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Strong Connectivity of Polyhedral Complexes

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Abstract. A classical theorem of Robbins states that the edges of a graph may be oriented, in such a way that an oriented path exists between any source and destination, if and only if the graph is both connected and two-connected (it cannot be disconnected by the removal of an edge). In this paper, an algebraic version of Robbins' result becomes a lemma on Hilbert bases for free abelian groups, which is then applied to generalize his theorem to higher dimensional complexes. An application to cycle bases for graphs is given, and various examples are presented.

Keywords: strong connectivity, Hilbert basis, homology

1. Introduction

Our aim is to generalize a classical theorem of Robbins of graph theory. A graph is a pair $G = (V, E)$ of vertices and edges, where $E \subset \binom{V}{2}$. A directed graph is a graph $G = (V, E)$ together with functions $\epsilon_e :\to \{s, t\}$ for each $e \in E$ (s = source, t = target). A directed graph is *strongly connected* if there is a directed path between any two vertices, in both directions. A graph is *k-connected* if it may not be disconnected by deletion of $(k - 1)$ edges. These notions are extensively studied in graph theory, and have numerous applications (see [3] for a survey of results).

Given a graph G, does there exist an orientation of its edges so that the resulting graph is strongly connected? An obvious necessary condition is G to be 2-connected and a classical theorem of Robbins asserts that this condition is also sufficient.

Theorem 1.1 [8] *There is a strong orientation of edges of a graph G if and only if G is 2-connected.*

A seminal graph-theoretic generalization of 1.1 is due to Nash-Williams.

Theorem 1.2 [7] *Let G be a graph and let $\lambda(a, b)$ denote the maximum number of edge—disjoint paths between vertices a, b of G. Then G has an orientation such that there are at least $\lfloor \frac{\lambda(x,y)}{2} \rfloor$ edge disjoint directed paths from any vertex x to any vertex y.*

Our aim is to present another kind of generalization: from strong connectivity for pairs of vertices to "strong connectivity" for oriented cycles. We recall that an n-polyhedron P is the convex closure of a finite set of points of R^N whose affine span has dimension n. The boundary ∂P of P is a union of $(n-1)$-polyhedrons. A 0-complex is a finite set K_0 of points. A j-complex $K = (K_j, \ldots, K_0)$ is obtained from a $(j-1)$-complex $K' = (K_{j-1}, \ldots, K_0)$ and a finite set K_j of j-polyhedrons by "glueings": for each $P \in K_j$ there is a map $\Phi_p \colon P \to K'$ which is an affine homeomorphism whenever restricted to a k-polyhedron, $k \leq (j-1)$, and K is obtained by identifying x with $\Phi_P(x)$, $x \in P$. Each j-polyhedron Q has two orientations $\vec{Q}$ and $-\vec{Q}$. We will consider formal sums of orientations of j-polyhedrons modulo the equations $\vec{Q} + (-\vec{Q}) = 0$. The orientation of Q induces the orientations to each $P \in \partial Q$. For $j > 0$ let $d_j\vec{Q} = \sum_{P\in\partial Q} \vec{P}$ be the formal sum of oriented polyhedrons of the boundary of $\vec{Q}$ with an orientation induced by $\vec{Q}$. Let $d_0\vec{v} = -d_0(-\vec{v}) = 1$ if $v \in K_0$. A key feature of the notion of orientation is

$$d_{j-1} \cdot d_j\vec{Q} = 0.$$

Let $K = (K_n, \ldots, K_0)$ be an n-complex. A *j-cycle* is a formal sum

$$R = \sum_{Q\in K_j} n_Q\vec{Q}, n_Q \in \mathbb{Z}$$

such that

$$\sum_{Q\in K_j} n_Q d_j\vec{Q} = 0.$$

Hence $d_j\vec{Q}$ is a $(j-1)$-cycle. An *orientation* of an n-complex is a choice of orientation $\vec{P}$ for each n-polyhedron P. Next, we introduce the notion of strong connectivity, state our main result and relate it back to graph theory.

Definition 1.3 Let $K = (K_n, \ldots, K_0)$ be an n-complex and let $\Phi = \{\vec{P}\}_{P\in K_n}$ be its orientation. We say that (K, Φ) is n-strongly connected if for each $(n-1)$-cycle R and for each $P \in K_n$ there are coefficients $n_P^R \in \mathbb{N} = \{0, 1, \ldots\}$ such that

$$R = \sum_{P\in K_n} n_P^R d_n\vec{P}.$$

Theorem 1.4 *Let $K = (K_n, \ldots, K_0)$ be an n-complex. There is an orientation Φ such that (K, Φ) is n-strongly connected if and only if the following two conditions hold:*

(Ai) *each $(n-1)$-cycle R is a formal sum*

$$R = \sum_{P\in K_n} m_P^R d_n\vec{P}, m_P^R \in \mathbb{Z}$$

(*i.e.* $\tilde{H}_{n-1}(K) = 0$).

