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The Spectra of Coxeter Graphs

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Abstract. We determine the spectra of the finite Coxeter graphs defined by a terminal node of the Coxeter diagram, and the spectra of their thick equivalents.

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F. Buekenhout asked for the spectra of the graphs $F_{4,1}$, $H_{4,1}$ and $H_{4,4}$ (for notation, see [2], Chapter 10). In this paper we give the spectra of these and similar graphs. Some of these spectra are very well known (and given here only for completeness); we refer to [4], although there may be older references. In the remaining cases the computations may be done using Proposition 2.2.2 in [2]; however, note that that proposition as formulated there is valid only when L_i has $d+1$ distinct eigenvalues. A version that is valid in general (even for non-symmetric association schemes) follows.

Theorem *Let $(X, \{R_0, \ldots, R_d\})$ be an association scheme with adjacency matrices $A_0, \ldots, A_d$ (where $A_0 = I$) and intersection numbers p_{ij}^k (so that $A_i A_j = \sum p_{ij}^k A_k$). Let L_i be the matrix of order $d+1$ defined by $(L_i)_{kj} = p_{ij}^k$. Let $A = A_1$ be the adjacency matrix of a symmetric relation R_1 in the scheme, and let $L := L_1$. Then the matrices A and L have the same eigenvalues. If λ is an eigenvalue of L, then there is a unique real eigenvector u of L such that $Lu = \lambda u$, $u_0 = 1$, $u^\top \Delta_n u$ minimal, where Δ_n is the diagonal matrix with the valencies $n_j = p_{jj'}^0$ on the diagonal, and the multiplicity of λ as an eigenvalue of A equals $v/(u^\top \Delta_n u)$.*

Proof: The map $A_i \mapsto L_i$ is an isomorphism from the Bose-Mesner algebra $\mathcal{A} = \langle A_i \mid 0 \le i \le d \rangle_F$ spanned by the matrices A_i (over some field F) onto the algebra $\langle L_i \mid 0 \le i \le d \rangle_F$. Indeed, it preserves multiplication, and both algebras have the same dimension since the L_i are independent (because $(L_i)_{k0} = \delta_{i,k}$). Now let $F = \mathbf{R}$. Both A and L are diagonalizable (since A and $\Delta_n L$ are symmetric); let $m(X) := \prod_{i=1}^s (X - \theta_i)$ be the minimal polynomial of A (and hence of L). Put $m_j(X) := m(X)/(X - \theta_j)$, and $E_j := m_j(A)/m_j(\theta_j)$. Then the E_j are mutually orthogonal idempotents in $\mathcal{A}$, and $AE_j = \theta_j E_j$. The multiplicity of θ_j as an eigenvalue of A equals $f_j := \operatorname{rk} E_j = \operatorname{tr} E_j$. Define a $(d+1) \times s$ matrix Q by $E_j = \frac{1}{v} \sum_{i=0}^d Q_{ij} A_i$ $(1 \le j \le s)$. Now $LQe_j = \theta_j Qe_j$ (where e_j is the jth unit vector) as one sees by comparing the coefficients of the A_i on both sides of $AE_j = \theta_j E_j$.

Furthermore, $(Qe_j)_0 = Q_{0j} = f_j$ follows by taking traces in the expression for E_j. Finally, $Q^\top \Delta_n Q = v\Delta_f$ follows from $\delta_{ij} f_i = \mathrm{tr}\, E_i E_j = \frac{1}{v}\sum Q_{li} Q_{lj} n_l$. (Note that if R_l and $R_{l'}$ are inverse relations, so that $A_{l'} = A_l^\top$, then $Q_{li} = Q_{l'i}$, since E_i is symmetric.) Thus, if we take $u = f_j^{-1}(Qe_j)$, then $u_0 = 1$ and $f_j = v/(u^\top \Delta_n u)$.

Remains to show that this u minimizes $u^\top \Delta_n u$. But if u is arbitrary with $u_0 = 1$, $Lu = \theta_j u$, then the matrix $U := \sum u_i A_i$ satisfies $AU = \theta_j U$ and hence $U = E_j M$ for some matrix M. Also $u^\top \Delta_n u = \frac{1}{v}\,\mathrm{tr}\, U^\top U$ and $u_0 = \frac{1}{v}\,\mathrm{tr}\, U$, so we have to minimize $\mathrm{tr}\, U^\top U$ (that is, the sum of the squares of the elements of U) given $\mathrm{tr}\, U$ (and $U = E_j M$). Diagonalising E_j we see immediately that U is uniquely determined, and corresponds to the u we had above. □

Thus, it suffices to work with L_i instead of A_i. The required parameters p_{ij}^k for the graphs $E_{6,*}$, $E_{7,*}$ and $E_{8,*}$ can be found in [1]. We give only the results of the computations.

1. $A_{n,1}$

The graph $A_{n,1}$ is the complete graph K_{n+1} with spectrum $n^1\ (-1)^n$, where the superscripts denote multiplicities (cf. [4], p. 72).

In the thick case $A_{n,1}(q)$ we find the complete graph on $v = \frac{q^{n+1}-1}{q-1}$ vertices with spectrum $(v-1)^1\ (-1)^{v-1}$.

2. $B_{n,1}$ and $B_{n,n}$

The graph $B_{n,1}$ is the complete n-partite graph $K_{n\times 2}$ with spectrum $(2n-2)^1 0^n (-2)^{n-1}$ (cf. [4], p. 73).

The graph $B_{n,n}$ is the n-cube with spectrum $n^1 (n-2)^n \ldots (n-2j)^{\binom{n}{j}} \ldots (-n)^1$ (cf. [4], p. 75).

In the thick case $B_{n,1}(q)$ we find the strongly regular polar graph $B_n(q)$ with eigenvalues $q\frac{q^{2n-2}-1}{q-1}$, $-q^{n-1}-1$, $q^{n-1}-1$ and multiplicities (respectively) 1, $\frac{1}{2}(\frac{q^{2n}-q^2}{q-1} + q^{n-1} + 1)$, $\frac{1}{2}(\frac{q^{2n}+q^2}{q-1} - q^{n-1} - 1) - \frac{q}{q-1}$. $B_{n,n}(q)$ is the distance-regular dual polar graph $[B_n(q)]$, with eigenvalues $q\left[{n-j \atop 1}\right]_q - \left[{j \atop 1}\right]_q$ and multiplicities: $q^j \left[{n \atop j}\right]_q \frac{1+q^{n+1-2j}}{1+q^{n+1-j}} (\prod_{i=1}^{j} \frac{1+q^{n+1-i}}{1+q^{i-1}})$ for $0 \le j \le n$ (cf. [2], p. 275).

3. $D_{n,1}$ and $D_{n,n}$

The graph $D_{n,1}$ is the same as $B_{n,1}$.

The graph $D_{n,n}$ is the halved n-cube $\frac{1}{2}2^n$ with eigenvalues $((n-2j)^2 - n)/2$ and multiplicities $\binom{n}{j}$ $(0 \le j < n/2)$, but multiplicity $\frac{1}{2}\binom{n}{j}$ if $j = n/2$ (cf. [2], p. 264).

In the thick case $D_{n,1}(q)$ is the strongly regular polar graph $D_n(q)$ with eigenvalues $q\frac{(q^{n-1}-1)(q^{n-2}+1)}{q-1}$, $q^{n-2}-1$, $-q^{n-1}-1$ and multiplicities (respectively) 1, $\frac{q^{2n}+2q^{n+2}-2q^{n+1}-q^2}{q^2-1}$, $\frac{q^{2n-1}-2q^{n+2}+3q^{n+1}-q^{n-1}-q}{q^2-1}$.

$D_{n,n}(q)$ is the halved graph of the dual polar graph $D_n(q)$ with eigenvalues $q^{2j+1}\left[{n-2j \atop 2}\right]_q - \frac{q^{2j}-1}{q^2-1}$ and multiplicities $\gamma_j q^j \left[{n \atop j}\right]_q \frac{1+q^{n-2j}}{1+q^{n-j}} \prod_{i=1}^{j} \frac{q^{m-i}+1}{q^i+1}$, where $0 \le j \le \frac{n}{2}$ and $\gamma_j = 1$ for $2j < m$ and $\gamma_n = \frac{1}{2}$ if n is even (cf. [2], p. 278).

4. $E_{6,1}$ and $E_{6,2}$

The graph $E_{6,1}$ is the Schläfli graph, strongly regular with parameters $(v, k, \lambda, \mu) = (27, 16, 10, 8)$ and spectrum $16^1\ 4^6\ (-2)^{20}$ (cf. [2], p. 312).

