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Schensted Algorithms for Dual Graded Graphs†

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Abstract. This paper is a sequel to [3]. We keep the notation and terminology and extend the numbering of sections, propositions, and formulae of [3].

The main result of this paper is a generalization of the Robinson-Schensted correspondence to the class of dual graded graphs introduced in [3]. This class extends the class of Y-graphs, or differential posets [22], for which a generalized Schensted correspondence was constructed earlier in [2].

The main construction leads to unified bijective proofs of various identities related to path counting, including those obtained in [3]. It is also applied to permutation enumeration, including rook placements on Ferrers boards and enumeration of involutions.

As particular cases of the general construction, we re-derive the classical algorithm of Robinson, Schensted, and Knuth [19, 12], the Sagan-Stanley [18], Sagan-Worley [16, 29] and Haiman's [11] algorithms and the author's algorithm for the Young-Fibonacci graph [2]. Some new applications are suggested.

The rim hook correspondence of Stanton and White [23] and Viennot's bijection [28] are also special cases of the general construction of this paper.

In [5], the results of this paper and the previous paper [3] were presented in a form of extended abstract.

Keywords: discrete algorithm, enumerative combinatorics, poset, Young diagram

3. Bijective correspondences

Recall from Definition 1.3.4 that two graded graphs $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ with the same set of vertices and same rank function are called r-dual if the up operator U of the first graph and the down operator D of the second one (see Definition 1.2.1) satisfy the commutation relation

$$DU = UD + rI; \tag{3.0.1}$$

the main special case is $r = 1$ with

$$DU = UD + I. \tag{3.0.2}$$

In what follows we always assume that G_1 and G_2 have a zero, and $r \geq 0$.

Various enumerative consequences of (3.0.1) were obtained in [3]. We shall demonstrate that all these formulae, and many others, can be derived in an entirely combinatorial way by establishing certain bijective correspondences between Hasse walks (i.e., paths

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in the oriented graded graph $G = (P, \rho, E_1, E_2))$ and permutations. These correspondences generalize the classical Robinson-Schensted construction for the Young tableaux (see, e.g., [12, 20, 17]). Each of these bijections also has a direct combinatorial interpretation in terms of certain permutation statistic that is peculiar to the particular pair of dual graphs. These statistics generalize the well-known invariant due to C. Greene [8].

To make our plans clear, let us consider the following typical case. It has been already shown that, for any pair of dual graded graphs,

$$\sum_{x \in P_k} e_1(x) e_2(x) = k! \tag{3.0.3}$$

(cf. (1.5.9)) where $e_1(x)$ and $e_2(x)$ denote the number of paths connecting $\hat{0}$ with x in G_1 and G_2, respectively. In this paper, we construct a bijective (though not canonical) correspondence between

(i) pairs (p_1, p_2) of paths of a fixed length k in G_1 and G_2, respectively, having a common endpoint, and

(ii) permutations of k elements.

Part 3 contains the main bijective construction of this paper that provides bijective proofs of (3.0.3) and many other combinatorial identities related to enumeration of Hasse walks. This construction gives a uniform interpretation of various Schensted-type algorithms. We show how to design such an algorithm for any pair of dual graded graphs. This extends the results of [2] (see also [15]) and, in turn, those of [1] to the case of dual graphs.

In **Part 4**, applications to concrete examples are given, including the Young, Young-Fibonacci, and Pascal graphs, the binary tree, etc. It is shown that the classical Robinson-Schensted-Knuth algorithm, the algorithms of Sagan and Stanley [18], Sagan [16], Worley [29], and Haiman [11], and the author's algorithm [2] for the Young-Fibonacci lattice are all special cases of the main construction of Part 3.

3.1. 2-Growth

The main concept of [3] was that of an oriented graded graph which is a pair of graded graphs sharing the same set of vertices. Now we introduce corresponding morphisms called *2-growths*.

Definition 3.1.1 Assume $G = (P, \rho, E_1, E_2)$ is an oriented graded graph (pair of graphs). Define

$$\begin{aligned}
\hat{E} &= E_1 \uplus E_2 \cup \{(x, x) : x \in P\}, \\
\hat{E}_1 &= E_1 \cup \{(x, x) : x \in P\}, \\
\hat{E}_2 &= E_2 \cup \{(x, x) : x \in P\}.
\end{aligned}$$

So $\hat{E}$ is the set of *generalized edges* each of which is either an ordinary *non-degenerate* edge of $E_1 \cup E_2$ or a *degenerate* edge joining a vertex with itself.

Let *start*(a) and *end*(a) denote the startpoint and the endpoint of a generalized edge a, respectively. Thus *start*(a) = *end*(a) if a is degenerate and $\rho(end(a)) = \rho(start(a)) + 1$ otherwise.

Definition 3.1.2 Assume $S = (T, \tau, F_1, F_2)$ and $G = (P, \rho, E_1, E_2)$ are oriented graded graphs. A map ϕ: $\hat{F} \to \hat{E}$ is a *2-growth* if the following conditions are satisfied:

(i) if $a, b \in \hat{F}$ and $end(a) = start(b)$, then $end(\phi(a)) = start(\phi(b))$;

(ii) if $a \in \hat{F}_i$, then $\phi(a) \in \hat{E}_i$, $\quad i = 1, 2$.

The rest of this section is devoted to technical preliminaries related to the notion of 2-growth.

Lemma 3.1.3 *Assume the conditions of* Definition 3.1.2 *hold. Then*

(i) *if two generalized edges $a, b \in \hat{F}$ have common startpoint (endpoint), then the same is true for $\phi(a)$ and $\phi(b)$;*

(ii) *the image of any degenerate edge is also a degenerate edge.*

In view of the latter statement, one can define a map $\tilde{\phi}$: $T \to P$ by

$$\phi((x, x)) = (\tilde{\phi}(x), \tilde{\phi}(x)). \tag{3.1.1}$$

Corollary 3.1.4 *Assume the conditions of* Definition 3.1.2 *hold; let $\tilde{\phi}$ be defined by* (3.1.1). *Then*

(i) *if y covers x in (T, τ, F_i), then either $\tilde{\phi}(y)$ covers $\tilde{\phi}(x)$ in (P, ρ, E_i) or $\tilde{\phi}(y) = \tilde{\phi}(x)$, $i = 1, 2$;*

(ii) *$\tilde{\phi}$ is bi-monotone, that is, $\tilde{\phi}$ is monotone with respect to partial orders on T and P induced by F_1 and E_1, and $\tilde{\phi}$ is also monotone regarding the F_2- and E_2-induced orderings;*

(iii) *if an edge $a \in \hat{F}_i$ joins vertices x and y, then $\phi(a) \in \hat{E}_i$ joins $\tilde{\phi}(x)$ and $\tilde{\phi}(y)$;*

(iv) *ϕ is uniquely determined by $\tilde{\phi}$ provided G has no multiple edges; namely,*

$$\phi((x, y)) = (\tilde{\phi}(x), \tilde{\phi}(y)). \tag{3.1.2}$$

So $\tilde{\phi}$ is a bi-monotone map $T \to P$ preserving both "cover-or-equal" relations, and ϕ is a map $\hat{F} \to \hat{E}$ consistent with $\tilde{\phi}$. This statement can also be used as an alternative definition of a 2-growth. Informally, 2-growth maps vertices to vertices, and the edges joining them—to the edges (maybe degenerate) joining their images, so that E_1-edges are mapped to F_1-edges, and E_2-edges—to F_2-edges. In a sense, T is time, and ϕ is a growth process in G that develops in course of the time.

Definition 3.1.5 A 2-growth $\phi : \hat{F} \to \hat{E}$ is called *strict* if $\phi(F) \subset \phi(E)$, i.e., a ϕ-image of any non-degenerate edge is non-degenerate.

3.2. Skew graphs

In a typical case, the "time graph" S is a so-called *skew graph.*

Definition 3.2.1 An oriented graded graph $S = (T, \tau, F_1, F_2)$ is called a *skew graph* if all the following conditions are satisfied:

(i) T is a finite convex subposet of the two-dimensional lattice $\mathbb{Z}^2$;

(ii) τ is a restriction of the standard rank function on $\mathbb{Z}^2$:

$$\tau((k,l)) = k + l; \tag{3.2.1}$$

(iii) F_1 contains the edges linking vertices (k,l) and $(k, l+1)$ both belonging to T;

(iv) F_2 contains the edges linking vertices (k,l) and $(k+1, l)$ both belonging to T.

In other words, T is a skew shape, F_1 is the set of horizontal edges of its Hasse diagram, and F_2 is the set of its vertical edges. Note that a skew graph is uniquely determined by the set of its vertices.

Sometimes it is convenient to make no distinction between skew graphs one of which is a translation of the other; for the same reason, a rank function

$$\tau((k,l)) = k + l + const \tag{3.2.2}$$

can be occasionally used instead of (3.2.1).

Definition 3.2.2 Let $S = (T, \tau, F_1, F_2)$ be a skew graph. Let the *lower* and *upper boundaries* of S to be skew graphs $\partial^- S = (T^-, \tau, F_1^-, F_2^-)$ and $\partial^+ S = (T^+, \tau, F_1^+, F_2^+)$ defined by

$$T^- = \{(k,l) \in T : (k-1, l-1) \notin T\} \tag{3.2.3}$$

and

$$T^+ = \{(k,l) \in T : (k+1, l+1) \notin T\}. \tag{3.2.4}$$

Let

$$F^- = F_1^- \cup F_2^-, \quad F^+ = F_1^+ \cup F_2^+; \tag{3.2.5}$$

thus, for example, F^+ is the set of edges lying on the upper boundary. In turn, $\hat{F}^+$ is the corresponding set of generalized edges.

Define the north-west and south-east corners of a skew graph S by

$$nw(S) = (k_{\min}, l_{\max}), \quad se(S) = (k_{\max}, l_{\min})$$

where $k_{\min} = \min\{k : \exists l : (k,l) \in T\}$ and so on. In other words, $nw(S)$ and $se(S)$ are the extremal elements of T according to the transversal ordering of $\mathbb{Z}^2$.

A skew graph $S = (T, \tau, F_1, F_2)$ is called *connected* if any two vertices of T can be joined by a path consisting of edges belonging to either F_1 or F_2. The lower and upper boundaries of a connected skew graph are simply *paths* in $\mathbb{Z}^2$ between $nw(S)$ and $se(S)$.

In what follows, some particular types of skew graphs play an important role.

Definition 3.2.3 Let $\square_{n\times m}$ denote the skew graph with the set of vertices

$$\{(k,l) \in \mathbb{Z}^2 : 0 \leq k \leq n, 0 \leq l \leq m\}.$$

The skew graph with the vertices

$$\{(k-1,l-1),(k-1,l),(k,l-1),(k,l)\} \tag{3.2.6}$$

is denoted $\square_{kl}$ (do not confuse with $\square_{k\times l}$!). Such a graph is called a *cell*. The notation $\square$ is used if the particular values of k and l are inessential.

