Journal of Algebraic Combinatorics **7** (1998), 115–126

Characteristic and Ehrhart Polynomials*

ANDREAS BLASS ablass@umich.edu
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1003

BRUCE E. SAGAN sagan@math.msu.edu
Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027

Received May 1, 1996; Revised December 17, 1996

Abstract. Let $\mathcal{A}$ be a subspace arrangement and let $\chi(\mathcal{A}, t)$ be the characteristic polynomial of its intersection lattice $L(\mathcal{A})$. We show that if the subspaces in $\mathcal{A}$ are taken from $L(\mathcal{B}_n)$, where $\mathcal{B}_n$ is the type B Weyl arrangement, then $\chi(\mathcal{A}, t)$ counts a certain set of lattice points. One can use this result to study the partial factorization of $\chi(\mathcal{A}, t)$ over the integers and the coefficients of its expansion in various bases for the polynomial ring $\mathbb{R}[t]$. Next we prove that the characteristic polynomial of any Weyl hyperplane arrangement can be expressed in terms of an Ehrhart quasi-polynomial for its affine Weyl chamber. Note that our first result deals with all subspace arrangements embedded in $\mathcal{B}_n$ while the second deals with all finite Weyl groups but only their hyperplane arrangements.

Keywords: Weyl group, hyperplane arrangement, subspace arrangement, Möbius function, characteristic polynomial, Ehrhart polynomial

1. Introduction and background

An *arrangement* is a finite set

$$\mathcal{A} = \{K_1, \ldots, K_m\} \tag{1}$$

of proper subspaces of Euclidean space $\mathbb{R}^n$. All the subspaces we consider will be linear and so go through the origin. If each K_i has dimension $n-1$, then $\mathcal{A}$ is called a *hyperplane arrangement*. We sometimes refer to general arrangements as *subspace arrangements* to emphasize that they need not be hyperplane arrangements. We write $\bigcup \mathcal{A}$ for the set-theoretic union of the subspaces in $\mathcal{A}$, i.e., $\bigcup_{i=1}^m K_i$.

The theory of hyperplane arrangements is a beautiful area of mathematics which brings together ideas from topology, algebra, and combinatorics. Its roots go back to the end of the 19th century but it is also an active area of research today. The recent book [15] of Orlik and Terao covers both classical work and recent developments in the field. Subspace arrangements, on the other hand, have received relatively little attention yet, as was noted in the recent survey article of Björner [2]. It is important to emphasize that in most cases it is *not* easy to generalize results from the hyperplane case to the subspace case. Particularly nicely

*AMS subject classification (1991): Primary 05A15; Secondary 05B35, 05E15, 20F55.

behaved hyperplane arrangements are those which are associated with finite Weyl groups (see, e.g., [14]). We wish to study these arrangements and certain subspace arrangements related to them. We begin by establishing some notation and terminology.

Let $\mathcal{A}$ be an arrangement as in (1) above, and assume, for simplicity, that there are no containments among the K_i. Let $L = L(\mathcal{A})$ be the set of all intersections of these subspaces, ordered by reverse inclusion, called the *intersection lattice*. (Concepts from lattice theory that are not explained here can be found in Stanley's text [17].) Note that L has a unique minimal element $\hat{0}$ corresponding to $\mathbb{R}^n$, an atom corresponding to each K_i, and a unique maximal element $\hat{1}$ corresponding to $\bigcap_{i=1}^m K_i$. If $\mathcal{A}$ is a hyperplane arrangement then $L(\mathcal{A})$ is a geometric lattice, but in general it is not even ranked. If $\mathcal{A}$ and $\mathcal{B}$ are subspace arrangements such that $\mathcal{A} \subseteq L(\mathcal{B})$, i.e., all the subspaces in $\mathcal{A}$ are intersections of subspaces in $\mathcal{B}$, then we say that $\mathcal{A}$ is *embedded* in $\mathcal{B}$.

Given an arrangement $\mathcal{A}$, let $\mu(X) = \mu(\hat{0}, X)$ denote the *Möbius function* of the lattice $L(\mathcal{A})$; it is uniquely defined by

$$\sum_{Y \leq X} \mu(Y) = \delta_{\hat{0},X}$$

where $\delta_{\hat{0},X}$ is the Kronecker delta. The Möbius function is one of the fundamental invariants of any partially ordered set; see the seminal article of Rota [16]. The *characteristic polynomial* of $\mathcal{A}$ is

$$\chi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}. \tag{2}$$

Since the characteristic polynomial is just the generating function for the Möbius function, it is also of prime importance. Our results in this paper give a combinatorial interpretation for the characteristic polynomials of hyperplane arrangements associated to Weyl groups and subspace arrangements embedded in some of these Weyl arrangements.

