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Vertex-Transitive Non-Cayley Graphs with Arbitrarily Large Vertex-Stabilizer

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Abstract. A construction is given for an infinite family $\{\Gamma_n\}$ of finite vertex-transitive non-Cayley graphs of fixed valency with the property that the order of the vertex-stabilizer in the smallest vertex-transitive group of automorphisms of Γ_n is a strictly increasing function of n. For each n the graph is 4-valent and arc-transitive, with automorphism group a symmetric group of large prime degree $p > 2^{2^n+2}$. The construction uses Sierpinski's gasket to produce generating permutations for the vertex-stabilizer (a large 2-group).

Keywords: symmetric graph, vertex-transitive, arc-transitive

1. Introduction

In this paper we provide a positive answer to the following question:
Does there exist an infinite family $\{\Gamma_n\}$ of finite vertex-transitive graphs of fixed valency such that if G_n is a vertex-transitive group of automorphisms of Γ_n of smallest possible order then the order of the stabilizer in G_n of a vertex of Γ_n increases as $n \to \infty$?

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Without the condition on the *smallest* vertex-transitive group of automorphisms, the construction of vertex-transitive graphs with automorphism group having an arbitrarily large vertex-stabilizer is relatively easy. For example, take the 4-valent graph with vertices $0, 1, 2, \ldots, 2n-1$, and edges joining each of $2i$ and $2i+1$ to each of $2i+2$ and $2i+3$ modulo $2n$ (for $0 \le i < n$). This is just a simple cycle of length n, with each vertex replaced by two vertices and each edge replaced by a $K_{2,2}$. Its automorphism group is the wreath product C_2 wr D_n of order $2^n \times 2n$. In particular, this group is transitive on the $2n$ vertices, with vertex-stabilizer of order 2^n. The automorphism group, however, contains a subgroup which acts regularly on vertices, and so the graph is a Cayley graph—in fact a Cayley graph for the dihedral group $D_n = \langle x, y \mid x^2 = y^n = (xy)^2 = 1 \rangle$ with edges corresponding to multiplication by x, y, y^{-1} and xy^2.

However, adding the condition that the vertex-stabilizer should act primitively on the set of neighbours of the fixed vertex makes a more difficult question, conjectured by Richard Weiss to have a negative answer for arc-transitive graphs (see [6, 8]).

We will construct for each positive integer n a finite arc-transitive 4-valent graph Γ_n, with full automorphism group S_N (the symmetric group of degree N), where $N = p(n)$ is a prime greater than 2^{2^n+2} and congruent to 1 modulo 4. The stabilizer in S_N of any vertex v of Γ_n will be a 2-group of order 2^{2^n+2}, acting transitively but imprimitively as the dihedral group D_4 on $\Gamma_n(v)$, the set of neighbours of the vertex v. By choice of N and the construction of Γ_n, the smallest group of automorphisms of Γ_n which acts transitively on vertices of Γ_n will be the alternating group A_N (since S_N has no other proper subgroup of index less than N), with vertex-stabilizer of order 2^{2^n+1}.

Before doing this in Section 3, we describe background material on arc-transitive graphs in Section 2, together with preliminaries on Sierpinski's gasket, which is used to define generating permutations for the vertex-stabilizer in the automorphism group S_N. Properties of Γ_n are verified in Section 4, and some concluding remarks are made in Section 5.

2. Preliminaries

Let Γ be an undirected simple graph. An *automorphism* of Γ is any permutation of the vertices of Γ preserving adjacency, and under composition the set of all such permutations of $V\Gamma$ forms a group known as the (full) automorphism group of Γ and denoted by $\mathrm{Aut}\,\Gamma$.

If $\mathrm{Aut}\,\Gamma$ acts transitively on $V\Gamma$, then Γ is said to be *vertex-transitive*. More generally, if G is any group of automorphisms of Γ which acts transitively on $V\Gamma$, then G is said to be vertex-transitive on Γ. Similarly, if G acts transitively on the set of arcs (ordered edges) of Γ, then G is said to be *arc-transitive* on Γ, and also Γ is said to be arc-transitive, or *symmetric*.

