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Asymmetric Combinatorially-Regular Maps*

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Abstract. It is shown that for every $g \geq 3$, there exists a combinatorially regular map M of type $\{3, 7\}$ on a closed orientable surface of genus g, such that M has trivial symmetry group. Such maps are constructed from Schreier coset graphs corresponding to permutation representations of the (2, 3, 7) triangle group.

1. Introduction

A map M is a 2-cell embedding of a connected graph (or multigraph) into some closed surface S without boundary, dividing S into simply-connected regions called *faces* of the map. The faces of M are of course the connected components of the space obtained by removing the embedded graph from the surface; alternatively, they may be viewed as the cycles of some permutation of the set of arcs (or directed edges) of the underlying graph.

We use $V(M)$, $E(M)$ and $F(M)$, to denote the sets of vertices, edges and faces of M, respectively. The Euler characteristic of the associated surface S may then be calculated using the Euler-Poincaré formula $\chi = |V(M)| - |E(M)| + |F(M)|$, and in the case where S is orientable, this is also equal to $2 - 2g$ where g is the (orientable) genus: the number of cylindrical handles glued to the sphere in order to obtain S; (see [2] Section 8.1).

An *automorphism* of a map M is any automorphism of the underlying graph (or multigraph) which preserves also the faces of the embedding—or, what is essentially equivalent, a homeomorphism of the surface S preserving $V(M)$ and $E(M)$ and necessarily also $F(M)$. As usual, these automorphisms form a group under composition, called the automorphism group of the map M, and denoted by Aut M.

By connectedness, the action of each automorphism of a map M is uniquely determined by its effect on any incident vertex-edge-face triple, or "flag". In particular, the order of Aut M is bounded above by the number of flags, and when this upper bound is attained, Aut M acts transitively, indeed regularly, on flags, and M is called a (*reflexible*) *regular* map. Examples include embeddings of the 1-skeletons of the Platonic solids on the sphere, and others on orientable surfaces of all possible genera—see [2, 4] and [5] for details of these and further background along with some of the history of regular maps.

Every regular map is combinatorially regular, in that each of its faces is bounded by the same number of edges, say p, and each of its vertices is incident with the same number of

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edges, say q. We say such a map M has type $\{p, q\}$. Counting the number of arcs of M in three different ways gives the well known identity $q|V(M)| = 2|E(M)| = p|F(M)|$, and the Euler characteristic χ becomes $2|E(M)|(1/p + 1/q - 1/2)$. Also the dual of M (whose vertices correspond to faces of M and whose faces correspond to vertices of M) is combinatorially regular of type $\{q, p\}$, lies on the same surface as M, and has the same automorphism group as M.

On the other hand, combinatorially regular maps are not necessarily regular (in the sense of having the largest possible number of automorphisms). In this paper we show that in fact they can go to the opposite extreme, of having no automorphisms other than the identity. We construct an infinite family of combinatorially regular maps of type $\{3, 7\}$, on orientable surfaces of every genus $g \geq 3$, and each with trivial automorphism group.

This answers a question raised by Paul Schmutz (in a private communication) based on his research on the lengths of shortest closed geodesics on Riemann surfaces; see [3]: *do there exist* $(2, 3, 7)$ *triangular tessellations of surfaces with automorphism group acting trivially on the triangles*? We show the answer is yes.

2. The construction

For each positive integer $n \geq 2$ we construct an orientable map $M(n)$ having $28n$ vertices, $42n$ edges, and $12n$ faces. Every face is bounded by 7 edges, and every vertex is incident with 3 edges, so that the dual of $M(n)$ is combinatorially regular of type $\{3, 7\}$, and lies on a surface of Euler characteristic $-2n$ and (orientable) genus $n + 1$.

To do this, we consider transitive permutation representations of the $(2, 3, 7)$ triangle group $\Delta = \langle x, y \mid x^2 = y^3 = (xy)^7 = 1 \rangle$. Corresponding to any such representation on a set Ω is a *Schreier coset diagram*, which is essentially a directed graph whose vertices are the points of Ω, and in which each vertex $\alpha \in \Omega$ is joined by an arc to each of the images α^x and α^y of α under the permutations induced by x and y.

