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On Symmetric Association Schemes with $k_1 = 3$

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Abstract. In this paper, we have a classification of primitive symmetric association schemes with $k_1 = 3$.

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1. Introduction

Let X be a finite set and R_i $(i = 0, 1, \ldots, d)$ be subsets of $X \times X$ with the properties that

(i) $R_0 = \{(x, x) \mid x \in X\}$;
(ii) $X \times X = R_0 \cup \cdots \cup R_d$, $R_i \cap R_j = \emptyset$ if $i \neq j$;
(iii) ${}^tR_i = R_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$, where ${}^tR_i = \{(x, y) \mid (y, x) \in R_i\}$;
(iv) for $i, j, k \in \{0, 1, \ldots, d\}$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is constant whenever $(x, y) \in R_k$. This constant is denoted by $p^k_{i,j}$.

Such a configuration $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is called an *association scheme* of class d on X. Association schemes with the additional properties

(v) $p^k_{i,j} = p^k_{j,i}$ for all i, j, k, and
(vi) for every i, $i' = i$, i.e., R_i is a symmetric relation

are called *commutative* and *symmetric*, respectively. Remark that if $\mathcal{X}$ is symmetric, then $\mathcal{X}$ is also commutative, and that a symmetric association scheme can be constructed from any commutative but non-symmetric association scheme by the *symmetrization* [1, p. 57].

The positive integer $k_i = p^0_{i,i'}$ is called the *valency* of R_i. It is clear that, for every i, the graph whose vertex set is X and edge set is R_i, is a k_i-regular graph, and, moreover, if $i = i'$, then it is undirected. We call it the R_i*-graph*. Note that, if Y is a connected component of the R_i-graph for some i, then $\mathcal{Y} = (Y, \{R_i \cap (Y \times Y)\}_{i \in \Lambda})$ is also an association scheme of class $|\Lambda| - 1$, where $\Lambda = \{i \mid (x, y) \in R_i,\ x, y \in Y\}$. If for any i with $1 \leq i \leq d$, R_i-graph is connected, we say that $\mathcal{X}$ is *primitive*.

Let Γ be a connected undirected finite simple graph. For $\alpha, \beta \in V(\Gamma)$, let $\partial(\alpha, \beta)$ be the distance between α and β. Let $d(\Gamma)$ be the maximal distance in Γ, called the *diameter*

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of Γ. Let $\Gamma_i(\alpha) = \{\beta \in V(\Gamma) \mid \partial(\alpha,\beta) = i\}$, and let $\Gamma(\alpha) = \Gamma_1(\alpha)$. For vertices α, β with $\partial(\alpha,\beta) = i$, let $C_i(\alpha,\beta) = \Gamma_{i-1}(\alpha) \cap \Gamma(\beta)$, $A_i(\alpha,\beta) = \Gamma_i(\alpha) \cap \Gamma(\beta)$ and $B_i(\alpha,\beta) = \Gamma_{i+1}(\alpha) \cap \Gamma(\beta)$. Let $c_i(\alpha,\beta) = |C_i(\alpha,\beta)|$, $a_i(\alpha,\beta) = |A_i(\alpha,\beta)|$ and $b_i(\alpha,\beta) = |B_i(\alpha,\beta)|$. If $c_i(\alpha,\beta)$, $a_i(\alpha,\beta)$ and $b_i(\alpha,\beta)$ depend only on $i = \partial(\alpha,\beta)$, we say c_i, a_i and b_i exist, respectively. Γ is said to be *distance-regular* if c_i, a_i and b_i exist for all i with $0 \le i \le d(\Gamma)$. It is clear that a distance-regular graph is b_0-regular. It is well known that, if Γ is distance-regular, then $\mathcal{X} = (V(\Gamma), \{R_i\}_{0\le i\le d(\Gamma)})$ is a symmetric association scheme (called P-polynomial with respect to R_1), where $R_i = \{(x,y) \in V(\Gamma) \times V(\Gamma) \mid \partial(x,y) = i\}$ $(0 \le i \le d(\Gamma))$.

In the study of association schemes, the following problems seem very important.

(1) Determine the graphs which can be the R_i-graphs of an association scheme.
(2) By giving a regular graph Γ, determine the association schemes such that Γ is the R_i-graph for some i.

In this paper, we study on the above problems, particularly (1), in the case when the case $\mathcal{X}$ is symmetric and Γ is a connected cubic (3-regular) graph. We remark that, if R_1-graph is a connected 2-regular graph, i.e., a polygon, then we easily see that $\mathcal{X}$ is P-polynomial with respect to R_1.

We shall show the following.

Theorem 1.1 *If $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ is a symmetric association scheme such that the R_1-graph Γ is a connected non-bipartite cubic graph, then $\mathcal{X}$ is P-polynomial with respect to R_1.*

This immediately implies the following.

Corollary 1.2 *If $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ is a primitive symmetric association scheme with $k_1 = 3$, then $\mathcal{X}$ is P-polynomial with respect to R_1.*

Remark that cubic distance-regular graphs have already been classified by several authors (see [5, 3, 2]). So, by the previous corollary, we have a classification of primitive symmetric association schemes with $k_1 = 3$.

2. R_1-graph

Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a symmetric association scheme of class d. For vertices $\alpha, \beta \in X$, let $\hat{\partial}(\alpha,\beta)$ be the index such that $(\alpha,\beta) \in R_{\hat{\partial}(\alpha,\beta)}$. Let $R_i(\alpha) = \{\beta \in X \mid \hat{\partial}(\alpha,\beta) = i\}$. Pick any t with $0 \le t \le d$ and let $\Gamma = (X, R_t)$ be the R_t-graph. For a pair of vertices (x,y) in R_i, $P^i_{j,l}(x,y) = \{z \in X \mid (x,z) \in R_j, (z,y) \in R_l\}$, and let $C(x,y) = \hat{C}_i(x,y) = C_{\partial(x,y)}(x,y)$, $A(x,y) = \hat{A}_i(x,y) = A_{\partial(x,y)}(x,y)$, $B(x,y) = \hat{B}_i(x,y) = B_{\partial(x,y)}(x,y)$. Let $c(x,y) = \hat{c}_i(x,y) = c_{\partial(x,y)}(x,y)$, $a(x,y) = \hat{a}_i(x,y) = a_{\partial(x,y)}(x,y)$ and $b(x,y) = \hat{b}_i(x,y) = b_{\partial(x,y)}(x,y)$.

Lemma 2.1 *Let $(X, \{R_i\}_{0\le i\le d})$ be a symmetric association scheme of class d. Pick any t with $1 \le t \le d$, and let Γ be the R_t-graph.*

For Γ, the following hold.

(1) *For any pair of vertices $\alpha, \beta \in X$, $\partial(\alpha,\beta)$ depends only on $\hat{\partial}(\alpha,\beta)$. In particular, if Γ is connected, there exists a surjection*

$$\varphi : \{0, 1, \ldots, d\} \to \{0, 1, \ldots, d(\Gamma)\}$$

such that, for all i with $0 \le i \le d$ and for all $x, y \in X$ with $\hat{\partial}(x, y) = i$, $\partial(x, y) = \varphi(i)$.

(2) *For any pair of vertices $\alpha, \beta \in X$, $c(\alpha,\beta)$, $a(\alpha,\beta)$ and $b(\alpha,\beta)$ depend only on $\hat{\partial}(\alpha,\beta)$. In particular, $c(\alpha,\beta) = c(\beta,\alpha)$, $a(\alpha,\beta) = a(\beta,\alpha)$ and $b(\alpha,\beta) = b(\beta,\alpha)$.*

(3) *Let α, β and γ be vertices in X with $\gamma \in B(\alpha,\beta)$. Then $c(\alpha,\beta) \le c(\alpha,\gamma)$ and $b(\alpha,\beta) \ge b(\alpha,\gamma)$.*

Proof: Straightforward. □

Remark From (1) and (2) in the previous lemma, we see that, if $d = d(\Gamma)$, i.e., if φ is bijective, then Γ is distance-regular. The converse does not necessarily hold.

We write $\hat{c}_i$, $\hat{a}_i$ and $\hat{b}_i$ for the parameters as in Lemma 2.1 (2).

From now on, let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a symmetric association scheme of class d such that R_1-graph Γ is a connected cubic graph.

Lemma 2.2

(1) *Let a_t exists and $a_t = 0$. Then there exist no vertices $x, y, z \in X$ such that $\partial(z, x) = \partial(z, y) = t + 1$, $\partial(x, y) = 1$ and that $c(z, x) = c(z, y) = 2$.*

(2) *Let j_1, j_2 be integers such that $\varphi(j_1) = \varphi(j_2) - 1$, $p^{j_2}_{1,j_1} = 2$, and that $p^{j_1}_{1,j_1} = 0$. Then $p^{j_2}_{1,j_2} = 0$.*

Proof:

(1) Suppose not. Note that, by Lemma 2.1(2), $c(x, z) = c(y, z) = 2$. As $k_1 = 3$, there must exist a vertex $u \in C_{t+1}(x, z) \cap C_{t+1}(y, z)$. However, this implies that $y \in A_t(u, x)$, which contradicts that $a_t = 0$.

(2) Similar to (1). □

Lemma 2.3

(1) *Let a_{t-2}, a_t exists and $a_{t-2} = a_t = 0$ for some $t \ge 2$. Then there exist no vertices x, y and z such that $\partial(x, y) = 1$, $\partial(z, x) = \partial(z, y) = t - 1$ and $c_{t-1}(z, x) = c_{t-1}(z, y) = b_{t-1}(z, x) = b_{t-1}(z, y) = 1$.*

(2) *Let j_1, j_2, j_3 be integers such that $\varphi(j_1) = \varphi(j_2) - 1 = \varphi(j_3) - 2$ and $p^{j_2}_{1,j_1} = p^{j_2}_{1,j_2} = p^{j_2}_{1,j_3} = 1$. Then $p^{j_1}_{1,j_1} \ne 0$ or $p^{j_3}_{1,j_3} \ne 0$.*

Proof:

(1) By Lemma 2.1 (2), $c_{t-1}(x,z) = c_{t-1}(y,z) = 1$. Let $\{u\} = C_{t-1}(x,z)$ and $\{v\} = C_{t-1}(y,z)$. We see that $u \neq v$. Otherwise, we have $y \in A_{t-2}(u,x)$, which contradicts that $a_{t-2} = 0$. Since $k_1 = 3$ and $b_{t-1}(x,z) = b_{t-1}(x,u) = 1$, there exists a vertex w in $B(x,z) \cap B(x,u)$. However, this implies that $y \in A_s(w,x)$, which contradicts that $a_t = 0$. Now we have the assertion.

(2) Similar to (1). □

For the convenience, we number some indices of relations. If $\#\{i \mid p^1_{1,i} \neq 0,\ i \notin \{0,1\}\} = 1$, let $p^1_{1,2} \neq 0$. If $\#\{i \mid p^2_{1,i} \neq 0,\ i \notin \{1,2\}\} = 1$, let $p^2_{1,3} \neq 0$. We repeat this, and let s be the maximal number such that, for $i \leq s$, $\#\{j \mid p^{i-1}_{1,j} \neq 0,\ j \notin \{i-2, i-1\}\} = 1$. Note that, for every vertex α in X and for any i with $0 \leq i \leq s$, $\Gamma_i(\alpha) = R_i(\alpha)$, and that c_i, a_i, b_i exist $(0 \leq i \leq s)$.

In this paper, the *distribution diagram* with respect to R_1 acts an important role. For the definition of it, see [4, Section 2.9].

It is well known that, if the distribution diagram with respect to R_1 is linear, i.e., $s = d$, then Γ is distance-regular with diameter d. See [4, Proposition 2.9.1 (ii)].

Lemma 2.4 *If $s \neq d$, then $s \geq 2$.*

Proof: Suppose $s = 1$. Let x, y be vertices in X with $\hat{\partial}(x,y) = 1$, $z \in P^1_{1,2}(x,y)$, and $u \in P^1_{1,3}(x,y)$. Note that $p^1_{1,2} = p^1_{1,3} = 1$. As $P^1_{1,3}(x,z) \neq \emptyset$, $\hat{\partial}(z,u) = 3$. However, this implies that $y, z \in P^1_{1,3}(x,u)$, which contradicts that $p^1_{1,3} = 1$. Now we have the assertion. □

Lemma 2.5 *Let $s \neq d$. Then $(c_i, a_i, b_i) = (1, 0, 2)$ for $0 \leq i \leq s$.*

Proof: If $s \neq d$, then we easily have $\hat{b}_s = b_s = 2$. Hence $\hat{c}_s = c_s = 1$. Therefore we have the assertion by Lemma 2.1 (3). □

The following lemma is clear.

Lemma 2.6 *Γ is bipartite if and only if a_i exists and $a_i = 0$ for any i with $0 \leq i \leq d(\Gamma)$.*

Let α and β be vertices in X with $\hat{\partial}(\alpha,\beta) = i$. Let $\Gamma(\alpha) = \{x_1, x_2, x_3\}$ and $\Gamma(\beta) = \{y_1, y_2, y_3\}$. Let $M_i(\alpha,\beta)$ be the 4×4-matrix whose rows and columns are indexed by $(\Gamma(\alpha) \cup \{\alpha\})$ and $(\Gamma(\beta) \cup \{\beta\})$, respectively, such that

$$(M_i(\alpha,\beta))_{u,v} = \hat{\partial}(u,v) - s,$$

where $u \in \Gamma(\alpha) \cup \{\alpha\}$ and $v \in \Gamma(\beta) \cup \{\beta\}$. We identify this up to the ordering of indices. If $M_i(\alpha,\beta)$ does not depend on the choice of $(\alpha,\beta) \in R_i$, we write $M_i = M_i(\alpha,\beta)$, and say M_i exists.

Suppose $s \neq d$, and let $p^s_{1,s+1} = p^s_{1,s+2} = 1$. Let α, β be vertices in X with $\hat{\partial}(\alpha, \beta) = s$. Let $\Gamma(\alpha) = \{x_1, x_2, x_3\}$ and $\Gamma(\beta) = \{y_1, y_2, y_3\}$. As $c_s = 1$, we may assume that $\hat{\partial}(\alpha, y_3) = \hat{\partial}(\beta, x_3) = s - 1$ and that $\hat{\partial}(x_3, y_3) = s - 2$.

