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A Homological Approach to Two Problems on Finite Sets

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Abstract. We propose a homological approach to two conjectures descended from the Erdős-Ko-Rado Theorem, one due to Chvátal and the other to Frankl and Füredi. We apply the method to reprove, and in one case improve, results of these authors related to their conjectures.

Keywords: extremal problem, finite set, Erdős-Ko-Rado Theorem

1. Introduction

The purpose of this paper is to propose a homological approach to two problems descended from the Erdős-Ko-Rado theorem [3], namely a conjecture of Chvátal [1], and another of Frankl and Füredi [4]. Our interest in these questions was prompted by [4], to which we refer for a more thorough discussion (and from which we borrow most of our terminology).

In what follows $\mathcal{F}$ will be a collection of k-element subsets of some finite set X of cardinality n. (Such a collection is often called a k-graph or k-uniform hypergraph.)

In our context a *d-simplex* is a collection $F_1, \ldots, F_{d+1}$ of sets such that

$$\bigcap_{i=1}^{d+1} F_i = \emptyset, \tag{1}$$

but

$$\cap\{F_i : 1 \le i \le d+1,\ i \ne j\} \ne \emptyset \quad \text{for each } j \in [d+1].$$

(We use $[s]$ for $\{1, \ldots, s\}$.)

A simplex is *special* if $|\bigcap_{i\in J} F_i| = d + 1 - |J|$ for all $J \subseteq [d+1]$ with $|J| \ge 2$ (equivalently, if $|\bigcup_{i<j}(F_i \cap F_j)| = d+1$).

We write $s(n, k, d)$ (resp. $s^*(n, k, d)$) for the maximum size of an $\mathcal{F} \subseteq \binom{[n]}{k}$ containing no d-simplex (resp. special d-simplex).

Then the Erdős-Ko-Rado theorem (actually only the best-known case thereof) says that $s(n, k, 1) = \binom{n-1}{k-1}$ for every $n \ge 2k$. Chvátal [1] proposed extending this to

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Conjecture 1.1 $s(n, k, d) = \binom{n-1}{k-1}$ *whenever* $d < k \leq \frac{dn}{d+1}$.

(Note that if $k > \frac{dn}{d+1}$ then one cannot even have (1).)

Chvátal proved his conjecture for $k = d + 1$. Frankl and Füredi [4] proved it for every (fixed) $k,\ d$ and $n > n_0(k, d)$, and showed that in this case one has equality only if $\mathcal{F} = \{F \in \binom{X}{k} : x \in F\}$, for some $x \in X$.

Here we give (in Section 3) an alternate, homological proof of Chvátal's result. We do not so far see how to push our approach to the general case, but hope it may eventually lead to more complete results.

For special simplices Frankl and Füredi [4] proved

Theorem 1.2 *Let* $k \geq d + 3$ *or* $d = 2$ *and* $n > n_0(k)$. *If* $\mathcal{F} \subseteq \binom{X}{k}$ *contains no special* d*-simplex, then* $|\mathcal{F}| \leq \binom{n-1}{k-1}$, *with equality iff* $\mathcal{F} = \{F \in \binom{X}{k} : x \in F\}$ *for some* $x \in X$.

They conjectured that this is actually true whenever $k \geq d + 1$, and in the case $k = d + 1$ proposed the more precise

Conjecture 1.3 *If* $n \geq 2k$ *and* $\mathcal{F} \subseteq \binom{X}{k}$ *contains no special* $(k - 1)$*-simplex, then* $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

As far as we can see, the natural generalization also seems plausible:

Conjecture 1.4 *If* $n \geq (d + 1)(k - d + 1)$ *and* $\mathcal{F} \subseteq \binom{X}{k}$ *contains no special* d*-simplex then* $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

Our second result is a proof (again homological) of Conjecture 1.3 for $k = 3$.

Theorem 1.5 *If* $n \geq 6,\ \mathcal{F} \subseteq \binom{X}{3}$, *and* $\mathcal{F}$ *contains no special triangle, then* $|\mathcal{F}| \leq \binom{n-1}{2}$.

This case ($d = 2,\ k = 3$) of Theorem 1.2 is proved in [4] provided $n \geq 75$; so we do add something here, though again we feel the approach is more interesting than the result.

For information on equality in Theorem 1.5 see the end of Section 4.