For any finite Weyl group, W, there is a corresponding hyperplane arrangement $\mathcal{W}$ whose elements are the reflecting hyperplanes of W. Initially we shall be interested in the case where W comes from one of the three infinite families A_n, B_n, D_n. (The arrangement for C_n is clearly the same as that for B_n.) In terms of the coordinate functions $x_1, \ldots, x_n$ in $\mathbb{R}^n$, the associated hyperplane arrangements can be defined as

$$\mathcal{A}_n = \{x_i = x_j \ : \ 1 \leq i < j \leq n\},$$
$$\mathcal{D}_n = \mathcal{A}_n \cup \{x_i = -x_j \ : \ 1 \leq i < j \leq n\},$$
$$\mathcal{B}_n = \mathcal{D}_n \cup \{x_i = 0 \ : \ 1 \leq i \leq n\}$$

so that $\mathcal{A}_n \subset \mathcal{D}_n \subset \mathcal{B}_n$. Note that n here refers to the dimension of the space, not the number of fundamental reflections (which is $n - 1$ for $\mathcal{A}_n$ and n for the other two).

2. Arrangements embedded in $\mathcal{B}_n$

We shall now give our first main result: a combinatorial interpretation for the characteristic polynomial of any subspace arrangement embedded in one of the three infinite families of

Weyl hyperplane arrangements. It was obtained in an attempt to generalize Zaslavsky's beautiful theory of signed graph coloring [23–25]. Given integers $r \leq s$, we let $[r, s] = \{r, r+1, \ldots, s\}$. Note that if $r = -s$ then $t = |[-s, s]|$ is odd, where $|\cdot|$ denotes cardinality. Note also that $[-s, s]^n$ is just the cube of points in $\mathbf{Z}^n$ centered at the origin with t points on a side. So $[-s, s]^n \setminus \bigcup \mathcal{A}$ is the set of points of $\mathbf{Z}^n$ that are in this cube but not on any subspace from $\mathcal{A}$.

Theorem 2.1 *If $\mathcal{A} \subseteq L(\mathcal{B}_n)$ then for any $t = 2s + 1$*

$$\chi(\mathcal{A}, t) = \left| [-s, s]^n \Big\backslash \bigcup \mathcal{A} \right|.$$

Note that the hypothesis of the theorem does not preclude the possibility that $\mathcal{A}$ may also be embedded in $\mathcal{A}_n$ or $\mathcal{D}_n$, as these are embedded in $\mathcal{B}_n$. Let us give a concrete example of this result before proving it. Let

$$\mathcal{A} = \mathcal{B}_2 = \{x = 0, y = 0, x = y, x = -y\}.$$

Also let $s = 2$ so that $t = 5$. Then $[-2, 2]^2$ and $\mathcal{B}_2$ are shown in figure 1. Removing the lines of $\mathcal{B}_2$ from the cube leaves 8 lattice points. On the other hand it is well known that $\chi(\mathcal{B}_2, t) = (t - 1)(t - 3)$; see Eq. (3). So $\chi(\mathcal{B}_2, 5) = 4 \cdot 2 = 8$ as expected.

Proof of Theorem 2.1: We construct two functions $f, g : L(\mathcal{A}) \to \mathbf{Z}$ by defining for each $X \in L(\mathcal{A})$

$$f(X) = |X \cap [-s, s]^n|,$$
$$g(X) = \left| \left(X \Big\backslash \bigcup_{Y > X} Y \right) \cap [-s, s]^n \right|.$$

Recall that $L(\mathcal{A})$ is ordered by *reverse* inclusion so that $\bigcup_{Y>X} Y \subset X$. In particular $g(\mathbb{R}^n) = |[-s, s]^n \setminus \bigcup \mathcal{A}|$. Note also that $X \cap [-s, s]^n$ is combinatorially just a cube of dimension $\dim X$ and side t so that $f(X) = t^{\dim X}$. Finally, $f(X) = \sum_{Y \geq X} g(Y)$ so by the Möbius

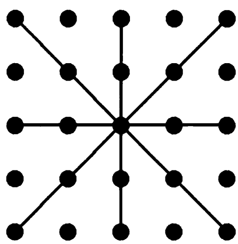

Figure 1. The lattice points of $[-2, 2]^2 \setminus \bigcup \mathcal{B}_2$.

Inversion Theorem [16]

$$\begin{aligned}\left|[-s,s]^n \Big\backslash \bigcup \mathcal{A}\right| &= g(\hat{0})\\ &= \sum_{X\in L(\mathcal{A})} \mu(X) f(X)\\ &= \sum_{X\in L(\mathcal{A})} \mu(X) t^{\dim X}\\ &= \chi(\mathcal{A},t)\end{aligned}$$

which is the desired result. □

In the proof of Theorem 2.1, it was crucial that each of the subspaces X under consideration had exactly $t^{\dim(X)}$ points in $[-s,s]^n$. In fact, the *only* subspaces of $\mathbb{R}^n$ with this property are those in $L(\mathcal{B}_n)$. So the method of proof of Theorem 2.1 cannot be applied directly to other arrangements.