In the latter case, the stabilizer $G_v = \{g \in G : v^g = v\}$ in G of a vertex $v \in V\Gamma$ acts transitively on the set $\Gamma(v)$ of vertices adjacent to v in Γ, or equivalently, on the set of arcs in Γ emanating from the vertex v. Further, if (v, w) is any one such arc, then by arc-transitivity there exists an automorphism $a \in G$ reversing (v, w), and then the structure of Γ may be defined completely in terms of a and G_v: vertices may be labelled with right cosets of G_v in G, and edges are the images under the action of G (by right multiplication) of the single edge $\{v, w\}$ labelled $\{G_v, G_v a\}$ in the natural order.

Conversely, given any group G containing a subgroup H and an element a such that $a^2 \in H$, we may construct a graph $\Gamma = \Gamma(G, H, a)$ on which G acts as an arc-transitive group of automorphisms, as follows: take as vertices of Γ the right cosets of H in G, and join two cosets Hx and Hy by an edge in Γ whenever $xy^{-1} \in HaH$. Defined in this way, Γ is an undirected graph on which the group G acts as a group of automorphisms under the action $g : Hx \to Hxg$ for each $g \in G$ and each coset Hx in G. The stabilizer in G of the vertex H is the subgroup H itself, and as this acts transitively on the set of neighbours of H (which are all of the form Hah for $h \in H$), it follows that Γ is symmetric.

The above construction is explained in more detail in [4] (and was used to answer similar questions in [1, 2]). The graph $\Gamma = \Gamma(G, H, a)$ is connected if and only if G is generated by HaH (or equivalently, by $H \cup \{a\}$), and is regular of degree d where $d = |H : H \cap a^{-1}Ha|$ is the number of right cosets of H contained in the double coset HaH. Similarly, other properties of Γ (such as its girth and diameter) depend on the choice of G, H and a, and in particular on relations satisfied in G by a and elements of H.

In what follows in this paper the group H will be a 2-group, generated by an element of order 4 and an increasing number of additional involutions. To define these involutions we use a modification of *Sierpinski's gasket*, or Pascal's triangle modulo 2, see [5; Section 2.2].

First let ${}^nC_k$ be the standard binomial coefficient, defined as the number of k-element subsets of an n-element set, and equal to the coefficient of x^k in the binomial expansion of $(1+x)^n$, for $0 \le k \le n$. Recall that these coefficients satisfy the additive identity ${}^nC_{k-1} + {}^nC_k = {}^{n+1}C_k$ for $1 \le k \le n$, which is a fundamental property of Pascal's triangle. The triangle's symmetry comes from the identity ${}^nC_k = {}^nC_{n-k}$, and from this it follows that ${}^nC_{n/2}$ is always even when n is even. However, when $n+1$ is a power of 2, every coefficient ${}^nC_k$ is odd; to see this, note that ${}^nC_k$ may be written as a product of rationals of the form $(n+1-j)/j$ for $1 \le j \le k$, and in each case the highest power of 2 dividing the numerator is equal to the highest power of 2 dividing the denominator.

Definition 2.1 For integers r and s satisfying $0 \le r \le 2s+1$, define $d_{rs} = {}^sC_{[r/2]}$ (where $[r/2]$ is the greatest integer not exceeding $r/2$).

Clearly, $d_{0s} = d_{1s} = d_{2s,s} = d_{2s+1,s} = 1$ for all $s \ge 0$, and $d_{rs} = d_{r+1,s}$ whenever r is even. Also the properties of binomial coefficients mentioned above imply the following Lemma.

Lemma 2.2 *$d_{r-1,s} + d_{r,s} \equiv d_{r,s+1} \bmod 2$ and $d_{rr} \equiv 0 \bmod 2$ whenever r is even.*

The proof is straightforward. A picture of the corresponding triangle of residues of these coefficients modulo 2 is given below for $s \le 31$, with x's for 1's and blanks for 0's:

