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On Residue Symbols and the Mullineux Conjecture

C. BESSENRODT christine.bessenrodt@mathematik.uni-magdeburg.de
Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg, 39016 Magdeburg, Germany

J.B. OLSSON olsson@math.ku.dk
Matematisk Institut, Københavns Universitet, Universitetsparken 5, 2100 Copenhagen Ø, Denmark

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Abstract. This paper is concerned with properties of the Mullineux map, which plays a rôle in p-modular representation theory of symmetric groups. We introduce the residue symbol for a p-regular partitions, a variation of the Mullineux symbol, which makes the detection and removal of good nodes (as introduced by Kleshchev) in the partition easy to describe. Applications of this idea include a short proof of the combinatorial conjecture to which the Mullineux conjecture had been reduced by Kleshchev.

Keywords: symmetric group, modular representation, Mullineux conjecture, signature sequence, good nodes in residue diagram

1. Introduction

It is a well-known fact that for a given prime p the p-modular irreducible representations D^λ of the symmetric group S_n of degree n are labelled in a canonical way by the p-regular partitions λ of n. When the modular irreducible representation D^λ of S_n is tensored by the sign representation we get a new modular irreducible representation D^{λ^P}. The question about the connection between the p-regular partitions λ and λ^P was answered in 1995 by the proof of the so-called "Mullineux Conjecture".

The importance of this result lies in the fact that it provides information about the decomposition numbers of symmetric groups of a completely different kind than was previously available. Also it is a starting point for investigations on the modular irreducible representations of the alternating groups. From a combinatorial point of view the Mullineux map gives a p-analogue of the conjugation map on partitions. The analysis of its fixed points has led to some interesting general partition identities [1, 2].

The origin of this conjecture was a paper by Mullineux [14], where he defined a bijective involutory map $\lambda \to \lambda^M$ on the set of p-regular partitions and conjectured that this map coincides with the map $\lambda \to \lambda^P$. The statement "$M = P$" is the Mullineux conjecture. To each p-regular partition Mullineux associated a double array of integers, known now as the Mullineux symbol and the Mullineux map is defined as an operation on these symbols. The Mullineux symbol may be seen as a p-analogue of the Frobenius symbol for partitions.

Before the proof of the Mullineux conjecture many pieces of evidence for it had been found, both of a combinatorial as well as of representation-theoretical nature. The breakthrough was a series of papers by Kleshchev [7–9] on “modular branching”, i.e., on the restrictions of modular irreducible representations from S_n to S_{n-1}. Using these results Kleshchev [9] reduced the Mullineux conjecture to a purely combinatorial statement about the compatibility of the Mullineux map with the removal of “good nodes” (see below). A long and complicated proof of this combinatorial statement was then given in a paper by Ford and Kleshchev [4].

In his work on modular branching Kleshchev introduced two important notions, normal and good nodes in p-regular partitions. Their importance has been stressed even further in recent work of Kleshchev [10] on modular restriction. Also these notions occur in the work of Lascoux et al. on Hecke Algebras at roots of unity and crystal bases of quantum affine algebras [11]; it was discovered that Kleshchev’s p-good branching graph on p-regular partitions is exactly the crystal graph of the basic module of the quantized affine Lie algebra $U_q(\widehat{sl}_p)$ which had been studied by Misra and Miwa [12].

From the above it is clear that a better understanding of the Mullineux symbols is desirable including their relation to the existence of good and normal nodes in the corresponding partition. In the present paper this relation will be explained explicitly. We introduce a variation of the Mullineux symbol called the residue symbol for p-regular partitions. In terms of these the detection of good nodes is easy and the removal of good nodes has a very simple effect on the residue symbol. In particular this implies a shorter and much more transparent proof of the combinatorial part of the Mullineux conjecture with additional insights (Section 4). We also note that the good behaviour of the residue symbols with respect to removal of good nodes allows one to give an alternative description of the p-good branching graph, and thus of the crystal graph mentioned above. Some further illustrations of the usefulness of residue symbols are given in Section 3. This includes combinatorial results on the fixed points of the Mullineux map.

2. Basic definitions and preliminaries

Let p be a natural number.

Let λ be a p-regular partition of n. The *p-rim* of λ is a part of the rim of λ ([6], p. 56), which is composed of *p-segments*. Each p-segment except possibly the last contains p points. The first p-segment consists of the first p points of the rim of λ, starting with the longest row. (If the rim contains at most p points it is the entire rim.) The next segment is obtained by starting in the row next below the previous p-segment. This process is continued until the final row is reached. We let a_1 be the number of nodes in the p-rim of $\lambda = \lambda^{(1)}$ and let r_1 be the number of rows in λ. Removing the p-rim of $\lambda = \lambda^{(1)}$ we get a new p-regular partition $\lambda^{(2)}$ of $n - a_1$. We let a_2, r_2 be the length of the p-rim and the number of parts of $\lambda^{(2)}$, respectively. Continuing this way we get a sequence of partitions $\lambda = \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)}$, where $\lambda^{(m)} \neq 0$ and $\lambda^{(m+1)} = 0$, and a corresponding *Mullineux symbol* of λ

$$G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ r_1 & r_2 & \cdots & r_m \end{pmatrix}.$$

The integer m is called the *length* of the symbol. For $p > n$, the well-known Frobenius symbol $F(\lambda)$ of λ is obtained from $G_p(\lambda)$ as above by

$$F(\lambda) = \begin{pmatrix} a_1 - r_1 & a_2 - r_2 & \cdots & a_m - r_m \\ r_1 - 1 & r_2 - 1 & \cdots & r_m - 1 \end{pmatrix}.$$

As usual, here the top and bottom line give the arm and leg lengths of the principal hooks.

It is easy to recover a p-regular partition λ from its Mullineux symbol $G_p(\lambda)$. Start with the hook $\lambda^{(m)}$, given by a_m, r_m, and work backwards. In placing each p-rim it is convenient to start from below, at row r_i. Moreover, by a slight reformulation of a result in [14], the entries of $G_p(\lambda)$ satisfy (see [1])

(1) $\varepsilon_i \leq r_i - r_{i+1} < p + \varepsilon_i,\ 1 \leq i \leq m-1;\ 1 \leq r_m < p + \varepsilon_m$
(2) $r_i - r_{i+1} + \varepsilon_{i+1} \leq a_i - a_{i+1} < p + r_i - r_{i+1} + \varepsilon_{i+1};\ 1 \leq i \leq m-1;\ r_m \leq a_m < p + r_m$
(3) $\sum_i a_i = n$

where $\varepsilon_i = 1$ if $p \nmid a_i$ and $\varepsilon_i = 0$ if $p \mid a_i$. If $p \mid a_i$, we call the corresponding column $\binom{a_i}{r_i}$ of the Mullineux symbol a *singular* column, otherwise the column is called *regular*.

If $G_p(\lambda)$ is as above then the Mullineux conjugate λ^M of λ is by definition the p-regular partition satisfying

$$G_p(\lambda^M) = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ s_1 & s_2 & \cdots & s_m \end{pmatrix} \quad \text{where } s_i = a_i - r_i + \varepsilon_i.$$

In particular, for $p > n$, this is just the ordinary conjugation of partitions.

Example Let $p = 5$, $\lambda = (8, 6, 5^2)$, then

$$\begin{array}{cccccccc} 4 & 4 & 3 & 2 & 2 & 1 & 1 & 1 \\ 4 & 3 & 3 & 2 & 1 & 1 & & \\ 3 & 3 & 2 & 2 & 1 & & & \\ 2 & 1 & 1 & 1 & 1 & & & \end{array} \qquad G_5(\lambda) = \begin{pmatrix} 10 & 6 & 5 & 3 \\ 4 & 4 & 3 & 2 \end{pmatrix}$$

$$\begin{array}{cccccccccc} 4 & 4 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 \\ 4 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & & \\ 2 & 1 & & & & & & & & \\ 1 & 1 & & & & & & & & \\ 1 & & & & & & & & & \\ 1 & & & & & & & & & \end{array} \qquad G_5(\lambda^M) = \begin{pmatrix} 10 & 6 & 5 & 3 \\ 6 & 3 & 2 & 2 \end{pmatrix}$$

(In both cases the nodes of the successive 5-rims are numbered 1, 2, 3, 4).

Thus $(8, 6, 5^2)^M = (10, 8, 2^2, 1^2)$.