(Aii) *for each* $Q \in K_n$,

$$0 = \sum_{P \in K_n} h_P^Q d_n \vec{P},\ h_P^Q \in \mathbb{Z},\ h_Q^Q \neq 0$$

(*i.e.* $\bar{H}_{n-1}(K - \operatorname{int} Q)$ *is torsion for each* $Q \in K_n$).

Remark 1.5 *(Relations to graph theory)* Let us observe how strong connectivity of graphs and Robbins theorem fit into this scheme. Graphs (undirected) coincide with 1-complexes, their vertices with 0-polyhedrons and their edges with 1-polyhedrons. An orientation of a 1-complex is a prescription of an orientation to each edge. All 0-cycles are generated by the pairs of oriented vertices $(w, -v)$. Hence by 1.3, an orientation of a graph is 1-strongly connected if and only if it has a directed path between any pair of vertices, in both directions. Hence 1-strong connectivity coincides with strong connectivity. Further, Theorem 1.4 is simply equivalent to Robbins' theorem. The condition (Ai) asserts the graph being connected, and (Aii) that each edge belongs to a circuit. These two conditions are trivially equivalent to 2-edge connectivity and actually give a geometrical form of the Robbins' theorem.

Further, let us see what 2-strong connectivity means. A pair (G, D), where G is a graph and each element of D is an orientation of a cycle of G, is 2-strongly connected if any orientation $\vec{C}$ of any cycle of G is obtained from a disjoint union of the arcs of some cycles of D (a cycle may be taken more than once) by deletion of pairs of oppositely directed arcs with the same vertices. Our theorem addresses the problem when is it possible to reorient cycles of D so that (G, D) is 2-strongly connected. Similar questions are studied extensively. Let us mention at least one open problem [10] called Cycle Double Cover Conjecture: Is it true that every 2-connected graph has a family of cycles such that each edge is in precisely two of them?

However, a relation of our result to the cycle double cover conjecture is not known. Let us remark finally that conditions (Ai) and (Aii) of 1.4 may be polynomially tested: To test (Ai) it suffices to solve a linear number of systems of linear diophantine equations with integer coefficients (for an efficient algorithm see [4]). Gaussian elimination may be applied to test (Aii).

The paper is organized as follows. In Section 2 we present examples. In Section 3 an algebraic version of Robbins' Theorem becomes the crucial lemma in the proof of 1.4. The lemma may be of independent interest. This raises the question of whether the geometric version of Robbins' Theorem (graphs whose edges lie on cycles are strongly connected) has higher-dimensional analogues; this issue is discussed in Section 4.

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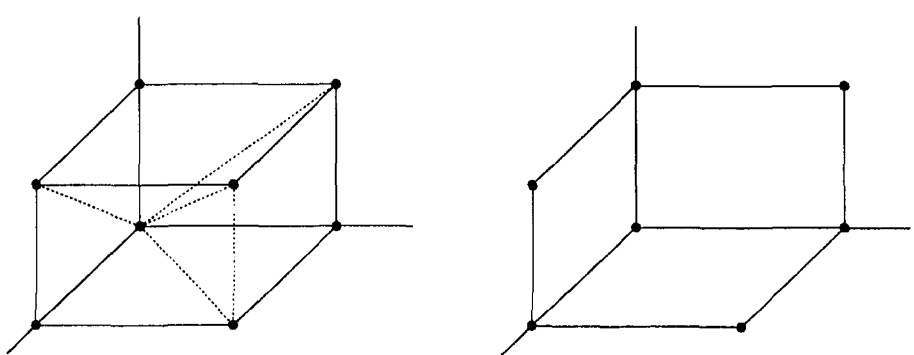
2. Examples

Example 1 Let $(P, \leq)$ be a partially ordered set (poset) and $m \in \mathbb{N}$. Consider an m-complex $C_m(P)$ defined as follows. Let $\{e_p;\ p \in P\}$ be a basis of R^P. For each strict chain $C = p_0 < p_1 < \cdots < p_k$, $k \leq m$, let Δ_C be the k-simplex associated with C, i.e., the convex closure of $\{e_{p_0}, \ldots, e_{p_k}\}$. Complex $C_m(P)$ is a union of all Δ_C, C being a strict chain of length at most m, modulo obvious glueings.