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                               xx
                              xxxx
                             xx  xx
                            xxxxxxxx
                           xx      xx
                          xxxx    xxxx
                         xx  xx  xx  xx
                        xxxxxxxxxxxxxxxx
                       xx              xx
                      xxxx            xxxx
                     xx  xx          xx  xx
                    xxxxxxxx        xxxxxxxx
                   xx      xx      xx      xx
                  xxxx    xxxx    xxxx    xxxx
                 xx  xx  xx  xx  xx  xx  xx  xx
                xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
               xx                              xx
              xxxx                            xxxx
             xx  xx                          xx  xx
            xxxxxxxx                        xxxxxxxx
           xx      xx                      xx      xx
          xxxx    xxxx                    xxxx    xxxx
         xx  xx  xx  xx                  xx  xx  xx  xx
        xxxxxxxxxxxxxxxx                xxxxxxxxxxxxxxxx
       xx              xx              xx              xx
      xxxx            xxxx            xxxx            xxxx
     xx  xx          xx  xx          xx  xx          xx  xx
    xxxxxxxx        xxxxxxxx        xxxxxxxx        xxxxxxxx
   xx      xx      xx      xx      xx      xx      xx      xx
  xxxx    xxxx    xxxx    xxxx    xxxx    xxxx    xxxx    xxxx
 xx  xx  xx  xx  xx  xx  xx  xx  xx  xx  xx  xx  xx  xx  xx  xx
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
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Figure 1. Modified Sierpinski gasket.

3. Construction of the graphs

Suppose n is any positive integer. Define $m = 2^n$, and let $N = p(n)$ be any prime such that $N \equiv 1 \bmod 4$ and $N > 2^{m+2}$. Note that since there are infinitely many primes congruent to 1 modulo 4, such a prime N can always be found.

Definition 3.1 Let G be the symmetric group S_N, in its natural action on the set $\{1, 2, \ldots, N\}$, and in this group define three elements a, b and c as follows:

$$\begin{aligned} a \;=\; & (1,3)(2,4)(5,7)(6,8)\cdots(2m-7,2m-5)(2m-6,2m-4) \\ & (2m-3,2m-1)(2m-2,2m)(2m+1,N)(2m+2,2m+3) \\ & (2m+4,2m+5)\cdots(N-5,N-4)(N-3,N-2), \\ b \;=\; & (1,2m+1,2,2m+2)(3,5)(4,6)(7,9)(8,10)\cdots(2m-5,2m-3) \\ & (2m-4,2m-2)(2m+3,2m+4)(2m+5,2m+6)\cdots(N-4,N-3) \\ & (N-2,N-1), \end{aligned}$$

and

$$\begin{aligned} c \;=\; & (1,2)(3,4)(5,6)(7,8)\cdots(2m-7,2m-6)(2m-5,2m-4) \\ & (2m-3,2m-2)(2m-1,2m). \end{aligned}$$

Note that a, b and c are even permutations of orders 2, 4 and 2, respectively, with b having a single 4-cycle, $(N-7)/2$ transpositions, and three fixed points (viz. $2m-1$, $2m$ and N). This is perhaps best seen with the help of the diagram below, in which tranpositions of a are represented by thin lines, while cycles of b are represented by heavy polygons and dots, and the effect of c corresponds to reflection in the vertical axis of symmetry:

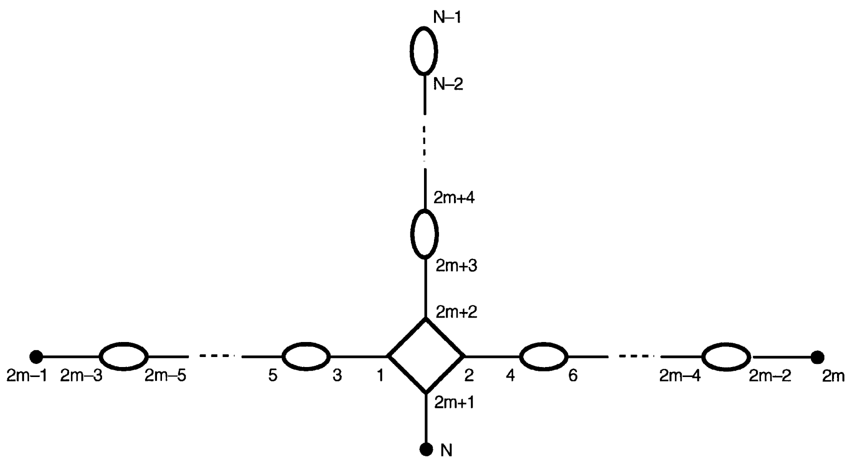

Figure 2. Generating permutations.