The graph $E_{6,2}$ is the root system graph of type E_6 on 72 vertices. It has spectrum $20^1\ 10^6\ 2^{20}\ (-2)^{30}\ (-4)^{15}$ (cf. [2], p. 313).

In the thick case $E_{6,1}(q)$ we find a strongly regular graph on $v = (q^8 + q^4 + 1)(q^9 - 1)/(q - 1)$ vertices, and with valency $q(q^8 - 1)(q^3 + 1)/(q - 1)$ and spectrum

eigenvalue: $q^{11} + q^{10} + q^9 + 2q^8 + 2q^7 + 2q^6 + 2q^5 + 2q^4 + q^3 + q^2 + q$
multiplicity: 1

eigenvalue: $q^8 + q^7 + q^6 + q^5 + q^4 - 1$
multiplicity: $q^{11} + q^8 + q^7 + q^5 + q^4 + q$

eigenvalue: $-q^3 - 1$
multiplicity: $q^{16} + q^{15} + q^{14} + q^{13} + 2q^{12} + q^{11} + 2q^{10} + 2q^9 + 2q^8 + q^7 + 2q^6 + q^5 + q^4 + q^3 + q^2$

In the thick case $E_{6,2}(q)$ we find a graph on $v = (q^3+1)(q^4+1)(q^6+1)(q^9-1)/(q-1)$ vertices, and with spectrum

eigenvalue: $q^{10} + q^9 + 2q^8 + 3q^7 + 3q^6 + 3q^5 + 3q^4 + 2q^3 + q^2 + q$
multiplicity: 1

eigenvalue: $q^8 + 2q^7 + 2q^6 + 3q^5 + 2q^4 + q^3 - 1$
multiplicity: $q^{11} + q^8 + q^7 + q^5 + q^4 + q$

eigenvalue: $q^7 + q^6 + q^5 + q^4 - q^2 - 1$
multiplicity: $q^{16} + q^{15} + q^{14} + q^{13} + 2q^{12} + q^{11} + 2q^{10} + 2q^9 + 2q^8 + q^7 + 2q^6 + q^5 + q^4 + q^3 + q^2$

eigenvalue: $q^5 - q^3 - q^2 - 1$
multiplicity: $\frac{1}{2}(q^{21} + q^{20} + 2q^{19} + 3q^{18} + 3q^{17} + 3q^{16} + 5q^{15} + 4q^{14} + 5q^{13} + 6q^{12} + 5q^{11} + 4q^{10} + 5q^9 + 3q^8 + 3q^7 + 3q^6 + 2q^5 + q^4 + q^3)$

eigenvalue: $-q^5 - q^3 - q^2 - 1$
multiplicity: $\frac{1}{2}(q^{21} + q^{20} + q^{18} + 3q^{17} + q^{16} + q^{15} + 4q^{14} + 3q^{13} + 3q^{11} + 4q^{10} + q^9 + q^8 + 3q^7 + q^6 + q^4 + q^3)$

5. $E_{7,1}$ and $E_{7,2}$ and $E_{7,7}$

The graph $E_{7,1}$ is the Gosset graph, distance-regular with intersection array {27, 10, 1; 1, 10, 27}. It has spectrum $27^1\ 9^7\ (-1)^{27}\ (-3)^{21}$.

The graph $E_{7,2}$ has spectrum $35^1\ 25^7\ 15^{27}\ 7^{56}\ 5^{21}\ 1^{120}\ (-1)^{35}\ (-3)^{189}\ (-5)^{105}\ (-7)^{15}$.

The graph $E_{7,7}$ is the root system graph of type E_7 on 126 vertices. It has spectrum $32^1\ 16^7\ 4^{27}\ (-2)^{56}\ (-4)^{35}$ (cf. [2], p. 313).

In the thick case $E_{7,1}(q)$ we find a distance-regular graph with spectrum

eigenvalue: $q^{17}+q^{16}+q^{15}+q^{14}+2q^{13}+2q^{12}+2q^{11}+2q^{10}+3q^9+2q^8+2q^7+2q^6+2q^5+q^4+q^3+q^2+q$
multiplicity: 1

eigenvalue: $q^{13}+q^{12}+q^{11}+q^{10}+2q^9+q^8+q^7+q^6+q^5-1$
multiplicity: $q^{17}+q^{13}+q^{11}+q^9+q^7+q^5+q$

eigenvalue: q^9-q^4-1
multiplicity: $q^{26}+q^{24}+2q^{22}+2q^{20}+3q^{18}+3q^{16}+3q^{14}+3q^{12}+3q^{10}+2q^8+2q^6+q^4+q^2$

eigenvalue: $-q^8-q^4-1$
multiplicity: $q^{27}+q^{25}+q^{23}+2q^{21}+2q^{19}+2q^{17}+3q^{15}+2q^{13}+2q^{11}+2q^9+q^7+q^5+q^3$

In the thick case $E_{7,2}(q)$ we find

eigenvalue: $q^{13}+q^{12}+2q^{11}+3q^{10}+4q^9+4q^8+5q^7+4q^6+4q^5+3q^4+2q^3+q^2+q$
multiplicity: 1

eigenvalue: $q^{11}+2q^{10}+3q^9+4q^8+5q^7+4q^6+4q^5+2q^4+q^3-1$
multiplicity: $q^{17}+q^{13}+q^{11}+q^9+q^7+q^5+q$

eigenvalue: $q^{10}+2q^9+3q^8+4q^7+3q^6+3q^5+q^4-q^2-1$
multiplicity: $q^{26}+q^{24}+2q^{22}+2q^{20}+3q^{18}+3q^{16}+3q^{14}+3q^{12}+3q^{10}+2q^8+2q^6+q^4+q^2$

eigenvalue: $q^{10}+q^9+q^8+2q^7+q^6+q^5-q^2-1$
multiplicity: $q^{27}+q^{25}+q^{23}+2q^{21}+2q^{19}+2q^{17}+3q^{15}+2q^{13}+2q^{11}+2q^9+q^7+q^5+q^3$

eigenvalue: $q^9+2q^8+3q^7+2q^6+2q^5-q^3-q^2-1$
multiplicity: $\frac{1}{2}(q^{33}+q^{32}+q^{31}+2q^{30}+2q^{29}+3q^{28}+3q^{27}+3q^{26}+4q^{25}+5q^{24}+5q^{23}+5q^{22}+6q^{21}+6q^{20}+6q^{19}+6q^{18}+6q^{17}+6q^{16}+6q^{15}+5q^{14}+5q^{13}+5q^{12}+4q^{11}+3q^{10}+3q^9+3q^8+2q^7+2q^6+q^5+q^4+q^3)$

eigenvalue: $q^9+q^7-q^3-q^2-1$
multiplicity: $\frac{1}{2}(q^{33}+q^{32}+q^{31}+2q^{29}+q^{28}+3q^{27}+q^{26}+4q^{25}+q^{24}+5q^{23}+q^{22}+6q^{21}+2q^{20}+6q^{19}+6q^{17}+2q^{16}+6q^{15}+q^{14}+5q^{13}+q^{12}+4q^{11}+q^{10}+3q^9+q^8+2q^7+q^5+q^4+q^3)$

eigenvalue: $q^8+2q^7+q^6+q^5-q^4-q^3-q^2-1$
multiplicity: $\frac{1}{2}(q^{38}+q^{37}+q^{36}+3q^{35}+3q^{34}+3q^{33}+5q^{32}+6q^{31}+6q^{30}+8q^{29}+$