We should also mention how our theorem is related to Zaslavsky's theory of signed graphs. Zaslavsky assigns to each hyperplane arrangement $\mathcal{A}$ contained (as a subset) in $\mathcal{B}_n$ a signed graph $G_{\mathcal{A}}$. The graph has vertices $1, 2, \ldots, n$ with a positive (respectively, negative) edge from vertex i to vertex j iff $x_i = x_j$ (respectively, $x_i = -x_j$) is in $\mathcal{A}$. The graph $G_{\mathcal{A}}$ also has a half-edge at vertex i iff $x_i = 0$ is in $\mathcal{A}$. He then defines a chromatic polynomial $P(G,t)$ for signed graphs (generalizing the one for ordinary graphs) and shows that $P(G_{\mathcal{A}},t) = \chi(\mathcal{A},t)$. If one thinks of the vertices of $G_{\mathcal{A}}$ as being coordinates, then a proper coloring of $G_{\mathcal{A}}$ in Zaslavsky's sense turns out to be just an element of $[-s,s]^n \backslash \bigcup \mathcal{A}$. The advantages of our viewpoint are that it applies to subspace arrangements embedded in $\mathcal{B}_n$ (not just hyperplane embeddings) and that it admits an analog for all Weyl hyperplane arrangements as we shall see in our second main theorem. We should mention that Stanley [18] has independently formulated a version of Theorem 2.1 for arrangements embedded in $\mathcal{A}_n$ using hypergraphs and symmetric functions.

3. Examples

First, let us show how Theorem 2.1 can be used to compute the well-known characteristic polynomials for the three infinite families of Weyl hyperplane arrangements. In the type A case we see that a point of $[-s,s]^n \backslash \bigcup \mathcal{A}_n$ must have all coordinates different. So there are $t = 2s+1$ choices for the first coordinate, $t-1$ for the second, etc. This gives a total of

$$\chi(\mathcal{A}_n,t) = t(t-1)\cdots(t-n+1).$$

It will be useful to have a notation for this falling factorial, so we will let $\langle t\rangle_n = t(t-1)\cdots(t-n+1)$.

For $\mathcal{B}_n$ the points in the cube minus the arrangement must all have different absolute values and must be nonzero. The first coordinate can be chosen in $t-1$ ways since zero is not allowed. The second coordinate can be anything except zero and plus or minus the value of the first, giving $t-3$ possibilities. Continuing in this way we see that

$$\chi(\mathcal{B}_n, t) = (t-1)(t-3)\cdots(t-2n+1). \tag{3}$$

We will let $\langle\langle t\rangle\rangle_n = t(t-2)\cdots(t-2n+2)$ so that $\chi(\mathcal{B}_n, t) = \langle\langle t-1\rangle\rangle_n$.

For the third family, note that any point of $[-s, s]^n \setminus \bigcup \mathcal{D}_n$ can have at most one zero coordinate. The points with no zero coordinate were counted in the $\mathcal{B}_n$ case. For those with one zero, there are n ways to pick this coordinate and the remaining nonzero ones are accounted for as in $\mathcal{B}_{n-1}$. The total is thus

$$\chi(\mathcal{D}_n, t) = \chi(\mathcal{B}_n, t) + n\chi(\mathcal{B}_{n-1}, t) = (t-1)(t-3)\cdots(t-2n+3)(t-n+1).$$

Notice that in all three of these examples χ factors over the integers. In fact for any Weyl hyperplane arrangement it is well known that the roots are just the exponents of the corresponding group [22]. The characteristic polynomial of a subspace arrangement $\mathcal{S}_n$ embedded in a Weyl hyperplane arrangement $\mathcal{H}_n$ from one of the three infinite families does not always have integral roots. But it can happen that it factors partially and is in fact divisible by the polynomial for a hyperplane arrangement $\mathcal{H}_m$, $m \leq n$. Further, when one expands $\chi(\mathcal{S}_n, t)$ in terms of the basis $\{\chi(\mathcal{H}_j, t) : j \geq 0\}$ for $\mathbb{R}[t]$ the coefficients vanish for small j, thus explaining the divisibility relation since for type A and B we have $\chi(\mathcal{H}_j, t) \mid \chi(\mathcal{H}_{j+1}, t)$. Finally, the coefficients in the basis expansion turn out to be nonnegative integers having a nice combinatorial interpretation which makes it obvious when they are zero. The next few results will illustrate this point. Other examples can be found in [7, 26] and are being pursued by Sagan.