We say that a cell $\square_{kl}$ is a cell of S if all the four vertices (3.2.6) belong to S. Denote

$$C(S) = \{(k,l) : \square_{kl} \text{ is a cell of } S\}. \tag{3.2.7}$$

Let $c = \square_{kl}$ be a cell and $t = (k', l')$ a point of $\mathbb{Z}^2$. Then we write $c \leq t$ if $k \leq k'$ and $l \leq l'$; similarly, $c \geq t$ means that $k-1 \geq k'$ and $l-1 \geq l'$.

Definition 3.2.4 A skew graph $S = (T, \tau, F_1, F_2)$ is a Ferrers graph if T is a *Ferrers diagram*, i.e., the point $(0,0)$ is the only minimal element of the poset T.

Definition 3.2.5 Let $S = (T, \tau, F_1, F_2)$ be a skew graph, and $\phi : \hat{F} \to \hat{E}$ a 2-growth with values in an oriented graded graph $G = (P, \rho, E_1, E_2)$. The following notation will be used throughout:

$$\begin{aligned}
\phi_1(k,l) &= \text{the edge [of } \hat{E}_1\text{] that } \phi \text{ assigns to the edge joining } (k-1,l) \text{ and } (k,l);\\
\phi_2(k,l) &= \text{the edge [of } \hat{E}_2\text{] that } \phi \text{ assigns to the edge joining } (k,l-1) \text{ and } (k,l);\\
\tilde{\phi}(k,l) &= \text{the vertex [of } G\text{] that } \tilde{\phi} \text{ assigns to the point } (k,l) \text{ (see (3.1.1))};\\
\Delta\phi_1(k,l) &= \rho(\tilde{\phi}(k,l)) - \rho(\tilde{\phi}(k-1,l)) = \begin{cases} 1, & \text{if } \phi_1(k,l) \text{ is not degenerate}\\ 0, & \text{otherwise;}\end{cases}\\
\Delta\phi_2(k,l) &= \rho(\tilde{\phi}(k,l)) - \rho(\tilde{\phi}(k,l-1)) = \begin{cases} 1, & \text{if } \phi_2(k,l) \text{ is not degenerate}\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

3.3. *Diagonal sets*

Definition 3.3.1 A set of cells is called *diagonal* if any two of them are situated neither in the same row nor in the same column; cf. Definition 2.6.1. A diagonal set containing n cells of the graph $\square_{n\times n}$ can be identified with a permutation of $\{1, 2, \ldots, n\}$.

Definition 3.3.2 Let S be a skew graph. A map $\alpha\colon C(S) \to \mathbb{Z}$ (see (3.2.7)) is called an *r-colored diagonal set* if

(i) for any $(k, l) \in C(S)$, $\quad 0 \le \alpha(k, l) \le r$;

(ii) the set $\sigma = \{(k, l) \in C(S) : \alpha(k, l) \ne 0\}$ is diagonal.

The set σ is called a *support* of α; we write $\sigma = \operatorname{supp}(\alpha)$. If σ is a permutation, α is called an *r-colored permutation*. Sometimes we will say that α or σ is contained in S. No distinction will be made between r-colored diagonal sets with a common support and the same coloring (i.e., the restriction of α to the support).

Definition 3.3.3 A 2-growth ϕ defined on a skew graph S and an r-colored diagonal set α with support σ (note that σ may contain cells lying outside S) are said to be *consistent* with each other if for any k, l

$$\Delta\phi_1(k, l) = \begin{cases} 0, & \text{if } \square_{kl'} \in \sigma \text{ for some } l' > l \\ 1, & \text{if } \square_{kl'} \in \sigma \text{ for some } l' \le l \\ \text{value not depending on } l, & \text{otherwise} \end{cases}$$

and for any k, l

$$\Delta\phi_2(k, l) = \begin{cases} 0, & \text{if } \square_{k'l} \in \sigma \text{ for some } k' > k \\ 1, & \text{if } \square_{k'l} \in \sigma \text{ for some } k' \le k \\ \text{value not depending on } k, & \text{otherwise} \end{cases}$$

(cf. Definition 3.2.5). If all the values appearing within "otherwise" options are 1, then ϕ and α are said to be *1-consistent* with each other.

Lemma 3.3.4 *Let $S = (T, \tau, F_1, F_2)$ be a connected skew graph, $\phi\colon \hat{F} \to \hat{E}$ a 2-growth, and $\alpha\colon C(S) \to \mathbb{Z}$ an r-colored diagonal set consistent with ϕ. Then the following statements are equivalent to each other:*

(a) *the restriction of ϕ to the upper boundary $\hat{F}^+$ is strict;*

(b) *the restriction of ϕ to the upper boundary $\hat{F}^+$ is 1-consistent with α;*

(c) *the restriction of ϕ to the lower boundary $\hat{F}^-$ is 1-consistent with α.*

Proof: The equivalence (a) $\Leftrightarrow$ (b) is trivial. The equivalence (b) $\Leftrightarrow$ (c) follows from the definition of 1-consistence and from the existence of a natural bijection between the edges of upper and lower boundaries F^+ and F^- (namely, edges crossing the same row/column correspond to each other). Details are left to the reader. $\square$

Definition 3.3.5 Let $S = (T, \tau, F_1, F_2)$ be a connected skew graph. Move along its upper boundary from $nw(S)$ to $se(S)$ and write, *from right to left*, a $\{U, D\}$-word, according to the following rule: write D while moving down and U while moving to the right. The resulting word is denoted $w^+(S)$. For example, if

$$T = \{(1,2),(1,3),(2,2),(2,3),(3,1),(3,2),(4,0),(4,1),(4,2)\}, \tag{3.3.1}$$

then $w^+(S) = D^2U^2DU$. The same operations with the lower boundary produce a $\{U, D\}$-word denoted $w^-(S)$. In the example (3.3.1), $w^-(S) = DUDU^2D$.

Now assume a diagonal set σ is given. Let us modify the last construction as follows: any letter corresponding to an edge of $\partial^-(S)$ that crosses a row or column containing a cell of σ should be omitted. Formally, if $\Box_{kl} \in \sigma$, then we omit the letters corresponding to the edges $((k-1,l'),(k,l'))$ and $((k',l-1),(k',l))$ belonging to F^-. The resulting word is denoted by $w^-(S,\sigma)$. Equivalently, $w^-(S,\sigma) = w^-(S')$ where S' is a skew graph obtained from S by cutting off the rows and columns containing the cells of σ. For example, if S is defined by (3.3.1) and $\sigma = \{\Box_{42}\}$, then $w^-(S,\sigma) = DU^2D$. If α is an r-colored diagonal set with a support σ, then, by definition, $w^-(S,\alpha) = w^-(S,\sigma)$.

Lemma 3.3.6 *Let $S = (T,\tau,F_1,F_2)$ be a connected skew graph, $G = (P,\rho,E_1,E_2)$ an oriented graded graph, and A and B two vertices of G. Then there is a canonical bijection between*

(a) *strict 2-growths $\phi^+ : \hat{F}^+ \to \hat{E}$ such that $\phi^+(nw(S)) = A$ and $\phi^+(se(S)) = B$ and*

(b) *paths from A to B having a structure $w^+(S)$.*

Given a diagonal set α, there is a canonical bijection between

(a) *2-growths $\phi^- : \hat{F}^- \to \hat{E}$ which are 1-consistent with α and satisfy $\phi^-(nw(S)) = A$ and $\phi^-(se(S)) = B$ and*

(b) *paths from A to B having a structure $w^-(S,\alpha)$.*

Proof: The first statement is trivial. To prove the second one, consider a 2-growth $\phi^- : \hat{F}^- \to \hat{E}$. It is 1-consistent with α if and only if for any edge $a \in F^-$ the image $\phi(a)$ is either degenerate or it does not depend on whether or not a crosses a row or column containing a cell of supp(α). Hence to define ϕ^- means to give a path having a structure $w^-(S,\alpha)$. The endpoints of such a path will certainly coincide with the values of ϕ^- at $nw(S)$ and $se(S)$. □

3.4. r-correspondence

Definition 3.4.1 Let $G = (P,\rho,E_1,E_2)$ be an oriented graded graph and A_r the set of all triples (a_1,a_2,α) such that

$$a_1 \in \hat{E}_1,\ a_2 \in \hat{E}_2; \tag{3.4.1}$$
$$start(a_1) = start(a_2); \tag{3.4.2}$$
$$\alpha \in \{0,\dots,r\}; \tag{3.4.3}$$
$$\text{if either } a_1 \text{ or } a_2 \text{ is not degenerate, then } \alpha = 0. \tag{3.4.4}$$

Let B be the set of all pairs (b_1, b_2) such that

$$b_1 \in \hat{E}_1,\ b_2 \in \hat{E}_2; \tag{3.4.5}$$
$$end(b_1) = end(b_2). \tag{3.4.6}$$

A *bijective* map $\Phi: B \to A_r$ is called an *r-correspondence* if the following conditions hold:

$$\Phi(b_1, b_2) = (a_1, a_2, \alpha) \text{ implies } end(a_1) = start(b_2) \text{ and } end(a_2) = start(b_1); \tag{3.4.7}$$
$$\text{if } b \text{ is degenerate, then } \Phi(b_1, b_2) = (b_1, b_2, 0). \tag{3.4.8}$$

Thus the four generalized edges a_1, b_2, b_1, and a_2 should form a *Hasse cycle* in G (see (3.4.2), (3.4.6), and (3.4.7)).

Lemma 3.4.2 *It is possible to define an r-correspondence in an oriented graded graph $G = (P, \rho, E_1, E_2)$ if and only if the graphs $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ are r-dual.*

Proof: Because of (3.4.7), an r-correspondence Φ should establish bijections between the sets $A_{r,x,y}$ and $B_{x,y}$ defined by:

$$A_{r,x,y} = \{(a_1, a_2, \alpha) \in A_r : end(a_1) = x,\ end(a_2) = y\};$$
$$B_{x,y} = \{(b_1, b_2) \in B : start(b_1) = y,\ start(b_2) = x\}.$$

Thus Φ exists if and only if $A_{r,x,y}$ and $B_{x,y}$ are equinumerous for every x and y.

Let us examine all possible cases. First note that if $|\rho(x) - \rho(y)| \geq 2$, then both $A_{r,x,y}$ and $B_{x,y}$ are empty.

Case 1. $\rho(x) = \rho(y) - 1$. Then necessarily $start(a_1) = start(a_2) = x$, $end(b_1) = end(b_2) = y$, $a_1 = (x, x)$, $b_1 = (y, y)$, $\alpha = 0$, and

$$\#A_{r,x,y} = \#\{e \in E : start(e) = x, end(e) = y,\} = \#B_{x,y}$$

for *any* graph G.

Case 2. $\rho(x) = \rho(y) + 1$. This case is quite similar to the previous one: interchange a_1 and a_2, b_1 and b_2, and x and y.

Case 3. $\rho(x) = \rho(y)$, $x \neq y$. The set $A_{x,y}$ contains all the triples $(a_1, a_2, 0)$ where a_1 and a_2 form a UD-path connecting x and y (cf. Definition 1.2.2). Similarly, $B_{x,y}$ is the set of pairs (b_1, b_2) forming DU-paths between these vertices. So the cardinalities of $A_{r,x,y}$ and $B_{x,y}$ are equal if and only if the condition (1) of Definition 1.3.3 holds with $q_n = 1$.