Now let p be a prime number and consider the modular representations of S_n in characteristic p; note that for all purely combinatorial results the condition of primality is not needed.

The modular irreducible representations D^λ of S_n may be labelled by *p-regular* partitions λ of n, a partition being p-regular if no part is repeated p (or more) times ([6], Section 6.1); this is the labelling we will consider in the sequel.

Tensoring the modular representation D^λ of S_n by the sign representation of S_n gives another modular irreducible representation, labelled by a p-regular partition λ^P. Mullineux has then conjectured [14]:

Conjecture *For any p-regular partition λ of n we have $\lambda^P = \lambda^M$.*

If λ is a p-regular partition we let as before

$$G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ r_1 & r_2 & \cdots & r_m \end{pmatrix}$$

denote its Mullineux symbol. We then define the *Residue symbol* $R_p(\lambda)$ of λ as

$$R_p(\lambda) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{Bmatrix}$$

where x_j is the residue of $a_{m+1-j} - r_{m+1-j}$ modulo p and y_j is the residue of $1 - r_{m+1-j}$ modulo p. Note that the Mullineux symbol $G_p(\lambda)$ can be recovered from the Residue symbol $R_p(\lambda)$ because of the strong restrictions on the entries in the Mullineux symbol. Also, it is very useful to keep in mind that for a residue symbol there are no restrictions except that $(x_1, y_1) \neq (0, 1)$ (which would correspond to starting with the p-singular partition (1^p)). We also note that a column $\binom{x_j}{y_j}$ in $R_p(\lambda)$ is a singular column in $G_p(\lambda)$ if and only if $x_j + 1 \equiv y_j \pmod p$.

Example $p = 5, \lambda = (10, 8, 7, 5, 3, 2^2)$, then

$$G_5(\lambda) = \begin{pmatrix} 15 & 12 & 7 & 3 \\ 7 & 6 & 3 & 2 \end{pmatrix} \quad \text{and} \quad R_5(\lambda) = \begin{Bmatrix} 1 & 4 & 1 & 3 \\ 4 & 3 & 0 & 4 \end{Bmatrix}.$$

Also for the residue symbol of a p-regular partition we have a good description of the residue symbol of its Mullineux conjugate; this is just obtained by translating the definition of the Mullineux map on the Mullineux symbol to the residue symbol notation.

Lemma 2.1 *Let the residue symbol of the p-regular partition λ be*

$$R_p(\lambda) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{Bmatrix}.$$

Then the residue symbol of λ^M *is*

$$R_p(\lambda^M) = \begin{Bmatrix} \delta_1 - y_1 & \cdots & \delta_m - y_m \\ \delta_1 - x_1 & \cdots & \delta_m - x_m \end{Bmatrix}$$

where

$$\delta_j = \begin{cases} 1 & \text{if } x_j + 1 = y_j \\ 0 & \text{otherwise} \end{cases}.$$

Notation. We now fix a p-regular partition λ. Then $\tilde{\lambda}$ denotes the partition obtained from λ by removing all those parts which are equal to 1. We will assume that λ has d such parts, $0 \leq d \leq p-1$. Moreover, we let μ be the partition obtained from λ by subtracting 1 from all its parts. We say that μ is obtained by removing the first column from λ. Unless otherwise specified we assume that the residue symbol $R_p(\lambda)$ for λ is as above.

For later induction arguments we formulate the connection between the residue symbols of λ and μ. First we consider the process of first column removal; this is an easy consequence of Proposition 1.3 in [3] and the definition of the residue symbol.

Lemma 2.2 *Suppose that*

$$R_p(\lambda) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{Bmatrix}.$$

Then

$$R_p(\mu) = \begin{Bmatrix} x'_1 & x'_2 & \cdots & x'_m \\ y'_1 & y'_2 & \cdots & y'_m \end{Bmatrix}$$

where for $1 \leq j \leq m$

$$x'_j = x_j - \nu_j$$
$$y'_j = y_{j-1} - \nu_j.$$

Here y_0 *is defined to be* 1 *and the* ν_j*'s are defined by*

$$\nu_j = \begin{cases} 0 & \text{if } x_j + 1 = y_{j-1} \\ 1 & \text{otherwise} \end{cases}.$$

Moreover, if $x_1 = 0$ *then the first column in* $R_p(\mu)$ *(consisting of* x'_1 *and* y'_1*) is omitted.*

Remark 2.3 In the notation of Lemma 2.2 the number d of parts equal to 1 in λ is determined by the congruence

$$d \equiv y'_m - y_m = y_{m-1} - y_m - \nu_m \pmod p$$

Moreover, since r_1 is the number of parts of λ and $y_m \equiv 1 - r_1$ it is clear that y_m is the p-residue of the lowest node in the first column of λ.

Next we consider the relationship between λ and μ from the point of adding a column to μ; this follows from Proposition 1.6 in [3].

Lemma 2.4 *Suppose that*

$$R_p(\mu) = \begin{Bmatrix} x'_1 & x'_2 & \cdots & x'_m \\ y'_1 & y'_2 & \cdots & y'_m \end{Bmatrix}.$$

Then

$$R_p(\lambda) = \begin{Bmatrix} x_0 & x_1 & \cdots & x_m \\ y_0 & y_1 & \cdots & y_m \end{Bmatrix},$$

where for $1 \leq j \leq m$

$$x_j = x'_j + \nu'_j, \quad y_{j-1} = y'_j + \nu'_j.$$

Here $x_0 = 0$, $y_m = y'_m - d$ *and the* ν'_j*'s are defined by*

$$\nu'_j = \begin{cases} 0 & \text{if } x'_j + 1 = y'_j \\ 1 & \text{otherwise} \end{cases}$$

Moreover, if $y'_1 = 0$ *and* $\nu'_1 = 1$, *then the first column in* $R_p(\lambda)$ (*consisting of* x_0 *and* y_0) *is omitted.*

Remark 2.5 In the notation of Lemma 2.2 and Lemma 2.4 we have

$$\nu_j = \nu'_j \quad \text{for } 1 \leq j \leq m.$$

Indeed,

$$\begin{aligned} \nu_j = 0 &\Leftrightarrow x_j + 1 = y_{j-1} \quad \text{(by definition of } \nu_j) \\ &\Leftrightarrow x'_j + \nu'_j + 1 = y'_j + \nu'_j \quad \text{(by Lemma 2.4)} \\ &\Leftrightarrow x'_j + 1 = y'_j \\ &\Leftrightarrow \nu'_j = 0 \quad \text{(by definition of } \nu'_j) \end{aligned}$$

3. Mullineux fixed-points in a *p*-block

The *p-core* $\lambda_{(p)}$ of a partition λ is obtained by removing p-hooks as much as possible; while the removal process is not unique the resulting p-regular partition is unique as can most easily be seen in the abacus framework introduced by James. The reader is referred to [6] or [17] for a more detailed introduction into this notion and its properties. We define the *weight* w of λ by $w = (|\lambda| - |\lambda_{(p)}|)/p$.

The representation-theoretic significance of the p-core is the fact that it determines the p-block to which an ordinary or modular irreducible character labelled by λ belongs. The *weight* of a p-block is the common weight of the partitions labelling the characters in the block.

Let $\lambda = (l_1 \geq l_2 \geq \cdots \geq l_k > 0)$ be a partition of n. Then

$$Y(\lambda) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq i \leq k, 1 \leq j \leq l_i\} \subset \mathbb{Z} \times \mathbb{Z}$$

is the Young diagram of λ, and $(i, j) \in Y(\lambda)$ is called a *node* of λ. If $A = (i, j)$ is a node of λ and $Y(\lambda)\backslash\{(i, j)\}$ is again a Young diagram of a partition, then A is called a *removable* node and $\lambda\backslash A$ denotes the corresponding partition of $n - 1$.

Similarly, if $A = (i, j) \in \mathbb{N} \times \mathbb{N}$ is such that $Y(\lambda) \cup \{(i, j)\}$ is the Young diagram of a partition of $n + 1$, then A is called an *indent* node of λ and the corresponding partition is denoted $\lambda \cup A$.

The *p-residue* of a node $A = (i, j)$ is defined to be the residue modulo p of $j - i$, denoted res $A = j - i \pmod p$. The p-residue diagram of λ is obtained by writing the p-residue of each node of the Young diagram of λ in the corresponding place.