For example, let us consider the poset $2^{[1,n]}$ of all subsets of $\{1, \ldots, n\}$. $C(2^{[1,n]})$ is homeomorphic to the n-cube C_n with vertices $(\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i \in \{0, 1\}$ in R^n (see figure 1). Further, let $2^{(1,n)}$ be the subposet of the nonempty proper subsets of $\{1, \ldots, n\}$. $C_{n-2}(2^{(1,n)})$ is the boundary of the subset $C'_n = \{(x_1, \ldots, x_n) \in C_n,\ \Pi_{x_i} = 0\}$ of C_n, hence $C_{n-2}(2^{(1,n)})$ is homeomorphic to an $(n-2)$-sphere (see figure 1). Now, let k be any field. *Tits' complex* $T(k, n)$ is the complex associated to the poset (k, n) of nonempty, proper subspaces of k^n. $T(k, n)$ is an $(n-2)$-complex, whose $(n-2)$-polyhedrons correspond to flags (maximal chains in (k, n)).

Tits' Theorem asserts that $T(k, n)$ is homotopically a wedge of $(n-2)$-spheres, so according to the remarks in the introduction one expects to find a natural $(n-2)$-strong orientation on $T(k, n)$.

Michel Brion provided the following one. Fix a basis $E = \{e_1, \ldots, e_n\}$ for k^n. A chain $V^{i_1} \subset \cdots \subset V^{i_k}$ $(i_1 = \dim V^{i_1})$ of subspaces is called *special* if each V^{i_1} is spanned by a subset of E. The subposet P of special chains is isomorphic to $2^{(1,n)}$ and thus $C(P)$ is a subcomplex of $T(k, n)$ which is an $(n-2)$-sphere. As a sphere, $C(P)$ has two orientations which make it an $(n-2)$-cycle. Choose one, and orient each simplex of a special flag according to it. Now observe that for any flat $V^1 \subset \cdots \subset V^{n-1}$ there is a special flag $F_V^1 \subset \cdots \subset F_V^{n-1}$ and an upper triangular matrix A satisfying $A\,V^k = F_V^k$, $1 \leq k \leq n-1$. For each flag V we will orient the associated $(n-2)$-simplex Δ_V of $T(k, n)$ by $\vec{\Delta}_V = A\vec{\Delta}_F$. Evidently every $\vec{\Delta}_V$ belongs to an $(n-2)$-cycle, hence the resulting orientation of $T(k, n)$ is $(n-2)$-strongly connected.



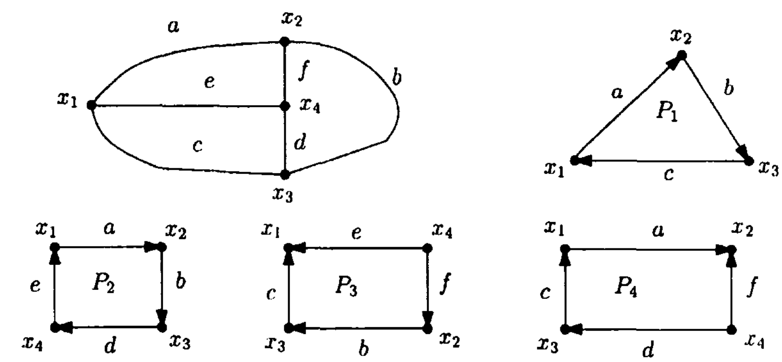
$C_3(2^{[1,3]})$ $\quad$ $C_1(2^{(1,3)})$

Figure 1.

To conclude this example, we remark that the property of being wedges of spheres is shared by many complexes defined in a combinatorial way, e.g., isthmus-free matroid complexes, order complexes of geometrical lattices and Tits buildings (see [1, 2]).

The next example provides a preparation to Section 4. Let K be a 2-complex and $\tilde{H}_1(K) = 0$. Is it true that K is 2-strongly connected if and only if each region of K_2 belongs to an orientable pseudosurface? The beetle provides a contraexample to this natural generalization of the geometrical Robbins' Theorem.

Example 2 (The beetle)



$K_2 = \{P_1, P_2, P_3, P_4\}$
$K_1 = \{a, b, c, d, e, f\} \cup \partial P_1 \cup \partial P_2 \cup \partial P_3 \cup \partial P_4$
$K_0 = \{x_1, x_2, x_3, x_4\} \cup \partial a \cup \partial b \cup \partial c \cup \partial d \cup \partial e \cup \partial f \cup \partial\partial P_1 \cup \partial\partial P_2 \cup \partial\partial P_3 \cup \partial\partial P_4.$

The glueings are indicated by arcs.