Within the alternating group A_N the permutations a and b generate a subgroup which is transitive (since the diagram is connected) and indeed primitive (since N is prime). Also it may be seen from the diagram that $c^{-1}ac = a$ while $c^{-1}bc = b^{-1}$.

Next, note that c is the product of transpositions $t_j = (2j-1, 2j)$ for $1 \le j \le m$. Letting this product correspond to the first m entries of the mth row of the modified Sierpinski gasket (as illustrated in figure 1), we work backwards to define a sequence of additional permutations $c_1, c_2, \ldots, c_m$ as follows:

Definition 3.2 For $1 \le i \le m$ define $c_i = \prod_{j=i}^{m} t_j^{d_{j-i,m-i}}$, where t_j is the transposition $(2j-1, 2j)$ in S_N, and $d_{j-i,m-i} = {}^{m-i}C_{[(j-i)/2]}$ (as given in Definition 2.1) for $i \le j \le m$.

For example, $c_1 = t_1 t_2 \cdots t_m = c$, while $c_2 = t_2\, t_3\, t_6\, t_7 \cdots t_{m-6}\, t_{m-5}\, t_{m-2}\, t_{m-1}$, and finally, $c_{m-1} = t_{m-1}t_m$ while $c_m = t_m = (2m-1, 2m)$. Note that each of the c_i other than c_m is even, since $d_{rs} = d_{r+1,s}$ and $d_{rr} \equiv 0 \bmod 2$ whenever r is even. Also note that the exponents $d_{j-i,m-i}$ may be taken modulo 2, and the only such exponents required for the definition of c_i are the ones which appear in the first half of the $(m-i+1)$st row of our modified Sierpinski gasket. The choice of these exponents is the key to our construction of the graph Γ_n, as will be seen later with the help of the two observations below.

Lemma 3.3 *For $1 \le j \le m$ we have*

$$\text{(a) } a^{-1}t_j a = \begin{cases} t_{j+1} & \text{if } j \text{ is odd} \\ t_{j-1} & \text{if } j \text{ is even} \end{cases} \qquad \text{(b) } b^{-1}t_j b = \begin{cases} b^2 t_1 & \text{if } j = 1 \\ t_{j+1} & \text{if } j \in \{2, 4, \ldots, m-2\} \\ t_{j-1} & \text{if } j \in \{3, 5, \ldots, m-1\} \\ t_m & \text{if } j = m. \end{cases}$$

Proof: This is a simple consequence of the definitions of the permutations a and b and the transpositions t_j. □

Lemma 3.4 *For $1 \le i \le m$ we have*

$$\text{(a) } a^{-1}c_i a = \begin{cases} c_i & \text{if } i \text{ is odd} \\ c_{i-1}c_i & \text{if } i \text{ is even} \end{cases} \qquad \text{(b) } b^{-1}c_i b = \begin{cases} b^2 c_1 & \text{if } i = 1 \\ c_i & \text{if } i \in \{2, 4, \ldots, m\} \\ c_{i-1}c_i & \text{if } i \in \{3, 5, \ldots, m-1\}. \end{cases}$$

Proof: First as $c_1 = c$ the cases where $i = 1$ for parts (a) and (b) are consequences of our earlier observation that $c^{-1}ac = a$ and $c^{-1}bc = b^{-1}$. Next if i is odd then we have

$$\begin{aligned} a^{-1}c_i a &= a^{-1}\left(\prod_{j=i}^{m} t_j^{d_{j-i,m-i}}\right)a = \prod_{j=i}^{m}(a^{-1}t_j a)^{d_{j-i,m-i}} && \text{by the definition of } c_i \\ &= \prod_{\text{odd } j=i}^{m-1} t_{j+1}^{d_{j-i,m-i}} \prod_{\text{even } j=i+1}^{m} t_{j-1}^{d_{j-i,m-i}} && \text{by Lemma 3.3 and since the } t_j \text{ commute} \\ &= \prod_{\text{odd } j=i}^{m-1} t_{j+1}^{d_{j+1-i,m-i}} \prod_{\text{even } j=i+1}^{m} t_{j-1}^{d_{j-1-i,m-i}} && \text{as } d_{rs} = d_{r+1,s} \text{ when } r \text{ is even} \end{aligned}$$