Thus $M_s(\alpha, \beta)$ can be written as follows:

	y_1	β	y_2	y_3
x_1	i_1	1	i_2	0
α	1	0	2	-1
x_2	i_4	2	i_3	0
x_3	0	-1	0	-2

We may assume that $0 \leq i_1, i_2, i_3 \leq 5$.

We have the following lemma.

Lemma 2.7 *Let $s \neq d$. For $M_s(\alpha, \beta)$ as above, the following hold.*

(1) $p^{s+1}_{1,s+i_1}, p^{s+1}_{1,s+i_2}, p^{s+1}_{1,s+i_4}, p^{s+2}_{1,s+i_2}, p^{s+2}_{1,s+i_3}, p^{s+2}_{1,s+i_4}$ *are all nonzero. In particular*, $i_2 = i_4$.

(2) $\#\{j \in \{1, 2\} \mid i_j = 0\} = \hat{c}_{s+1} - 1$ *and* $\#\{j \in \{2, 3\} \mid i_j = 0\} = \hat{c}_{s+2} - 1$.

(3) *If* $i_2 = 1$, *then* $i_1 = 2$. *Similarly, if* $i_2 = 2$, *then* $i_3 = 1$.

(4) *If* $i_1 = 2$, *then* $i_2 = 1$ *or* $i_3 = 1$. *Similarly, if* $i_3 = 1$, *then* $i_1 = 2$ *or* $i_2 = 2$.

Proof:

(1) The first claim is clear. The second immediately follows from $3 = k_1 = \sum_{j=0}^{d} p^{s+1}_{1,j}$.

(2) As $y_3 \in P^{s+1}_{s,1}(x_1, \beta) \cap P^{s+2}_{s,1}(x_2, \beta)$, it is clear.

(3) Let $i_2 = 1$. Then we see that $\alpha \in P^{s+1}_{1,s+2}(x_1, y_2)$, so that $p^{s+1}_{1,s+2} \neq 0$. Hence $P^{s+1}_{1,s+2}(\beta, x_1) \neq \emptyset$, so $i_1 = 2$. The latter assertion is proved similarly.

(4) Similar to the proof of (3). $\square$

By the same argument as in the previous lemma, we have the following lemma.

Lemma 2.8 *Let j_1, j_2, j_3 be distinct integers such that $\varphi(j_1) \geq 1$, $\varphi(j_1) = \varphi(j_2) - 1 = \varphi(j_3) - 1$, and that $p^{j_1}_{1,j_2} = p^{j_1}_{1,j_3} = 1$. Then the following hold.*

(1) *If* $\hat{c}_{j_2} = p^{j_2}_{1,j_1} = 1$, *then there exists an integer* j_4 $(\neq j_1)$ *such that both* $p^{j_2}_{1,j_4}$ *and* $p^{j_3}_{1,j_4}$ *are nonzero.*

(2) *If* $\hat{c}_{j_2} = p^{j_2}_{1,j_1}$, *and if there exists no integer* j_4 $(\neq j_1)$ *such that both* $p^{j_2}_{1,j_4}$ *and* $p^{j_3}_{1,j_4}$ *are nonzero, then* $p^{j_2}_{1,j_1} \geq 2$.

Proof:

(1) Let j_0 be the integer such that $p^{j_1}_{1,j_0} = 1$ and that $\varphi(j_0) = \varphi(j_1) - 1$. Let α, β be vertices in X with $\hat{\partial}(\alpha, \beta) = j_1$, $\Gamma(\alpha) = \{x_1, x_2, x_3\}$, and let $\Gamma(\beta) = \{y_1, y_2, y_3\}$. Then we may assume that $M_{j_1}(\alpha, \beta)$ is as follows:

	y_1	β	y_2	y_3
x_1	$*$	j_2-s	$*$	$*$
α	j_2-s	j_1-s	j_3-s	j_0-s
x_2	$*$	j_3-s	$*$	$*$
x_3	$*$	j_0-s	$*$	$*$

As $\hat{c}_{j_2}=p^{j_2}_{1,j_1}=1$, we have $\hat{\partial}(x_1,y_3)=j_1$ and $\hat{\partial}(x_1,y_2)\neq j_1$. Hence we may assume that $\hat{\partial}(x_1,y_2)=j_4\neq j_1$. It follows that $y_2\in P^{j_2}_{1,j_4}(\beta,x_1)$ and $x_1\in P^{j_3}_{1,j_4}(\alpha,y_2)$, so that $p^{j_2}_{1,j_4}\neq 0$ and $p^{j_3}_{1,j_4}\neq 0$.

(2) It follows directly from (1). □

By Lemma 2.7, we see that (i_1,i_2,i_3) is one of the following:

$(0,0,0)$, $(0,0,2)$, $(0,0,3)$, $(0,2,1)$, $(0,3,0)$, $(0,3,2)$, $(0,3,3)$, $(0,3,4)$, $(1,0,0)$, $(1,0,2)$, $(1,0,3)$, $(1,2,1)$, $(1,3,0)$, $(1,3,2)$, $(1,3,3)$, $(1,3,4)$, $(2,0,1)$, $(2,1,0)$, $(2,1,1)$, $(2,1,2)$, $(2,1,3)$, $(2,2,1)$, $(2,3,1)$, $(3,0,0)$, $(3,0,2)$, $(3,0,3)$, $(3,0,4)$, $(3,2,1)$, $(3,3,0)$, $(3,3,2)$, $(3,3,3)$, $(3,3,4)$, $(3,4,0)$, $(3,4,2)$, $(3,4,3)$, $(3,4,4)$, $(3,4,5)$, $(4,3,4)$.

Note that, for example, $(3,4,5)$, $(3,5,4)$, $(4,3,5)$ and $(4,5,3)$ can be regarded as the same type.

Lemma 2.9 *Let (i_1,i_2,i_3) be one of the above. Then the following hold.*

(1) *M_s exists, except the case $(i_1,i_2,i_3)=(0,3,0)$, $(3,0,3)$, $(1,2,1)$, $(2,1,2)$, $(3,4,3)$ or $(4,3,4)$. Moreover, $(i_1,i_2,i_3)=(0,3,0)$ or $(3,0,3)$ if and only if $(p^{s+1}_{1,s},p^{s+1}_{1,s+3},p^{s+2}_{1,s},p^{s+2}_{1,s+3})=(2,1,2,1)$, $(i_1,i_2,i_3)=(1,2,1)$ or $(2,1,2)$ if and only if $(p^{s+1}_{1,s},p^{s+1}_{1,s+1},p^{s+1}_{1,s+2},p^{s+2}_{1,s},p^{s+2}_{1,s+1},p^{s+2}_{1,s+2})=(1,1,1,1,1,1)$, and $(i_1,i_2,i_3)=(3,4,3)$ or $(4,3,4)$ if and only if $(p^{s+1}_{1,s},p^{s+1}_{1,s+3},p^{s+1}_{1,s+4},p^{s+2}_{1,s},p^{s+2}_{1,s+3},p^{s+2}_{s,s+4})=(1,1,1,1,1,1)$.*

(2) *If $i_1\neq i_3$, then (i_1,i_2,i_3) and (i'_3,i'_2,i'_1) can be regarded as the same type, where for $j=1,2,3$,*

$$i'_j=\begin{cases} i_j & \text{if } i_j\notin\{1,2\},\\ 2 & \text{if } i_j=1,\\ 1 & \text{if } i_j=2.\end{cases}$$

Proof:

(1) Straightforward. For example, $(i_1,i_2,i_3)=(0,3,4)$ if and only if $(p^{s+1}_{1,s},p^{s+1}_{1,s+3},p^{s+2}_{1,s},p^{s+2}_{1,s+3},p^{s+2}_{1,s+4})=(2,1,1,1,1)$.

(2) It is clear from the symmetry of indices of relations. □

By (1) in the previous lemma, the type of (i_1, i_2, i_3) can be regarded as of $\mathcal{X}$.
Thus, if $s \neq d$, the type of $\mathcal{X}$ is one of the following.

(I) $(i_1, i_2, i_3) = (0, 0, 0)$, $(1, 2, 1)$ or $(2, 1, 2)$, $(1, 3, 2)$, $(2, 3, 1)$, $(3, 3, 3)$,
$(0, 3, 0)$ or $(3, 0, 3)$,
(IIA) $(0, 2, 1)$, $(0, 3, 3)$, $(2, 1, 1)$,
(IIB) $(0, 0, 2)$, $(1, 0, 2)$, $(1, 0, 3)$, $(2, 0, 1)$,
(IIC) $(0, 3, 2)$, $(3, 3, 2)$,
(IID) $(0, 0, 3)$, $(3, 0, 4)$, $(0, 3, 4)$, $(3, 3, 4)$, $(3, 4, 3)$ or $(4, 3, 4)$,
(III) $(1, 3, 4)$,
(IV) $(2, 1, 3)$,
(V) $(3, 4, 5)$.

In the rest of this paper, we shall show that, if Γ is non-bipartite, then $\mathcal{X}$ is not of any type in (I)–(V).

3. Circuit and profile

For fixed vertices $u, v \in X$ with $(u, v) \in R_1$, let $D^i_j = D^i_j(u, v) = P^1_{i,j}(u, v)$. For subsets A, B in X, let $e(A, B)$ be the number of edges in Γ between A and B. For $x \in X$, write $e(x, A) = e(\{x\}, A)$.

We easily have the following.

Lemma 3.1 *Let $s \neq d$ and let (u, v) be any pair of adjacent vertices. Then the following hold.*

(1) *$D^i_j \neq \emptyset$ if and only if $p^i_{1,j} \neq 0$. In particular, $D^{i-1}_i \neq \emptyset$ $(1 \leq i \leq s+1)$, $D^s_{s+2} \neq \emptyset$, and $D^i_i = \emptyset$ $(1 \leq i \leq s)$.*
(2) *For $1 \leq i \leq s$, $e(D^{i-1}_i, D^{i-1}_i) = e(D^{i-1}_i, D^i_{i-1}) = e(D^i_{i+1}, D^i_{i+1}) = 0$.*
(3) *$e(D^s_{s+2}, D^s_{s+2}) = 0$.*
(4) *For every $x \in D^i_{i+1}$ $(1 \leq i \leq s)$, $e(x, D^{i-1}_i) = 1$.*
(5) *For every $x \in D^s_{s+2}$, $e(x, D^{s-1}_s) = 1$.*
(6) *For every $y \in D^{i-1}_i$ $(1 \leq i \leq s-1)$, $e(y, D^i_{i+1}) = 2$.*
(7) *For every $y \in D^{s-1}_s$, $e(y, D^s_{s+1}) = e(y, D^s_{s+2}) = 1$.*

Proof: Straightforward. □

For $x \in D^{s+i}_{s+j}$, let

$$E^i_j(x) = \left\{(i', j') \;\middle|\; e\left(x, D^{s+i'}_{s+j'}\right) \neq 0\right\},$$

where $D^m_n = D^m_n(u, v)$ for a given pair of adjacent vertices (u, v) in Γ. If $E^i_j(x)$ depend only on i and j, we write $E^i_j = E^i_j(x)$.

In this paper, a *circuit* in Γ is defined as a connected induced 2-regular subgraph.

Lemma 3.2 *Let* $s \neq d$ *and let* $\{x_0, x_1, \ldots, x_{t-1}, x_t = x_0\}$ *be a circuit of length* t *in* Γ. *Then, for any* i, j *with* $0 \leq i \leq t-1$ *and with* $1 \leq j \leq s$, $x_{i+j} \in D^{j}_{j-1}(x_i, x_{i+1})$ *and* $x_{i-j+1} \in D^{j-1}_{j}(x_i, x_{i+1})$, *where indices are given by modulo* t. *In particular*, $t \geq 2s+2$.

Proof: Immediate from Lemma 3.1. □

Let $C = \{x_0, x_1, \ldots, x_t\}$ be a circuit of length t in Γ. Note that $x_{u+s} \in D^{s}_{s-1}(x_u, x_{u+1})$ and $x_{u+t-s+1} \in D^{s-1}_{s}(x_u, x_{u+1})$, where indices are given by modulo t. If $x_{u+s+1} \in D^{s+j_1}_{s+l_1}(x_u, x_{u+1}), \ldots, x_{u+t-s} \in D^{s+j_{t-2s}}_{s+l_{t-2s}}(x_u, x_{u+1})$, we call

$$\begin{array}{cccccc} 0 & j_1 & j_2 & \ldots & j_{t-2s} & (-1) \\ (-1) & l_1 & l_2 & \ldots & l_{t-2s} & 0 \end{array}$$

the *profile* of C with respect to (x_u, x_{u+1}). Note that $l_1 = j_{t-2s} = 0$. Let

$$\sigma_C(x_u) = (j_1, j_2, \ldots, j_{t-2s}),$$

and, for $e_1, \ldots, e_n \in \{1, 2, \ldots, t-2s\}$, let

$$\sigma_C(x_u; e_1, \ldots, e_n) = \left(j_{e_1}, \ldots, j_{e_n}\right).$$

4. Type I, II

Lemma 4.1 *Suppose that* $\mathcal{X}$ *is of type* I. *Then* Γ *is distance-regular.*

Proof: Suppose that $(i_1, i_2, i_3) = (0, 0, 0)$. Then we have that, for every $\alpha \in X$, $\Gamma_{s+1}(\alpha) = R_{s+1}(\alpha) \cup R_{s+2}(\alpha)$, $d = s+2$ and that $c_{s+1} = p^{s+1}_{1,s} = p^{s+2}_{1,s} = 3$. This implies that Γ is a bipartite distance-regular graph with $d(\Gamma) = s+1$ and with the intersection array

$$\{3, 2, \ldots, 2; 1, \ldots, 1, 3\}.$$

Similarly, if $(i_1, i_2, i_3) = (1, 2, 1)$ or $(2, 1, 2)$, then Γ is a distance-regular graph with $d(\Gamma) = s+1$ and with the intersection array

$$\{3, 2, \ldots, 2; 1, \ldots, 1\}.$$

Suppose that $(i_1, i_2, i_3) = (0, 3, 0)$ or $(3, 0, 3)$. Then we see that $\Gamma_{s+2}(\alpha) = R_{s+3}(\alpha)$ for every $\alpha \in X$, and that c_i, a_i, and b_i exist for $0 \leq i \leq s+2$. Note that $(c_{s+1}, a_{s+1}, b_{s+1}) = (2, 0, 1)$ and that

$$k_{s+1} = k_{s+2} = k_{s+3} \cdot p^{s+3}_{1,s+1} = k_{s+3} \cdot p^{s+3}_{1,s+2},$$

that is, $c_{s+2} = p^{s+3}_{1,s+1} + p^{s+3}_{1,s+2} = 2$. If $d = s + 3$, then $p^{s+3}_{1,s+3} = 1$, and Γ is a non-bipartite distance-regular graph with $d(\Gamma) = s + 2$ and with the intersection array

$$\{3, 2, \ldots, 2, 1; 1, \ldots, 1, 2, 2\}.$$

Let $d > s + 3$ and $p^{s+i}_{1,s+i+1} \neq 0$ $(3 \leq i \leq d - s - 1)$. Then we have that $\Gamma_{s+i}(\alpha) = R_{s+i+1}(\alpha)$ $(3 \leq i \leq d - s - 1)$ for every $\alpha \in X$, and that Γ is a distance-regular graph with $d(\Gamma) = d - 1$. Moreover, by Lemma 2.2 (2), $(p^d_{1,d-1}, p^d_{1,d}) = (3, 0)$. Hence $a_i = 0$ for $0 \leq i \leq d(\Gamma)$ and Γ is bipartite. In the case $(i_1, i_2, i_3) = (3, 3, 3)$, the proof is similar.