Example $p = 5, \lambda = (6^2, 5, 4)$

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 0 \\ 4 & 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 0 & 1 & 2 & \\ 2 & 3 & 4 & 0 & & \end{array}$$

The *p-content* $c(\lambda) = (c_0, \ldots, c_{p-1})$ of a partition λ is defined by counting the number of nodes of a given residue in the p-residue diagram of λ, i.e., c_i is the number of nodes of λ of p-residue i. In the example above, the p-content of λ is $c(\lambda) = (c_0, \ldots, c_4) = (5, 3, 4, 4, 5)$.

It is important to note that the p-content determines the p-core of a partition. This can be explained as follows. First, for given $c = (c_0, c_1, \ldots, c_{p-1})$ we define the associated $\vec{n}$-vector by $\vec{n} = (c_0 - c_1, c_1 - c_2, \ldots, c_{p-2} - c_{p-1}, c_{p-1} - c_0)$. Now, for any vector

$$\vec{n} \in \left\{ (n_0, \ldots, n_{p-1}) \in \mathbb{Z}^p \;\middle|\; \sum_{i=0}^{p-1} n_i = 0 \right\}$$

there is a unique p-core μ with this $\vec{n}$-vector $\vec{n}$ associated to its p-content $c(\mu)$ (for short, we also say that $\vec{n}$ is associated to μ.) We refer the reader to [5] for the description of the explicit bijection giving this relation. From [5] we also have the following

Proposition 3.1 *Let μ be a p-core with associated $\vec{n}$-vector $\vec{n}$. Then*

$$|\mu| = p\frac{\|\vec{n}\|^2}{2} + \vec{b}\vec{n} = \frac{p}{2}\sum_{i=0}^{p-1} n_i^2 + \sum_{i=1}^{p-1} i n_i$$

with $\vec{b} = (0, 1, \ldots, p-1)$.

How do we obtain the $\vec{n}$-vector associated to λ from its Mullineux or residue symbol? This is answered by the following

Proposition 3.2 *Let λ be a p-regular partition whose Mullineux symbol and residue symbol are*

$$G_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ r_1 & r_2 & \cdots & r_m \end{pmatrix} \quad \text{and} \quad R_p(\lambda) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{Bmatrix},$$

respectively. Then the associated $\vec{n}$-vector $\vec{n} = (n_0, \ldots, n_{p-1})$ is given by

$$\begin{aligned} n_j &= |\{i \mid a_i - r_i \equiv j \bmod p\}| - |\{i \mid -r_i \equiv j \bmod p\}| \\ &= |\{i \mid x_i = j\}| - |\{i \mid y_i = j+1\}| \end{aligned}$$

Proof: In the residue symbol, singular columns do not contribute to the n-vector as they contain the same number of nodes for each residue. So let us consider a regular column $\binom{x}{y}$, respectively, $\binom{a}{r}$, in the Mullineux symbol and the corresponding p-rim in the p-residue diagram. In this case, the contribution only comes from the last section of the p-rim. The final node is in row r and column 1 so its p-residue is $1 - r \equiv y \pmod p$. What is the p-residue of the top node of this rim section? The length of this section is $\equiv a \pmod p$, hence we have to go $\equiv a - 1$ steps from the final node of residue y to the top node of the section, which hence has p-residue $\equiv y + a - 1 \equiv 1 - r + a - 1 \equiv a - r \equiv x$ (going one step northwards or eastwards always increases the p-residue by 1!). Thus going along the residues in the last section we have a strip $y, y+1, \ldots, x-1, x$. Now the contribution of the intermediate residues to the $\vec{n}$-vector cancel out, and we only have a contribution 1 for n_x and -1 for n_{y-1}, which proves the claim. □

First we use the preceding proposition to give a short proof of a relation already noticed by Mullineux [15]:

Corollary 3.3 *Let λ be a p-regular partition. Then*

$$(\lambda^M)_{(p)} = \lambda'_{(p)}.$$

Proof: Let the residue symbol of λ be $R_p(\lambda) = \left\{\begin{smallmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{smallmatrix}\right\}$.

So by Lemma 2.1 we have $R_p(\lambda^M) = \{ {\delta_1 - y_1 \cdots \delta_m - y_m \atop \delta_1 - x_1 \cdots \delta_m - x_m} \}$ with $\delta_j = 1$ if $x_j + 1 = y_j$ and 0 otherwise.

Now we consider the contributions of the entries in the residue symbol to the $\vec{n}$-vectors. If $x_i + 1 \neq y_i, x_i = j, y_i = k+1$, then we get a contribution 1 to $n_j(\lambda)$ and -1 to $n_k(\lambda)$ on the one hand, and a contribution 1 to $n_{-(k+1)}(\lambda^M)$ and -1 to $n_{-(j+1)}(\lambda^M)$ on the other hand. If $x_i + 1 = y_i$, then from column i in the residue symbol we get a contribution neither to $\vec{n}(\lambda)$ nor to $\vec{n}(\lambda^M)$. Hence $n_j(\lambda^M) = -n_{-(j+1)}(\lambda)$ for all j, i.e., if $\vec{n}(\lambda) = (n_0, \ldots, n_{p-1})$, then $\vec{n}(\lambda^M) = (-n_{p-1}, \ldots, -n_0)$.

Now let $c(\lambda) = (c_0, \ldots, c_{p-1})$ be the p-content of λ, then $c(\lambda') = (c_0, c_{p-1}, \ldots, c_1)$, and hence

$$\begin{aligned}\vec{n}(\lambda') &= (c_0 - c_{p-1}, c_{p-1} - c_{p-2}, \ldots, c_2 - c_1, c_1 - c_0)\\ &= (-n_{p-1}, -n_{p-2}, \ldots, -n_1, -n_0)\\ &= \vec{n}(\lambda^M)\end{aligned}$$

Thus $(\lambda^M)_{(p)} = (\lambda')_{(p)} = \lambda'_{(p)}$. □

Now we turn to Mullineux fixed-points.

Proposition 3.4 *Let p be an odd prime and suppose that λ is a p-regular partition with $\lambda = \lambda^M$. Then the representation D^λ belongs to a p-block of even weight w.*

Proof: If $\lambda = \lambda^M$, then its Mullineux symbol is of the form

$$G_p(\lambda) = G_p(\lambda^M) = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ \dfrac{a_1 + \varepsilon_1}{2} & \dfrac{a_2 + \varepsilon_2}{2} & \cdots & \dfrac{a_m + \varepsilon_m}{2} \end{pmatrix}$$

where as before $\varepsilon_i = 1$ if $p \nmid a_i$ and $\varepsilon_i = 0$ if $p \mid a_i$, and where a_i is even if and only if $p \mid a_i$.

Now by Proposition 3.1 we have

$$w = \frac{1}{p}\left(\sum_j a_j - p\frac{\|\vec{n}\|^2}{2} - \vec{b} \cdot \vec{n} \right)$$

where $\vec{n} = \vec{n}(\lambda) = (n_0, \ldots, n_{p-1})$ is the $\vec{n}$-vector associated to λ and $\vec{b} = (0, 1, \ldots, p-1)$. By Proposition 3.2 we have

$$n_j = \left| \left\{ i \;\middle|\; \frac{a_i - \varepsilon_i}{2} \equiv j \bmod p \right\} \right| - \left| \left\{ i \;\middle|\; \frac{-a_i - \varepsilon_i}{2} \equiv j \bmod p \right\} \right|$$

For $a_i \equiv 0 \pmod p$ we do not get a contribution to the $\vec{n}$-vector. For $a_i \not\equiv 0 \pmod p$ with $\frac{a_i - 1}{2} \equiv j \pmod p$ we get a contribution 1 to n_j and -1 to $n_{-(j+1)}$. Note that we cannot get any contribution to $n_{\frac{p-1}{2}}$. Thus we have

$$\vec{n} = \left(n_0, n_1, \ldots, n_{\frac{p-3}{2}}, 0, -n_{\frac{p-3}{2}}, \ldots, -n_0\right).$$

Now we obtain for the weight modulo 2:

$$\begin{aligned} w &\equiv \sum_j a_j + \sum_{i=0}^{\frac{p-3}{2}} n_i^2 + \sum_{i=0}^{\frac{p-3}{2}} n_i(i + (p-1-i)) \\ &\equiv |\{j \mid a_j \not\equiv 0 \bmod 2\}| + \sum_{i=0}^{\frac{p-3}{2}} n_i^2 \\ &\equiv \sum_{i=0}^{\frac{p-3}{2}} n_i + \sum_{i=0}^{\frac{p-3}{2}} n_i^2 \\ &\equiv 0 \end{aligned}$$

Hence the weight is even, as claimed. □

For the following theorem we recall the definition of the numbers $k(r, s)$:

$$k(r,s) = \left| \left\{ (\lambda^1, \ldots, \lambda^r) \mid \lambda^i \text{ is a partition for all } i, \text{ and } \sum_{i=1}^{r} |\lambda^i| = s \right\} \right|$$

In view of the now proved Mullineux conjecture, the following combinatorial result implies a representation-theoretical result in [16].