Case 4. $x = y$. In this case $A_{r,x,y}$ contains

(i) the triples $(a_1, a_2, 0)$ where a_1 and a_2 form a UD-loop from x to x;

(ii) the triples (e, e, α) where $e = (x, x)$ and $\alpha \in \{0, \ldots, r\}$.

Also, $B_{x,y}$ contains

(i) the pairs (b_1, b_2) where b_1 and b_2 form a DU-loop from x to x;

(ii) the degenerate pair (e, e).

Thus $A_{r,x,y}$ and $B_{x,y}$ have equal cardinalities if and only if the condition (2) of Definition 1.3.3 is satisfied with $q_n = 1$ and $r_n = r$. □

Informally, an r-correspondence is a constructive analogue of an r-duality; while the latter requires certain sets to be equinumerous, an r-correspondence establishes explicit bijections between them.

We give below a pseudo-language template defining an r-correspondence Φ and its inverse $\Psi = \Phi^{-1}$. The dots should be replaced by appropriate operators, peculiar to the particular choice of Φ.

Definition 3.4.3 $\{b_1$ and b_2 are generalized edges of G_1 and G_2, respectively; *end*(b_1) = *end*(b_2); returned parameters are $\alpha \in \{0, \ldots, r\}$ and generalized edges a_1 and a_2 of G_1 and G_2, satisfying *start*(a_1) = *start*(a_2), *end*(a_1) = *start*(b_2), and *end*(a_2) = *start*$(b_1)\}$

```
function Φ(b1, b2);
begin
   case
     b1 and b2 are degenerate ⇒ a1 := a2 := b1; α := 0;
     b1 is degenerate, b2 is not ⇒ a1 is degenerate; a2 := b2; α := 0;
     b2 is degenerate, b1 is not ⇒ a2 is degenerate; a1 := b1; α := 0;
     b1 and b2 are not degenerate ⇒ ...
   endcase;
   return(a1, a2, α)
end;
```

Definition 3.4.4 $\{a_1$ and a_2 are generalized edges of G_1 and G_2, respectively, such that *start*(a_1) = *start*(a_2); $\alpha \in \{0, \ldots, r\}$; (3.4.4) is satisfied; returned parameters are generalized edges b_1 and b_2 of G_1 and G_2, satisfying *end*(b_1) = *end*(b_2), *start*(b_2) = *end*(a_1), *start*(b_1) = *end*$(a_2)\}$

```
function Ψ(a1, a2, α);
begin
   case
     a1 and a2 are degenerate, α = 0 ⇒ b1 := b2 := a1;
     a1 is degenerate, a2 is not ⇒ b1 is degenerate; b2 := a2;
     a2 is degenerate, a1 is not ⇒ b2 is degenerate; b1 := a1;
     (a1 and a2 are not degenerate) or (a1 and a2 are degenerate and α ≠ 0) ⇒ ···
   endcase;
   return(b1, b2)
end;
```

Note that the operators replacing dots in the above definitions should be chosen so that the functions Φ and Ψ are inverse to each other.

3.5. Φ-growth

The following conventions are fixed throughout Secs. 3.5–3.7:

$G = (P, \rho, E_1, E_2)$ is an oriented graded graph (see Definition 1.2.1);

$G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ are r-dual graphs (see (3.0.1));

Φ is an r-correspondence in G (see Definition 3.4.1);

$\Psi = \Phi^{-1}$;

$S = (T, \tau, F_1, F_2)$ is a skew graph (see Definition 3.2.1).

See also Definition 3.2.5 for the notation associated with any 2-growth $\phi : \hat{F} \to \hat{E}$.

Definition 3.5.1 2-growth $\phi : \hat{F} \to \hat{E}$ is said to be a *ϕ-growth* if there exists a function $\alpha : C(S) \to \mathbb{Z}$ such that, for any cell $\square_{kl}$ of S,

$$(\phi_1(k, l-1), \phi_2(k-1, l), \alpha(k, l)) = \Phi(\phi_1(k, l), \phi_2(k, l)). \tag{3.5.1}$$

Equivalently,

$$(\phi_1(k, l), \phi_2(k, l)) = \Psi(\phi_1(k, l-1), \phi_2(k-1, l), \alpha(k, l)). \tag{3.5.2}$$

Informally, the values of a Φ-growth at the left and the bottom edges of every cell are uniquely determined by its values at the right and top edges—and vice versa, provided that $\alpha(k, l)$ is given. Therefore in order to define a Φ-growth it suffices to set its restriction to the upper boundary F^+ of S (see (3.2.5)). One can also define a Φ-growth by setting its values on the lower boundary F^- together with the function α. To express these observations formally, we use the algorithmic notation.

Definition 3.5.2 In the following procedures, 2-growth $\phi: \hat{F} \to \hat{E}$ and function $\alpha: C(S) \to \mathbb{Z}$ are treated as global unprotected variables (i.e., their values may be modified):

procedure *LeftDown*(k, l: *integer*);
begin
  $(\phi_1(k, l-1), \phi_2(k-1, l), \alpha(k, l)) := \Phi(\phi_1(k, l), \phi_2(k, l))$;
  $\tilde{\phi}(k-1, l-1) :=$ *start*$(\phi_1(k, l-1))$
end;

procedure *RightUp*(k, l: *integer*);
begin
  $(\phi_1(k, l), \phi_2(k, l)) := \Psi(\phi_1(k, l-1), \phi_2(k-1, l), \alpha(k, l))$;
  $\tilde{\phi}(k, l) :=$ *end*$(\phi_1(k, l))$
end;

cf. (3.5.1)–(3.5.2); the functions Φ and Ψ are to be defined,—e.g., by fixing appropriate versions of Definitions 3.4.3–3.4.4.

Algorithm 3.5.3

Input: the values of a Φ-growth ϕ on the upper boundary $\hat{F}^+$.
Output: (i) all the values of ϕ; (ii) a function α: $C(S) \to \mathbb{Z}$.
 begin
  for all $(k, l) \in C(S)$ **do southwestbound** $LeftDown(k, l)$
 end

The word "southwestbound" means that in the course of an execution of the algorithm the pairs (k, l) should be taken in an order anti-compatible with the usual ordering on $\mathbb{Z}^2$. In other words, the only requirement is:

$$\boxed{\text{if } k_1 \leq k_2 \text{ and } l_1 \leq l_2 \text{, then } (k_1, l_1) \text{ should be processed after } (k_2, l_2).} \qquad (3.5.3)$$

Note that the **for**-cycle may be executed *in parallel*, obeying the rule (3.5.3).

Algorithm 3.5.4

Input: (i) the values of a ϕ-growth ϕ on the lower boundary $\hat{F}^-$; (ii) a function α: $C(S) \to \mathbb{Z}$.
Output: all the values of ϕ at non-degenerate edges of S.
 begin
  for all $(k, l) \in C(S)$ **do northeastbound** $RightUp(k, l)$
 end

The word "northeastbound" means that, analogously to (3.5.3),

$$\boxed{\text{if } k_1 \leq k_2 \text{ and } l_1 \leq l_2 \text{, then } (k_1, l_1) \text{ should be processed before } (k_2, l_2).} \qquad (3.5.4)$$

This algorithm can be executed in parallel as well, taking into account the rule (3.5.4).

Now we proceed to the analysis of Algorithms 3.5.3–3.5.4. First we state explicitly the restrictions on the input of each of the algorithms.

Lemma 3.5.5 *Let ϕ: $\hat{F} \to \hat{E}$ be a Φ-growth and α the corresponding function defined by* (3.5.1). *Then α is an r-colored diagonal set consistent with ϕ.*

(Recall definitions 3.3.2 and 3.3.3.)

Proof: First we prove that $\sigma = supp(\alpha)$ is indeed a diagonal set. Suppose this is not the case; say, there exist two cells $\square_{kl_1}$ and $\square_{kl_2}$ belonging to σ; let $l_1 < l_2$. Since $\alpha(k, l_i) \neq 0,\ i = 1, 2$, then

$$\Delta\phi_1(k, l_i - 1) = \Delta\phi_2(k - 1, l_i) = 0 \qquad (3.5.5)$$

(cf. Definition 3.2.5, (3.4.4), and (3.5.1)). On the other hand,

$$\Delta\phi_1(k, l_i) = \Delta\phi_2(k, l_i) = 1 \tag{3.5.6}$$

(cf. (3.4.8) and (3.5.1)). Thus the sequence

$$\Delta\phi_1(k, l_1), \quad \Delta\phi_1(k, l_1 + 1), \ldots, \Delta\phi_1(k, l_2 - 1) \tag{3.5.7}$$

begins with 1 and ends with 0. Hence, for some l,

$$\Delta\phi_1(k, l-1) = 1, \quad \Delta\phi_1(k, l) = 0.$$

On the other hand, (3.5.1), (3.4.2), (3.4.6), and (3.4.7) imply

$$\Delta\phi_1(k, l-1) + \Delta\phi_2(k, l) = \Delta\phi_2(k-1, l) + \Delta\phi_1(k, l).$$

Hence (recall the values of $\Delta\phi_i$ are 0 and 1) $\Delta\phi_2(k, l) = 0$ and $\Delta\phi_2(k-1, l) = 1$ which contradicts (3.4.8). Therefore α is an r-colored (see (3.4.3)) diagonal set.

Now let us prove the consistence of α and ϕ. If any of the conditions of Definition 3.3.3 does not hold, then the same arguments as in the previous paragraph lead to a contradiction. For instance, if for a certain $l' > l$ we have $\Delta\phi_1(k, l) = 1$ where $\Box_{kl'} \in supp(\alpha)$, then examine the sequence

$$\Delta\phi_1(k, l), \ \Delta\phi_1(k, l+1), \ldots, \Delta\phi_1(k, l'-1)$$

just as was done with (3.5.5). □

The following simple observation is rather useful. An r-colored permutation α of Lemma 3.5.5 is clearly consistent with any restriction of a Φ-growth ϕ to any skew subgraph of S,—for example, with the restriction to its lower or upper boundary.

Lemma 3.5.6 *Any 2-growth $\phi^+ \colon \hat{F}^+ \to \hat{E}$ can be uniquely extended to a Φ-growth $\phi : \hat{F} \to \hat{E}$.*

Proof: Examine Algorithm 3.5.3 to verify that the only condition the restriction of a Φ-growth should satisfy is that it is a 2-growth. □

Therefore the following algorithm constructs ϕ from ϕ^+.

Algorithm 3.5.7

Input: 2-growth $\phi^+ \colon \hat{F}^+ \to \hat{E}$.
Output: (i) Φ-growth $\phi \colon \hat{F} \to \hat{E}$ extending ϕ^+; (ii) r-colored diagonal set α contained in S and consistent with ϕ. (Actually, α is determined by ϕ—see (3.5.1).)

```
begin
    for all a ∈ F̂⁺ do φ(a) := φ⁺(a);
    for all (k, l) ∈ C(S) do southwestbound LeftDown(k, l)
end
```

(See the comments after Algorithm 3.5.3.)