Suppose that $(i_1, i_2, i_3) = (1, 3, 2)$. Then we see that $\Gamma_{s+2}(\alpha) = R_{s+3}(\alpha)$ for every $\alpha \in X$, and that c_i, a_i, b_i exist for $0 \leq i \leq s + 2$. Note that $(c_{s+1}, a_{s+1}, b_{s+1}) = (1, 1, 1)$ and $c_{s+2} = p^{s+3}_{1,s+1} + p^{s+3}_{1,s+2} = 2$. Note that there exist $x, y, z \in X$ such that $\partial(z, x) = \partial(z, y) = s + 1$ and $\partial(x, y) = 1$. Since $c_{s+1} = b_{s+1} = 1$ and $a_s = 0$, it follows from Lemma 2.3 (1) that $a_{s+2} = p^{s+3}_{1,s+3} = 1$, so that Γ is a non-bipartite distance-regular graph with $d(\Gamma) = s + 2 = d - 1$ and with the intersection array

$$\{3, 2, \ldots, 2, 1; 1, \ldots, 1, 2\}.$$

The proof for the case $(i_1, i_2, i_3) = (2, 3, 1)$ is similar. □

Lemma 4.2 *Suppose that $\mathcal{X}$ is of type* I. *Then Γ is bipartite.*

Proof: By the list of cubic distance-regular graphs in [3], if $(i_1, i_2, i_3) = (1, 2, 1)$ or $(2, 1, 2)$, then Γ has the intersection array

$$\{3, 2; 1, 1\},$$

which contradicts Lemma 2.4. □

If $(i_1, i_2, i_3) = (0, 3, 0), (3, 0, 3)$ or $(3, 3, 3)$ and $d = s + 3$ (see the proof of Lemma 4.1), then we cannot find the appropriate graphs in the list in [3].

Suppose that $(i_1, i_2, i_3) = (1, 3, 2)$. Then, by the list in [3], Γ has the intersection array

$$\{3, 2, 2, 1; 1, 1, 1, 2\},$$

so that $s = 2$ and $d = d(\Gamma) + 1 = 5$. We need two sublemmas as follows.

Sublemma 4.2.1 *Suppose $(i_1, i_2, i_3) = (1, 3, 2)$. Then M_{s+1} and M_{s+2} exist as follows, respectively.*

	y_1	β	y_2	y_3
x_1	0	1	3	1
α	1	1	3	0
x_2	3	3	1	2
x_3	1	0	2	−1

	y_1	β	y_2	y_3
x_1	0	2	3	2
α	2	2	3	0
x_2	3	3	2	1
x_3	2	0	1	−1

Proof: Let $\alpha, \beta \in X$ with $\hat{\partial}(\alpha, \beta) = s + 1$, $R_1(\alpha) = \{x_1, x_2, x_3\}$, and let $R_1(\beta) = \{y_1, y_2, y_3\}$. As $c_{s+1} = p^{s+1}_{1,s} = 1$, we may assume that $\hat{\partial}(\alpha, y_3) = \hat{\partial}(\beta, x_3) = s$ and $\hat{\partial}(x_3, y_3) = s - 1$. As $p^{s+1}_{1,s+1} = p^{s+1}_{1,s+3} = 1$, let $\hat{\partial}(\alpha, y_1) = \hat{\partial}(\beta, x_1) = s + 1$ and $\hat{\partial}(\alpha, y_2) = \hat{\partial}(\beta, x_2) = s + 3$. As $p^{s+2}_{1,s+1} = 0$, we have $\hat{\partial}(x_1, y_3) = \hat{\partial}(x_3, y_1) = s + 1$ and $\hat{\partial}(x_2, y_3) = \hat{\partial}(x_3, y_2) = s + 2$. As $p^{s+3}_{1,s} = 0$, we have $\hat{\partial}(x_1, y_2) = \hat{\partial}(x_2, y_1) = s + 3$. This immediately implies that $\hat{\partial}(x_1, y_1) = s$ and $\hat{\partial}(x_2, y_2) = s + 1$. Thus we see that M_{s+1} exists as above.

Similarly, for M_{s+2}. □

Sublemma 4.2.2 *Suppose* $(i_1, i_2, i_3) = (1, 3, 2)$. *Then the following hold.*
(1) $E^0_1 = \{(-1, 0), (1, 1), (2, 3)\}$.
(2) $E^0_2 = \{(-1, 0), (1, 3), (2, 2)\}$.
(3) $E^1_1 = \{(0, 1), (1, 0), (3, 3)\}$.
(4) $E^2_3 = \{(0, 1), (2, 3), (3, 2)\}$.
(5) $E^1_3 = \{(0, 2), (1, 3), (3, 1)\}$.
(6) $E^2_2 = \{(0, 2), (2, 0), (3, 3)\}$.
(7) $E^3_3 = \{(1, 1), (2, 2), (3, 3)\}$.

Proof:

(1) Let $x \in D^s_{s+1}$. Then, for M_s, we may assume that $u = \alpha$, $v = x_1$ and $x = \beta$. Then we see that $y_3 \in D^{s-1}_s$, $y_1 \in D^{s+1}_{s+1}$ and $y_2 \in D^{s+2}_{s+3}$. Thus we have the assertion.
(2)–(6) Similar to (1).
(7) Let $x \in D^{s+3}_{s+3}$. As $p^{s+3}_{1,s+1} = p^{s+3}_{1,s+2} = p^{s+3}_{1,s+3} = 1$, we may assume that

$$E^3_3(x) = \{(1, j_1), (2, j_2), (3, j_3)\} = \{(j_4, 1), (j_5, 2), (j_6, 3)\}.$$

From (1) to (6), it must hold that $j_1 = j_4 = 1$, $j_2 = j_5 = 2$ and $j_3 = j_6 = 3$, as desired. □

Now we shall show that it is impossible that $(i_1, i_2, i_3) = (1, 3, 2)$. We use circuit chasing technique. Let $C = \{x_0, x_1, \ldots, x_9, x_{10} = x_0\}$ be a circuit of length $2s + 6 = 10$ in Γ such that the profile with respect to (x_0, x_1) is as follows:

$$\begin{array}{ccccccccc} 0 & 1 & 1 & 3 & 3 & 1 & 0 & & \\ & 0 & 1 & 3 & 3 & 1 & 1 & 0. \end{array}$$

Note that the existence of this circuit is guaranteed by Lemma 3.1 and Sublemma 4.2.2, $\hat{\partial}(x_0, x_3) = s + 1 = 3$, and that $x_4 \in D^{s+1}_s(x_1, x_2) = D^3_2(x_1, x_2)$. By Sublemma 4.2.2 (1), (4) and (3), we easily have that the profile of C with respect to (x_1, x_2) is

$$\begin{array}{ccccccccc} 0 & 1 & 3 & 3 & 1 & 1 & 0 & & \\ & 0 & 2 & 2 & 0 & 1 & 1 & 0. \end{array}$$

Similarly, the profile of C with respect to (x_2, x_3) is

$$\begin{array}{cccccccc} 0 & 2 & 2 & 0 & 1 & 1 & 0 & \\ & 0 & 2 & 2 & 3 & 3 & 2 & 0. \end{array}$$

This implies that $\hat{\partial}(x_3, x_0) = s + 2 = 4$, which is a contradiction. Thus we have the assertion.

Finally, we shall show that the case $(i_1, i_2, i_3) = (2, 3, 1)$ is impossible. In this case, Γ has the same intersection array as the one in the case $(i_1, i_2, i_3) = (1, 3, 2)$. For the proof, we need the following.

Sublemma 4.2.3 *Suppose* $(i_1, i_2, i_3) = (2, 3, 1)$. *Then the following hold.*

(1) $E_1^0 = \{(-1, 0), (1, 2), (2, 3)\}$.
(2) $E_2^0 = \{(-1, 0), (1, 3), (2, 1)\}$.
(3) $E_2^1 = \{(0, 1), (2, 0), (3, 3)\}$.
(4) $E_3^1 = \{(0, 2), (2, 3), (3, 1)\}$.
(5) $E_3^2 = \{(0, 1), (1, 3), (3, 2)\}$.
(6) $E_3^3 = \{(1, 2), (2, 1), (3, 3)\}$.

Proof: Similar to the proof of Sublemma 4.2.2. □

We use circuit chasing technique again. Let $C = \{x_0, x_1, \ldots, x_9, x_{10} = x_0\}$ be a circuit of length $2s + 6 = 10$ in Γ such that the profile with respect to (x_0, x_1) is as follows:

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 3 & 1 & 0 & \\ & 0 & 1 & 3 & 3 & 2 & 1 & 0. \end{array}$$

Note that $\hat{\partial}(x_0, x_3) = s + 1 = 3$. By Sublemma 4.2.3, we see that the profile of C with respect to (x_2, x_3) is

$$\begin{array}{cccccccc} 0 & 2 & 1 & 0 & 2 & 1 & 0 & \\ & 0 & 2 & 1 & 3 & 3 & 2 & 0. \end{array}$$

This implies that $\hat{\partial}(x_3, x_0) = s + 2 = 4$, which is a contradiction. Thus we have the assertion.

Now we conclude the proof of Lemma 4.2.

Lemma 4.3 *It is impossible that $\mathcal{X}$ is of type* IIA.

Proof: Suppose that $(i_1, i_2, i_3) = (0, 2, 1)$. Then we see that $p_{1,s}^{s+1} = 2$ and $p_{1,s+2}^{s+1} = p_{1,s}^{s+2} = p_{1,s+1}^{s+2} = 1$. By using the formula $k_l \cdot p_{i,j}^l = k_i \cdot p_{l,j}^i$, these imply that $k_s = 2 \cdot k_{s+1} = k_{s+2}$ and $k_{s+1} = k_{s+2}$. These are impossible.

Suppose that $(i_1, i_2, i_3) = (2, 1, 1)$. Then we see that $p^{s+2}_{1,s+1} = 2$ and $p^{s+1}_{1,s} = p^{s+1}_{1,s+2} = p^{s+2}_{1,s} = 1$. These are also impossible.

Suppose that $(i_1, i_2, i_3) = (0, 3, 3)$. Then we see that $p^{s+1}_{1,s} = p^{s+2}_{1,s+3} = 2$ and $p^{s+1}_{1,s+3} = p^{s+2}_{1,s} = 1$. Hence we have $k_s = 2k_{s+1} = k_{s+2}$, $k_{s+1} = k_{s+3} \cdot p^{s+3}_{1,s+1}$, and $2k_{s+2} = k_{s+3} \cdot p^{s+3}_{1,s+2}$. These imply that $p^{s+3}_{1,s+2} = 4p^{s+3}_{1,s+1} \geq 4$, which is a contradiction. □

Lemma 4.4 *It is impossible that $\mathcal{X}$ is of type* IIB.

Proof: In this case, we have vertices $x, y, z \in X$ such that $\partial(z, x) = \partial(z, y) = s + 1$, $\partial(x, y) = 1$ and that $c(z, x) = c(z, y) = 2$. On the other hand, we have $a_s = 0$, which contradicts Lemma 2.2. □

Lemma 4.5 *It is impossible that $\mathcal{X}$ is of type* IIC.

Proof: In the case $(i_1, i_2, i_3) = (0, 3, 2)$, we see that $p^{s+1}_{1,s} = 2$ and $p^{s+1}_{1,s+3} = p^{s+2}_{1,s} = p^{s+2}_{1,s+2} = p^{s+2}_{1,s+3} = 1$. These imply that $k_s = 2 \cdot k_{s+1} = k_{s+2}$, $k_{s+1} = k_{s+3} \cdot p^{s+3}_{1,s+1}$ and that $k_{s+2} = k_{s+3} \cdot p^{s+3}_{1,s+2}$. Hence we have $(p^{s+3}_{1,s+1}, p^{s+3}_{1,s+2}) = (1, 2)$ and $p^{s+3}_{1,s+3} = 0$. In the case $(i_1, i_2, i_3) = (3, 3, 2)$, we see that $p^{s+1}_{1,s+3} = 2$ and $p^{s+1}_{1,s} = p^{s+2}_{1,s} = p^{s+2}_{1,s+2} = p^{s+2}_{1,s+3} = 1$. These imply that $(p^{s+3}_{1,s+1}, p^{s+3}_{1,s+2}) = (2, 1)$ and that $p^{s+3}_{1,s+3} = 0$. Note that, in both cases, $\varphi(s+3) = s+2$ and $p^{s+3}_{1,s+3} = 0$. However, as $p^{s+2}_{1,s} = p^{s+2}_{1,s+2} = p^{s+2}_{1,s+3} = 1$, this contradicts Lemma 2.3 (2). □

Lemma 4.6 *Let $\mathcal{X}$ be of type* IID. *Then Γ is bipartite.*

Proof: Let $(i_1, i_2, i_3) = (0, 0, 3)$. Note that $p^{s+1}_{1,s+1} = 3$, $p^{s+2}_{1,s} = 2$ and that $a_i = 0$ for $0 \leq i \leq s+1$. By Lemma 2.1 (3), we have that, for every vertices $\alpha, \beta \in \Gamma$ with $\partial(\alpha, \beta) \geq s+1$, $c(\alpha, \beta) \geq 2$. Therefore, by Lemma 2.2 (1), we have $a_i = 0$ for $s + 1 \leq i \leq d(\Gamma)$. It follows from Lemma 2.3 that Γ is bipartite. In the case $(i_1, i_2, i_3) = (3, 0, 4)$, $(3, 4, 3)$ or $(4, 3, 4)$, similarly.