$9\,q^{28}+9\,q^{27}+11\,q^{26}+11\,q^{25}+11\,q^{24}+13\,q^{23}+13\,q^{22}+12\,q^{21}+13\,q^{20}+13\,q^{19}+11\,q^{18}+11\,q^{17}+11\,q^{16}+9\,q^{15}+9\,q^{14}+8\,q^{13}+6\,q^{12}+6\,q^{11}+5\,q^{10}+3\,q^{9}+3\,q^{8}+3\,q^{7}+q^{6}+q^{5}+q^{4})$

eigenvalue: $q^7-q^4-q^3-q^2-1$
multiplicity: $q^{41}+2\,q^{39}+4\,q^{37}+6\,q^{35}+9\,q^{33}+12\,q^{31}+15\,q^{29}+17\,q^{27}+19\,q^{25}+19\,q^{23}+19\,q^{21}+17\,q^{19}+15\,q^{17}+12\,q^{15}+9\,q^{13}+6\,q^{11}+4\,q^{9}+2\,q^{7}+q^{5}$

eigenvalue: $-q^6-q^4-q^3-q^2-1$
multiplicity: $q^{42}+q^{40}+2\,q^{38}+4\,q^{36}+5\,q^{34}+6\,q^{32}+9\,q^{30}+9\,q^{28}+10\,q^{26}+11\,q^{24}+10\,q^{22}+9\,q^{20}+9\,q^{18}+6\,q^{16}+5\,q^{14}+4\,q^{12}+2\,q^{10}+q^{8}+q^{6}$

eigenvalue: $-q^8-q^6-q^5-q^4-q^3-q^2-1$
multiplicity: $\frac{1}{2}(q^{38}-q^{37}+q^{36}-q^{35}+3\,q^{34}-3\,q^{33}+5\,q^{32}-4\,q^{31}+6\,q^{30}-6\,q^{29}+9\,q^{28}-7\,q^{27}+11\,q^{26}-9\,q^{25}+11\,q^{24}-9\,q^{23}+13\,q^{22}-10\,q^{21}+13\,q^{20}-9\,q^{19}+11\,q^{18}-9\,q^{17}+11\,q^{16}-7\,q^{15}+9\,q^{14}-6\,q^{13}+6\,q^{12}-4\,q^{11}+5\,q^{10}-3\,q^{9}+3\,q^{8}-q^{7}+q^{6}-q^{5}+q^{4})$

In the thick case $E_{7,7}(q)$ we find

eigenvalue: $q^{16}+q^{15}+q^{14}+2\,q^{13}+2\,q^{12}+3\,q^{11}+3\,q^{10}+3\,q^{9}+3\,q^{8}+3\,q^{7}+3\,q^{6}+2\,q^{5}+2\,q^{4}+q^{3}+q^{2}+q$
multiplicity: 1

eigenvalue: $q^{13}+q^{12}+2\,q^{11}+2\,q^{10}+2\,q^{9}+3\,q^{8}+2\,q^{7}+2\,q^{6}+q^{5}+q^{4}-1$
multiplicity: $q^{17}+q^{13}+q^{11}+q^{9}+q^{7}+q^{5}+q$

eigenvalue: $q^{11}+q^{10}+q^{9}+q^{8}+q^{7}+q^{6}-q^{3}-1$
multiplicity: $q^{26}+q^{24}+2\,q^{22}+2\,q^{20}+3\,q^{18}+3\,q^{16}+3\,q^{14}+3\,q^{12}+3\,q^{10}+2\,q^{8}+2\,q^{6}+q^{4}+q^{2}$

eigenvalue: $q^8-q^5-q^3-1$
multiplicity: $\frac{1}{2}(q^{33}+q^{32}+q^{31}+2\,q^{30}+2\,q^{29}+3\,q^{28}+3\,q^{27}+3\,q^{26}+4\,q^{25}+5\,q^{24}+5\,q^{23}+5\,q^{22}+6\,q^{21}+6\,q^{20}+6\,q^{19}+6\,q^{18}+6\,q^{17}+6\,q^{16}+6\,q^{15}+5\,q^{14}+5\,q^{13}+5\,q^{12}+4\,q^{11}+3\,q^{10}+3\,q^{9}+3\,q^{8}+2\,q^{7}+2\,q^{6}+q^{5}+q^{4}+q^{3})$

eigenvalue: $-q^8-q^5-q^3-1$
multiplicity: $\frac{1}{2}(q^{33}+q^{32}+q^{31}+2\,q^{29}+q^{28}+3\,q^{27}+q^{26}+4\,q^{25}+q^{24}+5\,q^{23}+q^{22}+6\,q^{21}+2\,q^{20}+6\,q^{19}+6\,q^{17}+2\,q^{16}+6\,q^{15}+q^{14}+5\,q^{13}+q^{12}+4\,q^{11}+q^{10}+3\,q^{9}+q^{8}+2\,q^{7}+q^{5}+q^{4}+q^{3})$

6. $E_{8,1}$ and $E_{8,2}$ and $E_{8,8}$

The graph $E_{8,1}$ has spectrum $56^1\ 28^8\ 8^{35}\ (-2)^{112}\ (-4)^{84}$.

The graph $E_{8,2}$ has spectrum $56^1\ 49^8\ 41^{35}\ \frac{1}{2}(45+\sqrt{409})^{112}\ 25^{210}\ \frac{1}{2}(25+3\sqrt{41})^{84}\ 17^{560}\ \frac{1}{2}(45-\sqrt{409})^{112}\ \frac{1}{2}(9+\sqrt{145})^{700}\ 9^{567}\ 5^{1400}\ 4^{400}\ \frac{1}{2}(25-3\sqrt{41})^{84}\ 1^{1960}\ (-1)^{1344}\ \frac{1}{2}(9-\sqrt{145})^{700}\ (-3)^{3240}\ (-4)^{448}\ (-5)^{2240}\ (-7)^{2900}\ (-8)^{175}$.

(Here the nonintegral eigenvalues are approximately 32.611874, 22.104686, 12.388126, 10.520797, 2.895314 and -1.520797. In this case the association scheme has $d+1=35$ relations, while L_1 has only 21 distinct eigenvalues. Indeed, the nonintegral eigenvalues and 17 and 4 all have multiplicity 2 in L_1, and 1 and -7 have multiplicities 3 and 5, respectively.)

The graph $E_{8,8}$ is the root system graph of type E_8 on 2160 vertices. It has spectrum $64^1\ 48^8\ 32^{35}\ 18^{112}\ 8^{210}\ 4^{84}\ 0^{560}\ (-4)^{700}\ (-6)^{400}\ (-8)^{50}$ (cf. [2], pp. 313-314).

In the thick case $E_{8,1}(q)$ we find

eigenvalue: $q^{28}+q^{27}+q^{26}+q^{25}+q^{24}+2q^{23}+2q^{22}+2q^{21}+2q^{20}+3q^{19}+3q^{18}+3q^{17}+3q^{16}+3q^{15}+3q^{14}+3q^{13}+3q^{12}+3q^{11}+3q^{10}+2q^9+2q^8+2q^7+2q^6+q^5+q^4+q^3+q^2+q$
multiplicity: 1

eigenvalue: $q^{23}+q^{22}+q^{21}+q^{20}+2q^{19}+2q^{18}+2q^{17}+2q^{16}+2q^{15}+3q^{14}+2q^{13}+2q^{12}+2q^{11}+2q^{10}+q^9+q^8+q^7+q^6-1$
multiplicity: $q^{29}+q^{23}+q^{19}+q^{17}+q^{13}+q^{11}+q^7+q$

eigenvalue: $q^{19}+q^{18}+q^{17}+q^{16}+q^{15}+q^{14}+q^{13}+q^{12}+q^{11}+q^{10}-q^5-1$
multiplicity: $q^{46}+q^{42}+q^{40}+q^{38}+2q^{36}+2q^{34}+q^{32}+3q^{30}+2q^{28}+2q^{26}+3q^{24}+2q^{22}+2q^{20}+3q^{18}+q^{16}+2q^{14}+2q^{12}+q^{10}+q^8+q^6+q^2$

eigenvalue: $q^{14}-q^9-q^5-1$
multiplicity: $\frac{1}{2}(q^{57}+q^{56}+q^{55}+q^{54}+q^{53}+2q^{52}+2q^{51}+2q^{50}+2q^{49}+3q^{48}+3q^{47}+3q^{46}+4q^{45}+4q^{44}+4q^{43}+5q^{42}+5q^{41}+6q^{40}+6q^{39}+5q^{38}+6q^{37}+7q^{36}+7q^{35}+6q^{34}+7q^{33}+7q^{32}+7q^{31}+8q^{30}+7q^{29}+7q^{28}+7q^{27}+6q^{26}+7q^{25}+7q^{24}+6q^{23}+5q^{22}+6q^{21}+6q^{20}+5q^{19}+5q^{18}+4q^{17}+4q^{16}+4q^{15}+3q^{14}+3q^{13}+3q^{12}+2q^{11}+2q^{10}+2q^9+2q^8+q^7+q^6+q^5+q^4+q^3)$

eigenvalue: $-q^{14}-q^9-q^5-1$
multiplicity: $\frac{1}{2}(q^{57}+q^{56}+q^{55}+q^{54}+q^{53}+2q^{51}+2q^{50}+2q^{49}+q^{48}+3q^{47}+q^{46}+4q^{45}+4q^{44}+4q^{43}+q^{42}+5q^{41}+2q^{40}+6q^{39}+5q^{38}+6q^{37}+q^{36}+7q^{35}+4q^{34}+7q^{33}+5q^{32}+7q^{31}+7q^{29}+5q^{28}+7q^{27}+4q^{26}+7q^{25}+q^{24}+6q^{23}+5q^{22}+6q^{21}+2q^{20}+5q^{19}+q^{18}+4q^{17}+4q^{16}+4q^{15}+q^{14}+3q^{13}+q^{12}+2q^{11}+2q^{10}+2q^9+q^7+q^6+q^5+q^4+q^3)$