Lemma 3.5.8 *Let $\phi^-: \hat{F}^- \to \hat{E}$ be a 2-growth and $\alpha: C(S) \to \mathbb{Z}$ an r-colored diagonal set consistent with ϕ^-. Then ϕ^- can be uniquely extended to a Φ-growth $\phi: \hat{F} \to \hat{E}$ satisfying* (3.5.1).

Proof: The only problem that can arise while applying Algorithm 3.5.4 to ϕ^- and α concerns the condition (3.4.4). That is, at every step of the algorithm we should be able to apply the function Ψ to the triple $(\phi_1(k, l-1), \phi_2(k-1, l), \alpha(k, l))$; in other words, we have to prove that

$$\alpha(k, l) \neq 0 \Rightarrow \Delta\phi_1(k, l-1) = \Delta\phi_2(k-1, l) = 0.$$

Suppose it is not the case; e.g., at some point

$$\alpha(k, l) \neq 0, \quad \Delta\phi_1(k, l-1) = 1 \tag{3.5.8}$$

occurred. Since α and ϕ^- are consistent, we have $\Delta\phi_1(k, l') = 0$ for the only edge $((k-1, l'), (k, l')) \in F^-$ (cf. Definition 3.3.3). Every execution of $DownUp(k, l'')$ for $l'' > l'$ results, by induction, in $\Delta\phi_1(k.l'') = 0$ since

(i) if $\Delta\phi_2(k-1, l'') = 0$ then, by $\alpha(k, l'' = 0$ and (3.4.8), $\Delta\phi_1(k, l'') = \Delta\phi_1(k, l''-1)$;

(ii) if $\Delta\phi_2(k-1, l'') = 1$ then $\Delta\phi_1(k, l'') = 0$ because of

$$\begin{aligned}\Delta\phi_1(k, l''-1) = 0 \quad \text{and} \quad & \Delta\phi_2(k-1, l'') + \Delta\phi_1(k, l'') \\ &= \Delta\phi_2(k, l'') + \Delta\phi_1(k, l''-1).\end{aligned}$$

So $\Delta\phi_1(k, l-1) = 0$ which contradicts (3.5.8). □

Thus the following algorithm constructs ϕ from ϕ^- and α.

Algorithm 3.5.9

Input: (i) 2-growth $\phi^-: \hat{F}^- \to \hat{E}$; (ii) r-colored diagonal set α contained in S and con sistent with ϕ^-.
Output: Φ-growth $\phi: \hat{F} \to \hat{E}$ extending ϕ^-, satisfying (3.5.1) and thus consistent with α. (Actually, α is determined by ϕ—see (3.5.1).)

```
begin
    for all a ∈ F̂⁻ do φ(a) := φ⁻(a);
    for all (k, l) ∈ C(S) do northeastbound RightUp(k, l)
end
```

(See the comments after Algorithm 3.5.4.)

3.6. Main bijection

Combining Lemmas 3.5.6 and 3.5.8 results in the following statement. (Recall the conventions fixed at the beginning of Section 3.5.)

Theorem 3.6.1 *There exist bijective correspondences between any two of the following:*

(a) *the 2-growths* $\phi^+ \colon \hat{F}^+ \to \hat{E}$;

(b) *the pairs* (ϕ^-, α) *where* $\phi^- \colon \hat{F}^- \to \hat{E}$ *is a 2-growth and* α *an* r*-colored diagonal set contained in* S *and consistent with* ϕ^-;

(c) *the* Φ*-growths* $\phi \colon \hat{F} \to \hat{E}$.

These bijections can be realized by the following algorithms:
(a) → (c) *Algorithm* 3.5.7
(b) → (c) *Algorithm* 3.5.9
(c) → (a) *Restricting* ϕ *to* $\hat{F}^+$
(c) → (b) *Restricting* ϕ *to* $\hat{F}^-$ *and finding* α *from* (3.5.1)
Thus ϕ *extends both* ϕ^+ *and* ϕ^-; α *is consistent with* ϕ, ϕ^-, *and* ϕ^+.

Comments 3.6.2

1. The bijections of Theorem 3.6.1 do depend on Φ; hence the whole construction is not canonically determined by the graph G.

2. The most interesting bijective correspondence is between (a) and (b). We shall see later that this is actually a generalization of the Robinson-Schensted bijection.

3. Since ϕ^- and ϕ^+ are both restrictions of ϕ, the values of ϕ^- and ϕ^+ at generalized edges belonging to $\hat{F}^- \cap \hat{F}^+$ should coincide. In particular, they coincide at the points $nw(S)$ and $se(S)$. So we can fix a priori the values of $\tilde{\phi}^-$, $\tilde{\phi}^+$, and $\tilde{\phi}$ at $nw(S)$ and/or $se(S)$ and obtain bijections between corresponding sets of objects (a), (b), and (c).

4. Another restriction one can impose on these objects is their 1-consistence (see Definition 3.3.3). By Lemma 3.3.4, the following conditions are equivalent to each other:

   (a) ϕ^+ is strict;

   (b) ϕ^- is 1-consistent with α.

Now use Lemma 3.3.6 and Comments 3.6.2.3 to obtain the following result.

Theorem 3.6.3 *Let* $A, B \in P$. *There exists a bijective correspondence between*

(a) *the paths in* G *from* A *to* B *with structure* $w^+(S)$ *and*

(b) *the pairs (r-colored diagonal set α contained in S, path in G from A to B with structure $w^-(S,\alpha)$).*

In particular, these sets have the same cardinality.

The algorithms establishing the bijections in both directions are given below. They depend on the choice of an r-correspondence Φ.

Algorithm 3.6.4 (G, Φ, and S are fixed.)

Input: path p^+ having a structure $f^+(S)$.
Output: (i) r-colored diagonal set α contained in S; (ii) path p^- with structure $f^-(S,\alpha)$.
$\{p^+$ and p^- connect the same pair of vertices$\}$

```
var φ: 2-growth F̂ → Ê
begin
    define the restriction of φ to F̂⁺ according to p⁺;
    find all the values of φ and α using Algorithm 3.5.3;
    move along ∂⁻S from nw(S) to se(S), including non-degenerate
    values of φ into p⁻
end
```

Algorithm 3.6.5 (G, Φ, and S are fixed.)

Input = Output of Algorithm 3.6.4.
Output = Input of Algorithm 3.6.4.

```
var φ: 2-growth F̂ → Ê
begin
    define the restriction of φ to F̂⁻ according to p⁻ and α
    (see proof of Lemma 3.3.6);
    find all the values of φ using Algorithm 3.5.4;
    move along ∂⁺S from nw(S) to se(S), including the values of φ into p⁺
end
```

3.7. *Generalized Schensted*

The above-stated theorems and algorithms become much simpler in the case of Ferrers graphs. Definition 3.3.5 provides a natural one-to-one correspondence between $\{U,D\}$-words and Ferrers graphs; namely, a Ferrers graph S corresponds to the $\{U,D\}$-word that is patterned after the upper boundary of S. Assume w is a $\{U,D\}$-word with m occurrences of D and n occurrences of U; let $S = S(w)$ be the corresponding Ferrers graph. Then $w^+(S) = w$, $w^-(S) = U^nD^m$, and $w^-(S,\alpha) = U^{n-k}D^{m-k}$ if α contains k (colored) cells of S. Hence in this case Theorem 3.6.3 can be restated as follows.

Corollary 3.7.1 *Let w be a $\{U, D\}$-word with m entries of D and n entries of U. Let $A, B \in P$. Then Algorithms* 3.6.4–3.6.5 *establish a bijective correspondence between*

(a) *w-paths from A to B and*

(b) *pairs of the form (r-colored diagonal set α contained in $S(w)$, path from A to B with structure $U^{n-k}D^{m-k}$ where $k = \#supp(\alpha)$).*

Assume, in addition, that $A = \hat{0}$ is the zero of G. Then there is no $U^{n-k}D^{m-k}$-paths from A to B unless $k = m$.

Corollary 3.7.2 *Let w be a $\{U, D\}$-word with m entries of D and n entries of U. Let $B \in P$, $\rho(B) = n - m$. Then Algorithms* 3.6.4–3.6.5 *establish a bijective correspondence between*

(a) *w-paths from $\hat{0}$ to B and*

(b) *pairs (r-colored m-cell diagonal set contained in $S(w)$, path in G_1 from $\hat{0}$ to B).*

In case $A = B = \hat{0}$ (hence $n = m$), a diagonal set consisting of n cells of $S(w)$ is actually a permutation of n elements (cf. Definition 3.3.1).

Corollary 3.7.3 *Let w be a <u>balanced</u> U, D-word of length $2n$. Then Algorithms* 3.6.4–3.6.5 *establish a bijective correspondence between*

(a) *w-paths (loops) starting at $\hat{0}$ and*

(b) *r-colored n-cell permutations contained in the Ferrers graph $S(w)$.*

Now consider a particular case of a rectangular Ferrers graph $\square_{n\times m}$ that corresponds to the word $w = D^m U^n$. In this case, Corollaries 3.7.1–3.7.3 turn into the following statements.

Corollary 3.7.4 *Let $A, B \in P$ and $\rho(B) - \rho(A) = n - m$. Then Algorithms* 3.6.4–3.6.5 *establish a bijective correspondence between*

(a) *$D^m U^n$-paths from A to B and*

(b) *pairs of the form*

(r-colored diagonal set α contained in $\square_{n\times m}$, $U^{n-k}D^{m-k}$-path from A to B where $k = \#supp(\alpha)$).

Corollary 3.7.5 *Let $B \in P$; $\rho(B) = n - m$. Then Algorithms* 3.6.4–3.6.5 *establish a bijective correspondence between*

(a) *$D^m U^n$-paths from $\hat{0}$ to B and*

(b) *pairs (r-colored m-cell diagonal set contained in $\square_{n\times m}$, path in G_1 from $\hat{0}$ to B).*

Corollary 3.7.6 *Algorithms* 3.6.4–3.6.5 (*see also Algorithms* 3.7.7–3.7.8) *establish a bijective correspondence between*

(a) *D^nU^n-loops starting at $\hat{0}$ and*

(b) *r-colored n-cell permutations.*

The last bijection is the *generalized Robinson-Schensted correspondence.* It is described by the appropriate specializations of Algorithms 3.6.4–3.6.5 (see Algorithms 3.7.7–3.7.8 below.) The classical Schensted bijection appears when $G_1 = G_2 = \mathbb{Y}$, $r = 1$; see Section 4.2.

For the particular cases of the Young graph and the graph of shifted shapes, the bijection of Corollary 3.7.4 was given by R.Stanley and B.Sagan [18].

Comments 3.7.6.1 Specializing Corollary 3.7.5 to $m = 1$ produces bijections between

(a) DU^n-paths from $\hat{0}$ to B and

(b) pairs (r-colored cell in $\square_{n\times 1}$, path in G_1 from $\hat{0}$ to B).

If $r = 1$ and G_2 has no multiple edges, then this gives bijections between

(a) paths w' of length n in G_1 and

(b) pairs (integer $k \in \{1,\ldots,n\}$, path w of length $n-1$ in G_1).

In the conventional tableau slang, inserting k into w results in w',—or, equivalently, w is obtained by deleting k from w' (to make it precise, the standardization procedure should also be used). Cf. Definitions 2.5.4, 2.5.10, and Section 4.8.