Let $(i_1, i_2, i_3) = (0, 3, 4)$. Note that $k_s = 2k_{s+1} = k_{s+2}$ by counting, and that $p^{s+3}_{1,s+1}$, $p^{s+3}_{1,s+2}$, $p^{s+4}_{1,s+2}$ are all nonzero. We easily have $k_{s+1} = k_{s+3} \cdot p^{s+3}_{1,s+1}$ and $k_{s+2} = k_{s+3} \cdot p^{s+3}_{1,s+2}$, which imply that $(p^{s+3}_{1,s+1}, p^{s+3}_{1,s+2}) = (1, 2)$. Hence, by Lemma 2.8 (2), $p^{s+4}_{1,s+2} \geq 2$. It follows from Lemma 2.2 that Γ is bipartite. In the case $(i_1, i_2, i_3) = (3, 3, 4)$, similarly. □

5. Type III

In this section we show the following lemma.

Lemma 5.1 *It is impossible that $\mathcal{X}$ is of type* III.

Note that $\mathcal{X}$ is of type III iff $p^{s+1}_{1,s} = p^{s+1}_{1,s+1} = p^{s+1}_{1,s+3} = p^{s+2}_{1,s} = p^{s+2}_{1,s+3} = p^{s+2}_{1,s+4} = 1$.

By the same argument as in the proof of Sublemma 4.2.1, we easily see that M_s, M_{s+1}, M_{s+2} exist, and can be written as follows, respectively.

	y_1	β	y_2	y_3
x_1	1	1	3	0
α	1	0	2	-1
x_2	3	2	4	0
x_3	0	-1	0	-2

	y_1	β	y_2	y_3
x_1	0	1	3	1
α	1	1	3	0
x_2	3	3	1	2
x_3	1	0	2	-1

	y_1	β	y_2	y_3
x_1	3	3	2	1
α	3	2	4	0
x_2	2	4	ε	2
x_3	1	0	2	-1

By these, we see that $p^{s+3}_{1,s+1} = p^{s+3}_{1,s+2} = p^{s+3}_{1,s+3} = 1$ and $p^{s+4}_{1,s+2} \geq 2$. As $p^{s+2}_{1,s+2} = 0$, it follows from Lemma 2.2 (2) that $p^{s+4}_{1,s+4} = 0$, so $\varepsilon = 2$ or 5.

Lemma 5.2 *Suppose that X is of type* III. *Then the following hold.*
(1) $E^0_1 = \{(-1, 0), (1, 1), (2, 3)\}$.
(2) $E^0_2 = \{(-1, 0), (1, 3), (2, 4)\}$.
(3) $E^1_1 = \{(0, 1), (1, 0), (3, 3)\}$.
(4) $E^2_3 = \{(0, 1), (3, 3), (4, 2)\}$.
(5) $E^1_3 = \{(0, 2), (1, 3), (3, 1)\}$.
(6) $E^3_3 = \{(1, 1), (2, 3), (3, 2)\}$.
(7) $E^2_4 = \begin{cases} \{(0, 2), (3, 2), (4, 2)\} & \textit{if } p^{s+4}_{1,s+2} = 3, \\ \{(0, 2), (3, 2), (4, 5)\} & \textit{if } p^{s+4}_{1,s+2} = p^{s+4}_{1,s+5} + 1 = 2. \end{cases}$

Proof: Similar to the proof of Sublemma 4.2.2. □

Proof of Lemma 5.1: We consider two cases; $\varepsilon = 2$ or 5.

Case 1. $\varepsilon = 5$, i.e., $p^{s+4}_{1,s+2} = 2$ and $p^{s+4}_{1,s+5} = 1$.

Let $C = \{x_0, x_1, x_2, \ldots, x_{2s+4}, x_{2s+5} = x_0\}$ be a circuit of length $2s + 5$ in Γ such that the profile of C with respect to (x_0, x_1) is as follows:

$$\begin{array}{ccccccc} 0 & 1 & 1 & 3 & 2 & 0 & \\ & 0 & 1 & 3 & 3 & 1 & 0. \end{array}$$

Note that the existence of such a circuit is guaranteed by Lemma 5.2 (1), (3), (6) and (4).
By using circuit chasing technique, the profile of C with respect to $(x_0, x_1), (x_1, x_2), \ldots, (x_5, x_6)$ are tabulated as follows:

$$\begin{array}{ccccccccccccc} 0 & 1 & 1 & 3 & 2 & 0 & & & & & & & \\ & 0 & 1 & 3 & 3 & 1 & 0 & & & & & & \\ & & 0 & 2 & 3 & 1 & 1 & 0 & & & & & \\ & & & 0 & 1 & 3 & 3 & 2 & 0 & & & & \\ & & & & 0 & 2 & 3 & 3 & 1 & 0 & & & \\ & & & & & 0 & 1 & 1 & 3 & 2 & 0 & & \\ & & & & & & 0 & 1 & 3 & 3 & 1 & 0, & \end{array}$$

where the ith and $(i+1)$th rows indicate the profile of C with respect to (x_{i-1}, x_i). We see that the profile with respect to (x_5, x_6) is the same as the one with respect to (x_0, x_1). It follows that $2s+5 \equiv 0 \pmod 5$, i.e., $s \equiv 0 \pmod 5$.

Let $C' = \{y_0, y_1, y_2, \ldots, y_{2s+6}, y_{2s+7} = y_0\}$ be a circuit of length $2s+7$ in Γ such that the profile of C' with respect to (y_0, y_1) is as follows:

$$\begin{array}{ccccccccc} 0 & 2 & 3 & 3 & 2 & 4 & 2 & 0 & \\ & 0 & 1 & 1 & 0 & 2 & 3 & 1 & 0. \end{array}$$

Then we see that the profile with respect to (y_5, y_6) is the same as the one with respect to (y_0, y_1). This implies that $2s+7 \equiv 0 \pmod 5$, i.e., $s \equiv -1 \pmod 5$, a contradiction. Now we have the assertion in Case 1.

Case 2. $\varepsilon = 2$, i.e., $p^{s+4}_{1,s+2} = 3$.

Let $C = \{x_0, x_1, x_2, \ldots, x_{2s+5}, x_{2s+6} = x_0\}$ be a circuit of length $2s+6$ in Γ such that the profile of C with respect to (x_0, x_1) is as follows:

$$\begin{array}{cccccccc} 0 & 1 & 3 & 2 & 4 & 2 & 0 & \\ & 0 & 2 & 4 & 2 & 3 & 1 & 0. \end{array}$$

Note that $x_{s+2} \in D^{s+2}_{s}(x_1, x_2)$. By Lemma 5.2 (2) and (7), we see that $x_{s+3} \in D^{s+4}_{s+2}(x_1, x_2)$, and $x_{s+4} \in D^{s+2}_{s+3}(x_1, x_2) \cup D^{s+2}_{s+4}(x_1, x_2)$. Suppose $x_{s+4} \in D^{s+2}_{s+4}(x_1, x_2)$. Then $\{x_0, x_2\} \subset P^{s+2}_{1,s+4}(x_1, x_{s+4})$. However, this contradicts that $p^{s+2}_{1,s+4} = 1$. Hence $x_{s+4} \in D^{s+2}_{s+3}(x_1, x_2)$. Thus, by using circuit chasing technique, we find the profile with respect to (x_4, x_5) is the same as one with respect to (x_0, x_1). Hence $2s+6 \equiv 0 \pmod 4$, i.e., $s \equiv 1 \pmod 2$.

Let $C' = \{y_0, y_1, y_2, \ldots, y_{2s+3}, y_{2s+4} = y_0\}$ be a circuit of length $2s+4$ in Γ such that the profile of C' with respect to (y_0, y_1) is as follows:

$$\begin{array}{cccccc} 0 & 2 & 4 & 2 & 0 & \\ & 0 & 2 & 3 & 1 & 0. \end{array}$$

By using circuit chasing technique, we easily see that the profile with respect to (y_2, y_3) is as follows.

$$\begin{array}{cccccc} 0 & 1 & 3 & 2 & 0 & \\ & 0 & 2 & 4 & 2 & 0. \end{array}$$

Note that $y_{s+4} \in D^{s+3}_{s+2}(y_2, y_3)$ and $y_{s+5} \in D^{s+2}_{s+4}(y_2, y_3)$. Let $z_1 \in \Gamma(y_{s+5}) \cap D^{s+4}_{s+2}(y_2, y_3)$ and $z_2 \in \Gamma(z_1) \cap D^{s+2}_{s+3}(y_2, y_3)$. Let $\{z_2, z_3, \ldots, z_{s+3} = y_2\}$ be the shortest path between z_2 and y_2. Then, for the circuit $\{y_2, y_3, \ldots, y_{s+5}, z_1, z_2, \ldots, z_{s+3} = y_2\}$, the profile with respect to (y_2, y_3) is the same as the one of C with respect to (x_0, x_1). Hence we have $z_1 \in D^{s+2}_{s+3}(y_3, y_4)$. As $y_{s+5} \in D^{s+4}_{s+2}(y_3, y_4)$, we have $y_{s+6} \in D^{s+2}_{s+4}(y_3, y_4)$. Thus the profile of C' with respect to (y_3, y_4) is as follows:

$$\begin{array}{cccccc} 0 & 2 & 4 & 2 & 0 & \\ & 0 & 2 & 4 & 2 & 0. \end{array}$$

Note that $y_{s+5} \in D^{s+4}_{s+2}(y_3, y_4)$, $y_{s+6} \in D^{s+2}_{s+4}(y_3, y_4)$ and, by Lemma 5.2 (2), $y_{s+6} \in D^{s+4}_{s+2}(y_4, y_5)$. Let $u_1 \in \Gamma(y_{s+6}) \cap D^{s+3}_{s+2}(y_3, y_4)$, and let $\{u_1, u_2, \ldots, u_{s+1}, y_4\}$ be the shortest path between u_1 and y_4. Note that $u_{s+1} \neq y_5$ as $\hat{c}_{s+2} = 1$. Now we see that, the profile of the circuit $\{y_3, y_4, u_{s+1}, \ldots, u_1, y_{s+6}, y_{s+7}, \ldots, y_3\}$ with respect to (y_3, y_4) is the same as the one of C' with respect to (y_2, y_3). Hence we have $y_{s+7} \in D^{s+2}_{s+4}(y_4, u_{s+1})$. Since $(y_4, y_{s+7}) \in R_{s+2}$, we have $(y_5, y_{s+7}) \in R_{s+3}$. Thus the profile of C' with respect to (y_4, y_5) is as follows:

$$\begin{array}{cccccc} 0 & 2 & 4 & 2 & 0 & \\ & 0 & 2 & 3 & 1 & 0, \end{array}$$

and we find that this is the same as the one with respect to (y_0, y_1). Therefore, we have $2s + 4 \equiv 0 \pmod 4$, i.e., $s \equiv 0 \pmod 2$. This is a contradiction. Now we have the assertion in Case 2. □

6. Type IV

In this section we show the following.

Lemma 6.1 *It is impossible that $\mathcal{X}$ is of type* IV.

Note that $\mathcal{X}$ is of type IV iff $p^{s+1}_{1,s} = p^{s+1}_{1,s+1} = p^{s+1}_{1,s+2} = p^{s+2}_{1,s} = p^{s+2}_{1,s+1} = p^{s+2}_{1,s+3} = 1$. We easily see that M_s, M_{s+1}, M_{s+2} exist, and can be written as follows, respectively.

	y_1	β	y_2	y_3
x_1	2	1	1	0
α	1	0	2	−1
x_2	1	2	3	0
x_3	0	−1	0	−2

	y_1	β	y_2	y_3
x_1	1	1	0	2
α	1	1	2	0
x_2	0	2	3	1
x_3	2	0	1	−1

	y_1	β	y_2	y_3
x_1	0	1	2	1
α	1	2	3	0
x_2	2	3	ε	2
x_3	1	0	2	−1

Note that $p^{s+3}_{1,s+2} \geq 2$. As $p^{s+2}_{1,s+2} = 0$, it follows from Lemma 2.2 (2) that $p^{s+3}_{1,s+3} = 0$, so that $\varepsilon = 2$ or 4.

Lemma 6.2 *Suppose that $\mathcal{X}$ is of type* IV. *Then the following hold.*
(1) $E^0_1 = \{(-1, 0), (1, 2), (2, 1)\}$.
(2) $E^0_2 = \{(-1, 0), (1, 1), (2, 3)\}$.
(3) $E^1_2 = \{(0, 1), (1, 0), (2, 3)\}$.
(4) $E^1_1 = \{(0, 2), (1, 1), (2, 0)\}$.

(5) $E_3^2 = \begin{cases} \{(0,2),(1,2),(3,2)\} & \textit{if } p^{s+3}_{1,s+2} = 3, \\ \{(0,2),(1,2),(3,4)\} & \textit{if } p^{s+3}_{1,s+2} = p^{s+3}_{1,s+4} + 1 = 2. \end{cases}$

Proof: Immediate from M_s, M_{s+1} and M_{s+2}. □

Proof of Lemma 6.1: We consider two cases; $\varepsilon = 2$ or 4.

