereich 343, Universität Bielefeld, Preprint 92-083 (1992).

21. C.A. Robinson and S. Whitehouse, "The tree representation of Σ_{n+1}," *J. Pure and Appl. Algebra* **111**(1–3) (1996), 245–253.
22. L. Solomon, "A decomposition of the group algebra of a finite Coxeter group," *J. Algebra* **9** (1968), 220–239.
23. R.P. Stanley, "Balanced Cohen-Macaulay complexes," *Trans. Amer. Math. Soc.* **249** (1979), 139–157.
24. R.P. Stanley, "Some aspects of groups acting on finite posets," *J. Comb. Theory (A)* **32**(2) (1982), 132–161.
25. R.P. Stanley, *Enumerative Combinatorics*, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986, Vol. 1.
26. S. Sundaram, "The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice," *Adv. in Math.* **104**(2) (1994), 225–296.
27. S. Sundaram, "Homotopy and homology of non-modular partitions and related posets, extended abstract," *Proceedings of the 9th Formal Power Series and Algebraic Combinatorics Conference*, Vienna, July 1997.
28. M.L. Wachs, "A basis for the homology of the d-divisible partition lattice," *Adv. in Math.* **117**(2) (1996), 294–318.
29. J.W. Walker, "Topology and combinatorics of ordered sets," Ph.D. Thesis, Massachusetts Institute of Technology, 1981.
30. J.W. Walker, "Homotopy type and Euler characteristic of partially ordered sets," *Europ. J. Combinatorics* **2** (1981), 373–384.
31. S.A. Whitehouse, "Γ-(Co)homology of commutative algebras and some related representations of the symmetric group," Ph.D. Thesis, University of Warwick, 1994.