In the thick case $E_{8,8}(q)$ we find

eigenvalue: $q^{22}+q^{21}+q^{20}+2q^{19}+2q^{18}+3q^{17}+4q^{16}+4q^{15}+4q^{14}+5q^{13}+5q^{12}+5q^{11}+5q^{10}+4q^9+4q^8+4q^7+3q^6+2q^5+2q^4+q^3+q^2+q$
multiplicity: 1

eigenvalue: $q^{19}+q^{18}+2q^{17}+3q^{16}+3q^{15}+4q^{14}+5q^{13}+5q^{12}+5q^{11}+5q^{10}+4q^9+4q^8+3q^7+2q^6+q^5+q^4-1$
multiplicity: $q^{29}+q^{23}+q^{19}+q^{17}+q^{13}+q^{11}+q^7+q$

eigenvalue: $q^{17}+2q^{16}+2q^{15}+3q^{14}+4q^{13}+4q^{12}+5q^{11}+4q^{10}+3q^9+3q^8+2q^7+q^6-q^3-1$

multiplicity: $q^{46}+q^{42}+q^{40}+q^{38}+2q^{36}+2q^{34}+q^{32}+3q^{30}+2q^{28}+2q^{26}+3q^{24}+2q^{22}+2q^{20}+3q^{18}+q^{16}+2q^{14}+2q^{12}+q^{10}+q^{8}+q^{6}+q^{2}$

eigenvalue: $q^{16}+q^{15}+2q^{14}+3q^{13}+3q^{12}+3q^{11}+3q^{10}+2q^{9}+2q^{8}+q^{7}-q^{5}-q^{3}-1$
multiplicity: $\frac{1}{2}(q^{57}+q^{56}+q^{55}+q^{54}+q^{53}+2q^{52}+2q^{51}+2q^{50}+2q^{49}+3q^{48}+3q^{47}+3q^{46}+4q^{45}+4q^{44}+4q^{43}+5q^{42}+5q^{41}+6q^{40}+6q^{39}+5q^{38}+6q^{37}+7q^{36}+7q^{35}+6q^{34}+7q^{33}+7q^{32}+7q^{31}+8q^{30}+7q^{29}+7q^{28}+7q^{27}+6q^{26}+7q^{25}+7q^{24}+6q^{23}+5q^{22}+6q^{21}+6q^{20}+5q^{19}+5q^{18}+4q^{17}+4q^{16}+4q^{15}+3q^{14}+3q^{13}+3q^{12}+2q^{11}+2q^{10}+2q^{9}+2q^{8}+q^{7}+q^{6}+q^{5}+q^{4}+q^{3})$

eigenvalue: $q^{16}+q^{15}+q^{13}+q^{12}+q^{11}+q^{10}+q^{7}-q^{5}-q^{3}-1$
multiplicity: $\frac{1}{2}(q^{57}+q^{56}+q^{55}+q^{54}+q^{53}+2q^{51}+2q^{50}+2q^{49}+q^{48}+3q^{47}+q^{46}+4q^{45}+4q^{44}+4q^{43}+q^{42}+5q^{41}+2q^{40}+6q^{39}+5q^{38}+6q^{37}+q^{36}+7q^{35}+4q^{34}+7q^{33}+5q^{32}+7q^{31}+7q^{29}+5q^{28}+7q^{27}+4q^{26}+7q^{25}+q^{24}+6q^{23}+5q^{22}+6q^{21}+2q^{20}+5q^{19}+q^{18}+4q^{17}+4q^{16}+4q^{15}+q^{14}+3q^{13}+q^{12}+2q^{11}+2q^{10}+2q^{9}+q^{7}+q^{6}+q^{5}+q^{4}+q^{3})$

eigenvalue: $q^{14}+2q^{13}+2q^{12}+3q^{11}+2q^{10}+q^{9}+q^{8}-q^{6}-q^{5}-q^{3}-1$
multiplicity: $\frac{1}{2}(q^{68}+q^{65}+2q^{64}+q^{63}+2q^{62}+q^{61}+3q^{60}+3q^{59}+4q^{58}+2q^{57}+6q^{56}+3q^{55}+5q^{54}+6q^{53}+9q^{52}+4q^{51}+9q^{50}+6q^{49}+11q^{48}+9q^{47}+12q^{46}+6q^{45}+14q^{44}+9q^{43}+13q^{42}+11q^{41}+16q^{40}+7q^{39}+15q^{38}+11q^{37}+16q^{36}+11q^{35}+15q^{34}+7q^{33}+16q^{32}+11q^{31}+13q^{30}+9q^{29}+14q^{28}+6q^{27}+12q^{26}+9q^{25}+11q^{24}+6q^{23}+9q^{22}+4q^{21}+9q^{20}+6q^{19}+5q^{18}+3q^{17}+6q^{16}+2q^{15}+4q^{14}+3q^{13}+3q^{12}+q^{11}+2q^{10}+q^{9}+2q^{8}+q^{7}+q^{4})$

eigenvalue: $q^{13}+q^{12}+q^{11}+q^{10}-q^{6}-q^{5}-q^{3}-1$
multiplicity: $q^{73}+q^{71}+2q^{69}+4q^{67}+4q^{65}+7q^{63}+10q^{61}+10q^{59}+15q^{57}+18q^{55}+18q^{53}+25q^{51}+26q^{49}+26q^{47}+33q^{45}+31q^{43}+31q^{41}+36q^{39}+31q^{37}+31q^{35}+33q^{33}+26q^{31}+26q^{29}+25q^{27}+18q^{25}+18q^{23}+15q^{21}+10q^{19}+10q^{17}+7q^{15}+4q^{13}+4q^{11}+2q^{9}+q^{7}+q^{5}$

eigenvalue: $q^{11}-q^{9}-q^{6}-q^{5}-q^{3}-1$
multiplicity: $\frac{1}{2}(q^{78}+q^{77}+2q^{76}+q^{75}+2q^{74}+q^{73}+5q^{72}+3q^{71}+7q^{70}+3q^{69}+8q^{68}+4q^{67}+14q^{66}+7q^{65}+17q^{64}+7q^{63}+19q^{62}+9q^{61}+28q^{60}+13q^{59}+31q^{58}+12q^{57}+33q^{56}+15q^{55}+44q^{54}+19q^{53}+45q^{52}+17q^{51}+47q^{50}+21q^{49}+57q^{48}+23q^{47}+54q^{46}+20q^{45}+55q^{44}+24q^{43}+62q^{42}+24q^{41}+55q^{40}+20q^{39}+54q^{38}+23q^{37}+57q^{36}+21q^{35}+47q^{34}+17q^{33}+45q^{32}+19q^{31}+44q^{30}+15q^{29}+33q^{28}+12q^{27}+31q^{26}+13q^{25}+28q^{24}+9q^{23}+19q^{22}+7q^{21}+17q^{20}+7q^{19}+14q^{18}+4q^{17}+8q^{16}+3q^{15}+7q^{14}+3q^{13}+5q^{12}+q^{11}+2q^{10}+q^{9}+2q^{8}+q^{7}+q^{6})$

eigenvalue: $-q^{11}-q^{9}-q^{6}-q^{5}-q^{3}-1$
multiplicity: $\frac{1}{2}(q^{78}+q^{77}+q^{75}+2q^{74}+q^{73}+q^{72}+3q^{71}+3q^{70}+3q^{69}+4q^{68}+4q^{67}+6q^{66}+7q^{65}+5q^{64}+7q^{63}+11q^{62}+9q^{61}+8q^{60}+13q^{59}+13q^{58}+12q^{57}+15q^{56}+15q^{55}+16q^{54}+19q^{53}+17q^{52}+17q^{51}+23q^{50}+21q^{49}+17q^{48}+23q^{47}+24q^{46}+20q^{45}+23q^{44}+24q^{43}+22q^{42}+24q^{41}+23q^{40}+20q^{39}+24q^{38}+23q^{37}+17q^{36}+21q^{35}+23q^{34}+17q^{33}+17q^{32}+19q^{31}+16q^{30}+15q^{29}+15q^{28}+12q^{27}+13q^{26}+13q^{25}+8q^{24}+9q^{23}+11q^{22}+7q^{21}+5q^{20}+7q^{19}+6q^{18}+4q^{17}+4q^{16}+3q^{15}+3q^{14}+3q^{13}+q^{12}+q^{11}+2q^{10}+q^{9}+q^{7}+q^{6})$