Case 1. $\varepsilon = 4$, i.e., $p^{s+3}_{1,s+2} = 2$ and $p^{s+3}_{1,s+4} = 1$.

By applying circuit chasing technique to two circuits having the following profiles:

$$\begin{array}{ccccc} 0 & 1 & 1 & 0 & \\ & 0 & 2 & 1 & 0 \end{array}$$

and

$$\begin{array}{cccccc} 0 & 1 & 1 & 2 & 0 & \\ & 0 & 2 & 3 & 2 & 0, \end{array}$$

we have $2s + 3 \equiv 0 \pmod 3$ and $2s + 4 \equiv 0 \pmod 3$, respectively, a contradiction.

Case 2. $\varepsilon = 2$, i.e., $p^{s+3}_{1,s+2} = 3$.

Let $C_1 = \{x_0, x_1, x_2, \ldots, x_{2s+4}, x_{2s+5} = x_0\}$ be a circuit of length $2s + 5$ in Γ such that the profile of C_1 with respect to (x_0, x_1) is as follows:

$$\begin{array}{ccccccc} 0 & 2 & 3 & 2 & 1 & 0 & \\ & 0 & 2 & 3 & 2 & 1 & 0. \end{array}$$

Note that $x_{s+3} \in D^{s+3}_{s+2}(x_1, x_2)$. By Lemma 6.2 (5), we see that $x_{s+4} \in D^{s+2}_{s+1}(x_1, x_2) \cup D^{s+2}_{s+3}(x_1, x_2)$. Suppose $x_{s+4} \in D^{s+2}_{s+1}(x_1, x_2)$. Then we see that $\{x_0, x_2\} \subset P^{s+2}_{1,s+1}(x_1, x_{s+4})$, which contradicts that $p^{s+2}_{1,s+1} = 1$. Hence we have $x_{s+4} \in D^{s+2}_{s+3}(x_1, x_2)$. Then the profile with respect to (x_1, x_2) is

$$\begin{array}{ccccccc} 0 & 2 & 3 & 2 & 1 & 0 & \\ & 0 & 2 & 3 & 2 & 1 & 0, \end{array}$$

which is the same as the one with respect to (x_0, x_1).

Similarly, if $C_2 = \{x'_0, x'_1, \ldots, x'_{2s+4}, x'_{2s+5} = x'_0\}$ is a circuit of length $2s + 5$ such that the profile with respect to (x'_0, x'_1) is

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 2 & 0 & \\ & 0 & 1 & 2 & 3 & 2 & 0, \end{array}$$

then the one with respect to (x_1', x_2') is also as above.

Let $C_3 = \{y_0, y_1, y_2, \ldots, y_{2s+3}, y_{2s+4} = y_0\}$ be a circuit of length $2s + 4$ in Γ such that the profile with respect to (y_0, y_1) is as follows:

$$\begin{array}{cccccc} 0 & 2 & 3 & 2 & 0 & \\ & 0 & 2 & 3 & 2 & 0. \end{array}$$

Note that $y_{s+3} \in D^{s+2}_{s+3}(y_0, y_1) \cap D^{s+3}_{s+2}(y_1, y_2)$. Let $u_1 \in \Gamma(y_{s+3}) \cap D^{s+1}_{s+2}(y_0, y_1)$ and $u_2 \in \Gamma(u_1) \cap D^{s}_{s+1}(y_0, y_1)$. Let $\{u_2, u_3, \ldots, u_{s+1}, y_0\}$ be the shortest path between u_2 and y_0. Then the profile of the circuit $\{y_0, y_1, \ldots, y_{s+3}, u_1, u_2, \ldots, u_{s+1}, y_0\}$ with respect to (y_0, y_1) is the same as of C_1 with respect to (x_0, x_1). Hence we have $u_1 \in D^{s+2}_{s+3}(y_1, y_2)$, so that $y_{s+4} \in D^{s+2}_{s+1}(y_1, y_2)$. Thus the profile of C_3 with respect to (y_1, y_2) is

$$\begin{array}{cccccc} 0 & 2 & 3 & 2 & 0 & \\ & 0 & 2 & 1 & 1 & 0. \end{array}$$

By Lemma 6.2, we easily see that the profiles of C_3 with respect to (y_2, y_3), (y_3, y_4) are the following:

$$\begin{array}{ccccccc} 0 & 2 & 1 & 1 & 0 & & \\ & 0 & 1 & 1 & 2 & 0 & \\ & & 0 & 2 & 3 & 2 & 0. \end{array}$$

Note that $y_{s+6} \in D^{s+2}_{s+3}(y_3, y_4) \cap D^{s+3}_{s+2}(y_4, y_5)$. Let $v_1 \in \Gamma(y_{s+6}) \cap D^{s+3}_{s+2}(y_3, y_4)$, and let $\{v_1, v_2, \ldots, v_{s+1}, y_4\}$ be the shortest path between v_1 and y_4. Note that $\partial(y_5, y_{s+6}) = \partial(v_{s+1}, y_{s+6}) = s + 1$. As $c_{s+1} = 1$, we have $y_5 \neq v_{s+1}$. Now we can find that the profile of the circuit $\{y_3, y_4, v_{s+1}, \ldots, v_1, y_{s+6}, \ldots, y_3\}$ with respect to (y_3, y_4) is the same as of C_3 with respect to (y_0, y_1). Hence we have $\hat{\partial}(v_{s+1}, y_{s+7}) = s + 1$. This implies that $\hat{\partial}(y_5, y_{s+7}) = s + 3$, i.e., $y_{s+7} \in D^{s+2}_{s+3}(y_4, y_5)$. Thus the profile of C_3 with respect to (y_4, y_5) is

$$\begin{array}{cccccc} 0 & 2 & 3 & 2 & 0 & \\ & 0 & 2 & 3 & 2 & 0, \end{array}$$

which is the same as the one with respect to (y_0, y_1). Therefore $2s + 4 \equiv 0 \pmod 4$, i.e., $s \equiv 0 \pmod 2$.

Let $C_4 = \{z_0, z_1, \ldots, z_{2s+5}, z_{2s+6}\}$ be a circuit of length $2s + 6$ such that the profile with respect to z_0, z_1 is

$$\begin{array}{cccccccc} 0 & 1 & 1 & 2 & 3 & 2 & 0 & \\ & 0 & 2 & 3 & 2 & 1 & 1 & 0. \end{array}$$

Note that $z_{s+3} \in D^{s+2}_{s+3}(z_0, z_1) \cap D^{s+3}_{s+2}(z_1, z_2)$. Let $\{z_{s+3}, u_1, \ldots, u_s, z_0\}$ be the shortest path between z_{s+3} and z_0. Then the profile of the circuit $\{z_0, z_1, \ldots, z_{s+3}, u_1, \ldots, z_0\}$

with respect to (z_0, z_1) is the same as of C_3 with respect to (y_3, y_4). Hence we see that $u_1 \in D^{s+2}_{s+3}(z_1, z_2)$, so that $z_{s+4} \in D^{s+2}_{s+1}(z_1, z_2)$. Thus, by Lemma 6.2, the profile of C_4 with respect to (z_1, z_2) is

$$\begin{array}{cccccccc} 0 & 2 & 3 & 2 & 1 & 1 & 0 & \\ & 0 & 2 & 1 & 0 & 2 & 1 & 0. \end{array}$$

By Lemma 6.2, we immediately have that the profile of C_4 with respect to (z_2, z_3) is

$$\begin{array}{cccccccc} 0 & 2 & 1 & 0 & 2 & 1 & 0 & \\ & 0 & 1 & 2 & 3 & 2 & 1 & 0. \end{array}$$

Note that $z_{s+5} \in D^{s+2}_{s+1}(z_3, z_4)$ and $z_{s+6} \in D^{s+2}_{s+3}(z_2, z_3) \cap D^{s+3}_{s+2}(z_3, z_4)$. Let $v_1 \in \Gamma(z_{s+6}) \cap D^{s+3}_{s+2}(z_2, z_3)$, and let $\{v_1, \ldots, v_{s+1}, z_3\}$ be the shortest path between v_1 and z_3. Let $\{z_{s+5}, w_1, w_2, \ldots, w_{s-1}, z_2\}$ be the shortest path between z_{s+5} and z_2. Then the profile of the circuit $\{z_2, z_3, v_{s+1}, \ldots, v_1, z_{s+6}, z_{s+5}, w_1, \ldots, w_{s-2}, z_2\}$ with respect to (z_2, z_3) is the same as of C_3 with respect to (y_0, y_1). Hence $\hat{\partial}(v_{s+1}, z_{s+5}) = s + 1$. But, as $\hat{\partial}(z_4, z_{s+5}) = s + 1$, we have $z_4 = v_{s+1}$ and $\hat{\partial}(z_4, v_1) = s$. Therefore we have $v_1 \in D^{s+2}_{s}(z_3, z_4)$, so that $z_{s+7} \in D^{s+2}_{s+3}(z_3, z_4)$. Thus, by Lemma 6.2, the profile of C_4 with respect to (z_3, z_4) is

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 2 & 1 & 0 & \\ & 0 & 1 & 2 & 3 & 2 & 1 & 0. \end{array}$$

Note that $z_{s+6} \in D^{s+2}_{s+1}(z_4, z_5)$ and $z_{s+7} \in D^{s+2}_{s+3}(z_3, z_4) \cap D^{s+3}_{s+2}(z_4, z_5)$. Let $\{z_{s+7}, \delta_1, \ldots, \delta_{s-1}, z_3\}$ be the shortest path between z_{s+7} and z_3. Then the profile of the circuit $\{z_3, z_4, z_{s+7}, \delta_1, \ldots, z_3\}$ with respect to (z_3, z_4) is the same as of C_2 with respect to (x'_0, x'_1). Hence $\delta_1 \in D^{s+2}_{s+3}(z_4, z_5)$, so that $z_{s+8} \in D^{s+2}_{s}(z_4, z_5)$. Thus, by Lemma 6.2, the profile of C_4 with respect to (z_4, z_5) is

$$\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 2 & 1 & 0 & \\ & 0 & 1 & 2 & 0 & 1 & 2 & 0. \end{array}$$

By Lemma 6.2, we immediately have that the profile of C_4 with respect to (z_5, z_6), (z_6, z_7) are

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 0 & 1 & 2 & 0 & & \\ & 0 & 1 & 1 & 2 & 3 & 2 & 0 & \\ & & 0 & 2 & 3 & 2 & 1 & 1 & 0. \end{array}$$

Note that the profile of C_4 with respect to (z_6, z_7) is the same as the one with respect to (z_0, z_1).

Now we shall show $s \equiv 1 \pmod 2$, and induce a contradiction. For the circuit C_4, we see that $z_0 \in D^{s+2}_{s+3}(z_{s+3}, z_{s+4})$ and $z_1 \in D^{s+3}_{s+2}(z_{s+3}, z_{s+4})$. Hence we easily have that the profile of C_4 with respect to (z_{s+3}, z_{s+4}) is the same as the one with respect to (z_0, z_1). Thus $s + 3 \equiv 0 \pmod 6$, i.e., $s \equiv 1 \pmod 2$.

Now we have the assertion in Case 2. □

7. Type V

If $\mathcal{X}$ is of type V, then we easily see the following.

(1) For every $\alpha \in X$, $\Gamma_{s+1}(\alpha) = R_{s+1}(\alpha) \cup R_{s+2}(\alpha)$ and $\Gamma_{s+2}(\alpha) = R_{s+3}(\alpha) \cup R_{s+4}(\alpha) \cup R_{s+5}(\alpha)$.
(2) c_{s+1}, a_{s+1} and b_{s+1} exist and $(c_{s+1}, a_{s+1}, b_{s+1}) = (1, 0, 2)$.
(3) $p^{s+4}_{1,s+1} = p^{s+4}_{1,s+2} = 1$.
(4) $k_s = k_{s+1} = k_{s+2} = k_{s+4}$.
(5) $p^{s+4}_{1,s+4} = 0$.

Note that (5) follows from Lemma 2.2 (1).

We separate this type into four cases as follows:

(VA) $p^{s+4}_{1,s+3} \neq 0$ or $p^{s+4}_{1,s+5} \neq 0$;
(VB) $p^{s+4}_{1,s+3} = p^{s+4}_{1,s+5} = 0$ and $p^{s+3}_{1,s+5} \neq 0$;
(VC) $p^{s+4}_{1,s+3} = p^{s+4}_{1,s+5} = p^{s+3}_{1,s+5} = 0$, $p^{s+3}_{1,s+1} \geq 2$ and $p^{s+5}_{1,s+2} \geq 2$;
(VD) $p^{s+4}_{1,s+3} = p^{s+4}_{1,s+5} = p^{s+3}_{1,s+5} = 0$ and $p^{s+3}_{1,s+1} = 1$ or $p^{s+5}_{1,s+2} = 1$.

Firstly, we consider the type VA. By the symmetry, we may assume that $p^{s+4}_{1,s+3} = 1$. Then we easily have that $k_{s+1} = k_{s+4} = p^{s+3}_{1,s+1} \cdot k_{s+3}$, and that $k_{s+4} = p^{s+3}_{1,s+4} \cdot k_{s+3}$. As $k_1 = 3$, we have $p^{s+3}_{1,s+1} = p^{s+3}_{1,s+4} = 1$. Thus we see that $M_{s+1}, M_{s+2}, M_{s+3}$ exist, and can be written as follows, respectively.

	y_1	β	y_2	y_3
x_1	4	3	3	1
α	3	1	4	0
x_2	3	4	1	2
x_3	1	0	2	−1

	y_1	β	y_2	y_3
x_1	3	4	2	1
α	4	2	5	0
x_2	2	5	ε	2
x_3	1	0	2	−1

	y_1	β	y_2	y_3
x_1	3	3	1	4
α	3	3	4	1
x_2	1	4	2	3
x_3	4	1	3	0

Note that $p^{s+5}_{1,s+2} \geq 2$, and $\varepsilon = 2$ or 6.