Now, P_1 does not belong to a pseudosurface, but the beetle is still 2-strongly connected by Theorem 1.4, since

$$0 = 2d_2(P_1(abc)) + d_2(P_2(abde)) + d_2(P_3(bcef)) + d_2(P_4(afdc)).$$

The order of arcs in the brackets determines the orientations of P_1, P_2, P_3, P_4.

Note that the coefficient at $d_2(P_1(abc))$ must be at least 2; indeed $\tilde{H}_1(k - P_1) \simeq \mathbb{Z}/2$.

3. A lemma on Hilbert bases and a proof of Theorem 1.4

Our generalization is based on the following key Lemma 3.2, whose proof is an algebraic version of Robbins' classical proof.

Definition 3.1 Let A be a free abelian group. *Hilbert basis* of A is a subset $B = \{b_1, \ldots, b_n\}$ of A such that for each $v \in A$ there are numbers $n_j^v \in N$ satisfying $v = \sum_{j=1}^{n} n_j^v b_j$.

For example, if B' is a basis of A then both $B' \cup (-B')$ and $B' \cup \{-\sum_{b \in B'} b\}$ are Hilbert bases of A. Hilbert bases are studied extensively in combinatorial optimization, see e.g., [4].

Lemma 3.2 *Let A be a free abelian group and let B be a finite subset of A. There exist $\epsilon_b \in \{1, -1\}$, $b \in B$ such that $\epsilon B = \{\epsilon_b b;\, b \in B\}$ is a Hilbert basis if and only if*
(Bi) *$B \cup (-B)$ generate A.*
(Bii) *For each $\beta \in B$ there are $m_b^\beta \in \mathbb{Z}$, $m_\beta^\beta \neq 0$ such that*

$$\sum_{b \in B} m_b^\beta b = 0.$$

In Remark 1.5 we observed a reformulation of the Robbins' Theorem: A graph G has a strong orientation if and only if G is connected and each edge belongs to a cycle. Our proof of 3.2 is a mere algebraisation of a proof of this statement.

Let us see first that (Bi) and (Bii) are necessary. The necessity of (Bi) is trivial, for necessity of (Bii) assume the coefficients ϵ_b, $b \in B$ exist. W.l.o.g. assume $\epsilon_\beta = 1$. Then

$$-\beta = \sum_{b \in B} n_b^\beta \epsilon_b b,$$

hence

$$0 = \sum_{b \in B} \bar{n}_b^\beta \epsilon_b b$$

where $\bar{n}_\beta^\beta = n_\beta^\beta + 1 > 0$.
To show sufficiency, we indeed prove a stronger result. Assume that $B \cup (-B)$ generates A and ϵ is defined on a subset B' of B. It is possible to extend ϵ to ϵ' such that $\epsilon' B$ is a Hilbert bases of A if and only if

(Cii) For each $\beta \in B$ there are $m_b^\beta \in N$ and $\bar{\epsilon}_b^\beta \epsilon \{1, -1\}$ such that

$$0 = \sum_{b \in B} m_b^\beta \bar{\epsilon}_b^\beta b, \bar{\epsilon}_b^\beta = \epsilon_b$$

for $b \in B'$ and $m_\beta^\beta \neq 0$.

The proof of this statement goes as follows: Take $\alpha \in B - B'$ and assume for a contradiction that ϵ cannot be extended to $B' \cup \{\alpha\}$ so that (Cii) holds. It means that there are $x, y \in B$ so that "x must use α and y must use $-\alpha$". That is to say

a) if ϵ^x is any extension of ϵ to B, $m_b^x \in \mathbb{N}$ so that

$$0 = \sum_{b \in B} m_b^x \epsilon_b^x b \text{ and } m_x^x > 0 \text{ then } \epsilon_\alpha^x = 1.$$

b) if ϵ^y is any extension of ϵ to B, $m_b^y \in \mathbb{N}$ so that

$$0 = \sum_{b \in B} m_b^y \epsilon_b^y b \text{ and } m_y^y > 0 \text{ then } \epsilon_\alpha^y = -1.$$

We assume (Cii) hence there are $\bar{\epsilon}^x$, $\bar{\epsilon}^y$ extending ϵ and $m_b^x, m_b^y \in \mathbb{N}$ such that

$$0 = \sum_{b \in B} m_b^x \bar{\epsilon}_b^x b = \sum_{b \in B} m_b^y \bar{\epsilon}_b^y b, m_x^x > 0, m_y^y > 0,$$

and also by a) and b) $\bar{\epsilon}_\alpha^x = 1$, $\bar{\epsilon}_\alpha^y = -1$,

$$m_\alpha^x > 0, \quad m_y^y > 0, \quad m_y^x = m_x^y = 0.$$

Without loss of generality assume $m_\alpha^x \geq m_\alpha^y$. Then

$$0 = \sum_{\substack{b \in B \\ b \neq \alpha}} (m_b^x \bar{\epsilon}_b^x + m_b^y \bar{\epsilon}_b^y) b + (m_\alpha^x - m_\alpha^y)\alpha.$$

In this formal sum, the coefficient at α is nonnegative and the coefficient at y equals to $m_y^y \bar{\epsilon}_y^y \neq 0$ which violates b).