Theorem 3.5 *Let p be an odd prime. Let μ be a symmetric p-core and $n \in \mathbb{N}$ with $w = \frac{n-|\mu|}{p}$ even. Then*

$$k\left(\frac{p-1}{2}, \frac{w}{2}\right) = |\{\lambda \vdash n \mid \lambda = \lambda^M, \lambda_{(p)} = \mu\}|$$

Proof: We set

$$\mathcal{F}(\mu) = \{\lambda \vdash n \mid \lambda = \lambda^M, \lambda_{(p)} = \mu\}.$$

For $\lambda \in \mathcal{F}(\mu)$ we consider its Mullineux symbol; as λ is a Mullineux fixed-point this has the form

$$G_p(\lambda) = G_p(\lambda^M) = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \\ \frac{a_1+\varepsilon_1}{2} & \frac{a_2+\varepsilon_2}{2} & \cdots & \frac{a_m+\varepsilon_m}{2} \end{pmatrix}$$

with $\varepsilon_i = 1$ if $p \nmid a_i$ and $\varepsilon_i = 0$ if $p \mid a_i$, and a_i being even if and only if $p \mid a_i$.

In this special situation the general restrictions on the entries in Mullineux symbols stated at the beginning of Section 2 are now given by:

(i) $0 \leq a_i - a_{i+1} \leq 2p$ for all i.
(ii) If $a_i = a_{i+1}$ then a_i is even.
(iii) If $a_i - a_{i+1} = 2p$ then a_i is odd.
(iv) a_i is even if and only if $p \mid a_i$.
(v) $\sum_i a_i = n$.

We have already explained before how to read off the p-core of a partition from its Mullineux symbol by calculating the $\vec{n}$-vector. In the proof of the previous proposition we have already noticed that entries $a_i \equiv 0 \pmod{p}$ do not contribute to the $\vec{n}$-vector.

Now the partitions $(a_1, \ldots, a_m) \vdash n$ with properties (i) to (iv) above are just the partitions satisfying the special congruence and difference conditions for $N = 2p$ and the congruence set

$$\mathcal{C} = \left\{2j + 1 \mid j = 0, \ldots, \frac{p-3}{2}, \frac{p+1}{2}, \ldots, p-1\right\}$$

considered in [1, 2]. The bijection described there transforms the set of partitions above into the set

$$\mathcal{D} = \{b = (b_1, \ldots, b_l) \vdash n \mid b_1 > \cdots > b_l,\ \mathrm{mod}_N b_i \in \mathcal{C}\}$$

where $\mathrm{mod}_N b$ denotes the smallest positive number congruent to b mod N. Computing the $\vec{n}$-vector from the b_i's instead of the a_i's with the formula given in the previous proof then gives the same answer since the congruence sequence of the b_i's is the same as the congruence sequence of the *regular* a_i's. For a bar partition $b \in \mathcal{D}$ as above we then compute its so called *N-bar quotient*; since b has no parts congruent to 0 or p modulo N, the bar quotient is a $\frac{p-1}{2}$-tuple of partitions. For the properties of these objects we refer the reader to [13, 17]. It remains to check that the N-weight of b equals $\frac{w}{2}$, i.e., that the *N-bar core* $\rho = b_{(\bar{N})}$ of b satisfies $|\rho| = |\mu|$.

We recall from above that we have for the $\vec{n}$-vector of λ:

$$n_j = \left|\left\{i \;\middle|\; \frac{a_i - \varepsilon_i}{2} \equiv j \bmod p\right\}\right| - \left|\left\{i \;\middle|\; \frac{-a_i - \varepsilon_i}{2} \equiv j \bmod p\right\}\right|$$

and

$$\vec{n} = \left(n_0, n_1, \ldots, n_{\frac{p-3}{2}}, 0, -n_{\frac{p-3}{2}}, \ldots, -n_0\right).$$

Hence by Proposition 3.1 we obtain

$$|\lambda_{(p)}| = |\mu| = \sum_{i=0}^{\frac{p-3}{2}} p n_i^2 + \sum_{i=0}^{\frac{p-3}{2}} n_i(2i - p + 1).$$

As remarked before the bijection transforming $a = (a_1, \ldots, a_m)$ into $(b_1, \ldots, b_l)$ leaves the sequence of congruences modulo $N = 2p$ of the regular elements in a invariant. Now for determining the N-bar core of b we have to pair off b_i's congruent to $2j + 1$ modulo $N = 2p$ with b_i's congruent to $2p - (2j + 1)$, for each $j = 0, \ldots, \frac{p-3}{2}$, and only have to know for each such j the number

$$|\{i \mid b_i \equiv 2j + 1 \bmod 2p\}| - |\{i \mid b_i \equiv 2p - (2j + 1) \bmod 2p\}|.$$

But this is equal to

$$\left|\left\{i \;\middle|\; \frac{b_i - 1}{2} \equiv j \bmod p\right\}\right| - \left|\left\{i \;\middle|\; \frac{-b_i - 1}{2} \equiv j \bmod p\right\}\right|$$

which is same as

$$\left|\left\{i \;\middle|\; \frac{a_i - \varepsilon_i}{2} \equiv j \bmod p\right\}\right| - \left|\left\{i \;\middle|\; \frac{-a_i - \varepsilon_i}{2} \equiv j \bmod p\right\}\right|,$$

which finally is n_j.

Now the contribution to the $2p$-bar core from the conjugate runners $2j+1$ and $2p - (2j + 1)$ for $j = 0, \ldots, \frac{p-3}{2}$ is for any value of n_j easily checked to be

$$n_j(2j + 1) + n_j(n_j - 1)p = n_j(2j + 1 - p) + p\, n_j^2.$$

Thus the total contribution to the $2p$-bar core is exactly the same as the one calculated above, i.e., we have $|\mu| = |\rho|$ as was to be proved. □

4. The combinatorial part of the Mullineux conjecture

We are now going to introduce the main combinatorial concepts for our investigations. The concept of the node signature sequence and the definition of its good nodes have their origin in Kleshchev's definition of good nodes of a partition. First we recapitulate his original definition [8].

We write the given partition in the form

$$\lambda = \left(\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_k^{a_k}\right)$$

where $\lambda_1 > \lambda_2 > \cdots \lambda_k > 0$, $a_i > 0$ for all i.

For $1 \leq i \leq j \leq k$ we then define

$$\beta(i, j) = \lambda_i - \lambda_j + \sum_{t=i}^{j} a_t \quad \text{and} \quad \gamma(i, j) = \lambda_i - \lambda_j + \sum_{t=i+1}^{j} a_t.$$

Furthermore, for $i \in \{1, \ldots, k\}$ let

$$M_i = \{j \mid 1 \leq j < i, \beta(j, i) \equiv 0 \pmod p\}.$$

We then call i *normal* if and only if for all $j \in M_i$ there exists $d(j) \in \{j+1, \ldots, i-1\}$ satisfying $\beta(j, d(j)) \equiv 0 \pmod p$, and such that $|\{d(j) \mid j \in M_i\}| = |M_i|$.

We call i *good* if it is normal and if $\gamma(i, i') \not\equiv 0 \pmod p$ for all normal $i' > i$.

Let us translate this into properties of the nodes of λ in the Young diagram that can most easily be read off the p-residue diagram of λ. One sees immediately that $\beta(i, j)$ is just the length of the path from the node at the beginning of the ith block of λ to the node at the end of the jth block of λ. The condition $\beta(i, j) \equiv 0 \pmod p$ is then equivalent to the equality of the p-residue of the indent node in the outer corner of the ith block and the p-residue of the removable node at the inner corner of the jth block.