Lemma 7.1 *Suppose that $\mathcal{X}$ is of type* VA. *Then the following hold.*
(1) $E^0_1 = \{(-1, 0), (1, 3), (2, 4)\}$.
(2) $E^0_2 = \{(-1, 0), (1, 4), (2, 5)\}$.
(3) $E^1_3 = \{(0, 1), (3, 4), (4, 3)\}$.
(4) $E^2_4 = \{(0, 1), (4, 3), (5, 2)\}$.

(5) $E_4^1 = \{(0,2),(3,3),(4,1)\}$.
(6) $E_4^3 = \{(1,3),(3,1),(4,2)\}$.
(7) $E_3^3 = \{(1,4),(3,3),(4,1)\}$.
(8) $E_5^2 = \begin{cases} \{(0,2),(4,2),(5,2)\} & \textit{if } p^{s+5}_{1,s+2} = 3, \\ \{(0,2),(4,2),(5,6)\} & \textit{if } p^{s+5}_{1,s+2} = p^{s+5}_{1,s+6} + 1 = 2. \end{cases}$
(9) *Let* $p^{s+5}_{1,s+2} = p^{s+5}_{1,s+6} + 1 = 2$. *Then*

$$E_6^5 = \begin{cases} \{(2,5),(6,5)\} & \textit{if } p^{s+6}_{1,s+5} = 3, \\ \{(2,5),(6,7)\} & \textit{if } p^{s+6}_{1,s+5} = p^{s+6}_{1,s+7} + 1 = 2. \end{cases}$$

Moreover, for any pair of adjacent vertices (u, v) *and any* $x \in D^{s+5}_{1,s+6} = D^{s+5}_{1,s+6}(u, v)$, $e(x, D^{s+2}_{1,s+5}) = 2$.

Proof: Immediate from M_{s+1}, M_{s+2} and M_{s+3}. □

Lemma 7.2 *It is impossible that* $\mathcal{X}$ *is of type* VA.

Proof: We consider the following three cases.

Case 1. $p^{s+5}_{1,s+6} = p^{s+6}_{s+7} = 1$ and $p^{s+5}_{1,s+2} = p^{s+6}_{1,s+5} = 2$.

By applying circuit chasing technique to two circuits having the following profiles:

$$\begin{array}{ccccccc} 0 & 1 & 3 & 4 & 2 & 0 & \\ & 0 & 1 & 3 & 4 & 1 & 0 \end{array}$$

and

$$\begin{array}{cccccccc} 0 & 2 & 4 & 3 & 3 & 1 & 0 & \\ & 0 & 1 & 3 & 3 & 4 & 2 & 0, \end{array}$$

we have $2s + 5 \equiv 0 \pmod 5$ and $2s + 6 \equiv 0 \pmod 5$, respectively, a contradiction.

Case 2. $p^{s+5}_{1,s+2} = 2$, $p^{s+5}_{1,s+6} = 1$ and $p^{s+6}_{1,s+5} = 3$.

Let $C = \{x_0, x_1, x_2, \ldots, x_{2s+3}, x_{2s+4} = x_0\}$ be a circuit of length $2s + 4$ in Γ such that the profile of C with respect to (x_0, x_1) is as follows:

$$\begin{array}{cccccc} 0 & 1 & 4 & 2 & 0 & \\ & 0 & 2 & 5 & 2 & 0. \end{array}$$

Then we see that the profile with respect to (x_3, x_4) is the same as one with respect to (x_0, x_1). It follows that $2s + 4 \equiv 0 \pmod 3$, i.e., $s \equiv 1 \pmod 3$.

Let $C' = \{y_0, y_1, y_2, \ldots, y_{2s+7}, y_{2s+8} = y_0\}$ be a circuit of length $2s+8$ in Γ such that the profile of C' with respect to (y_0, y_1) is as follows:

$$\begin{array}{cccccccccc} 0 & 1 & 4 & 2 & 5 & 6 & 5 & 2 & 0 & \\ & 0 & 2 & 5 & 6 & 5 & 2 & 4 & 1 & 0. \end{array}$$

By Lemma 7.1, $y_{s+3} \in D^{s+5}_{s+2}(y_1, y_2)$, $y_{s+4} \in D^{s+6}_{s+5}(y_1, y_2)$, and $y_{s+5} \in D^{s+5}_{s+2}(y_1, y_2) \cup D^{s+5}_{s+6}(y_1, y_2)$. Suppose that $y_{s+5} \in D^{s+5}_{s+6}(y_1, y_2)$. Then $\{y_0, y_2\} \subset P^{s+5}_{1,s+6}(y_1, y_{s+5})$, which contradicts that $p^{s+5}_{1,s+6} = 1$. Hence $y_{s+6} \in D^{s+5}_{s+2}(y_1, y_2)$, and the profile is

$$\begin{array}{cccccccccc} 0 & 2 & 5 & 6 & 5 & 2 & 4 & 1 & 0 & \\ & 0 & 2 & 5 & 2 & 0 & 1 & 4 & 2 & 0 \end{array}$$

or

$$\begin{array}{cccccccccc} 0 & 2 & 5 & 6 & 5 & 2 & 4 & 1 & 0 & \\ & 0 & 2 & 5 & 2 & 4 & 3 & 3 & 1 & 0. \end{array}$$

By using circuit chasing technique, we see that the profile with respect to (y_6, y_7) is the same as the one with respect to (y_0, y_1) in both cases. Thus we have $2s + 8 \equiv 0 \pmod 6$, i.e., $s \equiv 2 \pmod 3$. This is a contradiction. Now we have the assertion in Case 2.

Case 3. $p^{s+5}_{1,s+2} = 3$.

In this case, the proof is similar to the one of Case 2 of Lemma 5.1. Indeed, by applying circuit chasing technique to two circuits having the following profile:

$$\begin{array}{cccccccc} 0 & 1 & 4 & 2 & 5 & 2 & 0 & \\ & 0 & 2 & 5 & 2 & 4 & 1 & 0 \end{array}$$

and

$$\begin{array}{cccccc} 0 & 2 & 5 & 2 & 0 & \\ & 0 & 2 & 4 & 1 & 0, \end{array}$$

we have $2s + 6 \equiv 0 \pmod 4$ and $2s + 4 \equiv 0 \pmod 4$, respectively, a contradiction. □

Consider the case VB. Let $p^{s+4}_{1,s+6} \neq 0$ with $\varphi(s+6) = s+3$.

Lemma 7.3 *It is impossible that $\mathcal{X}$ is of type* VB.

Proof: By counting, we easily have $k_{s+1} = k_{s+3} \cdot p^{s+3}_{1,s+1}$, $k_{s+2} = k_{s+5} \cdot p^{s+5}_{1,s+2}$ and $k_{s+3} \cdot p^{s+3}_{1,s+5} = k_{s+5} \cdot p^{s+5}_{1,s+3}$. These imply that $p^{s+3}_{1,s+1} = p^{s+5}_{1,s+2}$ and $p^{s+3}_{1,s+5} = p^{s+5}_{1,s+3}$. Suppose that $p^{s+3}_{1,s+1} = p^{s+5}_{1,s+2} = 2$. Then we have a contradiction by Lemma 2.2 (1). Hence $p^{s+3}_{1,s+1} = p^{s+5}_{1,s+2} = 1$. It follows from Lemma 2.8 (1) that $p^{s+6}_{1,s+3} = p^{s+6}_{1,s+4} = p^{s+6}_{1,s+5} = 1$. Note that $R_{s+6}(\alpha) = \Gamma_{s+3}(\alpha)$ for every $\alpha \in X$, and that $a_{s+3} = \hat{a}_{s+6} = 0$. However, this is a contradiction from Lemma 2.3. Now we have the assertion. □

In regard to the case VC, we note that c_{s+2} exists and $c_{s+2} = 2$. Thus, from Lemma 2.1 (3) and Lemma 2.2, we see the following lemma.

Lemma 7.4 *Suppose that $\mathcal{X}$ is of type* VC. *Then Γ is bipartite.*

Suppose that $\mathcal{X}$ is of type VD.

Now, we change the indices of relations. Let $0, 1, \ldots, s+2$ be as above. Let $p^{s+1}_{1,s+3} = p^{s+1}_{1,d} = p^{s+2}_{1,s+3} = p^{s+2}_{1,s+4} = p^{s+4}_{1,s+2} = 1$. Note that $p^{s+3}_{1,s+1} = p^{s+3}_{1,s+2} = 1$, $k_{s+3} = k_{s+4} = p^{d}_{1,s+1} \cdot k_d$, $p^{s+3}_{1,s+4} = p^{s+3}_{1,d} = p^{s+4}_{1,d} = p^{s+3}_{1,s+3} = 0$, and that $\hat{b}_{s+3} \neq 0$.

Let $d^* = d - s$.

For the convenience, if $p^{s+j}_{1,s+i} \neq 0$, then we write

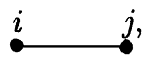

Figure 1.

and, in particular, if $p^{s+j}_{1,s+i} = 2, 3$, then we write

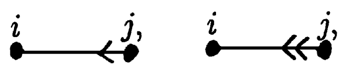

Figure 2.

respectively. If $p^{s+i}_{1,s+i} \neq 0$, we write

Figure 3.

We shall show the following.

Lemma 7.5 *Let $\mathcal{X}$ be of type* VD. *Then $\mathcal{X}$ is one of the following types.*

(i) $D_1(t)$ $\quad (6 \leq t \leq d^* - 2)$:

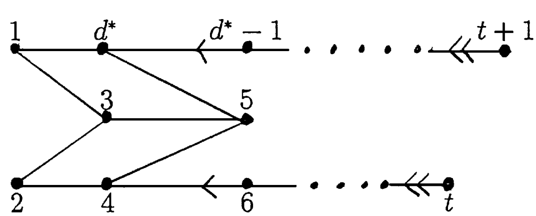

Figure 4.

(ii) $D_2(-1)$:

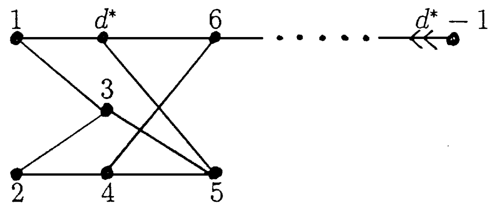

Figure 5.

(iii) $D_2(t)$ $(0 \leq t \leq d^*/2 - 5)$:

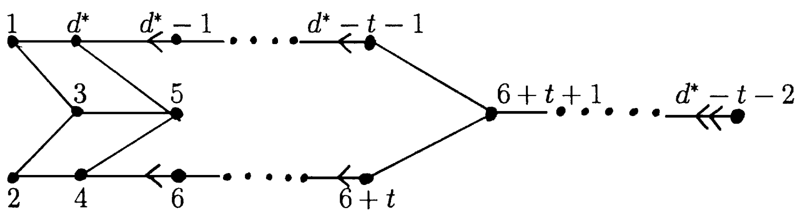

Figure 6.

(iv) D_3:

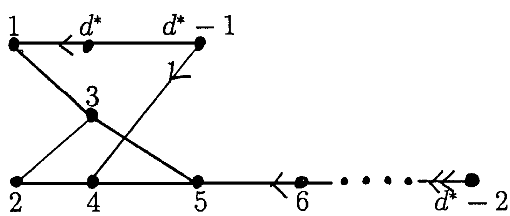

Figure 7.

(v) $D_4(t)$ $(1 \leq t \leq d^*/2 - 4)$:

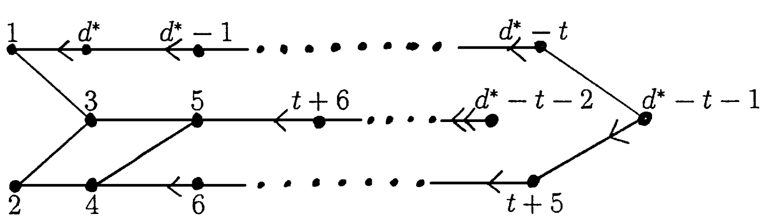

Figure 8.

(vi) $D_5(t_1, t_2)$ $(2 \le t_1 \le t_2 \le d^*/2 - 2)$:

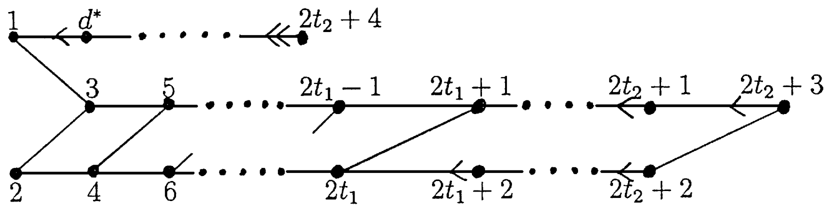

Figure 9.

(vii) $D_6(t)$ $(1 \le t \le d^*/2 - 2)$:

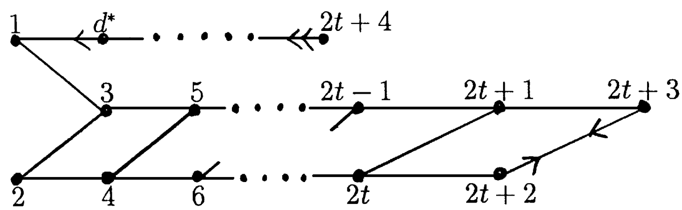

Figure 10.

(viii) $D_7(t_1, t_2, t_3)$ $(t_1 \ge 2, t_2 \ge 2t_1 + 2, t_2 + 1 \le t_3 \le d^* - 1)$:

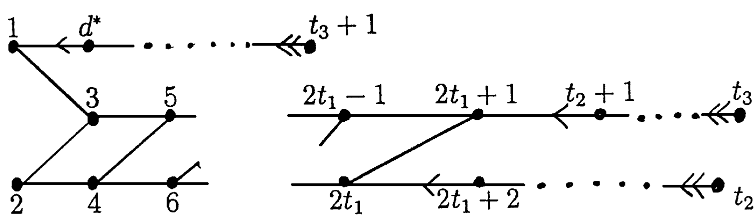

Figure 11.

(ix) $D_8(t)$ $(1 \le t \le t^*/2 - 2)$:

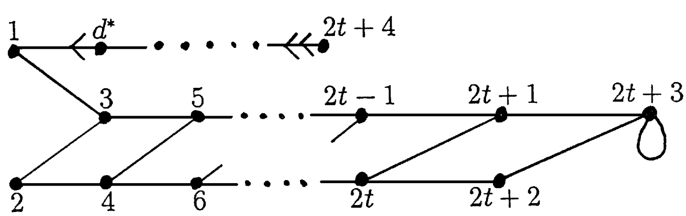

Figure 12.