Algorithm 3.7.7

Input: edges $a_1(1), a_1(2), \ldots, a_1(n)$ of G_1 and edges $a_2(1), a_2(2), \ldots, a_2(n)$ of G_2 which form two paths starting at $\hat{0}$ and having common endpoint;
Output: r-colored permutation α (an $n \times n$-matrix).

```
var
    φ1 : array [1..n, 0..n] of generalized G1-edges;
    φ2 : array [0..n, 1..n] of generalized G2-edges;
    α : array [1..n, 1..n] of integer;
begin
    for k := 1 to n do φ1(k, n) := a1(k);
    for l := 1 to n do φ2(n, l) := a2(l);
    for (k, l) := (n, n) downto (1, 1) do
        (φ1(k, l − 1), φ2(k − 1, l), α(k, l)) := Φ(φ1(k, l), φ2(k, l))
end
```

Comments: In the third **for**-cycle, a pair (k_1, l_1) should be treated later than (k_2, l_2) whenever $k_1 \le k_2$ and $l_1 \le l_2$. The calculations may be done in parallel as long as this condition is respected. Φ is to be defined by an appropriate version of Definition 3.4.3.

Algorithm 3.7.8

Input = Output of Algorithm 3.7.7.
Output = Input of Algorithm 3.7.7.

```
var φ1, φ2 : ...; {see Algorithm 3.7.7}
begin
  for k := 1 to n do φ1(k,0) := (0̂,0̂);
  for l := 1 to n do φ2(0,l) := (0̂,0̂);
  for (k,l) := (1,1) to (n,n) do
    (φ1(k,l), φ2(k,l)) := Ψ(φ1(k,l−1), φ2(k−1,l), α(k,l))
  for k := 1 to n do a1(k) := φ1(k,n);
  for l := 1 to n do a2(l) := φ2(n,l);
end
```

Comments: In the third **for**-cycle, a pair (k_1, l_1) should be treated later (k_2, l_2) whenever $k_1 \le k_2$ and $l_1 \le l_2$. The calculations may be done in parallel provided this condition is obeyed. Ψ is to be defined by an appropriate version of Definition 3.4.4.

Now assume that both G_1 and G_2 have *no multiple edges*. Then Φ-growth ϕ is uniquely determined by a function $\tilde{\phi}$ (see Corollary 3.1.4(iv)). Moreover, the basic procedures $LeftDown$ and $RightUp$ can be rewritten in terms of $\tilde{\phi}$. Namely, define functions ΦV and ΨV by

$$\Phi V(x, y, z) = (t, \alpha) \tag{3.7.1}$$

where $t = \mathit{start}(a_1) = \mathit{start}(a_2)$, $(a_1, a_2, \alpha) = \Phi(b_1, b_2)$, $b_1 = (y, z)$, $b_2 = (x, z)$, and

$$\Psi V(x, y, t, \alpha) = z \tag{3.7.2}$$

where $z = \mathit{end}(b_1) = \mathit{end}(b_2)$, $(b_1, b_2) = \Psi(a_1, a_2, \alpha)$, $a_1 = (t, x)$, $a_2 = (t, y)$. We conclude that, in the case of no multiple edges, Algorithms 3.7.7–3.7.8 can be rewritten as follows.

Algorithm 3.7.9 (G has no multiple edges)

Input: paths $(\hat{0} = v_0, v_1, \ldots, v_n)$ and $(\hat{0} = w_0, w_1, \ldots, w_n = v_n)$ in G_1 and G_2, respectively.
Output: r-colored permutation α (an $n \times n$-matrix).

```
var
  φ̃: array [0..n, 0..n] of vertices;
  α: array [1..n, 1..n] of integer;
begin
```

for $k := 0$ **to** n **do** $\tilde{\phi}(k,n) := v_k$;
for $l := 0$ **to** n **do** $\tilde{\phi}(n,l) := w_l$;
for $(k,l) := (n,n)$ **downto** $(1,1)$ **do**
$(\tilde{\phi}(k-1,l-1), \alpha(k,l)) := \Phi V(\tilde{\phi}(k,l-1), \tilde{\phi}(k-1,l), \tilde{\phi}(k,l))$
end

Comments: See the comments to Algorithm 3.7.7.

Algorithm 3.7.10 (G has no multiple edges)

Input = Output of Algorithm 3.7.9.
Output = Input of Algorithm 3.7.9.
var
$\tilde{\phi}$: array $[0..n, 0..n]$ of vertices;
begin
for $k := 0$ **to** n **do** $\tilde{\phi}(k,0) := \hat{0}$;
for $l := 0$ **to** n **do** $\tilde{\phi}(0,l) := \hat{0}$;
for $(k,l) := (1,1)$ **to** (n,n) **do**
$\tilde{\phi}(k,l) := \Psi V(\tilde{\phi}(k,l-1), \tilde{\phi}(k-1,l), \tilde{\phi}(k-1,l-1), \alpha(k,l))$;
for $k := 0$ **to** n **do** $v_k := \tilde{\phi}(k,n)$;
for $l := 0$ **to** n **do** $w_l := \tilde{\phi}(n,l)$;
end

Comments: See the comments to Algorithm 3.7.8.

Later on we shall use an algorithmic notation to define particular functions ΦV and ΨV in the following way (cf. Definitions 3.4.3–3.4.4).

Definition 3.7.11 (Dots are to be replaced by an appropriate operator.)
$\{x, y, z$ are vertices of G; z either covers x in G_2 or is equal to x; z either covers y in G_1 or is equal to y; returned parameters are $\alpha \in \{0, \ldots, r\}$ and a vertex $t\}$

function $\Phi V(x,y,z)$;
begin
case
$x = y = z \Rightarrow t := x; \alpha := 0$
$x = z \neq y \Rightarrow t := y; \alpha := 0$
$x \neq z = y \Rightarrow t := x; \alpha := 0$
$x \neq z \neq y \Rightarrow \ldots\ldots$
endcase;
return(t, α)
end;

Definition 3.7.12 (Dots are to be replaced by an appropriate operator.)
$\{x, y, t$ are vertices of G; x either covers t in G_1 or is equal to t; y either covers t in G_2 or is equal to t; $\alpha \in \{0, \ldots, r\}$; if $\alpha \neq 0 \Rightarrow x = y = t$; returned parameter a vertex $z\}$

```
function ΨV(x, y, t, α);
begin
   case
      x = t = y, α = 0 ⇒ z := x;
      x ≠ t = y ⇒ z := x;
      x = t ≠ y ⇒ z := y;
      (x = t = y and α ≠ 0) or (x ≠ t ≠ y) ⇒ ......
   endcase;
   return(z)
end;
```

The functions ΦV and ΨV should be inverse to each other for each pair (x, y).

3.8. Enumerative consequences

This section is devoted to deriving enumerative identities from the bijective correspondences of Secs. 3.6–3.7.

Recall from Section 1.4 that any $\{U, D\}$-word w can be naturally represented as a linear operator in the vector space of finitary functions on P. A matrix element of this operator that corresponds to a pair of vertices (x, y) is the number of w-paths from x to y (see the last paragraph of Section 1.2). Hence Theorem 3.6.3 has the following enumerative consequence.

Corollary 3.8.1 *Assume the up and down operators U and D in an oriented graded graph G satisfy* (3.0.1). *Then, for any skew graph S, the following operator identity holds*:

$$w^+(S) = \sum_\alpha w^-(S, \alpha) \tag{3.8.1}$$

where the sum is over all r-colored diagonal sets α contained in S.

This corollary can be easily derived directly from (3.0.1). We emphasize, however, that we gave a bijective proof of (3.8.1), viz., one associated with Theorem 3.6.3.

Since Corollaries 3.7.1–3.7.6 are special cases of Theorem 3.6.3, the corresponding enumerative identities follow from Corollary 3.8.1. In order to state these identities explicitly, the following notation is introduced.

Definition 3.8.2 Let S be a skew graph. The number of diagonal sets consisting of exactly k cells of S is denoted $d_k(S)$. The expression

$$R_S(t) = \sum d_k(S) t^k$$

is known as a *rook polynomial* of S (see, e.g., [14]). Thus $R_S(r)$ is the number of r-colored diagonal sets contained in S.

For a $\{U, D\}$-word w, put $d_k(w) = d_k(S)$ where $S = S(w)$ is a Ferrers graph naturally associated with w.

Now we are in a position to write enumerative formulae corresponding to Corollaries 3.7.1–3.7.6.

Corollary 3.8.3 *Assume the up and down operators U and D in an oriented graded graph G satisfy* (3.0.1). *In addition, the statements* (ii), (iii), (v), *and* (vi) *below require G to have a zero $\hat{0}$. Then*

(i) *for any $\{U, D\}$-word w with m occurrences of D and n occurrences of U,*

$$w = \sum_k r^k d_k(w) U^{n-k} D^{m-k}; \tag{3.8.2}$$

(ii) *for any $\{U, D\}$-word w with m occurrences of D and n occurrences of U and for any vertex x of rank $n - m$,*

$$\#\{w\text{-paths from } \hat{0} \text{ to } x\} = r^m d_m(w) e_1(x);$$

(iii) *for any balanced $\{U, D\}$-word w of length $2n$,*

$$\#\{w\text{-loops starting at } \hat{0}\} = r^n d_n(w);$$

(iv) *for any m and n,*

$$D^m U^n = \sum_k r^k \binom{m}{k} \binom{n}{k} k! U^{n-k} D^{m-k};$$

(v) *for any vertex x of rank $n - m$,*

$$\sum_{y \in P_n} e(\hat{0} \to y \to x) = r^m \binom{n}{m} m! e_1(x);$$

(vi) *for any n,*

$$\alpha(0 \to n \to 0) = r^n n!$$

(see Section 1.1 for the notation used).

Proof: Our general algorithmic construction provides unified bijective proofs to all these identities. The statements (i)–(vi) follow from Corollaries 3.7.1–3.7.6, respectively; in (iv)–(vi), the formula

$$d_k(D^m U^n) = \binom{m}{k}\binom{n}{k}k!$$

is used. Note that (ii)–(vi) follow directly from (i), just as Corollaries 3.7.2–3.7.6 follow from Corollary 3.7.1) □

The statement (ii) of Corollary 3.8.3 generalizes [St88, Theorem 3.7]. The statements (v) and (vi) coincide with (1.5.8) and (1.5.9), respectively.

For some types of Ferrers graphs, the rook polynomials can be computed explicitly.

Lemma 3.8.4 [14, Section 8.5]

(i) $d_k((UD)^n) = S(n, n-k)$ (*Stirling numbers of the second kind*)

(ii) $d_k(D^n U^{n-1}) = d_k(D(U^2D)^{n-1})$.

The word $(UD)^n$ corresponds to the isosceles staircase board, i.e., the Ferrers graph with the set of vertices $\{(k,l)\colon 0 \le k, 0 \le l, k+l \le n\}$, and $D(U^2D)^{n-1}$—to the staircase board with the vertices

$$\{(k,l)\colon 0 \le k \le 2n-2, 0 \le l, k+2l \le 2n\}.$$

Combine Lemma 3.8.4 and (3.8.2) to obtain the following identities.