To describe the subspace arrangements that we will consider, it is convenient to have some notation. Let $[n] = \{1, \ldots, n\}$. If $I = \{i, j, \ldots, k\} \subseteq [n]$ then let x_I stand for the equation $x_i = x_j = \cdots = x_k$. So $x_I = 0$ is the system of equations $x_i = 0$ for all $i \in I$. Also let $\pm x_I$ represent the set of all equations of the form

$$\epsilon_i x_i = \cdots = \epsilon_k x_k$$

for $\epsilon_i, \ldots, \epsilon_k \in \{\pm 1\}$. In each case we use the same symbol to denote the corresponding subspace(s). The *k-equal* and *k, h-equal* subspace arrangements are defined by

$$\begin{aligned}
\mathcal{A}_{n,k} &= \{x_I \ : \ I \subseteq [n] \text{ and } |I| = k\},\\
\mathcal{D}_{n,k} &= \{\pm x_I \ : \ I \subseteq [n] \text{ and } |I| = k\},\\
\mathcal{B}_{n,k,h} &= \mathcal{D}_{n,k} \cup \{x_J = 0 : J \subseteq [n] \text{ and } |J| = h\}.
\end{aligned}$$

The $\mathcal{A}_{n,k}$ arrangement first appeared in the work of Björner et al. [3], motivated by its relevance to a certain problem in computational complexity. Its study has been continued by these authors and Linusson, Sundaram, Wachs and Welker in various combinations [4–6, 8, 13, 20, 21]. The $\mathcal{B}_{n,k,h}$ and $\mathcal{D}_{n,k}$ were introduced by Björner and Sagan in a paper [7] about their combinatorial and homological properties. Note that each of these subspace arrangements is embedded in the hyperplane arrangement of the corresponding type and therefore in $\mathcal{B}_n$.

Consider the k-equal arrangement $\mathcal{A}_{n,k}$ embedded in $\mathcal{A}_n$ with $\chi(\mathcal{A}_n) = \langle t\rangle_n$. It will be convenient to let $S_k(n, j)$ denote the number of partitions of an n-element set into j subsets each of which is of size at most k. Thus these are generalizations of the Stirling numbers of the second kind.

Theorem 3.1 *We have the expansion*

$$\chi(\mathcal{A}_{n,k}, t) = \sum_j S_{k-1}(n, j)\langle t\rangle_j \tag{4}$$

and the divisibility relation

$$\langle t\rangle_{\lceil n/(k-1)\rceil} \mid \chi(\mathcal{A}_{n,k}, t). \tag{5}$$

Proof: To get the expansion, consider an arbitrary point $x \in [-s, s]^n \setminus \bigcup \mathcal{A}_{n,k}$. So x can have at most $k-1$ of its coordinates equal. Consider the x's with exactly j different coordinates. Then there are $S_{k-1}(n, j)$ ways to distribute the j values among the n coordinates with at most $k-1$ equal. We can then choose which values to use in $\langle t\rangle_j$ ways. Summing over all j gives the desired equation.

For the divisibility result, note that $S_{k-1}(n, j) = 0$ if $j < \lceil n/(k-1)\rceil$ because j sets of at most $k-1$ objects can partition a set of size of at most $n = j(k-1)$. Plugging this into (4) finishes the proof. □

We should note that expansion (4) was derived by Björner and Lovász [4] and by Sundaram [19] using formal power series techniques. Analogs of this expansion for type B and D can be found in a paper of Björner and Sagan [7] while applications to the Boolean algebra are in Zhang's thesis [26].

Theorem 3.2 *Let $\mathcal{A}$ be a subspace arrangement.*
(a) *If $\mathcal{A}$ is embedded in $\mathcal{A}_n$ and we write*

$$\chi(\mathcal{A}, t) = \sum_{j=0}^{n} a_j\langle t\rangle_j \tag{6}$$

then $a_j \in \mathbf{Z}_{\geq 0}$ for all j, $0 \leq j \leq n$. Furthermore if m is the largest index such that $a_m = 0$ then

$$\langle t\rangle_{m+1} \mid \chi(\mathcal{A}, t).$$

(b) *If $\mathcal{A}$ is embedded in $\mathcal{B}_n$ and we write*

$$\chi(\mathcal{A}, t) = \sum_{j=0}^{n} b_j\langle\langle t-1\rangle\rangle_j$$

then $b_j \in \mathbf{Z}_{\geq 0}$ *for all* j, $1 \leq j \leq n$. *Furthermore if* m *is the largest index such that* $b_m = 0$ *then*