2. Homological background

Write $\langle \mathcal{F} \rangle$ for the hereditary closure of $\mathcal{F}$: $\langle \mathcal{F} \rangle = \{A \subseteq X : \exists F \in \mathcal{F}, A \subseteq F\}$.

The (binary) *chain complex* belonging to $\mathcal{F} \subseteq \binom{X}{k}$ is $C_k(\mathcal{F}) \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots$, where C_i is the set of all formal $\mathbf{Z}_2$-sums of i-sets in $\langle \mathcal{F} \rangle$, and the *boundary maps* $\partial_i : C_i \to C_{i-1}$ are the linear maps defined by

$$\partial_i Y = \sum \{Z : Z \subset Y, |Z| = i - 1\} \quad \forall Y \in \langle \mathcal{F} \rangle, |Y| = i.$$

We will similarly write $C(\mathcal{G}) = C_l(\mathcal{G})$ for any $\mathcal{G} \subseteq \binom{X}{l}$. For background see [5].

Now $\partial_{i-1} \partial_i = 0$, so that, letting $Z_i = \ker \partial_i$, (the i-dimensional *cycles*), and $B_i = \operatorname{Im} \partial_{i+1}$, (the i-dimensional *boundaries*), we have $B_i \subseteq Z_i$.

It is often convenient to represent $\partial_l : \binom{X}{l} \to \binom{X}{l-1}$ by the incidence matrix $I(l, l-1) = I_n(l, l-1)$. (That is, the matrix indexed by $\binom{X}{l} \times \binom{X}{l-1}$ whose (A, B)-entry is $1_{\{A \supseteq B\}}$. To apply this matrix to $f \in C(\mathcal{G})$ ($\mathcal{G}$ again a subset of $\binom{X}{l}$) we interpret f in the natural way as a vector in $\mathbf{Z}_2^{\binom{X}{l}}$ with $f_A = 0$ if $A \notin \mathcal{G}$.) We write $\operatorname{rk} \mathcal{G}$ for $\dim \partial_l(C(\mathcal{G}))$, the rank of the submatrix consisting of the rows of $I(l, l-1)$ indexed by $\mathcal{G}$.

Our approach is motivated by the observation that the canonical families $\mathcal{F} = \{F \in \binom{X}{k} : F \ni x\}$ are acyclic, that is, $Z_k(\mathcal{F}) = (0)$, and that for any acyclic $\mathcal{F}$ we have

$$|\mathcal{F}| = \dim C_k(\mathcal{F}) = \dim B_{k-1}(\mathcal{F}) \le \dim B_{k-1}\left(\binom{X}{k}\right) = \binom{n-1}{k-1}. \tag{2}$$

Thus we can always assume that the family in question does contain cycles—that is, subsets $\mathcal{G}$ for which $\partial_k(\sum_{F \in \mathcal{G}} F) = 0$—and we expect that this assumption should imply even better bounds.

3. Proof of Chvátal's theorem

We assume $n = |X| \ge k+2$ and that $\mathcal{F} \subseteq \binom{X}{k}$ contains no $(k-1)$-simplex (henceforth just *simplex*), and must show

$$|\mathcal{F}| \le \binom{n-1}{k-1}. \tag{3}$$

As noted above, we may suppose $\mathcal{F}$ is not acyclic.

Claim 3.1 *Each minimal cycle of $\mathcal{F}$ is $\binom{Y}{k}$ for some $Y \in \binom{X}{k+1}$.*

Proof: Let $\mathcal{G}$ be a cycle of $\mathcal{F}$ and $F \in \mathcal{G}$, and suppose $\binom{F}{k-1} = \{A_1, \dots, A_k\}$. Since $\mathcal{G}$ is a cycle, it contains, for each $i \in [k]$, some F_i with $F_i \cap F = A_i$. But since $\{F_1, \dots, F_k\}$ is not a simplex, we must have $\bigcap_{i=1}^k F_i = \{x\}$ for some $x \notin F$, and then $\{F, F_1, \dots, F_k\} = \binom{F \cup \{x\}}{k}$ is a cycle contained in $\mathcal{G}$. □

Suppose then that the cycles of $\mathcal{F}$ are $\binom{Y_i}{k}$, $i \in [s]$. Since $\mathcal{F}$ contains no simplex we have