Corollary 3.8.5 *The relation* (3.0.1) *implies*:

$$\text{(i)} \quad (UD)^n = \sum_{k=0}^{n} r^k S(n, n-k) U^{n-k} D^{n-k}; \tag{3.8.3}$$

$$\text{(ii)} \quad U^{n-1}D^nU^{n-1} = D(U^2D)^{n-1}.$$

Identity (3.8.3) coincides with [22, Proposition 4.9, (46)]; in fact, it can be reduced to

$$(DU)^{n-1} = \sum_k r^k S(n, n-k) U^{n-k-1} D^{n-k-1}.$$

Subsequent computations make use of the following lemma.

Lemma 3.8.6 [6, case $n = m$] *For any n, m, and k,*

$$\sum_w d_k(w) = 2^{-k} k! \binom{m+n}{n}\binom{m}{k}\binom{n}{k}$$

where the sum is over all $\{U, D\}$-words w with m entries of D and n entries of U.

Comments: This lemma has an elegant probabilistic reformulation. Choose uniformly at random a diagonal set containing k cells of the rectangle $\square_{n\times m}$ (there are exactly $\binom{m}{k}\binom{n}{k}k!$

such sets). Choose independently a random shortest path in $\square_{n\times m}$ connecting the opposite corners $(0, m)$ and $(n, 0)$ (there are $\binom{m+n}{n}$ such paths). The lemma states that the probability of the entire diagonal set lying below the chosen path is exactly 2^{-k}. This fact is quite surprising for the following reasons. The probability of a single cell being under a random path is certainly $\frac{1}{2}$. Thus the lemma asserts that if one throws k random cells into the rectangle so that they form a diagonal set and wonders if they all lie under the same random path, then the probabilities of the k events "a cell lies under a path" are multiplied even though these events are strongly dependent. Nevertheless, the proof of Lemma 3.8.6 we give below is probabilistic.

Proof: Consider the following stochastic experiment. Let $x_1, \dots, x_n$, $y_1, \dots, y_m$ be independent random variables each having a uniform distribution on $[0, 1]$. Sort $x_1, \dots, x_n$ to obtain the order statistics $x_{(1)} < \cdots < x_{(n)}$; similarly, produce $y_{(1)} < \cdots < y_{(m)}$ from $y_1, \dots, y_m$. The set of cells $\{\square_{ij}\colon x_{(i)} + y_{(j)} < 1\}$ defines a Ferrers shape lying within the rectangle $\square_{n\times m}$. This shape is naturally associated with a path connecting the corners $(0, m)$ and $(n, 0)$, viz., the path along the upper boundary of the shape. All paths this experiment can produce are equally likely. Indeed, a path is uniquely determined by the ordering of the $n + m$ numbers $x_{(1)}, \dots, x_{(n)}, 1 - y_{(1)}, \dots, 1 - y_{(m)}$, and conversely (i.e., a path determines an ordering).

Now let us proceed with our experiment. Choose among nm points (x_i, y_j) some k points with distinct i and distinct j (let all the $\binom{m}{k}\binom{n}{k}k!$ options have the same probability). Such a choice produces k random points which are distributed just as if they have been thrown independently and uniformly into the square $[0, 1] \times [0, 1]$ (a uniform Poisson field). Hence the probability of all these points lying under the line $x + y = 1$ is 2^{-k}. On the other hand, this is just the probability of a random k-cell diagonal set in $\square_{n\times m}$ lying under the random path constructed at the first stage. $\square$

Corollary 3.8.7 *For any n and m, the relation* (3.0.1) *implies*

$$\sum w = \sum_k r^k 2^{-k} k! \binom{m+n}{n} \binom{m}{k} \binom{n}{k} U^{n-k} D^{m-k} \tag{3.8.4}$$

where the sum is over all $\{U, D\}$-words w with m occurrences of D and n occurrences of U.

Proof: Follows from (3.8.2) and Lemma 3.8.6. $\square$

Corollary 3.8.8 *For any p,* (3.0.1) *implies*

$$(U + D)^p = \sum_{n+m=p} \sum_k r^k 2^{-k} \frac{p!}{(m-k)!(n-k)!k!} U^{n-k} D^{m-k}. \tag{3.8.5}$$

Proof: Sum (3.8.4) over all n and m such that $n + m = p$. $\square$

Corollary 3.8.9 [22, Corollary 2.6(a)] *The relation* (3.0.1) *implies*

$$e^{t(U+D)} = e^{rt^2/2}e^{tD}$$

where t is a formal parameter.

Proof: Equate coefficients of t^p to get (3.8.5). □

Corollary 3.8.10 *Assume the up and down operators in an oriented graded graph G with zero* $\hat{0}$ *satisfy* (3.0.1). *Fix* n *and* $m < n$. *Then*

(i) *for any vertex* $x \in P_{n-m}$, *the number of oriented paths (Hasse walks) of length* $n+m$ *going from* $\hat{0}$ *to* x *equals*

$$r^m 2^{-m} \frac{(m+n)!}{(n-m)!m!} e_1(x); \tag{3.8.6}$$

(ii) *the number of closed Hasse walks* $\hat{0} = x_0, x_1, \ldots, x_{n+m} > x_{n+m+1} > \cdots > x_{2n} = \hat{0}$ *in G is equal to* $r^n 2^{-m}(m+n)!/m!$.

Proof:

(i) Apply (3.8.4) to $\hat{0}$. All terms on the right-hand side vanish except for the one corresponding to $k = m$.

(ii) Multiply (3.8.6) by $e_2(x)$, sum over all $x \in P_{n-m}$, and employ

$$\sum_{x \in P_{n-m}} e_1(x) e_2(x) = r^{n-m}(n-m)!.$$ □

The first statement of Corollary 3.8.10 generalizes the formula of S. Sundaram [25] (see also [17, Theorem 2.3.1]) for the Young lattice. The second statement of Corollary 3.8.10 coincides with [22, Corollary 3.16] in the self-dual case.

3.9. Self-dual graphs and involutions

Consider a self-dual case, i.e., the case $G_1 = G_2$. Assume, for simplicity, that G_1 has no multiple edges. Then it is easy to see that an r-correspondence can be chosen in such a way that the whole construction is *transpose-invariant*. Thus if a skew graph S is self-conjugate, then the bijection (a) $\leftrightarrow$ (c) of Theorem 3.6.1 assigns symmetric growths $\hat{F}^+ \to \hat{E}$ to symmetric growths $\hat{F}^- \to \hat{E}$ and symmetric diagonal sets α. The corresponding version of Corollary 3.7.1 will be stated after the following definition.

Definition 3.9.1 Let w be a $\{U, D\}$-word of length n. Define w^* to be a $\{U, D\}$-word obtained by writing w in the reverse order and replacing each and every U by a D and vice versa. Then $S(w^*w)$ is a self-conjugate Ferrers graph whose upper boundary has structure w^*w.

Theorem 3.9.2 *Let G be a (non-oriented, i.e., ordinary) graded graph without multiple edges; assume the condition* (3.0.1) *holds in G. Let A be a vertex of G and w a $\{U, D\}$-word of length n. Then there is a bijective correspondence between*

(a) *the w-paths in G starting at A and*

(b) *the pairs*

(symmetric r-colored diagonal set α contained in $S(w^*w)$, D^{n-k}-path starting at A) where $k = \#supp(\alpha)$.

To state the algebraic/enumerative version of this theorem we need the notation $\mathbf{P} = \sum_{x \in P} x$ introduced in Section 1.5.

Corollary 3.9.3 *Assume G is a (non-oriented) graded graph without multiple edges satisfying* (3.0.1). *Let w be a $\{U, D\}$-word of length n. Then*

$$w^*\mathbf{P} = \sum_k \tilde{d}_{k,r}(w^*w) U^{n-k} \mathbf{P}$$

where $\tilde{d}_{k,r}(w^*w)$ is the number of symmetric r-colored diagonal sets contained in $S(w^*w)$ (the coloring should also be symmetric).

In fact, the clause prohibiting multiple edges is unnecessary. Moreover, (3.9.1) has to be a formal algebraic consequence of (3.0.1) and the relation $(U + rI - D)\mathbf{P} = 0$ (cf. Section 1.5.13). Thus Corollary 3.9.3 is also valid for graded networks.

The case $A = \hat{0}$ of Theorem 3.9.2 corresponds to equating coefficients of $\hat{0}$ in (3.9.1). All the terms on the right-hand side vanish but the one corresponding to $k = n$.

Corollary 3.9.4 *Assume the conditions of Corollary 3.9.3 hold, and G has a zero $\hat{0}$. Then the number of w-paths starting at $\hat{0}$ is equal to the number of symmetric r-colored permutations (involutions) contained in the Ferrers graph $S(w^*w)$. (A coloring of an involution should also be symmetric.)*

Moreover, there is a bijection, based on the algorithms of Section 3.6, between w-paths and these r-colored involutions.

In the case of $w = U^n$, Corollary 3.9.4 can be restated as follows.

Corollary 3.9.5 *Let G be a self-r-dual [non-oriented] graded graph with zero $\hat{0}$. Then, for any n,*

$$\alpha(0 \to n) = \#\{r\text{-colored involutions of } n \text{ elements}\} .$$

(A coloring should also be symmetric.)

Corollaries 3.9.4–3.9.5 and Theorem 3.9.2 were proved in [2] for the case $r = 1$; Corollary 3.9.5 is well-known for the case of the Young graph (see, e.g., [21, Section 17]; cf. (2.1.1)).

4. Schensted algorithms: Examples

This part of the paper is devoted to exploring several special cases of the main bijective construction of Part 3. For each pair of dual graded graphs, we introduce a natural r-correspondence and study the corresponding specializations of Algorithms 3.7.7–3.7.10 (generalized Schensted). These examples include the classical RSK algorithm for the Young graph, the Sagan-Worley-Haiman algorithm for the graph of shifted shapes, the Schensted analogues for the Young-Fibonacci graph, the subword order, the Pascal graphs, and others.

The rim hook algorithm of Stanton and White [23] is a special case of our general construction; it is studied in [7] together with its analogue for the shifted shapes.

The following conventions are used throughout Part 4:

$G = (P, \rho, E_1, E_2)$ is an oriented graded graph;
$G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ are r-dual graphs;
Φ is an r-correspondence in G;
$\Psi = \Phi^{-1}$.

4.1. Functions of permutations

The generalized Schensted correspondence, as described in Corollary 3.7.6 and Algorithms 3.7.8 and 3.7.10, gives rise to a map $\alpha \mapsto v_\Phi(\alpha)$ that maps an r-colored permutation α to the common endpoint $v_\Phi(\alpha)$ of the two paths associated with α. The following definition will be used to explain the role played by the map v_Φ.

Definition 4.1.1 Let α be an r-colored diagonal set. For any integers k and l, define an r-colored set $\tilde{\alpha}_{kl}$ by

$$\tilde{\alpha}_{kl}(i,j) = \begin{cases} \alpha(i,j), & \text{if } i \le k, j \le l \\ 0, & \text{otherwise} \end{cases}$$

and let $\alpha_{k,l}$ be an r-colored permutation which is the *type* of $\tilde{\alpha}_{kl}$ in the sense of Definition 2.6.1. In other words, take all nonzero values of α in the quadrant $\{(i,j) : i \le k,\ \ j \le l\}$ and consider the rows and the columns containing corresponding points. The values of α at the intersections of these rows and columns give the elements of a matrix of an r-colored permutation α_{kl}.