$$\langle\langle t-1\rangle\rangle_{m+1} \mid \chi(\mathcal{A}, t).$$

Proof: We will do part (a) as (b) is similar. Consider any $X \in L(\mathcal{A}_n)$ and define $X^0 = (X \setminus \bigcup_{Y>X} Y) \cap [-s,s]^n$ where $Y \in L(\mathcal{A}_n)$. Then we have $X^0 \subseteq \bigcup \mathcal{A}$ if $X \subseteq K$ for some $K \in \mathcal{A}$. On the other hand we have $X^0 \subseteq [-s,s]^n \setminus \bigcup \mathcal{A}$ if there is no such K containing X. It follows that

$$[-s,s]^n \Big\backslash \bigcup \mathcal{A} = \biguplus_X X^0$$

where the disjoint union is over all X not contained in any subspace of $\mathcal{A}$. Taking cardinalities on both side of this equation and using the fact that $|X^0| = \langle t\rangle_{\dim X}$ shows that the a_j in (6) are nonnegative integers.

For the divisibility relation, it suffices to prove that $a_j = 0$ implies $a_{j-1} = 0$. But $a_j = 0$ implies that every $X \in L(\mathcal{A}_n)$ of dimension j is contained in some $K \in \mathcal{A}$. Thus any $Y > X$ is in a K and $a_{j-1} = 0$. □

4. Weyl hyperplane arrangements

In this section we confine our attention to hyperplane arrangements that consist of the reflecting hyperplanes of a Weyl group. For background information on Weyl groups, including any concepts that we use without explanation, see the book of Humphreys [11], whose notation we endeavor to follow. We shall obtain a combinatorial characterization of the characteristic polynomial of such an arrangement. In rough outline, the characterization is similar to Theorem 2.1, but the lattice $\mathbf{Z}^n$ will be replaced with another lattice, the cube of side $2s+1$ will be replaced with another polytope, and the restriction to odd values of t will be replaced with other congruences imposed on t.

Unfortunately, both of the (mathematical) meanings of "lattice"—a poset in which finite subsets have joins and meets, and a discrete subgroup of $\mathbb{R}^n$—are relevant to the present discussion. We rely on the context to make it clear which is meant.

Let W be a finite Weyl group, determined by a root system Φ spanning $\mathbb{R}^n$. The hyperplanes orthogonal to the roots constitute the *Weyl arrangement* $\mathcal{W}$ associated to W, and the reflections in these hyperplanes generate W. Throughout this section, we follow the convention of naming a Weyl arrangement by the script letter corresponding to the name of the Weyl group. This agrees with the notation in the preceding sections for $\mathcal{B}_n$ and $\mathcal{D}_n$, but what we now call $\mathcal{A}_n$ is the restriction, to the hyperplane $x_1 + x_2 + \cdots + x_{n+1} = 0$, of what was previously called $\mathcal{A}_{n+1}$.

Let $Z(\Phi)$ be the lattice in $\mathbb{R}^n$ consisting of those vectors x that satisfy $(\alpha, x) \in \mathbf{Z}$ for all roots $\alpha \in \Phi$. This is the coweight lattice associated to Φ, and it will play the role that $\mathbf{Z}^n$ played in Theorem 2.1.

Our analog of the cube $[-s, s]^n$ of lattice points is

$$P_t(\Phi) = \{x \in Z(\Phi) \mid (\alpha, x) < t \text{ for all } \alpha \in \Phi\}.$$

Of course we will be interested in counting the lattice points in $P_t(\Phi) \setminus \bigcup \mathcal{W}$.

Fix a simple system

$$\Delta = \{\sigma_1, \ldots, \sigma_n\}$$

in Φ. Thus, Δ is a basis for the vector space $\mathbb{R}^n$, and, when any root $\lambda \in \Phi$ is written as a linear combination,

$$\lambda = \sum_{i=1}^{n} c_i(\lambda)\sigma_i,$$

of Δ, the coefficients $c_i(\lambda)$ are integers and are either all ≥ 0 or all ≤ 0. The fact that the coefficients are integers implies that, if a vector x satisfies $(\alpha, x) \in \mathbf{Z}$ for all $\alpha \in \Delta$, then it automatically satisfies the same for all $\alpha \in \Phi$ and therefore belongs to $Z(\Phi)$. In other words, in defining the coweight lattice, we could have restricted attention to simple roots.

If Φ is irreducible then among all the roots there is a *highest* one, $\tilde{\alpha}$, characterized by the fact that, for all roots λ and all $i \in [n]$, $c_i(\tilde{\alpha}) \geq c_i(\lambda)$. We shall write simply c_i for $c_i(\tilde{\alpha})$. One final ingredient for our theorem is the *index of connection*, f, which we define for irreducible root systems as

$$f = \frac{|W|}{n! \cdot c_1 \cdots c_n}. \tag{7}$$

For an arbitrary root system, f is defined as the product of the indices of connection for each irreducible component. (Humphreys defines f [11, p. 40] as the index of the coroot lattice as a subgroup of the coweight lattice and derives (7) as his Proposition 4.9. Since this formula is all we need to know about f, we take it as the definition.)