(x) $D_9(t)$ $\quad (2 \leq t \leq (d^* - 3)/2)$:

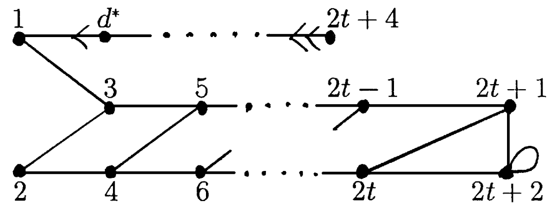

Figure 13.

In particular, in the case (i)–(viii) Γ *is bipartite, and in the case* (ix), (x) Γ *is non-bipartite.*

Proof: As $\hat{b}_{s+3} \neq 0$, let $p^{s+3}_{1,s+5} = 1$ with $\varphi(s+5) = s+3$. As $\hat{c}_{s+4} = p^{s+4}_{1,s+2} = 1$, we see that $p^{s+4}_{1,s+5} \neq 0$ by Lemma 2.8 (1).

Firstly, we assume that $p^{d}_{1,s+1} = 1$. Then, by Lemma 2.8 (1), we have $p^{s+5}_{1,s+3} = p^{s+5}_{1,s+4} = p^{s+5}_{1,d} = 1$. Note that $k_{s+3} = k_{s+5}$. As $k_{s+4} = k_d = k_{s+3}$, we have $p^{s+4}_{1,s+5} = p^{d}_{1,s+5} = 1$. By Lemma 2.3 (2), we have $p^{s+4}_{1,s+4} = p^{d}_{1,s+4} = p^{d}_{1,d} = 0$, so that $\hat{b}_{s+4} = \hat{b}_d = 2$.

Let $p^{s+4}_{1,s+6} = 1$ with $\varphi(s+6) = s+3$. Let $p^{s+6}_{1,d} \neq 0$. Then, as $k_{s+4} = k_d$, $p^{s+4}_{1,s+6} = p^{d}_{1,s+6} = 1$, and as $k_1 = 3$, we have $p^{s+6}_{1,s+4} = p^{s+6}_{1,d} = 1$. Then $\mathcal{X}$ is of type $D_2(-1)$. Let $p^{d}_{1,d-1} = 1$ with $d-1 \neq s+6$ and with $\varphi(d-1) = s+3$. By Lemma 2.8 (2), we have $p^{s+6}_{1,s+4} \geq 2$ and $p^{d-1}_{1,d} \geq 2$. It follows from Lemma 2.1 (3) and Lemma 2.2 that Γ is bipartite. Note that, if $p^{s+6}_{1,s+4} = p^{d-1}_{1,d}$, then $k_{s+6} = k_{d-1}$. Let $p^{s+6}_{1,s+7} = 1$ with $\varphi(s+7) = s+4$. Suppose that $p^{s+7}_{1,d-1} = 1$. Then, as $k_{s+6} = k_{d-1}$ and $k_1 = 3$, we have $p^{s+7}_{1,d-1} = p^{s+7}_{1,s+6} = 1$. Thus in this case $\mathcal{X}$ is of type $D_2(0)$. Similarly, we see that, if $p^{d}_{1,s+1} = 1$, then $\mathcal{X}$ is of type $D_1(t)$ $(t \geq 6)$ or $D_2(t)$ $(t \geq -1)$.

Now assume that $p^{d}_{1,s+1} \geq 2$. Note that, $k_d < k_{s+3} = k_{s+4}$.

Let $p^{s+4}_{1,s+5} = 2$. Since $2k_{s+4} = p^{s+5}_{1,s+4} \cdot k_{s+5}$ and $k_{s+4} = k_{s+3} = p^{s+5}_{1,s+3} \cdot k_{s+5}$, we have $(p^{s+5}_{1,s+3}, p^{s+5}_{1,s+4}) = (1, 2)$. Note that $\hat{a}_{s+5} = 0$. Thus, by Lemma 2.2 (2), Γ is bipartite, and $\mathcal{X}$ is of type $D_6(1)$.

Let $p^{s+4}_{1,s+5} = 1$. Then we easily have $p^{s+5}_{1,s+3} = p^{s+5}_{1,s+4} = 1$ and $k_{s+3} = k_{s+4} = k_{s+5}$. As $p^{s+5}_{1,d} \leq 1$ and $2k_d \leq k_{s+1} = k_{s+5}$, we have $p^{s+5}_{1,d} = p^{d}_{1,s+5} = 0$. Hence $\hat{c}_{s+5} = p^{s+5}_{1,s+3} + p^{s+5}_{1,s+4} = 2$.

Let $p^{s+4}_{1,s+5} = p^{s+4}_{1,s+4} = 1$. Then $\hat{a}_{s+4} = p^{s+4}_{1,s+4} = 1$, and Γ is non-bipartite. By Lemma 2.3 (2), it must hold that $p^{s+5}_{1,s+5} = 1$. In this case $\mathcal{X}$ is of type $D_8(1)$.

Let $p^{s+4}_{1,s+5} = p^{s+4}_{1,s+6} = 1$ with $\varphi(s+6) = s+3$. Let $p^{d}_{1,s+6} \neq 0$. Then we easily have $p^{s+6}_{1,s+4} = 2$ and $p^{s+6}_{1,d} = p^{d}_{1,s+6} = 1$. Note that $\hat{a}_{s+6} = 0$, and that, by Lemma 2.2 (1), $\hat{a}_{s+5} = 0$. Thus, by Lemma 2.1 (3) and Lemma 2.2 (2), Γ is bipartite and $\mathcal{X}$ is of type D_3.

Let $p^{d}_{1,s+6} = 0$ and $p^{s+6}_{1,s+4} \geq 2$. Note that, if $p^{d}_{1,d-1} = 1$ with $\varphi(d-1) = s+3$, then $p^{d-1}_{1,d} \geq 2$ by Lemma 2.1 (3). Hence we see that $c_{s+3}(x, y) \geq 2$ for any vertices x, y with $\partial(x, y) = s+3$. Hence, by Lemma 2.2 (1), Γ becomes bipartite. In particular,

$p^{s+5}_{1,s+5} = p^{s+6}_{1,s+6} = 0$. Let $p^{s+5}_{1,s+7} = 1$ with $\varphi(s+7) = s+4$. If $p^{s+6}_{1,s+7} \neq 0$, then we easily have $(p^{s+7}_{1,s+5}, p^{s+7}_{1,s+6}) = (2,1)$, so that $\mathcal{X}$ is of type $D_5(2,2)$. Suppose that $p^{d-1}_{1,s+7} \neq 0$. Then we have

$$k_{s+5} \leq 2k_{s+7} \leq 2k_{d-1} = 4k_d = 2k_{s+5},$$

a contradiction. Let $p^{d-1}_{1,d-2} = 1$ with $\varphi(d-2) = s+4$. If $p^{d-2}_{1,s+6} \neq 0$, then we easily have $(p^{d-2}_{1,s+6}, p^{d-2}_{1,d-1}) = (2,1)$, and $\mathcal{X}$ is of type $D_4(1)$.

Let $p^{d}_{1,s+6} = 0$ and $p^{s+6}_{1,s+4} = 1$. By Lemma 2.8 (1), $p^{s+6}_{1,s+6} = p^{s+6}_{1,s+5} = p^{s+5}_{1,s+6} = 1$, or $p^{s+5}_{1,s+7} \neq 0$ and $p^{s+6}_{1,s+7} \neq 0$ with $\varphi(s+7) = s+4$. In particular, in the first case, $\mathcal{X}$ is of type $D_9(2)$.

Thus, by repeating the same argument as above, we have this lemma. □

In the following, for the simplification of indices, we replace the indices $D_8(t)$ and $D_9(t)$ with $D'_0(t)$ and $D'_1(t)$, respectively.

To complete the proof of Theorem 1.1, it remains to show the following two lemmas.

Lemma 7.6 *It is impossible that $\mathcal{X}$ is of type $D'_0(t)$ $(t \geq 1)$.*

Lemma 7.7 *It is impossible that $\mathcal{X}$ is of type $D'_1(t)$ $(t \geq 2)$.*

In order to show these two lemmas, we need the following two lemmas.

Lemma 7.8 *Suppose that $\mathcal{X}$ is of type $D'_f(t)$ $(f = 0$ or $1)$. Then the following hold.*

(1) $E^0_1 = \{(-1,0), (1,d^*), (2,3)\}$.

(2) $E^0_2 = \{(-1,0), (1,3), (2,4)\}$.

(3) $E^1_3 = \{(0,2), (3,5), (d^*,1)\}$.

(4) *For* $1 \leq i \leq t-f$,
$E^{2i}_{2i+1} = \{(2i-2, 2i-1), (2i+1, 2i), (2i+2, 2i+3)\}$.

(5) *For* $2 \leq i \leq t-f$,
$E^{2i-1}_{2i+1} = \{(2i-3, 2i-1), (2i-2, 2i), (2i+1, 2i+3)\}$.

(6) *For* $1 \leq i \leq t-f$,
$E^{2i}_{2i+2} = \{(2i-2, 2i), (2i+1, 2i+3), (2i+2, 2i+4)\}$.

(7) *If* $f = 0$, *then*
$E^{2t}_{2t+2} = \{(2t-2, 2t), (2t+1, 2t+3), (2t+2, 2t+2)\}$,
$E^{2t+2}_{2t+3} = \{(2t, 2t+1), (2t+2, 2t+3), (2t+3, 2t+2)\}$,
$E^{2t+1}_{2t+3} = \{(2t-1, 2t+1), (2t, 2t+2), (2t+3, 2t+3)\}$,
$E^{2t+2}_{2t+2} = \{(2t, 2t+2), (2t+2, 2t), (2t+3, 2t+3)\}$,
$E^{2t+3}_{2t+3} = \{(2t+1, 2t+3), (2t+2, 2t+2), (2t+3, 2t+1)\}$.

(8) *If* $f = 1$, *then*
$E^{2t}_{2t+2} = \{(2t-2, 2t), (2t+1, 2t+2), (2t+2, 2t)\}$,
$E^{2t}_{2t+1} = \{(2t-2, 2t-1), (2t+1, 2t), (2t+2, 2t+2)\}$,

$$E_{2t+1}^{2t-1} = \{(2t-3, 2t-1), (2t-2, 2t), (2t+1, 2t+2)\},$$
$$E_{2t+2}^{2t+2} = \{(2t, 2t+1), (2t+1, 2t), (2t+2, 2t+2)\},$$
$$E_{2t+2}^{2t+1} = \{(2t-1, 2t+1), (2t, 2t+2), (2t+2, 2t)\}.$$

(9) $E_{d^*}^{s+1} = \begin{cases} \{(0,1), (3,1), (d^*, d^*-1)\} & \textit{if } p_{1,s+1}^{d} = p_{1,d-1}^{d} + 1 = 2, \\ \{(0,1), (3,1), (d^*, 1)\} & \textit{if } p_{1,s+1}^{d} = 3. \end{cases}$

(10) *If* $p_{1,s+1}^{d} = p_{1,d-1}^{d} + 1 = 2$, *then*

$$E_{d^*-1}^{d^*} = \begin{cases} \{(1, d^*), (d^*-1, d^*)\} & \textit{if } p_{1,d}^{d-1} = 3, \\ \{(1, d^*), (d^*-1, d^*-2)\} & \textit{if } p_{1,d}^{d-1} = p_{1,d-2}^{d-1} + 1 = 2. \end{cases}$$

Moreover, for any pair of adjacent vertices (u, v) *and any* $x \in D_{d-1}^{d} = D_{d-1}^{d}(u, v)$, $e(x, D_d^{s+1}) = 2$.

Proof: Similar to the proof of Sublemma 4.2.2. □

For $j \geq 0$, let $[i \nearrow i+2j]$ and $[i+2j \searrow i]$ be the sequences

$$i, i+2, i+4, \ldots, i+2j,$$

$$i+2j, i+2(j-1), \ldots, i,$$

respectively.

By Lemma 7.8, we immediately have the following.

Lemma 7.9 *Let* $\mathcal{X}$ *be of type* $D'_f(t)$ $(f = 0$ *or* $1)$. *Let* $C = \{x_0, x_1, \ldots, x_n = x_0\}$ *be a circuit of length* n. *Then the following hold.*

(1) *Let* $x_{s+i} \in D_{s+j_2}^{s+j_1}(x_0, x_1)$ *with* $j_1 \neq j_2$ *and* $j_1, j_2 \geq 0$. *If, for a positive integer* j_3 *with* $j_1 + 2j_3, j_2 + 2j_3 \leq 2t + 3 - f$,

$$\sigma_C(x_0; i, \ldots, i+j_3) = ([j_1 \nearrow j_1 + 2j_3]),$$

then

$$\sigma_C(x_1; i, \ldots, i+j_3-1) = ([j_2 + 2 \nearrow j_2 + 2j_3]).$$

(2) *Let* $x_{s+i} \in D_{s+j_2}^{s+j_1}(x_0, x_1)$ *with* $j_1 \neq j_2$ *and* $j_1, j_2 \leq 2t+3-f$. *If, for a positive integer* j_3 *with* $j_1 - 2j_3, j_2 - 2j_3 \geq 0$,

$$\sigma_C(x_0; i, \ldots, i+j_3) = ([j_1 \searrow j_1 - 2j_3]),$$

then

$$\sigma_C(x_1; i, \ldots, i+j_3-1) = ([j_2 - 2 \searrow j_2 - 2j_3]).$$

Proof of Lemma 7.6: It is enough to consider the following three cases.

Case 1. $d = s + 2t + 4$, i.e., $p^d_{1,s+1} = 3$.