Similarly, $\gamma(i, j) \equiv 0 \pmod p$ is equivalent to the equality of the p-residues of the removable nodes at the end of the ith and jth block.

We will say that a node $A = (i, j)$ is *above* the node $B = (i', j')$ or B is *below* A) if $i < i'$, and write this relation as $B \nearrow A$. Then a removable node A of λ is *normal* if for any $B \in \mathcal{M}_A = \{C \mid C$ indent node of λ above A with res $C =$ res $A\}$ we can choose a removable node C_B of λ with $A \nearrow C_B \nearrow B$ and res $C_B =$ res A, such that $|\{C_B \mid B \in \mathcal{M}_A\}| = |\mathcal{M}_A|$. A node A is *good* if it is the lowest normal node of its p-residue.

Consider the example $\lambda = (11, 9^2, 6, 4^2, 2, 1)$, $p = 5$. In the p-residue diagram below we have included the indent node at the beginning of the second block, marked $\underline{3}$, and we have also marked in boldface the removable node of residue 3 at the end of the fourth block. The equality of these residues corresponds to $\beta(2, 4) \equiv 0 \pmod 5$. We also see immediately from the diagram below that $\gamma(4, 6) \equiv 0 \pmod 5$.

0	1	2	3	4	0	1	2	3	4	0
4	0	1	2	3	4	0	1	2	$\underline{3}$	
3	4	0	1	2	3	4	0	1		
2	3	4	0	1	2					
1	2	3	4							
0	1	2	**3**							
4	0									
3										

The set M_i corresponds in this picture to taking the removable node, say A, at the end of the ith block and then collecting into M_i (respectively $\mathcal{M}_A$) all the indent nodes above this block of the same p-residue as A. For i (respectively A) being normal, we then have to check whether for any such indent node, B say, at the end of the jth block we can find a removable node $C = C_B$ between A and B of the same p-residue, and such that the collection of all these removable nodes has the same size as M_i (respectively $\mathcal{M}_A$). The node A (respectively i) is then good if A is the lowest normal node of its p-residue.

The critical condition for the normality of i (respectively A) above is just a lattice condition: it says that in any section above A there are at least as many removable nodes of the p-residue of A as there are indent nodes of the same residue.

With these notions the Mullineux conjecture was reduced by Kleshchev to combinatorial form as below:

Conjecture *Let* λ *be a* p*-regular partition*, A *a good node of* λ*. Then there exists a good node* B *of the Mullineux image* λ^M *such that* $(\lambda\backslash A)^M = \lambda^M\backslash B$.

Now we define signature sequences.

A (p)-*signature* is a pair $c\varepsilon$ where $c \in \{0, 1, \ldots, p-1\}$ is a residue modulo p and $\varepsilon = \pm$ is a sign. Thus $2+$ and $3-$ are examples of 5-signatures.

A (p)-*signature sequence* X is a sequence

$$X : c_1\varepsilon_1\ c_2\varepsilon_2\ \cdots\ c_t\varepsilon_t$$

where each $c_i\varepsilon_i$ is a signature.

Given such a signature sequence X we define for $0 \le i \le p-1$ and $1 \le j \le t$

$$\sigma_X(i, j) = \sigma(i, j) = \sum_{\substack{k\le j\\ c_k=i}} \varepsilon_k.$$

We make the conventions that an empty sum is 0 and that $+$ is counted as $+1$ and $-$ as -1 in the sum.

The i*th peak value* $\pi_i(X)$ for X is defined as

$$\pi_i(X) = \max\{0, \sigma(i, j) \mid 1 \le j \le t\}$$

and the i*th end value* $\omega_i(X)$ for X is defined as

$$\omega_i(X) = \sigma(i, t).$$

We call i a *good residue for* X if $\pi_i(X) > 0$. In that case let

$$k = \min\{j \mid \sigma(i, j) = \pi_i(X)\},$$

and we then say that the residue c_k at step k is i*-good* for X, for short: c_k is i*-good* for X. Let us note that if c_k is i-good for X then $c_k = i$ and $\varepsilon_k = +$. Indeed, if $k = 1$ this is clear since otherwise $\pi_i(X) \le \sigma(i, 1) \le 0$. Assume $k > 1$. If $c_k \ne i$ then $\sigma(i, k) = \sigma(i, k-1)$, contrary to the definition of c_k. If $c_k = i$ and $\varepsilon_k = -1$ then $\sigma(i, k-1) > \sigma(i, k) = \pi_i(X)$, contrary to the definition of $\pi_i(X)$.

The residue c_l is called i*-normal* if c_l is i-good for the truncated sequence

$$X : c_1\varepsilon_1\ c_2\varepsilon_2\ \cdots\ c_l\varepsilon_l$$

The following is quite obvious from the definitions.

Lemma 4.1 *Let* $X^* : c_1\varepsilon_1\, c_2\varepsilon_2 \cdots c_{t-1}\varepsilon_{t-1}$ *be a signature sequence and let* X *be obtained from* X^* *by adding a signature* $c_t\varepsilon_t$ *at the end. For* $0 \le i \le p-1$ *the following statements are equivalent*:

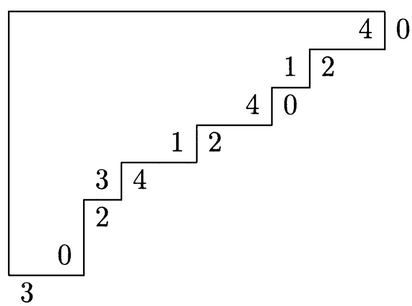
(1) $\pi_i(X) = \pi_i(X^*) + 1$.
(2) $\pi_i(X) \neq \pi_i(X^*)$.
(3) $c_t\varepsilon_t = i+$ *and* $\omega_i(X^*) = \pi_i(X^*)$.
(4) c_t *is* i*-good for* X.

We are going to define two signature sequences based on λ, the *node sequence* $N(\lambda)$ and the *Mullineux sequence* $M(\lambda)$. Although they are defined in very different ways we will show that they have the same peak and end value for each i.

The *node sequence* $N(\lambda)$ consists of the residues of the indent and removable nodes of λ, read from right to left, top to bottom in λ. For each indent residue the sign is $+$ and for each removable residue the sign is $-$.

Let us note that according to Remark 2.3 the final signature in $N(\lambda)$ is $(y_m - 1)-$.

Example Let $p = 5$, $\lambda = (10, 8, 7, 5, 3, 2^2)$. Below, we have only indicated the removable and indent nodes in the 5-residue diagram of λ.



$$N(\lambda): 0-\ \ 4+'\ \ 2-\ \ 1+'\ \ 0-\ \ \underline{4+}'\ \ 2-\ \ \underline{1+}'\ \ 4-\ \ \underline{3+}'\ \ 2-\ \ 0+\ \ 3-$$

Residue	0	1	2	3	4
End value	-1	2	-3	0	1
Peak value	0	2	0	1	2
Good?	N	Y	N	Y	Y

(The good signatures (peaks) are underlined and the normal signatures marked with a prime.)

In other words, in the node sequence $N(\lambda)$ defined before, if $c_m\varepsilon_m$ corresponds to the removable node A, then $c_m = \text{res } A$, $\varepsilon = +$, and A is normal if and only if the sequence of signs to the left of A belonging to c_j's with $c_j = \text{res } A$ is latticed read from right to left. Again, the node A (respectively c_m) is good if it is the last normal node of its residue respectively of its value. The peak value of the node sequence $N(\lambda)$ is the number of normal nodes of λ.