These diagrams, described more generally in [2], were used in [1] to investigate factor groups of Δ. In particular, in any coset diagram for Δ the 3-cycles of the permutation induced by the generator y may be represented by small triangles, with edges directed anticlockwise, and the double arcs representing 2-cycles of x may be replaced by single edges (whose end-points are interchanged by x). Also under certain circumstances, separate diagrams for Δ may be joined together to create a new diagram, corresponding to a transitive permutation representation of Δ of larger degree.

Now suppose D is any such diagram, corresponding to a representation of Δ in which none of the elements x, y, xy and $xy^{-1}xy$ has a fixed point. If the above simplifications are made, and then each small triangle representing a 3-cycle of y is shrunk to a single vertex, the diagram becomes a simple cubic graph. Further, this graph has a natural embedding in an orientable surface S, in which the edges incident to any given vertex are oriented anticlockwise in the same order as described above: if v is the vertex corresponding to the 3-cycle $(\alpha, \alpha^y, \alpha^{yy})$ of y, then the three edges incident to v are oriented anticlockwise in the order (α, α^x), (α^y, α^{yx}), $(\alpha^{yy}, \alpha^{yyx})$. Subject to this orientation, the faces of the resulting map M correspond precisely to the cycles of the product xy, and are therefore each bounded by 7 edges, so the map M is combinatorially regular of type $\{7, 3\}$.

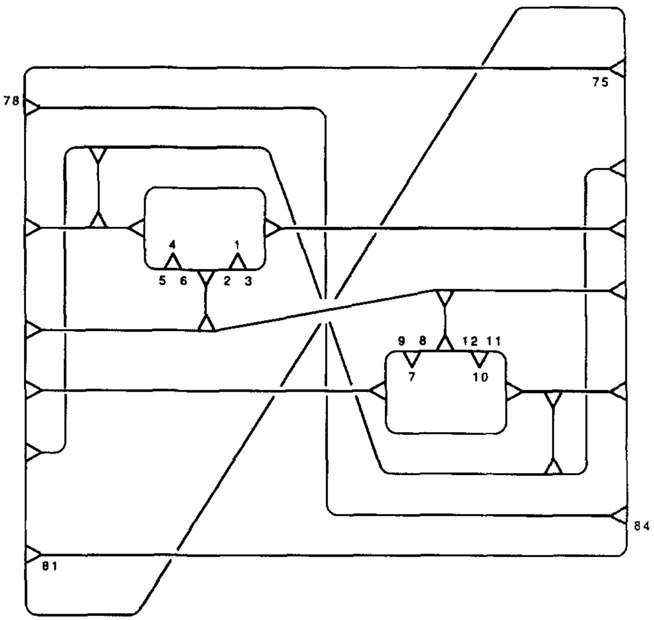

Diagram P

We apply this argument to a family of coset diagrams, all obtained by joining together a chain made up of one copy of a particular diagram P and a number of copies of a second diagram Q. These two basic diagrams P and Q correspond to the following transitive permutation representations of the (2, 3, 7) triangle group Δ, each on 84 points:

Diagram P:

$x \to$ (2,13)(3,16)(5,19)(6,14)(8,22)(9,25)(11,28)(12,23)(15,31)(17,34)(18,20)
(21,37)(24,40)(26,43)(27,29)(30,46)(32,49)(33,42)(35,52)(36,55)(38,58)(39,61)
(41,53)(44,51)(45,64)(47,67)(48,70)(50,63)(54,72)(56,73)(57,69)(59,68)(60,66)
(62,76)(65,79)(71,82)(74,80)(75,78)(77,83)(81,84),
$y \to$ (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)
(25,26,27)(28,29,30)(31,32,33)(34,35,36)(37,38,39)(40,41,42)(43,44,45)(46,47,48)
(49,50,51)(52,53,54)(55,56,57)(58,59,60)(61,62,63)(64,65,66)(67,68,69)(70,71,72)
(73,74,75)(76,77,78)(79,80,81)(82,83,84);