Theorem 4.1 *Let Φ be a root system for a finite Weyl group with associated arrangement $\mathcal{W}$. Let t be a positive integer relatively prime to all the coefficients $c_i = c_i(\tilde{\alpha})$. Then*

$$\chi(\mathcal{W}, t) = \frac{1}{f}\left|P_t(\Phi)\Big\backslash \bigcup \mathcal{W}\right|.$$

Proof: We may as well assume that Φ is irreducible since if it is not then both sides of the given equation decompose into a product of factors, one for each of the irreducible components. We begin by representing vectors in a form convenient for counting the points in $P_t(\Phi) \setminus \bigcup \mathcal{W}$. For any $x \in \mathbb{R}^n$, let x^* be the n-tuple consisting of the inner products of x with the simple roots, i.e., $x_i^* = (\sigma_i, x)$. So $x \in Z(\Phi)$ if and only if $x^* \in \mathbf{Z}^n$. Also, x lies in the open fundamental chamber C of W if and only if x^* lies in the open positive orthant $(\mathbb{R}_{>0})^n$.

Since $P_t(\Phi)$ and $\mathcal{W}$ are both invariant under the action of the group W, we can count the points of $P_t(\Phi)\setminus\bigcup\mathcal{W}$ by first counting the ones in C and then multiplying by the number of chambers (which equals the group's order $|W|$). To do the counting in C, we count instead the corresponding points x^* in the positive orthant of $\mathbf{Z}^n$ subject to the requirement $x \in P_t(\Phi)$. Note that since x^* is in the *open* positive orthant, x is automatically not in $\bigcup\mathcal{W}$. For x^* in $\mathbf{Z}^n$ the requirement that $x \in P_t(\Phi)$ is equivalent to the fact that, for all roots λ,

$$t > (\lambda, x) = \sum_i c_i(\lambda)x_i^*.$$

But since the x_i^* are all positive, these inequalities for all $\lambda \in \Phi$ follow from the one with the largest coefficients, namely the one for $\lambda = \tilde{\alpha}$. So our task is to count the number $\psi(t)$ of points $x^* \in (\mathbf{Z}_{>0})^n$ that satisfy the one linear inequality $\sum c_i x_i^* < t$. This $\psi(t)$ is known as the *Ehrhart quasi-polynomial* of the open simplex bounded by the coordinate hyperplanes and the hyperplane $\sum_i c_i x_i^* = 1$; see [17], page 235ff. It is also interesting to note that $P_1(\Phi)\bigcup c$ is just the fundamental chamber for the affine Weyl group corresponding to W.

Getting back to the task at hand, we must prove that $\psi(t)\cdot|W| = f\cdot\chi(\mathcal{W}, t)$ when t is relatively prime to all c_i. Using our definition (7) of f we see that this is equivalent to showing

$$\chi(\mathcal{W}, t) = \psi(t)\cdot n!\prod_i c_i$$

for the appropriate values of t and this is the form that we shall use in practice.

To compute $\psi(t)$, we use its generating function $\gamma(z) = \sum_t \psi(t)\cdot z^t$. It is easy to see that the generating function for n-tuples x^* of positive integers with $\sum c_i x_i^*$ equal to t is

$$\prod_{i=1}^{n}(z^{c_i} + z^{2c_i} + \cdots) = \prod_{i=1}^{n}\frac{z^{c_i}}{1 - z^{c_i}}.$$

To get the generating function for $\sum c_i x_i^*$ strictly smaller than t, one just multiplies this by $z + z^2 + z^3 + \cdots$, obtaining

$$\gamma(z) = \frac{z}{1 - z}\cdot\prod_{i=1}^{n}\frac{z^{c_i}}{1 - z^{c_i}}.$$

If we let m be the least common multiple of the c_i's, then all the fractions in this product can be written with denominator $1 - z^m$. It follows, by the general theory of rational generating functions (cf., [17], Chapter 4), that $\psi(t)$ is, for positive t, a quasi-polynomial with quasi-period m and degree n. This means that, when restricted to values of t in any one congruence class modulo m, ψ is a polynomial of degree n.

From here on, the proof is computational. One inserts into the formula for $\gamma(z)$ the coefficients c_i appropriate for a particular Φ (cf., page 98 of [11]), one obtains a polynomial formula for ψ on each congruence class modulo m (either by direct calculation or by computing enough values of ψ to uniquely interpolate polynomials of the right degree), and one verifies that, for the congruence classes prime to m (or equivalently prime to all the c_i), the polynomial so obtained, when multiplied by $|W|/f$, yields the (known) characteristic polynomial of $\mathcal{W}$. Here are some of the computations.