eigenvalue: $-q^{14}-q^{11}-q^9-q^8-q^6-q^5-q^3-1$
multiplicity: $\frac{1}{2}(q^{68}-q^{65}+2q^{64}-q^{63}+2q^{62}-q^{61}+3q^{60}-3q^{59}+4q^{58}-2q^{57}+6q^{56}-3q^{55}+5q^{54}-6q^{53}+9q^{52}-4q^{51}+9q^{50}-6q^{49}+11q^{48}-9q^{47}+12q^{46}-6q^{45}+14q^{44}-9q^{43}+13q^{42}-11q^{41}+16q^{40}-7q^{39}+15q^{38}-11q^{37}+16q^{36}-11q^{35}+15q^{34}-7q^{33}+16q^{32}-11q^{31}+13q^{30}-9q^{29}+14q^{28}-6q^{27}+12q^{26}-9q^{25}+11q^{24}-6q^{23}+9q^{22}-4q^{21}+9q^{20}-6q^{19}+5q^{18}-3q^{17}+6q^{16}-2q^{15}+4q^{14}-3q^{13}+3q^{12}-q^{11}+2q^{10}-q^9+2q^8-q^7+q^4)$

In the thick case $E_{8,2}(q)$ we find

eigenvalue: $q^{16}+q^{15}+2q^{14}+3q^{13}+4q^{12}+5q^{11}+6q^{10}+6q^9+6q^8+6q^7+5q^6+4q^5+3q^4+2q^3+q^2+q$
multiplicity: 1

eigenvalue: $q^{14}+2q^{13}+3q^{12}+5q^{11}+6q^{10}+7q^9+7q^8+7q^7+5q^6+4q^5+2q^4+q^3-1$
multiplicity: $q^{29}+q^{23}+q^{19}+q^{17}+q^{13}+q^{11}+q^7+q$

eigenvalue: $q^{13}+2q^{12}+4q^{11}+6q^{10}+7q^9+7q^8+7q^7+5q^6+3q^5+q^4-q^2-1$
multiplicity: $q^{46}+q^{42}+q^{40}+q^{38}+2q^{36}+2q^{34}+q^{32}+3q^{30}+2q^{28}+2q^{26}+3q^{24}+2q^{22}+2q^{20}+3q^{18}+q^{16}+2q^{14}+2q^{12}+q^{10}+q^8+q^6+q^2$

eigenvalue: $2q^{11}+4q^{10}+6q^9+7q^8+6q^7+3q^6+q^5-q^4-q^3-q^2-1$
multiplicity: $\frac{1}{2}(q^{68}+q^{65}+2q^{64}+q^{63}+2q^{62}+q^{61}+3q^{60}+3q^{59}+4q^{58}+2q^{57}+6q^{56}+3q^{55}+5q^{54}+6q^{53}+9q^{52}+4q^{51}+9q^{50}+6q^{49}+11q^{48}+9q^{47}+12q^{46}+6q^{45}+14q^{44}+9q^{43}+13q^{42}+11q^{41}+16q^{40}+7q^{39}+15q^{38}+11q^{37}+16q^{36}+11q^{35}+15q^{34}+7q^{33}+16q^{32}+11q^{31}+13q^{30}+9q^{29}+14q^{28}+6q^{27}+12q^{26}+9q^{25}+11q^{24}+6q^{23}+9q^{22}+4q^{21}+9q^{20}+6q^{19}+5q^{18}+3q^{17}+6q^{16}+2q^{15}+4q^{14}+3q^{13}+3q^{12}+q^{11}+2q^{10}+q^9+2q^8+q^7+q^4)$

eigenvalue: $q^{11}+3q^{10}+5q^9+5q^8+5q^7+2q^6-q^4-q^3-q^2-1$
multiplicity: $q^{73}+q^{71}+2q^{69}+4q^{67}+4q^{65}+7q^{63}+10q^{61}+10q^{59}+15q^{57}+18q^{55}+18q^{53}+25q^{51}+26q^{49}+26q^{47}+33q^{45}+31q^{43}+31q^{41}+36q^{39}+31q^{37}+31q^{35}+33q^{33}+26q^{31}+26q^{29}+25q^{27}+18q^{25}+18q^{23}+15q^{21}+10q^{19}+10q^{17}+7q^{15}+4q^{13}+4q^{11}+2q^9+q^7+q^5$

eigenvalue: $q^{11}+2q^{10}+3q^9+3q^8+3q^7+q^6-q^4-q^3-q^2-1$
multiplicity: $q^{74}+q^{72}+2q^{70}+3q^{68}+5q^{66}+7q^{64}+9q^{62}+11q^{60}+15q^{58}+17q^{56}+21q^{54}+23q^{52}+26q^{50}+29q^{48}+31q^{46}+32q^{44}+34q^{42}+33q^{40}+34q^{38}+32q^{36}+31q^{34}+29q^{32}+26q^{30}+23q^{28}+21q^{26}+17q^{24}+15q^{22}+11q^{20}+9q^{18}+7q^{16}+5q^{14}+3q^{12}+2q^{10}+q^8+q^6$

eigenvalue: $q^{11}+q^{10}+2q^9+2q^8+2q^7-q^4-q^3-q^2-1$
multiplicity: $\frac{1}{2}(q^{78}+q^{77}+q^{75}+2q^{74}+q^{73}+q^{72}+3q^{71}+3q^{70}+3q^{69}+4q^{68}+4q^{67}+6q^{66}+7q^{65}+5q^{64}+7q^{63}+11q^{62}+9q^{61}+8q^{60}+13q^{59}+13q^{58}+12q^{57}+15q^{56}+15q^{55}+16q^{54}+19q^{53}+17q^{52}+17q^{51}+23q^{50}+21q^{49}+17q^{48}+23q^{47}+24q^{46}+20q^{45}+23q^{44}+24q^{43}+22q^{42}+24q^{41}+23q^{40}+20q^{39}+24q^{38}+23q^{37}+17q^{36}+21q^{35}+23q^{34}+17q^{33}+17q^{32}+19q^{31}+16q^{30}+15q^{29}+15q^{28}+12q^{27}+13q^{26}+$

$13q^{25}+8q^{24}+9q^{23}+11q^{22}+7q^{21}+5q^{20}+7q^{19}+6q^{18}+4q^{17}+4q^{16}+3q^{15}+3q^{14}+3q^{13}+q^{12}+q^{11}+2q^{10}+q^{9}+q^{7}+q^{6})$

eigenvalue: $q^{11}+q^{10}+q^{9}+q^{8}+q^{7}-q^{4}-q^{3}-q^{2}-1$
multiplicity: $q^{73}+q^{71}+2q^{69}+4q^{67}+4q^{65}+7q^{63}+10q^{61}+10q^{59}+15q^{57}+18q^{55}+18q^{53}+25q^{51}+26q^{49}+26q^{47}+33q^{45}+31q^{43}+31q^{41}+36q^{39}+31q^{37}+31q^{35}+33q^{33}+26q^{31}+26q^{29}+25q^{27}+18q^{25}+18q^{23}+15q^{21}+10q^{19}+10q^{17}+7q^{15}+4q^{13}+4q^{11}+2q^{9}+q^{7}+q^{5}$

eigenvalue: $q^{10}+3q^{9}+3q^{8}+3q^{7}-q^{5}-q^{4}-q^{3}-q^{2}-1$
multiplicity: $\frac{1}{6}(q^{83}+3q^{82}+5q^{81}+6q^{80}+8q^{79}+9q^{78}+11q^{77}+15q^{76}+22q^{75}+24q^{74}+29q^{73}+33q^{72}+40q^{71}+45q^{70}+60q^{69}+63q^{68}+71q^{67}+75q^{66}+90q^{65}+93q^{64}+117q^{63}+120q^{62}+132q^{61}+132q^{60}+157q^{59}+153q^{58}+181q^{57}+180q^{56}+194q^{55}+183q^{54}+217q^{53}+207q^{52}+233q^{51}+225q^{50}+241q^{49}+216q^{48}+249q^{47}+234q^{46}+252q^{45}+234q^{44}+249q^{43}+216q^{42}+241q^{41}+225q^{40}+233q^{39}+207q^{38}+217q^{37}+183q^{36}+194q^{35}+180q^{34}+181q^{33}+153q^{32}+157q^{31}+132q^{30}+132q^{29}+120q^{28}+117q^{27}+93q^{26}+90q^{25}+75q^{24}+71q^{23}+63q^{22}+60q^{21}+45q^{20}+40q^{19}+33q^{18}+29q^{17}+24q^{16}+22q^{15}+15q^{14}+11q^{13}+9q^{12}+8q^{11}+6q^{10}+5q^{9}+3q^{8}+q^{7})$