$$= \prod_{\text{even } k=i+1}^{m} t_k^{d_{k-i,m-i}} \prod_{\text{odd } k=i}^{m-1} t_k^{d_{k-i,m-i}} \quad \text{after relabelling subscripts}$$

$$= \prod_{k=i}^{m} t_k^{d_{k-i,m-i}} = c_i.$$

A very similar argument shows $b^{-1}c_i b = c_i$ when i is even, noting that since $m-i$ is even we have $d_{m-i,m-i} = {}^{m-i}C_{(m-i)/2} \equiv 0 \bmod 2$ in this case.

On the other hand, if i is even then also

$$\begin{aligned}
c_{i-1}c_i &= \prod_{j=i-1}^{m} t_j^{d_{j-i+1,m-i+1}} \prod_{j=i}^{m} t_j^{d_{j-i,m-i}} = t_{i-1}^{d_{0,m-i+1}} \prod_{j=i}^{m} t_j^{d_{j-i+1,m-i+1}+d_{j-i,m-i}} \\
&= t_{i-1} \prod_{\text{even } j=i}^{m} t_j^{d_{j-i,m-i+1}+d_{j-i,m-i}} \prod_{\text{odd } j=i+1}^{m-1} t_j^{d_{j-i+1,m-i+1}+d_{j-i,m-i}} \\
&= t_{i-1} \prod_{\text{even } j=i+2}^{m} t_j^{d_{j-i-1,m-i}} \prod_{\text{odd } j=i+1}^{m-1} t_j^{d_{j-i+1,m-i}} \quad \text{by Lemma 2.2} \\
&= t_{i-1} \prod_{\text{odd } k=i+1}^{m-1} t_{k+1}^{d_{k-i,m-i}} \prod_{\text{even } k=i+2}^{m} t_{k-1}^{d_{k-i,m-i}} \quad \text{after relabelling subscripts} \\
&= \prod_{\text{even } k=i}^{m} t_{k-1}^{d_{k-i,m-i}} \prod_{\text{odd } k=i+1}^{m-1} t_{k+1}^{d_{k-i,m-i}} \quad \text{since } d_{0,m-i} = 1 \\
&= \prod_{k=i}^{m} (a^{-1}t_k a)^{d_{k-i,m-i}} = a^{-1}c_i a.
\end{aligned}$$

Finally, the same procedure shows that if i is odd and $i > 1$ then $c_{i-1}c_i = b^{-1}c_i b$, noting that $d_{m-i+1,m-i+1} \equiv 0 \bmod 2$ in this case and recalling that $b^{-1}t_m b = t_m$. □

We are now in a position to define the graph Γ_n.

Definition 3.5 Let Γ_n be the graph $\Gamma(G, H, a)$, where $G = S_N$ and $a \in G$ are as defined earlier, and H is the subgroup of S_N generated by b and the m involutions $c_1, c_2, \ldots, c_m$.

Recall that the vertices of $\Gamma(G, H, a)$ may be taken as the right cosets of H in G, with two cosets Hx and Hy joined by an edge whenever $xy^{-1} \in HaH$, and the group G acts as a group of automorphisms of this graph by right multiplication on cosets.

Before investigating its properties in the next Section, we make some observations about G, H and a.

Lemma 3.6 *The group $G = S_N$ is generated by the subgroup H and the element a.*

Proof: The subgroup generated by H and a contains b and a and is therefore transitive, of prime degree and therefore primitive, and contains the single 2-cycle $c_m = (2m-1, 2m)$ and is therefore equal to S_N (by Jordan's theorem [9; Section 13]). □

Lemma 3.7 *The subgroup generated by* $c_1, c_2, \ldots, c_m$ *is elementary abelian of order* 2^m.

Proof: First the c_i are products of commuting transpositions t_j, and therefore generate an elementary abelian 2-group. Also by definition of the c_i (and the properties of the Sierpinski gasket), for $1 \leq k \leq m$ the subgroup generated by $c_k, \ldots, c_m$ moves only the points $2k-1, 2k, \ldots, 2m-1, 2m$, and it follows that each such subgroup has order 2^{m-k+1}. □

Lemma 3.8 *The subgroup* H *is a* 2*-group of order* 2^{m+2}.