For A_n, the c_i are all 1, so

$$\gamma(z) = \frac{z^{n+1}}{(1-z)^{n+1}}.$$

Here the coefficients of the expansion are well known, and we find that $\psi(t) = \binom{t-1}{n}$. Multiplying by $n! \prod_i c_i = n!$ we get $\langle t-1\rangle_n$, the characteristic polynomial of $\mathcal{A}_n$. (This differs from the characteristic polynomial of $\mathcal{A}_n$ in the preceding section because what was there called $\mathcal{A}_n$ is the current $\mathcal{A}_{n-1}$ with all dimensions increased by 1.)

For B_n, the c_i are all 2 except for a single 1, so t is odd. The generating function is

$$\gamma(z) = \frac{z}{1-z} \cdot \left(\frac{z^2}{1-z^2}\right)^{n-1} \cdot \frac{z}{1-z} = \frac{z^{2n}(1+z)^2}{(1-z^2)^{n+1}}.$$

Here the expansion of $(1-z^2)^{-n-1}$ contains every even power z^{2k} of z with coefficient $\binom{k+n}{n}$ (and of course contains no odd powers of z). So, since t is odd, the coefficient of z^t in $\gamma(z)$ is

$$\psi(t) = 2 \cdot \binom{(t-1)/2}{n}.$$

Multiplying by $n! \prod_i c_i = 2^{n-1} n!$, we get

$$2^n \cdot \langle (t-1)/2\rangle_n = \langle\langle t-1\rangle\rangle_n,$$

the characteristic polynomial of $\mathcal{B}_n$. We note that when t is even a similar calculation gives

$$2^{n-1} n! \psi(t) = (t-2)(t-4)\cdots(t-2n+2)\cdot(t-n) = \chi(\mathcal{D}_n, t-1).$$

We do not know any reason for this coincidence.

The computations for C_n, D_n and the exceptional root systems follow the same pattern as those for A_n and B_n. The necessary information can be found in the following table. In it, the c_i are listed using the notation $1^{m_1}, \ldots, n^{m_n}$ which means that the value j appears with multiplicity m_j. Also for brevity $\chi(\mathcal{W}, t)$ is expressed by listing its roots which are just the exponents of W. □

W	Roots of $\chi(\mathcal{W}, t)$	$\gamma(z)$	c_i
A_n	$1, 2, \ldots, n$	$\frac{z^{n+1}}{(1-z)^{n+1}}$	1^n
B_n/C_n	$1, 3, \ldots, 2n-1$	$\frac{z^{2n}(1+z)^2}{(1-z^2)^{n+1}}$	$1, 2^{n-1}$
D_n	$1, 3, \ldots, 2n-3, n-1$	$\frac{z^{2n-2}(1+z)^4}{(1-z^2)^{n+1}}$	$1^3, 2^{n-3}$
E_6	1, 4, 5, 7, 8, 11	$\frac{z^{12}}{(1-z)^3(1-z^2)^3(1-z^3)}$	$1^2, 2^3, 3$
E_7	1, 5, 7, 9, 11, 13, 17	$\frac{z^{18}}{(1-z)^2(1-z^2)^3(1-z^3)^2(1-z^4)}$	$1, 2^3, 3^2, 4$
E_8	1, 7, 11, 13, 17, 19, 23, 29	$\frac{z^{30}}{(1-z)(1-z^2)^3(1-z^3)^2(1-z^4)^2(1-z^5)(1-z^6)}$	$2^2, 3^2, 4^2, 5, 6$
F_4	1, 5, 7, 11	$\frac{z^{12}}{(1-z)(1-z^2)^2(1-z^3)(1-z^4)}$	$2^2, 3, 4$
G_2	1, 5	$\frac{z^6}{(1-z)(1-z^2)(1-z^3)}$	2, 3

We should mention that Haiman [10, Section 7.4] independently discovered this theorem and gave a proof which is more uniform but less elementary. Very recently Christos Athanasiadis [1] has given another uniform demonstration. His main tool is the following result of Crapo and Rota [9] which is similar in statement and proof to Theorem 2.1 but replaces $[-s, s]^n$ by $\mathbb{F}_p^n$ where $\mathbb{F}_p$ is the finite field with p elements, p prime.