Remark 4.2 Let, as before, $\tilde{\lambda}$ denote the partition obtained from λ by removing all those parts which are equal to 1, and let μ be the partition obtained from λ by subtracting 1 from all its parts. From the definitions it is obvious that for all i

$$\pi_i(N(\tilde{\lambda})) = \pi_{i-1}(N(\mu)).$$

Proposition 4.3 *Let λ and μ be as above, and let d be the number of parts 1 in λ.*
(1) *If $i \neq y_m$ and $i \neq y_m - 1$ then*

$$\omega_i(N(\lambda)) = \omega_{i-1}(N(\mu)).$$

(2) *If $i = y_m$ then*

$$\omega_i(N(\lambda)) = \omega_{i-1}(N(\mu)) + 1$$

and if $i = y_m - 1$ then

$$\omega_i(N(\lambda)) = \omega_{i-1}(N(\mu)) - 1.$$

(3) *We have*

$$\pi_i(N(\lambda)) = \pi_{i-1}(N(\mu))$$

unless the following conditions are all fulfilled
(i) $i = y_m$
(ii) $d > 0$
(iii) $\omega_{i-1}(N(\mu)) = \pi_{i-1}(N(\mu))$.
In that case y_m is i-good for $N(\lambda)$ and

$$\pi_i(N(\lambda)) = \pi_{i-1}(N(\mu)) + 1.$$

Proof: Assume that $N(\lambda)$ consists of m' signatures (m' is odd). Then

$$N(\mu) \text{ consists of } \begin{cases} m' & \text{signatures when } d = 0 \\ m' - 2 & \text{signatures when } d \neq 0 \end{cases}$$

Suppose that $\underline{d = 0}$.
If

$$N(\lambda) = c_1\varepsilon_1 \quad c_2\varepsilon_2 \quad \cdots \quad c_{m'}\varepsilon_{m'}$$

then

$$N(\mu) = (c_1 - 1)\varepsilon_1 \quad (c_2 - 1)\varepsilon_2 \quad \cdots \quad (c_{m'-1} - 1)\varepsilon_{m'-1} \quad c_{m'}\varepsilon_{m'}$$

where in both sequences $c_{m'} = y_m - 1$, $\varepsilon_{m'} = -$. From this and the definition of end values, (1) and (2) follow easily. Also since the final sign is $-$ we have $\pi_i(N(\lambda)) = \pi_{i-1}(N(\mu))$ for all i, (by Lemma 4.1) proving (3) in this case.

Suppose $\underline{d \neq 0}$.

If again

$$N(\lambda): c_1\varepsilon_1 \quad c_2\varepsilon_2 \quad \cdots \quad c_{m'}\varepsilon_{m'}$$

then $c_{m'-1}\varepsilon_{m'-1} = y_m+$ and $c_{m'}\varepsilon_{m'} = (y_m - 1)-$ and

$$N(\mu): (c_1 - 1)\varepsilon_1 \quad (c_2 - 1)\varepsilon_2 \quad \cdots \quad (c_{m'-2} - 1)\varepsilon_{m'-2}$$

Again (1) and (2) follow easily. To prove (3) we consider the sequence

$$N^*(\lambda): c_1\varepsilon_1 \quad c_2\varepsilon_2 \quad \cdots \quad c_{m'-2}\varepsilon_{m'-2}$$

Obviously

$$(*) \quad \begin{cases} \pi_i(N^*(\lambda)) = \pi_{i-1}(N(\mu)) \\ \omega_i(N^*(\lambda)) = \omega_{i-1}(N(\mu)) \end{cases}$$

for all i. The final signature of $N(\lambda)$ has no influence on $\pi_i(N(\lambda))$, since the sign is $-$. Therefore, in order for $\pi_i(N^*(\lambda))$ to be different from $\pi_i(N(\lambda))$, we need $i = y_m$ and $\pi_i(N^*(\lambda)) = \omega_i(N^*(\lambda))$ by Lemma 4.1. Thus condition (i) of (3) is fulfilled and condition (iii) follows from $(*)$. Since by assumption $d \neq 0$ (ii) is also fulfilled. Thus (3) is proved in this case also. □

We proceed to prove an analogue of Proposition 4.3 for the *Mullineux* (signature) *sequence* $M(\lambda)$, which is defined as follows:

Let the residue symbol of λ be

$$R_p(\lambda) = \begin{Bmatrix} x_1 & \cdots & x_m \\ y_1 & \cdots & y_m \end{Bmatrix}.$$

Then

$$\begin{aligned} M(\lambda) = 0- \quad & x_1+ \quad (x_1+1)- \quad y_1+ \quad (y_1-1)- \\ & x_2+ \quad (x_2+1)- \quad y_2+ \quad (y_2-1)- \\ & \qquad\qquad \vdots \qquad\qquad \vdots \\ & x_m+ \quad (x_m+1)- \quad y_m+ \quad (y_m-1)- \end{aligned}$$

Starting with the signature $0-$ corresponds to starting with an empty partition at the beginning which just has the indent node $(1, 1)$ of residue 0.

Example $p = 5, \lambda = (10, 8, 7, 5, 3, 2^2)$ as before. Then

$$R_5(\lambda) = \begin{Bmatrix} 1 & 4 & 1 & 3 \\ 4 & 3 & 0 & 4 \end{Bmatrix}$$

and

$$M(\lambda) = 0- \quad 1+' \quad 2- \quad 4+' \quad 3- \quad \underline{4+}' \quad 0- \quad 3+ \quad 2- \quad \underline{1+}' \quad 2-$$
$$0+ \quad 4- \quad \underline{3+}' \quad 4- \quad 4+ \quad 3-$$

Residue	0	1	2	3	4
End value	−1	2	−3	0	1
Peak value	0	2	0	1	2
Good?	N	Y	N	Y	Y

(The good signatures in $M(\lambda)$ are again underlined and the normal signatures marked with a prime.)
The table above is identical with the one in the previous example.

Lemma 4.4 *Let λ and μ be as above. Let $M^*(\lambda)$ be the signature sequence obtained from $M(\lambda)$ by removing the two final signatures y_m+ and $(y_m - 1)-$. Then for all i we have*

$$\omega_i(M^*(\lambda)) = \omega_{i-1}(M(\mu))$$
$$\pi_i(M^*(\lambda)) = \pi_{i-1}(M(\mu))$$

Proof: We use the notation of Lemma 2.2 for $R_p(\lambda)$ and $R_p(\mu)$ and proceed by induction on m. First we study the beginnings of $M^*(\lambda)$ and $M(\mu)$. We compare

$$(1) \quad 0- \quad x_1+ \quad (x_1+1)- \quad (\text{from } M^*(\lambda))$$

with

$$(2) \quad 0-[x_1'+ \quad (x_1'+1)- \quad y_1'+ \quad (y_1'-1)-] \quad (\text{from } M(\mu))$$

We have put brackets [ ] around a part of (2), because these signatures do not occur when $x_1 = 0$ by Lemma 2.2.

If $x_1 = 0$ then (1) and (2) become

$$0- \quad 0+ \quad 1- \quad \text{and} \quad 0-$$

The former gives a contribution −1 to residue 1 and contributions 0 to all others, the latter a contribution −1 to residue 0 and 0 to all others.

If $x_1 \neq 0$ then $x_1 + 1 \neq 1 = y_0$, so by Lemma 2.2 $\delta_1 = 1$, and (2) becomes

$$(2)' \quad 0- \quad (x_1 - 1)+ \quad x_1- \quad 0+ \quad (p-1)-$$

The signatures $0-\ 0+$ in the latter sequence have no influence on the end values and peak values of $M(\mu)$, (even when $x_1 - 1 = 0$) and may be ignored. Then again we see that (1) gives the same contribution to residue i as $(2)'$ to residue $(i-1)$ for all i. Thus our result is true if $m = 1$.

We assume that the result is true for partitions whose Mullineux symbols have length $m - 1 \geq 1$, and we have to compare

$$(3) \quad y_{m-1}+ \quad (y_{m-1} - 1)- \quad x_m+ \quad (x_m + 1)- \quad (\text{from } M^*(\lambda))$$

with

$$(4) \quad x'_m+ \quad (x'_m + 1)- \quad y'_m+ \quad (y'_m - 1)- \quad (\text{from } M(\mu))$$

By Lemma 2.2, (4) may be written as

$$(4)' \quad (x_m - \delta_m)+ \quad (x_m - \delta_m + 1)- \quad (y_{m-1} - \delta_m)+ \quad (y_{m-1} - \delta_m - 1)-$$

We see that up to rearrangement the difference between the residues occurring in (3) and $(4)'$ is just δ_m. Whereas the rearrangement is irrelevant for the end values it could influence the peak value if signatures with same residue but different signs are interchanged. The possible coincidences of residues with different signs are

$$(\alpha) \quad y_{m-1} = x_m + 1 \quad (\text{first and fourth residue in (3)})$$

or

$$(\beta) \quad y_{m-1} - 1 = x_m \quad (\text{second and third residue in (3)})$$

But the equations (α) and (β) are equivalent, and by Lemma 2.2 they are fulfilled if and only if $\delta_m = 0$! If $y_{m-1} = x_m + 1$ (and thus $\delta_m = 0$) (3) and (4) becomes

$$y_{m-1}+ \quad (y_{m-1} - 1)- \quad (y_{m-1} - 1)+ \quad y_{m-1}-$$

and

$$(y_{m-1} - 1)+ \quad y_{m-1}- \quad y_{m-1}+ \quad (y_{m-1} - 1)-$$

In this case the difference between the occurring residues is 1 (without rearrangement) and our statement is true.