Proof: From part (b) of Lemma 3.4 we see that b^2 centralizes $\langle c_1, c_2, \ldots, c_m\rangle$, and further, that b normalizes $\langle b^2, c_1, c_2, \ldots, c_m\rangle$, therefore $H = \langle b, c_1, c_2, \ldots, c_m\rangle$ has order 2^{m+2}. □

Lemma 3.9 $H \cap a^{-1}Ha$ *has index* 4 *in* H.

Proof: From part (a) of Lemma 3.4 we see that a normalizes $\langle c_1, c_2, \ldots, c_m\rangle$, which is a subgroup of index 4 in H with transversal $\{1, b, b^2, b^3\}$. Since each of $a^{-1}ba$, $a^{-1}b^2a$ and $a^{-1}b^3a$ moves the point N, none of these three elements can lie in H and it follows that $H \cap a^{-1}Ha = \langle c_1, c_2, \ldots, c_m\rangle$. □

4. Properties of the graphs

We begin with some of the basic properties which follow from the construction described in the previous Section:

Proposition 4.1 *The graph* Γ_n *defined in Section* 3 *has* $N!/2^{m+2}$ *vertices, and is connected,* 4*-valent and arc-transitive, with* S_N *as an arc-transitive group of automorphisms. The girth of* Γ_n *is* 4.

Proof: Most of this follows from Lemmas 3.6, 3.8 and 3.9, and the fact that $G = S_N$ acts on $\Gamma_n = \Gamma(G, H, a)$ by right multiplication, with trivial kernel since A_N is simple. Finally, $(ab^2)^2 = (1,2)(3,4)(2m+1, 2m+2)(2m+3, N)$ has order 2, so Γ_n has a circuit of length 4 with vertices H, Hab^2, $H(ab^2)^2$ and Ha (in that order); and as no two of the four neighbours of H are adjacent, there is no circuit of length 3, and therefore Γ_n has girth 4. □

The stabilizer in $G = S_N$ of the vertex H is the subgroup H itself, of order 2^{m+2}. This group acts transitively on the neighbour-set $\Gamma_n(H) = \{Ha, Hab, Hab^2, Hab^3\}$, and the stabilizer of the arc (H, Ha) is the subgroup $H \cap a^{-1}Ha = \langle c_1, c_2, \ldots, c_m\rangle$. As the latter subgroup is centralized by b^2, it is also the stabilizer of the arc (H, Hab^2), while the element c_1 interchanges the other two arcs (H, Hab) and (H, Hab^3). It follows that H acts as the dihedral group D_4 of order 8 on the neighbour-set $\Gamma_n(H)$.

From Lemma 3.4 it is easy to see that every element $g \in G$ may be written in the form $g = hw$ where $h \in \langle c_1, c_2, \ldots, c_m\rangle$ and w is a word in the elements a and b. In particular, every vertex of Γ_n is of the form Hw where $w \in \langle a, b\rangle$, and it follows easily that the subgroup $\langle a, b\rangle$ is transitive on vertices of Γ_n. (In fact we will see in Proposition 4.4 that $\langle a, b\rangle = A_N$ and this is the smallest vertex-transitive group of automorphisms of Γ_n.)

Next for any positive integer s, an *s-arc* in a graph Γ is an ordered $(s+1)$-tuple of vertices $(v_0, v_1, v_2, \ldots, v_s)$ of Γ such that any two consecutive v_i are adjacent in Γ and any three consecutive v_i are distinct. From what we have seen above, the group S_N does not act transitively on 2-arcs (ordered paths of length 2) in Γ_n, because of the existence of circuits of length 4. More of the nature of the action of S_N on Γ_n is revealed below in Lemma 4.2.