eigenvalue: $q^{10}+q^{9}+q^{8}+q^{7}-q^{5}-q^{4}-q^{3}-q^{2}-1$
multiplicity: $\frac{1}{2}(q^{83}+q^{82}+q^{81}+2q^{80}+2q^{79}+3q^{78}+5q^{77}+5q^{76}+6q^{75}+8q^{74}+9q^{73}+11q^{72}+14q^{71}+15q^{70}+16q^{69}+21q^{68}+23q^{67}+25q^{66}+30q^{65}+31q^{64}+33q^{63}+40q^{62}+42q^{61}+44q^{60}+49q^{59}+51q^{58}+53q^{57}+60q^{56}+62q^{55}+61q^{54}+67q^{53}+69q^{52}+69q^{51}+75q^{50}+75q^{49}+72q^{48}+77q^{47}+78q^{46}+76q^{45}+78q^{44}+77q^{43}+72q^{42}+75q^{41}+75q^{40}+69q^{39}+69q^{38}+67q^{37}+61q^{36}+62q^{35}+60q^{34}+53q^{33}+51q^{32}+49q^{31}+44q^{30}+42q^{29}+40q^{28}+33q^{27}+31q^{26}+30q^{25}+25q^{24}+23q^{23}+21q^{22}+16q^{21}+15q^{20}+14q^{19}+11q^{18}+9q^{17}+8q^{16}+6q^{15}+5q^{14}+5q^{13}+3q^{12}+2q^{11}+2q^{10}+q^{9}+q^{8}+q^{7})$

eigenvalue: $q^{10}-q^{5}-q^{4}-q^{3}-q^{2}-1$
multiplicity: $\frac{1}{3}(q^{83}+2q^{81}+2q^{79}+5q^{77}+7q^{75}+8q^{73}+16q^{71}+18q^{69}+20q^{67}+33q^{65}+33q^{63}+39q^{61}+55q^{59}+49q^{57}+59q^{55}+73q^{53}+62q^{51}+76q^{49}+81q^{47}+66q^{45}+81q^{43}+76q^{41}+62q^{39}+73q^{37}+59q^{35}+49q^{33}+55q^{31}+39q^{29}+33q^{27}+33q^{25}+20q^{23}+18q^{21}+16q^{19}+8q^{17}+7q^{15}+5q^{13}+2q^{11}+2q^{9}+q^{7})$

eigenvalue: $2q^{9}+3q^{8}+2q^{7}-q^{6}-q^{5}-q^{4}-q^{3}-q^{2}-1$
multiplicity: $\frac{1}{6}(q^{88}+5q^{86}+11q^{84}+17q^{82}+27q^{80}+43q^{78}+61q^{76}+84q^{74}+114q^{72}+145q^{70}+180q^{68}+220q^{66}+258q^{64}+300q^{62}+344q^{60}+377q^{58}+408q^{56}+439q^{54}+457q^{52}+470q^{50}+478q^{48}+470q^{46}+457q^{44}+439q^{42}+408q^{40}+377q^{38}+344q^{36}+300q^{34}+258q^{32}+220q^{30}+180q^{28}+145q^{26}+114q^{24}+84q^{22}+61q^{20}+43q^{18}+27q^{16}+17q^{14}+11q^{12}+5q^{10}+q^{8})$

eigenvalue: $q^{9}+q^{8}+q^{7}-q^{6}-q^{5}-q^{4}-q^{3}-q^{2}-1$
multiplicity: $q^{89}+2q^{87}+4q^{85}+7q^{83}+11q^{81}+17q^{79}+24q^{77}+33q^{75}+44q^{73}+56q^{71}+70q^{69}+85q^{67}+100q^{65}+116q^{63}+131q^{61}+145q^{59}+158q^{57}+168q^{55}+176q^{53}+181q^{51}+182q^{49}+181q^{47}+176q^{45}+168q^{43}+158q^{41}+145q^{39}+131q^{37}+116q^{35}+100q^{33}+85q^{31}+70q^{29}+56q^{27}+44q^{25}+33q^{23}+24q^{21}+17q^{19}+11q^{17}+7q^{15}+4q^{13}+2q^{11}+q^{9}$

eigenvalue: $q^8-q^6-q^5-q^4-q^3-q^2-1$
multiplicity: $\frac{1}{2}(q^{92}+q^{91}+q^{90}+2q^{89}+2q^{88}+3q^{87}+5q^{86}+5q^{85}+6q^{84}+9q^{83}+10q^{82}+12q^{81}+16q^{80}+17q^{79}+19q^{78}+25q^{77}+27q^{76}+30q^{75}+37q^{74}+39q^{73}+42q^{72}+51q^{71}+54q^{70}+57q^{69}+66q^{68}+69q^{67}+72q^{66}+83q^{65}+86q^{64}+87q^{63}+97q^{62}+100q^{61}+101q^{60}+111q^{59}+113q^{58}+111q^{57}+120q^{56}+122q^{55}+119q^{54}+126q^{53}+126q^{52}+120q^{51}+126q^{50}+126q^{49}+119q^{48}+122q^{47}+120q^{46}+111q^{45}+113q^{44}+111q^{43}+101q^{42}+100q^{41}+97q^{40}+87q^{39}+86q^{38}+83q^{37}+72q^{36}+69q^{35}+66q^{34}+57q^{33}+54q^{32}+51q^{31}+42q^{30}+39q^{29}+37q^{28}+30q^{27}+27q^{26}+25q^{25}+19q^{24}+17q^{23}+16q^{22}+12q^{21}+10q^{20}+9q^{19}+6q^{18}+5q^{17}+5q^{16}+3q^{15}+2q^{14}+2q^{13}+q^{12}+q^{11}+q^{10})$

eigenvalue: $-q^8-q^6-q^5-q^4-q^3-q^2-1$
multiplicity: $\frac{1}{2}(q^{92}+q^{91}+q^{90}+3q^{88}+q^{87}+6q^{86}+3q^{85}+9q^{84}+q^{83}+15q^{82}+4q^{81}+23q^{80}+7q^{79}+31q^{78}+4q^{77}+44q^{76}+11q^{75}+59q^{74}+14q^{73}+73q^{72}+10q^{71}+94q^{70}+22q^{69}+115q^{68}+23q^{67}+134q^{66}+19q^{65}+159q^{64}+35q^{63}+182q^{62}+32q^{61}+200q^{60}+29q^{59}+223q^{58}+47q^{57}+241q^{56}+38q^{55}+251q^{54}+37q^{53}+265q^{52}+53q^{51}+272q^{50}+39q^{49}+269q^{48}+40q^{47}+270q^{46}+51q^{45}+263q^{44}+34q^{43}+247q^{42}+37q^{41}+236q^{40}+42q^{39}+218q^{38}+25q^{37}+193q^{36}+29q^{35}+176q^{34}+29q^{33}+153q^{32}+15q^{31}+127q^{30}+19q^{29}+110q^{28}+16q^{27}+89q^{26}+7q^{25}+68q^{24}+10q^{23}+56q^{22}+7q^{21}+41q^{20}+2q^{19}+28q^{18}+4q^{17}+22q^{16}+2q^{15}+14q^{14}+8q^{12}+q^{11}+6q^{10}+3q^8+q^6+q^4)$

eigenvalue: $-q^9-q^7-q^6-q^5-q^4-q^3-q^2-1$
multiplicity: $\frac{1}{3}(q^{88}-q^{86}+2q^{84}+2q^{82}+4q^{78}+7q^{76}+12q^{72}+10q^{70}+6q^{68}+19q^{66}+18q^{64}+9q^{62}+32q^{60}+20q^{58}+18q^{56}+37q^{54}+25q^{52}+20q^{50}+43q^{48}+20q^{46}+25q^{44}+37q^{42}+18q^{40}+20q^{38}+32q^{36}+9q^{34}+18q^{32}+19q^{30}+6q^{28}+10q^{26}+12q^{24}+7q^{20}+4q^{18}+2q^{14}+2q^{12}-q^{10}+q^8)$