Theorem 4.2 (Crapo and Rota) *Let $\mathcal{A}$ be any subspace arrangement in $\mathbb{R}^n$ defined over the integers and hence over $\mathbb{F}_p$. Then for large enough primes p we have*

$$\chi(\mathcal{A}, p) = \left|\mathbb{F}_p^n \Big\backslash \bigcup \mathcal{A}\right|. \qquad \square$$

It is interesting to note that this result can also be obtained from results of Lehrer [12] about the l-adic cohomology of hyperplane complements in $\mathbb{C}^n$. In fact Lehrer has an l-adic cohomological interpretation of the characteristic polynomial in the equivariant case. This suggests the problem of trying to find versions of our two main theorems when there is an automorphism g of $\mathbb{C}^n$ stabilizing $\mathcal{A}$ and one considers the poset of all elements of $L(\mathcal{A})$ fixed by g.

Acknowledgments

We would like to thank John Stembridge who suggested using the affine Weyl chamber to obtain the the set of points counted in Theorem 4.1. In addition, we thank Christos Athanasiadis and Arun Ram for interesting discussions and relevant references as well as the referees for helpful suggestions.

References

1. C.A. Athanasiadis, “Characteristic polynomials of subspace arrangements and finite fields,” *Adv. in Math.* **122** (1996), 193–233.

2. A. Björner, "Subspace arrangements," in *Proc. 1st European Congress Math.* Paris, 1992, A. Joseph and R. Rentschler (Eds.), *Progress in Math.*, Birkhäuser, Boston, MA, 1994, Vol. 122, pp. 321–370.
3. A. Björner, L. Lovász, and A. Yao, "Linear decision trees: Volume estimates and topological bounds," *Proceedings 24th ACM Symp. on Theory of Computing*, ACM Press, New York, NY, 1992, pp. 170–177.
4. A. Björner and L. Lovász, "Linear decision trees, subspace arrangements and Möbius functions," *J. Amer. Math. Soc.* **7** (1994), 667–706.
5. A. Björner and V. Welker, "The homology of "k-equal" manifolds and related partition lattices," *Adv. in Math.* **110** (1995), 277–313.
6. A. Björner and M. Wachs, "Shellable nonpure complexes and posets I," *Trans. Amer. Math. Soc.* **348** (1996), 1299–1327.
7. A. Björner and B. Sagan, "Subspace arrangements of type B_n and D_n," *J. Algebraic Combin.*, submitted.
8. A. Björner and M. Wachs, "Shellable nonpure complexes and posets II," *Trans. Amer. Math. Soc.*, to appear.
9. H. Crapo and G.-C. Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, M.I.T. Press, Cambridge, MA, 1970.
10. M. Haiman, "Conjectures on the quotient ring of diagonal invariants," *J. Alg. Combin.* **3** (1994), 17–76.
11. J. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Press, Cambridge, England, 1990.
12. G. Lehrer, "The l-adic cohomology of hyperplane complements," *Bull. London Math. Soc.* **24** (1992), 76–82.
13. S. Linusson, "Partitions with restricted block sizes, Möbius functions and the k-of-each problem," *SIAM J. Discrete Math.*, to appear.
14. P. Orlik and L. Solomon, "Coxeter arrangements," *Proc. Symp. Pure Math.*, Amer. Math. Soc., Providence, RI, 1983, Vol. 40, Part 2, pp. 269–291.
15. P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Grundlehren 300, Springer-Verlag, New York, NY, 1992.
16. G.-C. Rota, "On the foundations of combinatorial theory I. Theory of Möbius functions," *Z. Wahrscheinlichkeitstheorie* **2** (1964), 340–368.
17. R.P. Stanley, *Enumerative Combinatorics*, Wadsworth and Brooks/Cole, Pacific Grove, CA, Vol. 1, 1986.
18. R.P. Stanley, "Graph colorings and related symmetric functions: Ideas and applications," *Discrete Math.*, to appear.
19. S. Sundaram, "Applications of the Hopf trace formula to computing homology representations," *Contemp. Math.* **178** (1994), 277–309.
20. S. Sundaram and M. Wachs, "The homology representations of the k-equal partition lattice," *Trans. Amer. Math. Soc.*, to appear.
21. S. Sundaram and V. Welker, "Group actions on linear subspace arrangements and applications to configuration spaces," *Trans. Amer. Math. Soc.*, to appear.
22. H. Terao, "Generalized exponents of a free arrangement of hyperplanes and the Shepherd-Todd-Brieskorn formula," *Invent. Math.* **63** (1981), 159–179.
23. T. Zaslavsky, "The geometry of root systems and signed graphs," *Amer. Math. Monthly* **88** (1981), 88–105.
24. T. Zaslavsky, "Signed graph coloring," *Discrete Math.* **39** (1982), 215–228.
25. T. Zaslavsky, "Chromatic invariants of signed graphs," *Discrete Math.* **42** (1982), 287–312.
26. P. Zhang, "Subposets of Boolean Algebras," Ph.D. thesis, Michigan State University, 1994.