Lemma 4.2 *For $1 \le k \le m$, the subgroup of S_N generated by $c_k, \ldots, c_m$ is the stabilizer in S_N of the k-arc $(H(ab)^{[\frac{k}{2}]}, H(ab)^{[\frac{k}{2}]-1}, \ldots, Hab, H, Ha, Haba, \ldots, H(ab)^{[\frac{k-1}{2}]}a)$ of Γ_n. Moreover, this subgroup fixes all vertices at distance up to $[\frac{k}{2}]$ from H but not all vertices at distance $[\frac{k}{2}]+1$ from H in Γ_n.*

Proof: We use the following corollary of Lemma 3.4: if $h \in \langle c_i, c_{i+1}, \ldots, c_m\rangle$ where $i \ge 3$, and e is any integer, then $ab^e h = h' ab^e$ for some $h' \in \langle c_{i-2}, c_{i-1}, \ldots, c_m\rangle$. Now consider any s-arc in Γ_n of the form $(H, Hab^{e_1}, Hab^{e_2}ab^{e_1}, \ldots, Hab^{e_s}\ldots ab^{e_2}ab^{e_1})$, where $1 \le s \le m/2$ and $e_1 \in \{0, 1, 2, 3\}$ while $e_i \in \{1, 2, 3\}$ for $2 \le i \le s$. The stabilizer in S_N of the initial 1-arc (H, Hab^{e_1}) is either $\langle c_1, c_2, \ldots, c_m\rangle$ or $\langle b^2c_1, c_2, \ldots, c_m\rangle$, depending on whether $e_1 \in \{0, 2\}$ or $e_1 \in \{1, 3\}$. By induction on s (and using Lemma 3.4) it follows that the stabilizer of the given s-arc contains $\langle c_{2s-1}, c_{2s}, \ldots, c_m\rangle$ or $\langle c_{2s}, c_{2s+1}, \ldots, c_m\rangle$, again depending on whether $e_1 \in \{0, 2\}$ or $e_1 \in \{1, 3\}$. However, in the case where $e_i = 1$ for all $i \ge 2$, Lemma 3.4 shows that c_{2s-2} moves the final vertex $H(ab)^{s-1}a$ of the s-arc $(H, Ha, Haba, \ldots, H(ab)^{s-2}a, H(ab)^{s-1}a)$, to $Hab^3(ab)^{s-2}a$, and similarly c_{2s-1} moves the final vertex of the s-arc $(H, Hab, H(ab)^2, \ldots, H(ab)^{s-1}, Hab^s)$, to $Hab^3(ab)^{s-1}$. The result follows by taking $s = [\frac{k}{2}]$. □

Proposition 4.3 *The symmetric group S_N is the full automorphism group of Γ_n.*

Proof: Assume the contrary, and let J be a subgroup of $\operatorname{Aut}\Gamma_n$ which properly contains $G = S_N$, and let K be the stabilizer in J of the vertex labelled H in Γ_n. Then K contains H, and since Γ_n contains circuits of length 4, the subgroup K acts as D_4 on $\Gamma_n(H)$ in the same way as H, and by induction on the length of a stabilizer sequence it follows that K is also a 2-group. Now let $\theta \in K\backslash H$ be an automorphism of Γ_n, chosen so that H has index 2 in $\langle H, \theta\rangle$, in which case θ normalizes H. By Lemma 4.2, and multiplying by a suitable element of H if necessary, we may suppose that θ fixes every vertex at distance up to $m/2$ from H in Γ_n, and further, that θ fixes the $(m+1)$-arc $(H(ab)^{m/2}, H(ab)^{m/2-1}, \ldots, Hab, H, Ha, Haba, \ldots, H(ab)^{m/2}a)$. In particular, we may suppose that $\langle\theta\rangle$ is the stabilizer in J of this $(m+1)$-arc, which will be denoted by M.

Now M is fixed by $a\theta a$, and so $a\theta a \in \langle\theta\rangle\backslash H$, which in turn implies $\theta^{-1}a\theta a \in H$ and therefore $\theta^{-1}a\theta = a$ (since the stabilizer in H of M is trivial). It follows that θ normalizes $\langle H, a\rangle = G = S_N$, and then since $\operatorname{Aut} S_N = S_N$ (see [3; Section II.5]) there must be a permutation $\sigma \in S_N$ corresponding to θ which normalizes H and centralizes a. But further, $\theta^{-1}b\theta b^{-1}$ lies in H and also fixes every vertex Hx at distance up to $m/2$ from H in Γ_n (since θ fixes Hx and Hxb), so from Lemma 4.2 we deduce that $\theta^{-1}b\theta b^{-1} \in \langle c_m\rangle$, and therefore $\sigma^{-1}b\sigma = \theta^{-1}b\theta \in \{b, c_m b\}$. The case $\sigma^{-1}b\sigma = c_m b$ is impossible since b is even while $c_m b$ is odd, and therefore σ centralizes b. Consequently, θ centralizes both a and b, from which it follows that θ fixes every vertex of Γ_n, a contradiction. □