eigenvalue: $\frac{q^{13}}{2}+q^{12}+\frac{3q^{11}}{2}+\frac{5q^{10}}{2}+\frac{5q^9}{2}+\frac{5q^8}{2}+\frac{5q^7}{2}+\frac{3q^6}{2}+\frac{q^5}{2}-\frac{q^3}{2}-q^2-1+\frac{1}{2}\sqrt{a(q)}$
multiplicity: $\frac{1}{2}(q^{57}+q^{56}+q^{55}+q^{54}+q^{53}+2q^{51}+2q^{50}+2q^{49}+q^{48}+3q^{47}+q^{46}+4q^{45}+4q^{44}+4q^{43}+q^{42}+5q^{41}+2q^{40}+6q^{39}+5q^{38}+6q^{37}+q^{36}+7q^{35}+4q^{34}+7q^{33}+5q^{32}+7q^{31}+7q^{29}+5q^{28}+7q^{27}+4q^{26}+7q^{25}+q^{24}+6q^{23}+5q^{22}+6q^{21}+2q^{20}+5q^{19}+q^{18}+4q^{17}+4q^{16}+4q^{15}+q^{14}+3q^{13}+q^{12}+2q^{11}+2q^{10}+2q^9+q^7+q^6+q^5+q^4+q^3)$

eigenvalue: $\frac{q^{13}}{2}+q^{12}+\frac{3q^{11}}{2}+\frac{5q^{10}}{2}+\frac{5q^9}{2}+\frac{5q^8}{2}+\frac{5q^7}{2}+\frac{3q^6}{2}+\frac{q^5}{2}-\frac{q^3}{2}-q^2-1-\frac{1}{2}\sqrt{a(q)}$
multiplicity: $\frac{1}{2}(q^{57}+q^{56}+q^{55}+q^{54}+q^{53}+2q^{51}+2q^{50}+2q^{49}+q^{48}+3q^{47}+q^{46}+4q^{45}+4q^{44}+4q^{43}+q^{42}+5q^{41}+2q^{40}+6q^{39}+5q^{38}+6q^{37}+q^{36}+7q^{35}+4q^{34}+7q^{33}+5q^{32}+7q^{31}+7q^{29}+5q^{28}+7q^{27}+4q^{26}+7q^{25}+q^{24}+6q^{23}+5q^{22}+6q^{21}+2q^{20}+5q^{19}+q^{18}+4q^{17}+4q^{16}+4q^{15}+q^{14}+3q^{13}+q^{12}+2q^{11}+2q^{10}+2q^9+q^7+q^6+q^5+q^4+q^3)$

eigenvalue: $\frac{q^{11}}{2}+\frac{3q^{10}}{2}+2q^9+\frac{5q^8}{2}+2q^7+\frac{q^6}{2}-\frac{q^5}{2}-q^4-q^3-q^2-1+\frac{1}{2}\sqrt{b(q)}$
multiplicity: $\frac{1}{2}(q^{78}+q^{77}+2q^{76}+q^{75}+2q^{74}+q^{73}+5q^{72}+3q^{71}+7q^{70}+3q^{69}+8q^{68}+4q^{67}+14q^{66}+7q^{65}+17q^{64}+7q^{63}+19q^{62}+9q^{61}+28q^{60}+13q^{59}+31q^{58}+12q^{57}+33q^{56}+15q^{55}+44q^{54}+19q^{53}+45q^{52}+17q^{51}+47q^{50}+21q^{49}+57q^{48}+23q^{47}+54q^{46}+20q^{45}+55q^{44}+24q^{43}+62q^{42}+24q^{41}+55q^{40}+20q^{39}+54q^{38}+23q^{37}+57q^{36}+21q^{35}+47q^{34}+17q^{33}+45q^{32}+19q^{31}+44q^{30}+15q^{29}+33q^{28}+12q^{27}+31q^{26}+13q^{25}+28q^{24}+9q^{23}+19q^{22}+7q^{21}+17q^{20}+7q^{19}+14q^{18}+4q^{17}+8q^{16}+3q^{15}+7q^{14}+3q^{13}+5q^{12}+q^{11}+2q^{10}+q^9+2q^8+q^7+q^6)$

eigenvalue: $\frac{q^{11}}{2}+\frac{3q^{10}}{2}+2q^9+\frac{5q^8}{2}+2q^7+\frac{q^6}{2}-\frac{q^5}{2}-q^4-q^3-q^2-1-\frac{1}{2}\sqrt{b(q)}$
multiplicity: $\frac{1}{2}(q^{78}+q^{77}+2q^{76}+q^{75}+2q^{74}+q^{73}+5q^{72}+3q^{71}+7q^{70}+3q^{69}+8q^{68}+4q^{67}+14q^{66}+7q^{65}+17q^{64}+7q^{63}+19q^{62}+9q^{61}+28q^{60}+13q^{59}+31q^{58}+12q^{57}+33q^{56}+15q^{55}+44q^{54}+19q^{53}+45q^{52}+17q^{51}+47q^{50}+21q^{49}+57q^{48}+23q^{47}+54q^{46}+20q^{45}+55q^{44}+24q^{43}+62q^{42}+24q^{41}+55q^{40}+20q^{39}+54q^{38}+23q^{37}+57q^{36}+21q^{35}+47q^{34}+17q^{33}+45q^{32}+19q^{31}+44q^{30}+15q^{29}+33q^{28}+12q^{27}+31q^{26}+13q^{25}+28q^{24}+9q^{23}+19q^{22}+7q^{21}+17q^{20}+7q^{19}+14q^{18}+4q^{17}+8q^{16}+3q^{15}+7q^{14}+3q^{13}+5q^{12}+q^{11}+2q^{10}+q^9+2q^8+q^7+q^6)$

eigenvalue: $\frac{q^{13}}{2}+q^{12}+\frac{5q^{11}}{2}+\frac{7q^{10}}{2}+\frac{9q^9}{2}+\frac{9q^8}{2}+\frac{9q^7}{2}+\frac{5q^6}{2}+\frac{3q^5}{2}-\frac{q^3}{2}-q^2-1+\frac{1}{2}\sqrt{c(q)}$
multiplicity: $\frac{1}{2}(q^{57}+q^{56}+q^{55}+q^{54}+q^{53}+2q^{52}+2q^{51}+2q^{50}+2q^{49}+3q^{48}+3q^{47}+3q^{46}+4q^{45}+4q^{44}+4q^{43}+5q^{42}+5q^{41}+6q^{40}+6q^{39}+5q^{38}+6q^{37}+7q^{36}+7q^{35}+6q^{34}+7q^{33}+7q^{32}+7q^{31}+8q^{30}+7q^{29}+7q^{28}+7q^{27}+6q^{26}+7q^{25}+7q^{24}+6q^{23}+5q^{22}+6q^{21}+6q^{20}+5q^{19}+5q^{18}+4q^{17}+4q^{16}+4q^{15}+3q^{14}+3q^{13}+3q^{12}+2q^{11}+2q^{10}+2q^9+2q^8+q^7+q^6+q^5+q^4+q^3)$

eigenvalue: $\frac{q^{13}}{2}+q^{12}+\frac{5q^{11}}{2}+\frac{7q^{10}}{2}+\frac{9q^9}{2}+\frac{9q^8}{2}+\frac{9q^7}{2}+\frac{5q^6}{2}+\frac{3q^5}{2}-\frac{q^3}{2}-q^2-1-\frac{1}{2}\sqrt{c(q)}$
multiplicity: $\frac{1}{2}(q^{57}+q^{56}+q^{55}+q^{54}+q^{53}+2q^{52}+2q^{51}+2q^{50}+2q^{49}+3q^{48}+3q^{47}+3q^{46}+4q^{45}+4q^{44}+4q^{43}+5q^{42}+5q^{41}+6q^{40}+6q^{39}+5q^{38}+6q^{37}+7q^{36}+7q^{35}+6q^{34}+7q^{33}+7q^{32}+7q^{31}+8q^{30}+7q^{29}+7q^{28}+7q^{27}+6q^{26}+7q^{25}+7q^{24}+6q^{23}+5q^{22}+6q^{21}+6q^{20}+5q^{19}+5q^{18}+4q^{17}+4q^{16}+4q^{15}+3q^{14}+3q^{13}+3q^{12}+2q^{11}+2q^{10}+2q^9+2q^8+q^7+q^6+q^5+q^4+q^3)$,

where

$$\begin{aligned}
a(q) &= q^{26}+2q^{24}+2q^{23}+7q^{22}+12q^{21}+21q^{20}+30q^{19}+39q^{18}+46q^{17}\\
&\quad +49q^{16}+46q^{15}+39q^{14}+30q^{13}+21q^{12}+12q^{11}+7q^{10}+2q^9+2q^8+q^6\\
b(q) &= q^{22}-2q^{21}+q^{20}+6q^{19}+18q^{18}+30q^{17}+37q^{16}+30q^{15}+18q^{14}+6q^{13}\\
&\quad +q^{12}-2q^{11}+q^{10}\\
c(q) &= q^{26}-2q^{24}-2q^{23}-q^{22}+4q^{21}+17q^{20}+34q^{19}+51q^{18}+66q^{17}+73q^{16}\\
&\quad +66q^{15}+51q^{14}+34q^{13}+17q^{12}+4q^{11}-q^{10}-2q^9-2q^8+q^6
\end{aligned}$$

7. $F_{4,1}$

The graph $F_{4,1}$ is the 1-skeleton of the 24-cell on 24 vertices, with spectrum $8^1\ 4^4\ 0^9\ (-2)^8\ (-4)^2$ (cf. [2], p. 314).