If $y_{m-1} \neq x_m + 1$ (and thus $\delta_m = 1$) then the difference between the occurring residues is again $1 (= \delta_m)$ and since there is no coincidence for residues with different signs we may apply Lemma 4.1 and the induction hypotheses to prove the statement in this case too. □

Lemma 4.5 *Suppose that in the notation as above we have for* $i = y_m$

$$\omega_{i-1}(M(\mu)) = \pi_{i-1}(M(\mu)).$$

Then $d \neq 0$.

Proof: Suppose $d = 0$. Then by Remark 2.3, $y'_m = y_m = i$, and hence $M(\mu)$ ends on $(i-1)-$. But then clearly $\omega_{i-1}(M(\mu)) \neq \pi_{i-1}(M(\mu))$. □

Lemma 4.6 *Let the notation be as in Lemma* 4.4.
(1) *For* $1 \leq j \leq m-1$ *we have*:

$$y_j \text{ is } i\text{-good for } M^*(\lambda) \Leftrightarrow \begin{cases} \delta_{j+1} = 1 \text{ and } y'_{j+1} \text{ is } (i-1)\text{-good for } M(\mu) \\ \text{or} \\ \delta_{j+1} = 0 \text{ and } x'_{j+1} \text{ is } (i-1)\text{-good for } M(\mu). \end{cases}$$

(2) *For* $1 \leq j \leq m$ *we have*:

$$x_j \text{ is } i\text{-good for } M^*(\lambda) \Leftrightarrow \delta_j = 1 \text{ and } x'_j \text{ is } (i-1)\text{-good for } M(\mu).$$

Proof: This follows immediately from the proof of Lemma 4.4. It should be noted that x_1 cannot be 0-good for $M^*(\lambda)$ since $M^*(\lambda)$ starts by $0-$. Moreover, the proof of Lemma 4.4 shows that if x_j is i-good for $M^*(\lambda)$, then we cannot have $\delta_j = 0$, since otherwise $x_j- = (y_{j-1} - 1)-$ proceeds x_j+. □

Proposition 4.7 *Let* λ *and* μ *be as above.*
(1) *If* $i \neq y_m$ *and* $i \neq y_m - 1$ *then*

$$\omega_i(M(\lambda)) = \omega_{i-1}(M(\mu)).$$

(2) *If* $i = y_m$ *then*

$$\omega_i(M(\lambda)) = \omega_{i-1}(M(\mu)) + 1$$

and if $i = y_m - 1$ *then*

$$\omega_i(M(\lambda)) = \omega_{i-1}(M(\mu)) - 1.$$

(3) *We have*

$$\pi_i(M(\lambda)) = \pi_{i-1}(M(\mu))$$

unless $i = y_m$ *and* $\omega_{i-1}(M(\mu)) = \pi_{i-1}(M(\mu))$. *In that case* y_m *is* i*-good for* $M(\lambda)$ *and*

$$\pi_i(M(\lambda)) = \pi_{i-1}(M(\mu)) + 1.$$

Note. There is a strict analogy between the Propositions 4.3 and 4.7. In part (3) the assumption $d \neq 0$ is not necessary in Proposition 4.7 due to Lemma 4.5.

Proof: By Lemma 4.4

$$\omega_i(M^*(\lambda)) = \omega_{i-1}(M(\mu))$$
$$\pi_i(M^*(\lambda)) = \pi_{i-1}(M(\mu))$$

If we add y_m+ and $(y_m - 1)-$ to $M^*(\lambda)$ we get $M(\lambda)$. Therefore, an argument completely analogous to the one used in the case $d \neq 0$ in the proof of Proposition 4.3 may be applied. □

Theorem 4.8 *Let λ be a p-regular partition.*
Then for all i, $0 \leq i \leq p - 1$

$$\omega_i(M(\lambda)) = \omega_i(N(\lambda))$$
$$\pi_i(M(\lambda)) = \pi_i(N(\lambda))$$

Proof: We use induction on the number ℓ of columns in λ. For $\ell = 1$, i.e., $\lambda = (1^d)$ we have $G_p(\lambda) = \binom{d}{d}$ and $R_p(\lambda) = \{ {0 \atop 1-d} \}$. Thus

$$N(\lambda) : 1- \quad (1-d)+ \quad (-d)-$$
$$M(\lambda) : 0- \quad 0+ \quad 1- \quad (1-d)+ \quad (-d)-$$

and the result is clear. Assume the result has been proved for partitions with $\ell - 1$ columns, $\ell \geq 2$. Let μ be obtained by removing the first column from λ. By the induction hypothesis we have

$$\omega_{i-1}(M(\mu)) = \omega_{i-1}(N(\mu))$$
$$\pi_{i-1}(M(\mu)) = \pi_{i-1}(N(\mu))$$

for all i. Using Propositions 4.3 and 4.7 (see also the note to Proposition 4.7) we get the result. □

Theorem 4.9 *The following statements are equivalent for a p-regular partition λ and i,* $0 \leq i \leq p - 1$.
(1) *There is a good node of residue i in λ.*
(2) *$M(\lambda)$ has i as a good residue.*
(3) *$N(\lambda)$ has i as a good residue.*

Proof: (1)⇔(3): See the beginning of this section.
(2)⇔(3): Theorem 4.8. □

Finally, we describe the effect of the removal of a good node on the residue symbol (or equivalently on the Mullineux symbol). First we prove a lemma.

Lemma 4.10 *Suppose that there is a good node of residue i in λ. Then the following statements are equivalent*:
(1) *The good node of residue i occurs in the first column of λ.*
(2) *y_m is i-good for $M(\lambda)$.*

Proof: The statement (1) clearly is equivalent to

$$(1)' \quad \pi_i(N(\lambda)) \neq \pi_i(N(\tilde{\lambda}))$$

(where, as before, $\tilde{\lambda}$ is obtained from λ by removing all parts equal to 1) We now have

$$\begin{aligned} &\pi_i(N(\lambda)) \neq \pi_i(N(\tilde{\lambda})) \\ &\quad \Leftrightarrow \ \pi_i(N(\lambda)) \neq \pi_{i-1}(N(\mu)) \quad \text{(by Remark 4.2)} \\ &\quad \Leftrightarrow \ \pi_i(M(\lambda)) \neq \pi_{i-1}(M(\mu)) \quad \text{(by Theorem 4.8)} \\ &\quad \Leftrightarrow \ y_m \ \text{ is } i\text{-good for } M(\lambda) \quad \text{(by Proposition 4.7)} \end{aligned}$$ □

Theorem 4.11 *Suppose that the p-regular partition λ has a good node A of residue i. Let*

$$R_p(\lambda) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_m \end{Bmatrix}.$$

Then for some j, $1 \leq j \leq m$, one of the following occurs:
(1) *x_j is i-good for $M(\lambda)$ and*

$$R_p(\lambda\backslash A) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_j - 1 & \cdots & x_m \\ y_1 & y_2 & \cdots & y_j & \cdots & y_m \end{Bmatrix}.$$

(2) *y_j is i-good for $M(\lambda)$ and*

$$R_p(\lambda\backslash A) = \begin{Bmatrix} x_1 & x_2 & \cdots & x_j & \cdots & x_m \\ y_1 & y_2 & \cdots & y_j + 1 & \cdots & y_m \end{Bmatrix} \quad \text{if } (j, i) \neq (1, 0),$$

$$R_p(\lambda\backslash A) = \begin{Bmatrix} x_2 & \cdots & x_m \\ y_2 & \cdots & y_m \end{Bmatrix} \quad \text{if } j = 1,\ i = 0\ .$$

Proof: The proof is by induction on $|\lambda|$. Suppose first that A occurs in the first column of λ. Then the first column in $G_p(\lambda\backslash A)$ is obtained from the first column in $G_p(\lambda)$ by subtracting 1 in each entry and all other entries are unchanged; note that in the case where $G_p(\lambda) = \binom{1}{1}$, we have a degenerate case and $G_p(\lambda\backslash A)$ is the empty symbol. By definition of the residue symbol this means that y_m in $R_p(\lambda)$ is replaced by $y_m + 1$ in $R_p(\lambda\backslash A)$; of course, in the degenerate case also $R_p(\lambda\backslash A)$ is the empty residue symbol. On the other hand y_m is i-good for $M(\lambda)$ by Lemma 4.10, and in the degenerate case y_1 is 0-good for $M(\lambda)$, and so we are done in this case.