Proposition 4.4 *The smallest vertex-transitive group of automorphisms of Γ_n is the alternating group A_N, and in this group the stabilizer of a vertex of Γ_n has order 2^{m+1}.*

Proof: First, A_N contains the vertex-transitive subgroup $\langle a, b\rangle$, since a and b are even, and hence A_N itself is vertex-transitive on Γ_n. On the other hand, in the group S_N the stabilizer of a vertex of Γ_n has order 2^{m+2}, and therefore any vertex-transitive subgroup has index at most 2^{m+2} in S_N. But S_N has no proper subgroup of index less than N other than A_N (see [3; Section II.5]), however, so by choice of $N > 2^{m+2}$ the alternating group A_N is the smallest vertex-transitive subgroup of $\mathrm{Aut}\,\Gamma_n$. In particular, since each of the permutations c_i other than c_m is even, we find $\langle a, b\rangle = \langle a, b, c_1, c_2, \ldots, c_{m-1}\rangle = A_N$, and the stabilizer in A_N of a vertex of Γ_n is $\langle b, c_1, c_2, \ldots, c_{m-1}\rangle$, which has order 2^{m+1}. □

5. Final remarks

Thus, we have proved the following theorem.

Theorem 5.1 *For every positive integer n there exists a finite arc-transitive 4-valent graph Γ_n, with the property that in the smallest vertex-transitive group of automorphisms of Γ_n, the stabilizer of a vertex has order 2^{2^n+1}.*

In fact for each n there are infinitely many such graphs, because in our construction there are infinitely many possibilities for the prime degree N. Moreover, the construction works also when N is not prime, as primitivity of the group generated by H and a may be proved using conjugates of the 2-cycle c_m or the quadruple transposition $(ab^2)^2$.

Since producing this family of graphs it has been pointed out to us by Brendan McKay and a referee that a very different family of graphs also provides an answer to the question raised by Chris Godsil. This family of graphs, named $C(p, r, s)$ where p, r and s are positive integers with $p \geq 2$ and $r \geq 3$, were constructed by Praeger and Xu [7]. The graph $C(p, r, s)$ has $p^s r$ vertices and valency $2p$, and except for small values of the parameters, the automorphism group of $C(p, r, s)$ is the wreath product $S_p \mathrm{wr}\, D_r$, of order $2r(p!)^r$ (see Theorem 2.13 of [7]). In particular, $C(p, r, s)$ is vertex-transitive whenever $r \geq s$.

Now if, for example, $s = 2$ while p and r are primes with $r > p$, then a very easy group-theoretic argument shows that a minimal transitive subgroup G of the automorphism group of $C(p, r, 2)$ is of the form $G = PR$ where P is an elementary Abelian p-group and $R = Z_r$. Moreover, G is contained in the subgroup QR where $Q = Z_p^r$ is a Sylow p-subgroup of $S_p\,\mathrm{wr}\, D_r$, and Q is the $Z_p R$ permutation module for the regular representation of R on r points. The irreducible constituents of R in Q consist of one trivial submodule, and $(r-1)/e$ of dimension e where e is the order of p modulo r. Since G is transitive on the $p^2 r$ vertices of $C(p, r, 2)$, it follows that $|P| \geq p^e$. Choosing an infinite sequence of primes $r_1 < r_2 < \cdots$ all greater than p such that the order of p modulo r_i is e_i and $e_1 < e_2 < \cdots$, one finds a minimal vertex-transitive subgroup of automorphisms of $C(p, r_i, 2)$ has order at least $p^{e_i} r$, and so the vertex-stabilizer in such a subgroup has order at least $p^{e_i - 2}$.

Unfortunately, in both our family and this family the action of the vertex-stabilizer is imprimitive on the neighbour-set, so neither construction has much effect on progress towards settling Weiss's conjecture (see [6, 8]). We believe that this conjecture remains open.

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