Claim 3.2 *For all $i \in [s]$ and $F \in \mathcal{F}$, either $F \subseteq Y_i$ or $|F \cap Y_i| \le k-2$. In particular, for each $1 \le i < j \le s$, $|Y_i \cap Y_j| \le k-2$.*

Let

$$\mathcal{F}' = \mathcal{F} \Big\backslash \bigcup_{i=1}^{s} \binom{Y_i}{k}, \quad E' = \binom{X}{k-1} \Big\backslash \bigcup_{i=1}^{s} \binom{Y_i}{k-1},$$

$$\mathcal{F}'' = \bigcup_{i=1}^{s} \binom{Y_i}{k}, \quad E'' = \bigcup_{i=1}^{s} \binom{Y_i}{k-1}.$$

Then $\mathcal{F}'$ is acyclic and by Claim 3.2, $\partial_k C(\mathcal{F}') \subseteq C(E')$ (i.e., no member of $\mathcal{F}'$ contains a member of E''), whence

$$\begin{aligned}|\mathcal{F}'| &= \dim C(\mathcal{F}') = \dim \partial_k C(\mathcal{F}') \le \dim Z_{k-1}(E') \\ &= |E'| - \operatorname{rk} E' = \binom{n}{k-1} - |E''| - \operatorname{rk} E'.\end{aligned} \tag{4}$$

Thus (3) will follow from

$$\operatorname{rk} E' \ge \binom{n-1}{k-2} - |E''| + |\mathcal{F}''|. \tag{5}$$

Now rk E' is also the rank of E' in the binary matroid M given by the rows of $I(k-1, k-2)$. (For instance, if $k = 3$ this is the ordinary polygon matroid of the graph E'. For matroid background see [6].)

The dual of this matroid, M^*, is the matroid given by the columns of $I(k, k-1)$. By the rank formula for dual matroids (with ground set E),

$$\operatorname{rk}^* E'' = |E''| - \operatorname{rk} E + \operatorname{rk} E' = |E''| - \binom{n-1}{k-2} + \operatorname{rk} E',$$

so (5) is equivalent to

$$\operatorname{rk}^* E'' \ge |\mathcal{F}''| = (k+1)s. \tag{6}$$

Proof of (6): Suppose $x \in X$ belongs to precisely t of $Y_1, \ldots, Y_s$, say $x \in \bigcap_{i=1}^t Y_i \setminus (\bigcup_{i=t+1}^s Y_i)$. Then the columns of $I(k, k-1)$ corresponding to

$$E''' := \bigcup_{i=1}^{t} \binom{Y_i \setminus \{x\}}{k-1} \cup \bigcup_{i=t+1}^{s} \binom{Y_i}{k-1}$$

are independent, since their restriction to the rows indexed by $\{Z \cup \{x\} : Z \in E'''\}$ is a diagonal matrix.

Thus (using Claim 3.2) $\operatorname{rk}^* E'' \ge tk + (s-t)\binom{k+1}{2}$, which gives (6) provided

$$t \le \frac{(k+1)(k-2)}{k(k-1)} s. \tag{7}$$

But the average number of Y_i containing an element of X is $s(k+1)/n$, so we have (7) provided

$$\left\lfloor \frac{s(k+1)}{n} \right\rfloor \le \frac{(k+1)(k-2)}{k(k-1)} s, \tag{8}$$

which is true. (In fact, our assumption $n \ge k+2$ gives (8) without the "$\lfloor\ \rfloor$" except in the trivial cases $k \le 2$ and the case $k = 3$, $n = 5$, $s = 1$, for which the left-hand side of (8) is zero.)

4. Proof of Theorem 1.5

We suppose $\mathcal{F}$ is as in Theorem 1.5 and, as above, may assume $\mathcal{F}$ contains cycles.

Claim 4.1 *Each minimal cycle of $\mathcal{F}$ is either $\binom{Y}{3}$ for some $Y \in \binom{X}{4}$ or isomorphic to*

$$\{vab, vbc, vcd, vda, abc, acd\}. \tag{9}$$

We call cycles of these two types 4- and 5-*cycles*, respectively.

Proof: Let $\mathcal{G}$ be a cycle of $\mathcal{F}$. As usual, the *link* in $\mathcal{G}$ of $W \subseteq X$ is $L_{\mathcal{G}}(W) = \{F \backslash W : W \subseteq F \in \mathcal{G}\}$.