We shall prove it by the same way as in Case 2 of the proof of Lemma 5.1. Indeed, by applying circuit chasing technique to two circuits having the following profiles:

$$\begin{array}{cccccccc} 0 & 2 & 3 & 1 & d^* & 1 & 0 & \\ & 0 & 1 & d^* & 1 & 3 & 2 & 0, \end{array}$$

and

$$\begin{array}{cccccc} 0 & 1 & d^* & 1 & 0 & \\ & 0 & 1 & 3 & 2 & 0, \end{array}$$

we have $2s + 6 \equiv 0 \pmod 4$ and $2s + 4 \equiv 0 \pmod 4$, respectively, a contradiction.

Case 2. $d - 1 = s + 2t + 4$, i.e., $(p^d_{1,s+1}, p^d_{1,d-1}, p^{d-1}_{1,d}) = (2, 1, 3)$.

In this case we proof by the same way as in Case 2 of the proof of Lemma 7.2. Indeed, by applying circuit chasing technique to two circuits having the following profiles:

$$\begin{array}{cccccc} 0 & 2 & 3 & 1 & 0 & \\ & 0 & 1 & d^* & 1 & 0. \end{array}$$

and

$$\begin{array}{cccccccccc} 0 & 2 & 3 & 1 & d^* & d^*-1 & d^* & 1 & 0 & \\ & 0 & 1 & d^* & d^*-1 & d^* & 1 & 3 & 2 & 0, \end{array}$$

we have $2s + 4 \equiv 0 \pmod 3$ and $2s + 8 \equiv 0 \pmod 6$, respectively, a contradiction.

Case 3. $d - 2 \geq s + 2t + 4$, i.e., $p^d_{1,d-1} = p^{d-1}_{1,d-2} = 1$ with $\varphi(d) = \varphi(d-1) - 1 = \varphi(d-2) - 2$.

Let $C = \{x_0, x_1, \ldots, x_{2s+2t+5} = x_0\}$ be a circuit of length $2s + 2t + 5$ such that

$$\sigma_C(x_0) = ([2 \nearrow 2t+2], 2t+3, [2t+3 \searrow 1], 0),$$
$$\sigma_C(x_1) = ([2 \nearrow 2t], [2t+1 \nearrow 2t+3], [2t+3 \searrow 3], [2 \searrow 0]).$$

Note that

$$\begin{aligned} \sigma_C(x_1; 1, \ldots, t) &= ([2 \nearrow 2t]), \\ \sigma_C(x_1; t+1, t+2) &= ([2t+1 \nearrow 2t+3]), \\ \sigma_C(x_1; t+3, \ldots, 2t+3) &= ([2t+3 \searrow 3]), \end{aligned}$$

$\sigma_C(x_1; 2t+4, 2t+5) = ([2 \searrow 0])$, and that $x_{s+2} \in D_s^{s+2}(x_1, x_2)$. By Lemma 7.9 (1), we have

$$\sigma_C(x_2; 1, \dots, t-1) = ([2 \nearrow 2t-2]).$$

Note that $x_{s+t+1} \in D_{s+2t-2}^{s+2t}(x_1, x_2)$ and $\hat{\partial}(x_1, x_{s+t+2}) = s + 2t + 1$. By Lemma 7.8, we have $x_{s+t+2} \in D_{s+2t-1}^{s+2t+1}(x_1, x_2)$, $x_{s+t+3} \in D_{s+2t+1}^{s+2t+3}(x_1, x_2)$, $x_{s+t+4} \in D_{s+2t+3}^{s+2t+3}(x_1, x_2)$, and $x_{s+t+5} \in D_{s+2t+3}^{s+2t+1}(x_1, x_2)$. By Lemma 7.9 (2), $\sigma_C(x_2; t+3, \dots, 2t+2) = ([2t+3 \searrow 5])$. By Lemma 7.8, $\sigma_C(x_2; 2t+3, 2t+4, 2t+5) = (4, 2, 0) = ([4 \searrow 0])$. Thus, we see that

$$\sigma_C(x_2) = ([2 \nearrow 2t-2], [2t-1 \nearrow 2t+3], [2t+3 \searrow 5], [4 \searrow 0]).$$

By repeating the same argument, we see that

$$\sigma_C(x_i) = ([2 \nearrow 2t+2-2i], [2t+3-2i \nearrow 2t+3], [2t+3 \searrow 1+2i], [2i \searrow 0])$$

for $i = 3, 4, \dots, t$,

$$\begin{aligned}
\sigma_C(x_{t+1}) &= ([1 \nearrow 2t+3], 2t+3, [2t+2 \searrow 0]),\\
\sigma_C(x_{t+2}) &= ([2 \nearrow 2t+2], 2t+2, [2t+3 \searrow 1], 0),\\
\sigma_C(x_{t+j}) &= ([2 \nearrow 2t+8-2j], [2t+9-2j \nearrow 2t+3], [2t+3 \searrow 2j-3],\\
&\qquad [2j-4 \searrow 0])
\end{aligned}$$

for $j = 3, 4, \dots, t+3$,

$$\sigma_C(x_{2t+4}) = ([1 \nearrow 2t+3], 2t+2, [2t+2 \searrow 0]),$$

$\sigma_C(x_{2t+5}) = \sigma_C(x_0)$, and that $\sigma_C(x_{2t+6}) = \sigma_C(x_1)$.

Thus we have $2s + 2t + 5 \equiv 0 \pmod{2t+5}$, i.e., $2s \equiv 0 \pmod{2t+5}$.

Let $C' = \{y_0, y_1, \dots, y_{2s+2t+7} = y_0\}$ be a circuit of length $2s + 2t + 7$ such that

$$\begin{aligned}
\sigma_{C'}(y_0) &= ([2 \nearrow 2t+2], 2t+3, 2t+1, [2t \nearrow 2t+2], [2t+2 \searrow 0]),\\
\sigma_{C'}(y_1) &= ([2 \nearrow 2t], 2t+1, 2t-1, [2t-2 \nearrow 2t+2], [2t+2 \searrow 0]).
\end{aligned}$$

Then, by using the same argument as above, we see that

$$\begin{aligned}
\sigma_{C'}(y_i) = ([2 \nearrow 2t+2-2i],&\\
&2t+3-2i, 2t+1-2i, [2t-2i \nearrow 2t+2], [2t+2 \searrow 0])
\end{aligned}$$

for $i = 2, \dots, t$,

$$\begin{aligned}
\sigma_{C'}(y_{t+1}) &= (1, d^*, [1 \nearrow 2t+3], [2t+3 \searrow 1], 0),\\
\sigma_{C'}(y_{t+2}) &= ([1 \nearrow 2t+3], [2t+3 \searrow 1], d^*, 1, 0),\\
\sigma_{C'}(y_{t+j}) &= ([2 \nearrow 2t+2], [2t+2 \searrow 2j-6], 2j-5, 2j-3, [2j-4 \searrow 0])
\end{aligned}$$

for $j = 3, \dots, t+3$,

$$\sigma_{C'}(y_{2t+4}) = ([2 \nearrow 2t+2], 2t+2, 2t+3, 2t+3, 2t+2, [2t+2 \searrow 0]),$$

$\sigma_{C'}(y_{2t+5}) = \sigma_{C'}(y_0)$, and that $\sigma_{C'}(y_{2t+6}) = \sigma_{C'}(y_1)$.

Thus we have $2s + 2t + 7 \equiv 0 \pmod{2t+5}$, i.e., $2s \not\equiv 0 \pmod{2t+5}$. This is a contradiction.

Now we conclude the proof of Lemma 7.6.

Proof of Lemma 7.7:

Case 1. $d - 1 \leq s + 2t + 3$. Similar to Cases 1 and 2 in the proof of Lemma 7.6.

Case 2. $d - 2 \geq s + 2t + 3$, i.e., $p^d_{1,d-1} = p^{d-1}_{1,d-2} = 1$ with $\varphi(d) = \varphi(d-1) - 1 = \varphi(d-2) - 2$.

We prove it by the same way as in Case 3 of the proof of Lemma 7.6.

Let $C = \{x_0, x_1, \dots, x_{2s+2t+3} = x_0\}$ be a circuit of length $2s + 2t + 3$ such that

$$\sigma_C(x_0) = ([2 \nearrow 2t+2], [2t+1 \searrow 1], 0),$$
$$\sigma_C(x_1) = ([2 \nearrow 2t+2], [2t+1 \searrow 3], [2 \searrow 0]).$$

Then, by using circuit chasing technique, we see that $\sigma_C(x_{2t+3}) = \sigma_C(x_0)$, and that $\sigma_C(x_{2t+4}) = \sigma_C(x_1)$.

Thus we have $2s + 2t + 3 \equiv 0 \pmod{2t+3}$, i.e., $2s \equiv 0 \pmod{2t+3}$.

Let $C' = \{y_0, y_1, \dots, y_{2s+2t+7} = y_0\}$ be a circuit of length $2s + 2t + 7$ such that

$$\sigma_{C'}(y_0) = (1, d^*, [1 \nearrow 2t+1], [2t+2 \searrow 2t], [2t+1 \searrow 1], 0),$$
$$\sigma_{C'}(y_1) = ([1 \nearrow 2t+1], [2t+2 \searrow 2t-2], [2t-1 \searrow 1], d^*, 1, 0).$$

Then, in conclusion, we see that the profile of C' with respect to $(y_{(t+1)(2t+5)+1}, y_{(t+1)(2t+5)+2})$ is the same as the one with respect to (y_0, y_1), which implies that $2s \not\equiv 0 \pmod{2t+3}$, and we have a contradiction. The author is afraid that it is very hard for any reader to check it, but the argument used for it (circuit chasing technique) is routine.

Now we conclude the proof of Lemma 7.7. □

Proof of theorem 1.1: Suppose that Γ is a connected cubic graph. Then, by Lemmas 4.2, 4.3, 4.4, 4.5, 4.6, 5.1, 6.1, 7.2, 7.3, 7.4, 7.5, 7.6 and 7.7, it must hold that Γ is bipartite. Thus we have the assertion. □

8. On bipartite case

In the preceding sections, we avoided considering on the case when Γ is bipartite. But we see the following:

Proposition 8.1 *If Γ is a non-distance-regular bipartite graph, then $\mathcal{X}$ is one of the following types.*

(i) $D_{10}(t)$ $(0 \leq t \leq d^*/2 - 1)$:

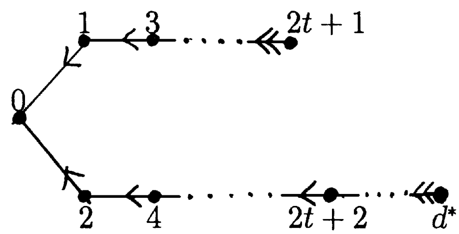

Figure 14.

(ii) D_{11}:

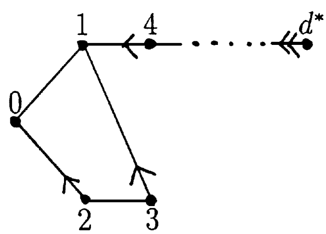

Figure 15.

(iii) D_{12}:

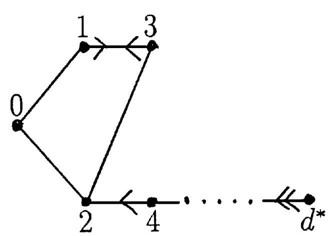

Figure 16.

(iv) $D_{13}(t)$ $(2 \leq t \leq d^*/2 - 1)$:

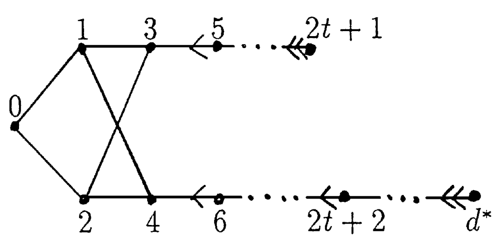

Figure 17.

(v) $D_{14}(t)$ $(1 \le t \le d^*/2 - 2)$:

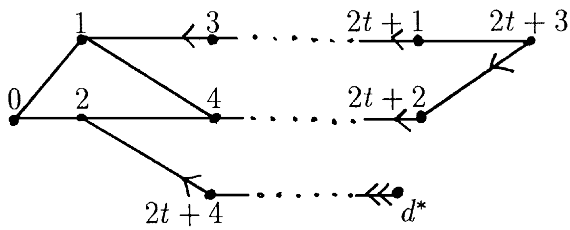

Figure 18.

(vi) $D_{15}(t_1, t_2)$ $(t_1 \ge 1, 2t_1 + 4 \le t_2 \le d^* - 2)$:

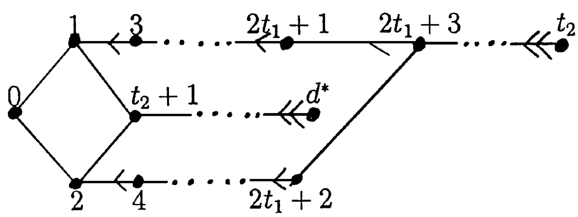

Figure 19.

(vii) $D_{16}(t_1, t_2)$ $(t_1 \ge 1, 2t_1 + 2 \le t_2 \le d^* - 2)$:

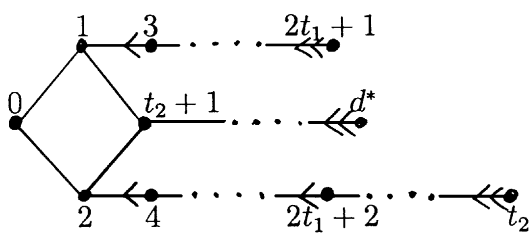

Figure 20.

(viii) (i)–(viii) *in Lemma 7.5.*

Proof: It suffices to consider the type IID or VC. If $\mathcal{X}$ is of type IID, then we easily see that $\mathcal{X}$ is of type (i), (ii), (iii) or (iv). If $\mathcal{X}$ is of type VC, then we easily see that $\mathcal{X}$ is of type (v), (vi) or (vii).

Thus we have the assertion. $\square$

Remark The author knows one example of symmetric association scheme such that Γ is a connected non-distance-regular bipartite cubic graph, which is of type $D_{10}(0)$ with $s = 2$ and $d = 5$. Namely, this is constructed from one connected component of $\Gamma_3'(x)$ $(x \in \Gamma')$ of Γ' when Γ' is the generalized hexagon of $(2, 2)$ (one of two graphs). See [4, p. 384].

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