The following result is simple but important for our further considerations.

Lemma 4.1.2 *Let α be an r-colored permutation and ϕ the corresponding ϕ-growth. Then the function $\tilde{\phi}$ associated with ϕ (see Definition* 3.2.5) *is given by* $\tilde{\phi}(k, l) = v_\Phi(\alpha_{kl})$.

Thus the function v_Φ: *Perm* $\to P$ (permutations to vertices) gives an alternative description of a Φ-growth provided G has no multiple edges (cf. Corollary 3.1.4(iv)). In the author's opinion, a natural choice of an r-correspondence should produce a function v_Φ having a reasonable intrinsic (direct, non-recursive) definition. In other words, $v_\Phi(\alpha)$ should be a meaningful statistic of a permutation α provided that Φ has been properly chosen. A well-known result in the theory of the Robinson-Schensted correspondence describes the shape $v_\Phi(\alpha)$ in terms of increasing and decreasing subsequences of α (Greene's theorem [8]). Other pairs of dual graphs and respective natural r-correspondences give rise to other permutation statistics. Note that each of these can be computed by means of an appropriate version of generalized Schensted algorithm, i.e., Algorithm 3.7.7 and/or 3.7.9.

4.2. The Young graph: RSK

In the case of the Young graph (see Example 2.1.2), a natural choice of an r-correspondence converts Algorithms 3.7.7–3.7.10 into certain parallel versions of the Robinson-Schensted algorithm (see [19, 12, 20, etc.]); these versions initially appeared in [1] and then in [2]; see also [15] and [26, 27].

First we introduce an r-correspondence.

Lemma 4.2.1 *Let G be the Young graph (so $r = 1$). Define functions ΦV and ΨV as follows*:

$\boxed{\Phi V}$: *replace the dots in Definition* 3.7.11 *by*

case
- $x \neq y \Rightarrow t := x \cap y;\ \alpha := 0;$
- $x = y,\ z = x \cup$ {*box in the kth row* }, $k \geq 2 \Rightarrow$
  - $t := x \cup$ {*box in the* $(k-1)$*st row*}; $\alpha := 0;$
- $x = y,\ z = x \cup$ {*box in the first row*} $\Rightarrow t := x;\ \alpha := 1;$

endcase

$\boxed{\Psi V}$: *replace the dots in Definition* 3.7.12 *by*

case
- $x \neq y \Rightarrow z := x \cup y;$
- $x = y = t \cup$ {*box in the kth row*} $\Rightarrow$
  - $z := x \cup$ {*box in the* $(k+1)$*st row*};
- $x = y = t,\ \alpha = 1 \Rightarrow z := x \cup$ {*box in the first row*}

endcase

Then ΦV and ΨV define an r-correspondence in $G = \mathbb{Y}$ (cf. (3.7.1)–(3.7.2)).

Proof: A straightforward verification shows that the corresponding functions Φ and Ψ are indeed inverse bijections between appropriate sets. □

Each edge of the Young graph adds a box to a certain Young diagram; say, this box lies in the kth row (if an edge is degenerate, let $k = 0$). The procedures ΦV and ΨV of Lemma 4.2.1 can be entirely rewritten in terms of the parameters k (i.e., row numbers). Thus Algorithms 3.7.9–3.7.10 can process these parameters instead of the vertices of $\mathbb{Y}$. For instance, in the input of Algorithm 3.7.9 the sequences $\{v_i\}$ and $\{w_i\}$ may be substituted by respective *Yamanouchi symbols* $\{Y_1(i)\}$ and $\{Y_2(i)\}$, i.e., the sequences of integers indicating which row is a box placed into at each step; thus, e.g., $Y_1(i)$ is the row number for the box $v_i \setminus v_{i-1}$.

Now we are prepared to write the versions of Algorithms 3.7.9–3.7.10 for the Young graph.

Algorithm 4.2.2 (cf. Algorithm 3.7.9)

Input: Yamanouchi symbols $Y_1(1), Y_1(2), \ldots, Y_1(n)$ and $Y_2(1), Y_2(2), \ldots, Y_2(n)$ of two standard Young tableaux of the same shape.
Output: $n \times n$ permutation matrix α.

```
var
  φ1 : array [1..n, 0..n] of integer;
  φ2 : array [0..n, 1..n] of integer;
  α: array [1..n, 1..n] of integer;
  c: integer;
begin
  for k := 1 to n do φ1(k, n) := Y1(k);
  for l := 1 to n do φ2(n, l) := Y2(l);
  for (k, l) := (n, n) downto (1, 1) do
    begin
      if φ1(k, l) = φ2(k, l) ≠ 0 then c := 1 else c := 0;
      φ1(k, l − 1) := φ1(k, l) − c;
      φ2(k − 1, l) := φ2(k, l) − c;
      if c = 1 and φ1(k, l) = 1 then α(k, l) := 1 else α(k, l) := 0
    end
end
```

Algorithm 4.2.3 (cf. Algorithm 3.7.10)

Input: $n \times n$ permutation matrix α.
Output: Yamanouchi symbols $Y_1(1),\ Y_1(2), \ldots, Y_1(n)$ and $Y_2(1),\ Y_2(2), \ldots, Y_2(n)$ of two standard Young tableaux of the same shape.

```
var
  φ1 : array [1..n, 0..n] of integer;
  φ2 : array [0..n, 1..n] of integer;
  c: integer;
begin
  for k := 1 to n do φ1(k, 0) := 0;
  for l := 1 to n do φ2(0, l) := 0;
  for (k, l) := (1, 1) to (n, n) do
    begin
      if φ1(k, l − 1) = φ2(k − 1, l) ≠ 0 or α(k, l) = 1 then c := 1 else c := 0;
      φ1(k, l) := φ1(k, l − 1) + c;
      φ2(k, l) := φ2(k − 1, l) + c
```

```
        end
      for k := 1 to n do Y_1(k) := φ_1(k, n);
      for l := 1 to n do Y_2(l) := φ_2(n, l)
    end
```

Definition 4.2.4 Algorithms 4.2.2–4.2.3 establish bijective correspondence between pairs of standard Young tableaux of the same shape and permutations. This is the *Robinson-Schensted correspondence.*

To demonstrate that this definition coincides with the traditional one, consider sequential versions of Algorithms 4.2.2–4.2.3. In other words, replace, e.g.,

for $(k, l) := (1, 1)$ **to** (n, n) **do**

by

for $k := 1$ **to** n **do**
 for $l := 1$ **to** n **do**

and verify that the interior **for**-loop is just the usual Schensted insertion. Note that the parallel version makes transparent the well-known symmetry of the entire construction: the inverse permutation corresponds to the same pair of tableaux switched with each other.

Algorithms 4.2.2–4.2.3 can be viewed as "Yamanouchi versions" of the Robinson-Schensted bijections. These versions seem to provide the most convenient techniques for the actual computation of the Schensted correspondence. By the way, their parallel computational complexity is slightly smaller than that of the usual "bumping" versions.

Theorem 4.2.5 *Both Robinson-Schensted correspondences (i.e., constructing a permutation from a pair of tableaux and constructing tableaux from a permutation) can be realized by algorithms running in $O(n)$ time using $O(n)$ processors; or by a circuit with $O(n^2)$ nodes and $O(n)$ depth.*

Since the algorithmic constructions we use apply to any pair of dual graded graphs, results analogous to Theorem 4.2.5 are also valid for other Schensted-type correspondences, e.g., for the examples given in the next sections.

Let us now describe the function v_Φ (see Section 4.1). In the case under consideration this function assigns Ferrers shapes to permutations: in the notation of Algorithm 3.7.9, $v_\Phi(\alpha) = v_n$. According to the main message of Section 4.1, there should be a direct (not recurrent) definition of the map v_Φ. Such a definition is provided by the Greene-Kleitman duality theorem for finite posets.

Theorem 4.2.6 [9, 10] *Let A be a finite poset. Let R_k (Q_k, respectively) denote the maximal number of elements in a union of k chains (antichains, respectively) in A. Denote $x_k = R_k - R_{k-1}$, $y_k = Q_k - Q_{k-1}$. Then*

(i) $x_1 \geq x_2 \geq \cdots$; $y_1 \geq y_2 \geq \cdots$;

(ii) *the Young diagram with row lengths* $x_1, x_2, \ldots$ *has column lengths* $y_1, y_2, \ldots$

Let $\lambda(A)$ denote this Young diagram.

Any permutation σ can be regarded as a poset with the ordering induced from $\mathbb{Z}^2$. Thus one can consider the corresponding diagram $\lambda(\sigma)$.

Theorem 4.2.7 [8] $v_\Phi(\sigma) = \lambda(\sigma)$.

This theorem can also be proved by demonstrating that the correspondence $\sigma \mapsto \lambda(\sigma)$ respects the rule $\boxed{\Phi V}$ of Lemma 4.2.1; see [2].

4.3. General plan

We have now a general scheme that can be used to study a graded graph; this scheme has been already applied to the Young graph. Let us outline the main elements of this scheme.

Let $G_1 = (P, \rho, E_1)$ be a graded graph. Define the up operator U: $KP \to KP$ (recall K is the ground field) by

$$Ux = \sum_y a_1(x, y)\, y$$

where $a_1(x, y)$ is the multiplicity (weight) of the edge (x, y) in G_1.

1. Find a graph $G_2 = (P, \rho, E_2)$ which is dual (r-dual) to G_1. This means that the down operator D: $KP \to KP$ of G_2 defined by

$$Dy = \sum_x a_2(x, y)\, x$$

   (here $a_2(x, y)$ is the multiplicity of the edge (x, y) in G_2) satisfies the commutation relation $DU = UD + rI$.

2. Define an oriented graded graph $G = (P, \rho, E_1, E_2)$ by reversing the direction of the G_2-edges (so one can only move "up" in G_1 and "down" in G_2; see Definition 1.2.1). Introduce an r-correspondence Φ in G (see Section 3.4). Generally, there are various ways of defining such a correspondence; choose a "natural" one.

3. Once Φ has been chosen, certain parallel algorithms based on the concept of a Φ-growth arise. They establish bijective correspondences between permutations (or diagonal sets; in case $r \neq 1$ they are r-colored) and oriented paths in G with prescribed structure. These algorithms can be presented in either "vertex version" (cf. Algorithms 3.7.9–3.7.10) or "edge version" (cf. Algorithms 3.6.4–3.6.5, 3.7.7–3.7.8). Sequential versions can be sometimes restated in terms of tableaux and insertion-replacement procedures.

   Bijective correspondences of this kind can be used for obtaining various formulae related to path enumeration.

4. The main particular case of the construction is the case of a square Ferrers graph $\square_{n\times n}$ (cf. Corollary 3.7.6)); it leads to Schensted-type algorithms. Such an algorithm constructs paths p_1 and p_2 in G_1 and G_2, respectively, for a given permutation σ; these paths have a common endpoint denoted $v_\Phi(\sigma)$. The map $\sigma \mapsto (p_1, p_2)$ is a bijection; the inverse map can also be presented in a similar algorithmic form.