Diagram Q:

$x \to$ (2,13)(3,16)(5,19)(6,14)(8,22)(9,25)(11,28)(12,23)(15,31)(17,34)(18,20)
(21,37)(24,40)(26,43)(27,29)(30,46)(32,49)(33,52)(35,55)(36,58)(38,61)(39,64)
(41,67)(42,63)(44,70)(45,51)(47,73)(48,76)(50,66)(53,79)(54,56)(57,82)(59,80)
(60,72)(62,71)(65,74)(68,78)(69,84)(75,83)(77,81),

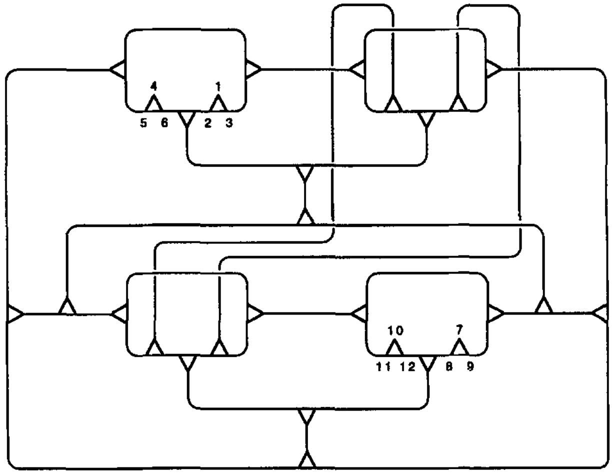

Diagram Q

$y \to$ (1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)
(25,26,27)(28,29,30)(31,32,33)(34,35,36)(37,38,39)(40,41,42)(43,44,45)(46,47,48)
(49,50,51)(52,53,54)(55,56,57)(58,59,60)(61,62,63)(64,65,66)(67,68,69)(70,71,72)
(73,74,75)(76,77,78)(79,80,81)(82,83,84).

Note that in both representations, the permutations induced by y and xy have no fixed points, while on the other hand, x fixes each of the four points 1, 4, 7 and 10. Moreover, $(xy)^3$ takes 1 to 4, and 7 to 10, in both cases. In the terminology of [1], this means these four points form a pair of (3)-handles: [1, 4] and [7, 10], each of which can be used to join the coset diagram to any other one containing the same sort of configuration.

We take one copy of the diagram P, and $n - 1$ copies of the diagram Q, and join them together in a circular chain, by attaching the (3)-handle [1, 4] of each diagram to the (3)-handle [7, 10] of the next. Specifically, we insert two new x-edges, from the points 1 and 4 of one diagram to the points 7 and 10, respectively, of the next. Overall these joins correspond to the addition of $2n$ disjoint transpositions to the cycle structure of the permutation induced by x on the union of the associated vertex-sets, in a way that preserves the cycle structures of the permutations induced by y and xy. The result is a diagram which corresponds to a transitive permutation representation of Δ on $84n$ points.

By our choice of the diagrams P and Q and our use of all (3)-handles, the permutations induced by x, y and xy in this representation are easily seen to have no fixed points. Also it is not difficult to verify that $xy^{-1}xy$ has no fixed points; indeed $xy^{-1}xy$ induces a permutation with cycle structure $2^2 3^4 4^2 21^2 51^2$ when $n = 2$, or $2^2 3^{2n} 4^2 21^2 28^2 39^{2(n-3)} 62^2$ when $n \geq 3$.

As explained earlier, this is enough to ensure that a combinatorially regular map of type $\{7, 3\}$ may be formed from the diagram by shrinking each of its small triangles to a single vertex, and so we obtain such a map $M(n)$ having $28n$ vertices, $28n \times 3/2 = 42n$ edges, and $42n \times 2/7 = 12n$ faces, as claimed.