In the thick case $F_{4,1}(q)$ we find one of the metasymplectic spaces, described in [3]. The spectrum is:

eigenvalue: $q^7+q^6+q^5+2q^4+q^3+q^2+q$
multiplicity: 1

eigenvalue: $q^5+2q^4+q^3+q^2-1$
multiplicity: $\frac{q^{11}}{2}+q^8+\frac{q^7}{2}+\frac{q^5}{2}+q^4+\frac{q}{2}$

eigenvalue: q^4-1
multiplicity: $q^{14}+q^{12}+2q^{10}+q^8+2q^6+q^4+q^2$

eigenvalue: $-q^3 - 1$
multiplicity: $q^{15} + q^{13} + q^{11} + 2\,q^9 + q^7 + q^5 + q^3$

eigenvalue: $-q^5 - q^3 - q^2 - 1$
multiplicity: $\frac{q^{11}}{2} + \frac{q^7}{2} + \frac{q^5}{2} + \frac{q}{2}$

8. $H_{3,1}$ and $H_{3,3}$

The graph $H_{3,1}$ is the 1-skeleton of the icosahedron, with spectrum $5^1\ \sqrt{5}^3\ (-1)^5\ (-\sqrt{5})^3$ (cf. [4], p. 310).

The graph $H_{3,3}$ is the 1-skeleton of the dodecahedron, with spectrum $3^1\ \sqrt{5}^3\ 1^5\ 0^4\ (-2)^4\ (-\sqrt{5})^3$ (cf. [4], p. 308).

9. $H_{4,1}$ and $H_{4,4}$

The graph $H_{4,1}$ is the 1-skeleton of the 600-cell on 120 vertices. It has spectrum $12^1\ (3+3\sqrt{5})^4\ (2+2\sqrt{5})^9\ 3^{16}\ 0^{25}\ (-2)^{36}\ (2-2\sqrt{5})^9\ (-3)^{16}\ (3-3\sqrt{5})^4$.

Note that the vertices of $H_{4,1}$ may be identified with the elements of the group $G = SL(2,5)$ (cf. [2], p. 315), and the graph is invariant under right multiplication by elements of G, so that we have the right regular representation of G. This explains why the multiplicities are the squares of the degrees of the irreducible characters of $SL(2,5)$.

The graph $H_{4,4}$ is the 1-skeleton of the 120-cell on 600 vertices. (The vertices can be regarded as 4-cliques in $H_{4,1}$, adjacent when they have a triangle in common.)

It has spectrum

approx. ev.	eigenvalue	mult.	approx. ev.	eigenvalue	mult.
4.000000	4	1	0.000000	0	18
3.854102	$3\tau - 1$	4	−0.302776	$\frac{1}{2}(3-\sqrt{13})$	16
3.618034	$\tau + 2$	9	−0.381966	$\tau - 2$	30
3.302776	$\frac{1}{2}(3+\sqrt{13})$	16	−0.618034	$1 - \tau$	24
2.925423	γ	25	−1.000000	-1	8
2.518199	ζ	36	−2.000000	-2	8
2.236068	$\sqrt{5}$	24	−2.236068	$-\sqrt{5}$	24
1.791288	$\frac{1}{2}(-1+\sqrt{21})$	16	−2.414214	$-\sqrt{2}-1$	48
1.618034	τ	24	−2.477352	ϵ	25
1.381966	$3 - \tau$	9	−2.618034	$-1-\tau$	30
1.178194	η	36	−2.696393	θ	36
1.000000	1	40	−2.791288	$\frac{1}{2}(-1-\sqrt{21})$	16
0.551929	δ	25	−2.854102	$2 - 3\tau$	4
0.414214	$\sqrt{2}-1$	48			

Here γ, δ, ϵ are the three roots of $x^3 - x^2 - 7x + 4 = 0$, and ζ, η, θ are the three roots of $x^3 - x^2 - 7x + 8 = 0$, and $\tau = (1+\sqrt{5})/2$.

In this case the association scheme has $d+1 = 45$ relations, while L_1 has only 27 distinct eigenvalues. Indeed, every eigenvalue of L_1 that has degree m over $\mathbf{Q}(\tau)$ occurs in L_1 with multiplicity m.

10. $I_{2,1}^m$

The graph $I_{2,1}^m$ is the m-gon, with spectrum $2\cos(2\pi j/m)$ $(0 \le j \le m-1)$ (cf. [4], p. 53).

In the thick case we only have to look at $m = 6$ or $m = 8$.

In the case $I_{2,1}^6(q) = G_{2,1}(q)$ we find a generalized hexagon which is distance-regular with spectrum:

eigenvalue: $q^2 + q$
multiplicity: 1

eigenvalue: $-1 + 2q$
multiplicity: $\frac{q^5}{6} + \frac{q^4}{2} + \frac{2q^3}{3} + \frac{q^2}{2} + \frac{q}{6}$

eigenvalue: $-q - 1$
multiplicity: $\frac{q^5}{3} + \frac{q^3}{3} + \frac{q}{3}$

eigenvalue: -1
multiplicity: $\frac{q^5}{2} + \frac{q^4}{2} + \frac{q^2}{2} + \frac{q}{2}$

In the case $I_{2,1}^8(q)$ with $q = 2^{k+\frac{1}{2}}$ we find a generalized octagon which is distance-regular with spectrum:

eigenvalue: $q^6 + q^2$
multiplicity: 1

eigenvalue: $q^2 - 1$
multiplicity: $\frac{q^{20}}{2} + \frac{q^{16}}{2} + \frac{q^8}{2} + \frac{q^4}{2}$

eigenvalue: $-q^4 - 1$
multiplicity: $q^{14} - q^{12} + q^8 - q^4 + q^2$

eigenvalue: $-1 + q^2 - \sqrt{2}\,q^3$
multiplicity: $2^{4k} + 2^{5k+1} + 2^{6k+2} - 2^{8k+2} - 2^{9k+4} + 2^{11k+4} + 2^{12k+6} + 2^{13k+5} - 2^{15k+7} - 2^{16k+6} + 2^{18k+8} + 2^{19k+8} + 2^{20k+8}$

eigenvalue: $-1 + q^2 + \sqrt{2}q^3$
multiplicity: $2^{4k} - 2^{5k+1} + 2^{6k+2} - 2^{8k+2} + 2^{9k+4} - 2^{11k+4} + 2^{12k+6} - 2^{13k+5} + 2^{15k+7} - 2^{16k+6} + 2^{18k+8} - 2^{19k+8} + 2^{20k+8}$

11. Remark

This research has been done in 1993. We found that Yasushi Gomi ([5, 6]) computed some of the results mentioned here in the context of computing character tables of (commutative) Hecke algebras. His results agree with ours.

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References

1. A.E. Brouwer and A.M. Cohen, “Computation of some parameters of Lie geometries,” *Annals of Discr. Math*. **26** (1983), 1–48.
2. A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer, Heidelberg, 1989.
3. A.M. Cohen, “Points and lines in metasymplectic spaces,” *Annals of Discrete Mathematics* **18** (1983), 193–196.
4. D.M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs*, V.E.B. Deutscher Verlag der Wissenschaften, Berlin, 1979 and Academic Press, New York, 1980.
5. Yasushi Gomi, “Character tables of commutative Hecke algebras associated with exceptional Weyl groups,” *Comm. in Alg.* **22**(1) (1994), 123–138.
6. Yasushi Gomi, “Character tables of commutative Hecke algebras associated with finite Chevalley groups of exceptional type,” *Comm. in Alg.* **22**(11) (1994), 4361–4372.