Now we assume that A does *not* occur in the first column of λ. Let B be the node of μ corresponding to A. Clearly B is a good node of residue $i-1$ for μ. We may apply the induction hypothesis to μ and B. Suppose that

$$R_p(\mu) = \begin{Bmatrix} x'_1 & x'_2 & \cdots & x'_m \\ y'_1 & y'_2 & \cdots & y'_m \end{Bmatrix}$$

By the induction hypothesis we know that one of the following cases occurs:

Case I. x'_j in $R_p(\mu)$ is replaced by $x'_j - 1$ in $R_p(\mu\backslash B)$ and x'_j is $(i-1)$-good for $M(\mu)$.
Case II. y'_j in $R_p(\mu)$ is replaced by $y'_j + 1$ in $R_p(\mu\backslash B)$ and y'_j is $(i-1)$-good for $M(\mu)$, respectively in the degenerate case y'_1 is 0-good for $M(\mu)$, and then the first column $\begin{smallmatrix} x'_1 \\ y'_1 \end{smallmatrix}$ in $R_p(\mu)$ is omitted in $R_p(\mu\backslash B)$.

We treat Case I in detail. Case II is treated in a similar way.

Case I: By Lemma 4.6 we have one of the following cases:

Case Ia: y_{j-1} is i-good for $M^*(\lambda)$ and $\delta_j = 0$
Case Ib: x_j is i-good for $M^*(\lambda)$ and $\delta_j = 1$

We add a first column to $\mu\backslash B$ to get $\lambda\backslash A$. Then $R_p(\lambda\backslash A)$ is obtained from $R_p(\mu\backslash B)$ using Lemma 2.4. We fix the notation

$$R_p(\mu\backslash B) = \begin{Bmatrix} x''_1 & \cdots & x''_m \\ y''_1 & \cdots & y''_m \end{Bmatrix}$$

and

$$R_p(\lambda\backslash A) = \begin{Bmatrix} \bar{x}_0 & \bar{x}_1 & \cdots & \bar{x}_m \\ \bar{y}_0 & \bar{y}_1 & \cdots & \bar{y}_m \end{Bmatrix}$$

Case Ia. We know $x'_j = i-1$ since we are in Case I and $y_{j-1} = i$, since we are in Case Ia. Moreover since $\delta_j = \delta'_j = 0$ (see Remark 2.5) we have $y'_j = x'_j = i$. Also $x'_j = x_j$ and $y'_j = y_{j-1}$ by Lemma 2.2. By Lemma 2.4

$$\bar{y}_{j-1} = y''_j + \delta''_j$$

where

$$\delta''_j = \begin{cases} 0 & \text{if } x''_j + 1 = y''_j \\ 1 & \text{otherwise} \end{cases}$$

But $x''_j + 1 = (x'_j - 1) + 1 = x'_j = i - 1$ and $y''_j = y'_j = y_{j-1} = i$ by the above. Thus $\delta''_j = 1$ and $\bar{y}_{j-1} = y'_j + 1 = i + 1$. It is readily seen that all other entries in $R_p(\lambda\backslash A)$ coincide with those of $R_p(\lambda)$. Thus possibility (2) occurs in the theorem.

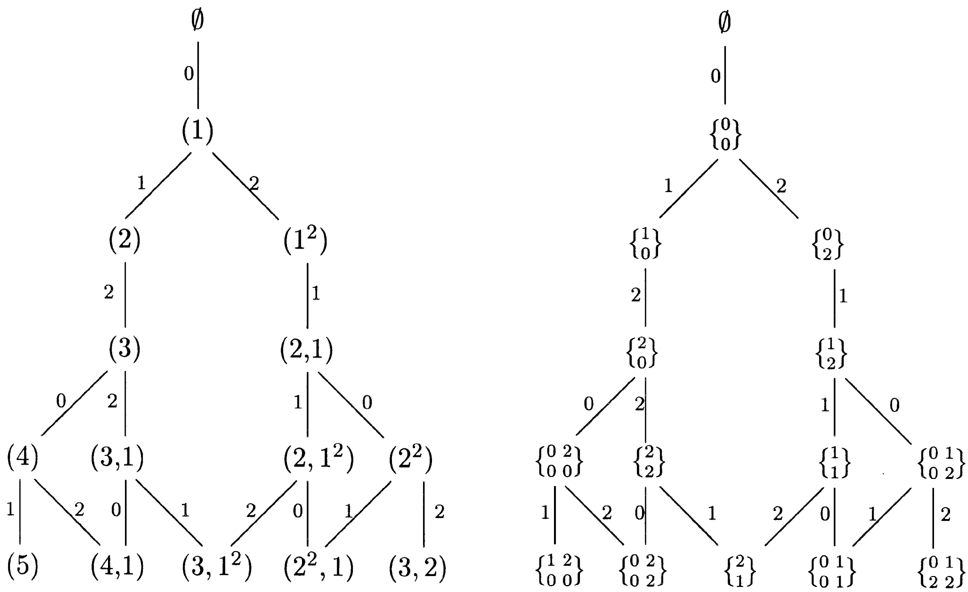
Case Ib. We know $x'_j = i - 1$ since we are in Case I and $x_j = i$ since we are in Case Ib. Also since $\delta_j = 1$, $y'_j = y_{j-1} - 1$. Moreover $y''_j = y'_j$ and $x''_j = x'_j - 1$, i.e., $x''_j = i - 2$. Let δ''_j again be defined by

$$\delta''_j = \begin{cases} 0 & \text{if } x''_j + 1 = y''_j \\ 1 & \text{otherwise} \end{cases}.$$

Then by Lemma 2.4 $\bar{x}_j = x''_j + \delta''_j = i - 2 + \delta''_j$. We claim that $\delta''_j = 1$. Otherwise $i - 1 = x''_j + 1 = y''_j = y'_j$ and we also know that $x'_j = i - 1$. But if $x'_j = y'_j$ then by definition of $M(\mu)$ x'_j is not a peak, contrary to our assumption that we are in Case I. Thus $\delta''_j = 1$ and $\bar{x}_j = i - 1$, as desired. Again it is easily seen that all other entries of $R_p(\lambda\backslash A)$ coincide with those of $R_p(\lambda)$. Thus possibility (1) occurs in the theorem. □

We illustrate the theorem above by giving Kleshchev's p-good branching graph for p-regular partitions for $p = 3$ up to $n = 5$ in both the usual and the residue symbol notation; we recall that the p-good branching graph for p-regular partitions is also the crystal graph for the basic representation of the quantum affine algebra (see [11] for these connections).

Below, an edge labelled r is drawn from a partition λ of m to a partition μ of $m - 1$ if μ is obtained from λ by removing a node of residue r.



We can now easily deduce the combinatorial conjecture to which the Mullineux conjecture had been reduced by Kleshchev:

Corollary 4.12 *Suppose that the p-regular partition* λ *has a good node* A *of residue* i. *Then its Mullineux conjugate* λ^M *has a good node* B *of residue* $-i$ *satisfying*

$$(\lambda \backslash A)^M = \lambda^M \backslash B.$$

Proof: Considering the residue symbol of λ it is easily seen that the Mullineux sequence of λ and its conjugate λ^M are very closely related. Indeed, the peak and end values for each residue i in $M(\lambda)$ equal the corresponding values for the residue $-i$ in $M(\lambda^M)$, and if there is a i-good node at column k in the residue symbol of λ, then there is a $-i$-good node at column k in the residue symbol of λ^M. More precisely, in the regular case these good nodes are one at the top and one at the bottom of the column, whereas in the singular case both are at the top. Comparing this with Theorem 4.11 implies the result. □

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