If $L_{\mathcal{G}}(x)$ ($x \in X$) is nonempty then it contains a cycle, say $\{x_1, \ldots, x_t\}$ (actually $L_{\mathcal{G}}(x)$ is an Eulerian graph). Choose $x \in \mathcal{G}$, such that t is maximal. Set $F_i = \{x, x_i, x_{i+1}\}$ (subscripts modulo t) and let

$$G_i = \{x_i, x_{i+1}, y_i\} \qquad \text{with } y_i \in L_{\mathcal{G}}(\{x_i, x_{i+1}\}) \backslash \{x\}.$$

(Note there must be such a G_i.)

Suppose first that $t \geq 4$. Then for each i we must have $y_i \in \{x_{i-1}, x_{i+2}\}$, since otherwise $\{F_{i-1}, F_{i+1}, G_i\}$ is a special triangle. But then: if $t \geq 5$ and (say) $y_i = x_{i+2}$, then $\{F_{i-1}, F_{i+2}, G_i\}$ is a special triangle; while if $t = 4$, it is easy to see that there are i, j with $G_i \cup G_j = \{x_1, \ldots, x_4\}$, and then $\{G_i, G_j, F_1, \ldots, F_4\}$ is a 5-cycle in $\mathcal{G}$.

Now suppose $t = 3$. Then by the maximality of t, $\mathcal{G}$ contains the cycle $\binom{Y}{3}$ with $Y = \{x, x_1, x_2, x_3\}$. □

In what follows, for $\mathcal{K} \subseteq \binom{X}{3}$, we take $\partial\mathcal{K} = \langle\mathcal{K}\rangle \cap \binom{X}{2}$. We also set $\binom{X}{2} = E$.

We will associate with each cycle $\mathcal{G}$ of $\mathcal{F}$ a set $H = H(\mathcal{G}) \subseteq X$.

(a) If $\mathcal{G}$ is a 5-cycle, then $H(\mathcal{G})$ is just the vertex set of $\mathcal{G}$. Note that in this case with labels as in (9),

$$|T \cap H| \neq 2 \quad \forall T \in \mathcal{F} \tag{10}$$

and

$$\binom{H}{2} \Big\backslash \partial\left(\mathcal{F} \cap \binom{H}{3}\right) \subseteq \{\{b, d\}\}.$$

Now suppose $\mathcal{G} = \binom{Y}{3}$ is a 4-cycle. Notice that if $T_1, T_2 \in \mathcal{F}$ satisfy $|T_i \cap Y| = 2$ and $|T_i \cap T_j \cap Y| = 1$, then necessarily $T_1 \backslash Y = T_2 \backslash Y$ (or we have a special triangle). We therefore have one of the following.

(b) There are at most two (opposite) edges $\{x, y\}$ of Y for which there exists $T \in \mathcal{F}$ with $T \cap Y = \{x, y\}$. In this case we take $H(\mathcal{G}) = Y$.
(c) There exist $v \in X \setminus Y$ and $a, c, d \in Y = \{a, b, c, d\}$ such that $\{v, a, c\}, \{v, a, d\} \in \mathcal{F}$. In this case we take $H = H(\mathcal{G}) = Y \cup \{v\}$ and observe that the absence of special triangles implies

$$T \in \mathcal{F},\ |T \cap H| = 2 \ \Rightarrow\ T \cap H = \{v, a\}, \tag{11}$$

$$\binom{H}{2} \Big\backslash \partial\left(\mathcal{F} \cap \binom{H}{3}\right) \subseteq \{\{v, b\}\}.$$

It is also easy to see that

$$T \in \mathcal{F} \cap \binom{H}{3} \ \Rightarrow\ \left|\partial(\mathcal{F} \setminus \{T\}) \cap \binom{T}{2}\right| \geq 2 \tag{12}$$

(i.e., at least two of the pairs from T are covered by triangles of $\mathcal{F}$ other than T).

Let $\mathcal{C}$ be the collection of minimal cycles, and $\mathcal{H} = \{H(\mathcal{G}) : \mathcal{G} \in \mathcal{C}\} = \mathcal{H}_4 \cup \mathcal{H}_5$, where $\mathcal{H}_i = \{H \in \mathcal{H} : |H| = i\}$.