We have a contradiction, hence ϵ can always be extended so that (Cii) is satisfied and thus lemma follows.

We get Theorem 1.4 as a corollary.

Proof of Theorem 1.4: Let $A = \mathbb{Z}_{n-1}(K)$ be the group of $(n-1)$-cycles of K and let $B = \{d_n \vec{P}\}_{P \in K_n}$. By 4.2 we may change an orientation of K_n to an orientation for which $B' = \{d_n \vec{P}\}_{P \in K_n}$ is a Hilbert basis of A if and only if $\bar{H}_{n-1}(K) = 0$ and B satisfies (Bii) which is equivalent to (Aii). Hence the theorem is proved.

4. Some geometrical observations

Is there a geometrical form of (Aii)? Example 2 shows that a natural condition is strictly stronger than (Aii) even for 2-complexes. Hence, the geometrical analogue turns out to be not very geometrical, although it is illustrative at least for 2-complexes.

Definition 4.1 *A geometric n-cycle* is a triple $L = \{L_n, L_{n-1}, H\}$ where L_n is a set of n-polyhedrons L_{n-1} is a set of $(n-1)$-polyhedrons containing ∂P for each $P \in L_n$ and H is a set of affine "glueing" homeomorphisms between elements of L_{n-1} closed under inverse, composition and so that $P \xrightarrow{\text{id}} P$ is the only glueing from P to P. We write $P \sim Q$ if $h(P) = Q$ for some $h \in H$. Moreover, a geometric n-cycle satisfies that for each $P \in L_{n-1}/\sim$ there exist exactly two $P_1, P_2 \in L_n$ and $Q_i \in \partial P_i$, $i = 1, 2$, such that $Q_1 \neq Q_2$ in L_{n-1} and $P = Q_1 = Q_2$ in $L_{n-1}/\sim$.

An orientation of a geometric n-cycle is an orientation of L_n which satisfies $\sum_{P \in L_n} d_n \vec{P} = 0$. L is orientable if it possesses an orientation.

A geometric 1-cycle is a graph consisting of a cycle. It seems to be a well-known fact that the geometric 2-cycles are the pseudosurfaces and the orientable geometric 2-cycles are the orientable pseudosurfaces. However, in general, geometric n-cycles do not posses a "nice" geometrical realization.

Definition 4.2 Let K be an n-complex and let $P \in K_n$. We say that P *belongs to a ramified geometric n-cycle* if there exists an oriented geometric n-cycle $L = (\vec{L}_n, L_{n-1})$ and a function l: $L \to K$ such that

(i) $L_j = \cup\{Q_1, \ldots, Q_{m_Q}\} m_Q \geq 0$, $j = n, n-1$ and $m_P > 0$.
(ii) $1/Q_i = \mathrm{id}/_Q$ for each $Q \in L_n, L_{n-1}$ and $i \in [1, m_Q]$
(iii) For each $Q \in K_n$, the orientations of Q induced by all the corresponding Q_i of L_n are the same.

Condition (Aii) is now equivalent to

(Gii) Each $P \in K_n$ belongs to a ramified geometric n-cycle
which for $n = 2$ has form
(G′ii) Each region of K_2 belongs to a ramified orientable pseudosurface.

Concluding remarks

1. Given a graph G and a family D of its cycles, when is it possible to orient cycles of D so that any orientation $\vec{C}$ of any cycle of D may be obtained from a disjoint union of the arcs of some oriented cycles of D (each oriented cycle may be taken at most once this time) by deletion of pairs of opposity directed arcs with the same vertices? The answer to this problem is not known. It is not difficult to observe that the natural condition that each cycle (region) of D belongs to an orientable pseudosurface is again stronger.
2. Given 2-connected graph G, does there exist a family D of its cycles such that each edge of G belongs to at most two of them and (G, D) is 2-strongly connected? This question is equivalent to the cycle double cover conjecture and thus probably hard, but we also do not know the answer for a weaker problem when the condition "each edge belongs to at most 2 cycles" is replaced by "each edge belongs to at most C cycles, C being a constant".

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