3. Asymmetry

We now show that the maps constructed in the previous section have no automorphisms other than the identity. To do this, we use the fact that the underlying graph of the map $M(n)$ in each case has exactly one circuit of length 4, which must then be preserved by every automorphism. For notational convenience, let $\overline{\alpha}$ denote the vertex of $M(n)$ obtained by shrinking the small triangle containing the point α of the diagram P.

First observe that any circuit of length 4 corresponds to the existence of a fixed point of some element of the form $xy^pxy^qxy^rxy^s$ (with $p, q, r, s \in \{1, 2\}$) in the group Δ, and that every element of this form is conjugate to either a non-trivial power of xy, or $(xy^{-1}xy)^2$, or $xy^{-1}xy^{-1}xyxy$. (Note, for example: $xyxyxyxy^{-1} = (yxy)^{-1}xy(yxy)$ since $(xy)^7 = 1$.) But we know that xy has no fixed points, while $(xy^{-1}xy)^2$ has exactly four fixed points: these are the points labelled 75, 78, 81 and 84 in the single copy of the diagram P, and they make up the two transpositions in the cycle structure of $xy^{-1}xy$. On the other hand, $xyxyxy^{-1}xy^{-1}$ has cycle structure $3^2 10^2 22^2 49^2$ when $n = 2$, and $3^2 7^{2(n-2)} 18^2 21^{2(n-3)} 52^2 (7n-4)^2 (7n+8)^2$ when $n \geq 3$, and hence has no fixed points.

In particular, there is just one circuit of length 4 in $M(n)$, made up of the vertices $\overline{75}$, $\overline{78}$, $\overline{81}$ and $\overline{84}$ in the single copy of the diagram P.

Next, there are four vertices at distance 1 from this circuit in $M(n)$, namely $\overline{57}$, $\overline{63}$, $\overline{66}$ and $\overline{72}$, and a further eight vertices at distance 2, namely $\overline{36}$, $\overline{69}$, $\overline{39}$, $\overline{51}$, $\overline{45}$, $\overline{60}$, $\overline{48}$ and $\overline{54}$. Of these eight, all but the two vertices $\overline{60}$ and $\overline{69}$ are adjacent to a vertex of distance 3 from the circuit (while $\overline{60}$ and $\overline{69}$ are adjacent only to vertices at distance 1 or 2).

It follows that if θ is any automorphism of the map $M(n)$, then θ must preserve the circuit of length 4, and either fix or interchange the vertices $\overline{60}$ and $\overline{69}$ of the diagram P. If θ fixes them, then θ fixes all vertices close to the 4-circuit, and it follows easily that θ is trivial. Otherwise θ induces the following partial permutation on those vertices of $M(n)$ obtained from triangles in diagram P, corresponding geometrically to a 180-degree rotation of diagram P:

$$\theta \to (\overline{3}, \overline{9})(\overline{6}, \overline{12})(\overline{15}, \overline{24})(\overline{18}, \overline{27})(\overline{21}, \overline{30})(\overline{33}, \overline{42})(\overline{36}, \overline{45})(\overline{39}, \overline{48})(\overline{51}, \overline{54})(\overline{57}, \overline{66})$$
$$(\overline{60}, \overline{69})(\overline{63}, \overline{72})(\overline{75}, \overline{81})(\overline{78}, \overline{84}).$$

This partial permutation, however, does not extend to the whole of $M(n)$, because the rotational symmetry of the diagram P does not carry over to the copy or copies of the diagram Q to which diagram P is joined. Thus $M(n)$ has no non-trivial automorphisms, as required.

Note: There is no such asymmetric map of type $\{7, 3\}$ on a surface of genus 2. This follows from the fact that although the the $(2, 3, 7)$ triangle group has 100 different transitive permutation representations of degree 84 in which none of the elements x, y, xy and $xy^{-1}xy$ has a fixed point, the corresponding map in each case has automorphism group of order 2, 4, 8, 12, 24 or 32. Further details are available from the author upon request.

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