From the preceding observations we have

$$|H \cap H'| \leq 2 \quad \text{for all distinct } H, H' \in \mathcal{H}. \tag{13}$$

To see this note that we cannot have $T, T' \in \mathcal{F}$ with $|T \cap H| = |T' \cap H| = 2$ and $|T \cap T' \cap H| = 1$; on the other hand, if $|H \cap H'| = \{x, y, z\}$, then H' contains triangles of $\mathcal{F}$ other than $\{x, y, z\}$ covering at least two of the pairs from $\{x, y, z\}$. (In (a), (b)—with H' in place of H—there is at most one pair in H' not covered by at least *two* triangles of $\mathcal{F}$ contained in H'. In (c) no two such pairs can lie in a common triangle of $\mathcal{F}$ (this takes care of the case $\{x, y, z\} \in \mathcal{F}$), and there is at most one pair (namely $\{v, b\}$) which may not lie in any triangle of $\mathcal{F} \cap \binom{H'}{3}$ (this covers the case $\{x, y, z\} \notin \mathcal{F}$).)

Now let

$$\mathcal{F}'' = \mathcal{F} \cap \bigcup_{H \in \mathcal{H}} \binom{H}{3}, \quad \mathcal{F}' = \mathcal{F} \setminus \mathcal{F}'',$$

$$E''(H) = \partial\left(\mathcal{F} \cap \binom{H}{3}\right) \Big\backslash \partial\left(\mathcal{F} \Big\backslash \binom{H}{3}\right)$$

(where $H \in \mathcal{H}$), and

$$E'' = \bigcup_{H \in \mathcal{H}} E''(H), \quad E' = \partial \mathcal{F}'.$$

By the discussion in (a)–(c) and (13) we have, for all distinct $H, H' \in \mathcal{H}$,

$$|E''(H)| \geq \left|\mathcal{F} \cap \binom{H}{3}\right|, \quad E''(H) \cap E''(H') = \emptyset,$$

so that $|E''| \geq |\mathcal{F}''|$.

It is thus enough to show

$$|\mathcal{F}'| \leq \binom{n-1}{2} - |E''| = |E \setminus E''| - (n-1). \tag{14}$$

Set $E_0 = E \setminus E' \setminus E''$. As earlier (see (4)), acyclicity of $\mathcal{F}'$ gives

$$|\mathcal{F}'| \leq |E'| - \operatorname{rk} E', \tag{15}$$

so (14) follows from

$$\operatorname{rk} E' \geq n - 1 - |E_0|. \tag{16}$$

Proof of (16): Fix $H \in \mathcal{H}$. Let

$$Z_i = \{w \in X \setminus H,\ |E(w, H) \cap \partial\mathcal{F}| = i\}$$

for $3 \leq i \leq |H|$ (where $E(w, H) = \{\{w, a\} : a \in H\}$). Also let

$$Z = \bigcup_{i=3}^{|H|} Z_i.$$

We assert that if $Z \neq \emptyset$, then

$$\operatorname{rk} E' \geq |Z| + \max\{i : Z_i \neq \emptyset\} - 1. \tag{17}$$

In view of the definition of Z, (17) follows from

$$E(w, H) \cap \partial\mathcal{F} \subset E' \quad \text{for all } w \in Z$$

(since if $w \in Z_t$ with t the maximum in (17), then adding to $E(w, H) \cap \partial\mathcal{F}$ one edge of $E(w', H)$ for each $w' \in Z \setminus \{w\}$ gives an independent subset of E' whose size is the right-hand side of (17)).

Proof: Let $w \in Z$. We distinguish two cases.

Case 1. $H = \{a, b, c, d\} \in \mathcal{H}_4$.

If there exists $T \in \mathcal{F}$, such that $w \in T$ and $|T \cap H| = 2$, say $T = \{w, a, b\}$, then there is no $T' \in \mathcal{F}$ with $w \in T'$ and $T' \cap H \in \{\{c\}, \{d\}\}$ (since this would give a special triangle).

The definition of Z thus requires $T' := \{w, c, d\} \in \mathcal{F}$. Now $T, T' \in \mathcal{F}'$, since if, say, $T' \in \mathcal{F}''$, then there is a $T'' \in \mathcal{F}$ with $w \in T''$ and $T'' \cap H = \{c\}$ or $\{d\}$ (using (13) and (12)), which we have just seen to be impossible.

Suppose on the other hand that there is no $T \in \mathcal{F}$ with $w \in T$ and $|T \cap H| = 2$. Then for each $x \in H$ with $\{w, x\} \in \partial\mathcal{F}$, there exists $T_x \in \mathcal{F}$ with $w \in T_x$ and $T_x \cap H = \{x\}$. Moreover, the absence of special triangles implies that $T_x \setminus \{w, x\} = T_y \setminus \{w, y\} = \{z\}$, say, whenever T_x, T_y are as just described. This gives $T_x \in \mathcal{F}'$; for if $T_x \subseteq H' \in \mathcal{H}$, then at least one of $\{w, y\}, \{z, y\}$ is contained in a triangle of $\mathcal{F} \cap \binom{H}{3}$ other than T_x (see (12)), and this with any other T_y (and the triangles in H) gives a special simplex.

Case 2. $H = \{v, a, b, c, d\} \in \mathcal{H}_5$ (with labels as in Claim 4.1 or (c) as appropriate).

Here we can only have $w \in T \in \mathcal{F}$ and $|T \cap H| = 2$ if H is as in (c) and $T = \{w, v, a\}$ (see (10)). But in this case we cannot have any of $\{w, b\}$, $\{w, c\}$, $\{w, d\}$ in $\partial\mathcal{F}$ without creating a special triangle, so cannot have $w \in Z$.

So as in Case 1, for each $x \in H$ with $\{w, x\} \in \partial\mathcal{F}$, there exists $T_x \in \mathcal{F}$ with $w \in T_x$ and $T_x \cap H = \{x\}$. This again gives $T_x \in \mathcal{F}'$ via the argument of Case 1 applied with some y for which (T_y exists and) $\{x, y\} \in E''(H)$ (noting that there is always at least one such y). $\square$

We can now complete the proof of (16). For $w \in X \setminus (Z \cup H)$, $|E(w, H) \cap E_0| \geq |H| - 2$, and for $w \in Z_i$, $|E(w, H) \cap E_0| = |H| - i$ for $3 \leq i \leq |H|$. Thus we have in Case 1,

$$|E_0| \geq 2(n - |Z| - 4) + |Z_3|,$$

and in Case 2,

$$|E_0| \geq 3(n - |Z| - 5) + 2|Z_3| + |Z_4|.$$

These in conjunction with (17) give (16) whenever $Z \neq \emptyset$, and also when $Z = \emptyset$ provided $n \geq 7$. The remaining case $n = 6$ is easily disposed of; for completeness: if there exists $H \in \mathcal{H}_5$ then (10) and (11) show that there is at most one triangle of $\mathcal{F}$ not contained in H, and none if $\binom{H}{3} \subseteq \mathcal{F}$; and if there exists $H \in \mathcal{H}_4$, then $Z = \emptyset$ implies that $|\mathcal{F} \setminus \binom{H}{3}| \leq 2$. $\square$

Regarding cases of equality in Theorem 1.5, we have the following result, whose proof, omitted here, is given in [2].

Theorem 4.2 *Suppose* $\mathcal{F} \subseteq \binom{X}{3}$, $|\mathcal{F}| = \binom{n-1}{2}$, *and* $\mathcal{F}$ *contains no special triangle.*

(a) *If* $n \geq 8$, *then* $\mathcal{F} = \{F \in \binom{X}{3} : x \in F\}$ *for some* $x \in X$.
(b) *If* $n = 7$, *then* $\mathcal{F}$ *is either as in* (a) *or is isomorphic to* $\{F \in \binom{[7]}{3} : |F \cap \{1, 2\}| \neq 1\}$.
(c) *If* $n = 6$, *then* $\mathcal{F}$ *is either as in* (a) *or is* $\binom{Y}{3}$ *for some* $Y \in \binom{X}{5}$.

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References

1. V. Chvátal, "An extremal set-intersection theorem," *J. London Math. Soc.* **9** (1974), 355–359.
2. R. Csákány, Ph.D. Thesis, Rutgers University, 1997.
3. P. Erdős, C. Ko, and R. Rado, "Intersection theorems for systems of finite sets," *Quart. J. Math. Oxford (2)* **12** (1961), 313–320.
4. P. Frankl and Z. Füredi, "Exact solution of some Turán-type problems," *J. Comb. Theory Ser. A* **45** (1987), 226–262.
5. P.J. Hilton and S. Wylie, *Homology Theory*, Cambridge University Press, Cambridge, 1965.
6. D.J.A. Welsh, *Matroid Theory*, Academic Press, London, 1976.