If an r-correspondence Φ is properly chosen, the map v_Φ has an intrinsic definition (usually not easy to find). In other words, $v_\Phi(\sigma)$ is a reasonable permutation statistic. Hence we may use the above mentioned algorithms for computing these statistics.

4.4. The Young-Fibonacci graph

The Young-Fibonacci graph $\mathbb{YF}$ was defined in Example 2.1.4. Similarly to the case of the Young graph, we begin by introducing an r-correspondence via functions ΦV and ΨV. The descriptions of these functions are similar to those of Lemma 4.2.1.

Lemma 4.4.1 *Let G be the Young-Fibonacci graph (so $r = 1$). Define functions ΦV and ΨV as follows:*

$\boxed{\Phi V}$: *replace the dots in Definition* 3.7.11 *by*

case

$z = 1u$ *for some* $u \Rightarrow t := u;\ \alpha := 1;$

$z = 2u$ *for some* $u \Rightarrow t := u;\ \alpha := 0;$

endcase

$\boxed{\Psi V}$: *replace the dots in Definition* 3.7.12 *by*

case

$x \neq t \neq y \Rightarrow z := 2t;$

$x = y = t,\ \alpha = 1 \Rightarrow z := 1t;$

endcase

This gives an r-correspondence in $G = \mathbb{YF}$ (cf. Lemma 4.2.1).

Proof: Straightforward verification. □

Now we follow the example of the Young graph and introduce certain numerical characteristics of edges which can be processed by the Young-Fibonacci Schensted algorithms instead of edges themselves (cf. Algorithms 4.2.2–4.2.3).

Suppose (s, t) is an edge of $\mathbb{YF}$. This implies that there is a single well-defined entry t_i of t which either has to be removed (if it is 1) or replaced by 1 (if it is 2) in order to obtain s. Let k be the sum of all entries of t which precede t_i, including t_i itself. Then we will say that t is obtained from s by adding a box into the kth position. For example, if $s = 22121$, then a box can be inserted into the 1st, 3rd, 5th, or 6th position, hence getting $122121, 212121, 221121$, or 22221, respectively. This parameter plays the same role for the Young-Fibonacci graph as the row number plays for the Young lattice. In particular, we can work with Young-Fibonacci analogues of the Yamanouchi words instead of the standard

Young-Fibonacci tableaux of Example 2.5.7. For example, the tableau (see Example 2.5.7) is represented by the sequence 112433 which simply encodes the positions where new boxes are inserted. We do not discuss here the simple restrictions such a sequence should satisfy.

3			4
2	6	5	1

Let us now write the versions of Algorithms 3.7.9–3.7.10 for the Young-Fibonacci graph.

Algorithm 4.4.2 (cf. Algorithms 3.7.9, 4.2.2)

Input: Sequences $Y_1(1),\ Y_1(2), \ldots, Y_1(n)$ and $Y_2(1),\ Y_2(2), \ldots, Y_2(n)$ defining two standard Young-Fibonacci tableaux of the same shape.
Output: $n \times n$ permutation matrix α.

    var
      ϕ_1: array $[1..n, 0..n]$ of integer;
      ϕ_2: array $[0..n, 1..n]$ of integer;
      α: array $[1..n, 1..n]$ of integer;
    begin
      for $k := 1$ **to** n **do** $\phi_1(k,n) := Y_1(k)$;
      for $l := 1$ **to** n **do** $\phi_2(n,l) := Y_2(l)$;
      for $(k,l) := (n,n)$ **downto** $(1,1)$ **do**
        begin
          if $\phi_1(k,l) \neq 0 \neq \phi_2(k,l)$ **then**
            begin
              $\phi_1(k, l-1) := \phi_2(k,l) - 1$;
              $\phi_2(k-1, l) := \phi_1(k,l) - 1$;
            end
          else
            begin
              $\phi_1(k, l-1) := \phi_1(k,l)$;
              $\phi_2(k-1, l) := \phi_2(k,l)$;
            end;
          if $\phi_1(k,l) = \phi_2(k,l) = 1$ **then** $\alpha(k,l) := 1$ **else** $\alpha(k,l) := 0$
        end
    end

Algorithm 4.4.3 (cf. Algorithm 3.7.10, 4.2.3)

Input = Output of Algorithm 4.4.2.
Output = Input of Algorithm 4.4.2.

```
var
  φ1: array [1..n, 0..n] of integer;
  φ2: array [0..n, 1..n] of integer;
begin
  for k := 1 to n do φ1(k, 0) := 0;
  for l := 1 to n do φ2(0, l) := 0;
  for (k, l) := (1, 1) to (n, n) do
    begin
      if φ1(k, l − 1) ≠ 0 ≠ φ2(k − 1, l) or α(k, l) = 1 then
        begin
          φ1(k, l) := φ2(k − 1, l) + 1;
          φ2(k, l) := φ1(k, l − 1) + 1;
        end
      else
        begin
          φ1(k, l) := φ1(k, l − 1);
          φ2(k, l) := φ2(k − 1, l);
        end
      end
  for k := 1 to n do Y1(k) := φ1(k, n);
  for l := 1 to n do Y2(l) := φ2(n, l)
end
```

Algorithms 4.4.2–4.4.3 establish bijective correspondences (originally appeared in [2]; see also [15]) between pairs of Young-Fibonacci tableaux of the same shape and permutations. These bijections possess all the main features of RSK.

Sequential versions of Algorithms 4.4.2–4.4.3 give rise to corresponding insertion procedures which play the same role as the Schensted insertion. The description of these procedures in conventional "bumping" terms results precisely in the construction of Definition 2.5.10. (See also [15].) Details are left to the reader.

Now we describe a "Greene analogue" for the Young-Fibonacci lattice, i.e., a permutation statistic v_Φ that assigns vertices of $\mathbb{YF}$ to permutations (cf. Theorem 4.2.7). The following result was independently obtained by T. Roby [15] and the author.

Theorem 4.4.4 *Let σ be an $(n \times n)$-permutation. Then the corresponding $\{1, 2\}$-word (a Young-Fibonacci shape) $v_\Phi(\sigma)$ can be defined as follows.*

Take the uppermost and the rightmost elements of σ. (Or, if you prefer to write σ in a one-line notation, take the last and the largest elements of σ.) If they coincide, write 1; otherwise write 2. Then remove these elements from σ and repeat this operation until σ is empty. The resulting $\{1, 2\}$-word is $v_\Phi(\sigma)$.

Proof: Induction on the size of σ based on Algorithm 4.4.3. □

4.5. Shifted Shapes

This is the first example with $G_1 \neq G_2$. Recall the definition of the graph of shifted shapes $\mathbb{SY}$ and its dual from *Example* 2.2.8. Let $G_1 = \mathbb{SY}$. Since the dual graph G_2 does have multiple edges, the "vertex version" of the generalized Schensted algorithm (Algorithms 3.7.8–3.7.9) does not apply and the "edge version" will only be given.

To describe an r-correspondence, color the edges of $\mathbb{SY}$ in black, blue, and red as follows. Each edge a adds a box B to a certain shifted shape λ. If B lies on the main diagonal (cf. (2.2.8)) then a is colored black. Otherwise color one of the edges which add B to λ blue and another one red (this corresponds to two kinds of entries in the usual shifted P-tableaux; cf. [18] or [17]). Now we can define an r-correspondence for this dual pair.

Lemma 4.5.1 *Let G_1 be the graph of shifted shapes $\mathbb{SY}$ and G_2 a dual graph defined as in Example* 2.2.8. *Define the functions Φ and Ψ as follows:*

$\boxed{\Phi}$: replace the dots in Definition 3.4.3 by
case
- b_1 and b_2 add different boxes $\Rightarrow$
  - a_1 and a_2 add the same boxes as b_1 and b_2, respectively;
  - a_2 has the same color as b_2; $\alpha := 0$;
- b_1 and b_2 add a box into the kth row, $k \geq 2$, b_2 is blue or black $\Rightarrow$
  - a_1 and a_2 add a box into the $(k-1)$st row; a_2 is blue; $\alpha := 0$;
- b_1 and b_2 add a box into the first row, b_2 is blue or black $\Rightarrow$
  - a_1 and a_2 are degenerate; $\alpha := 1$;
- b_1 and b_2 add a box into the kth column, b_2 is red $\Rightarrow$
  - a_1 and a_2 add a box into the $(k-1)$st column; a_2 is red or black; $\alpha := 0$

endcase

$\boxed{\Psi}$: replace the dots in Definition 3.4.4 by
case
- a_1 and a_2 add different boxes $\Rightarrow$
  - b_1 and b_2 add the same boxes as a_1 *and* a_2, respectively;
  - b_2 has the same color as a_2;
- a_1 and a_2 add the same box, a_2 is blue $\Rightarrow$
  - b_1 and b_2 add a box into the next row; b_2 is blue or black;
- a_1 and a_2 add the same box, a_2 is red or black $\Rightarrow$
  - b_1 and b_2 add a box into the next column; b_2 is red;
- a_1 and a_2 are degenerate, $\alpha = 1 \Rightarrow$
  - b_1 and b_2 add a box into the first row; b_2 is blue or black

endcase

Then Φ is an r-correspondence in G, and $\Psi = \Phi^{-1}$.

Proof: It is straightforward to verify that these procedures are well-defined and that Φ and Ψ are inverse bijections between appropriate sets. □

Thus we obtain Robinson-Schensted analogues for the dual pair (G_1, G_2) which are specializations of Algorithms 3.7.7–3.7.8 with Φ and Ψ defined as above.

Proposition 4.5.2 *The sequential versions of Algorithms* 3.7.7–3.7.8 *for the graph of shifted shapes and an* r*-correspondence defined in Lemma* 4.5.1 *coincide with algorithms of Sagan-Worley* [16, 29] *and Haiman* [11].

(*Comments*: These sequential versions are the row-by-row and column-by-column versions which appear when the main **for**-cycle

for $(k,l) := (1,1)$ **to** (n,n) **do**

is transformed into either

for $k := 1$ **to** n **do**
  for $l := 1$ **to** n**do**

or

for $l := 1$ **to** n **do**
  for $k := 1$ **to** n **do**;

similar substitutions can be done in the **downto**-cycles.)

A **proof** of this statement reduces to a formal verification.

Proposition 4.5.2 clarifies the well-known fact (see [17]): Sagan-Worley and Haiman's algorithms produce the same pairs of tableaux when applied to inverse permutations. In fact, these algorithms are just different versions of one and the same parallel algorithm.

A Greene analogue $v_\Phi(\sigma)$ for the shifted case was found by D. Worley [29] and B. Sagan [16]. It can be expressed in terms of the ordinary Greene invariant. Namely, for $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, compute the Young diagram $\lambda(\sigma_n\cdots\sigma_2\sigma_1\sigma_1\sigma_2\cdots\sigma_n)$ and partition it in halves along the main diagonal (the diagonal itself is contained in the lower part). These halves always have the same shifted shape, and this is $v_\Phi(\sigma)$.

4.6. Binary tree and subword order

The lifted binary tree and the dual graph *BinWord* (the binary subword order) were defined in Examples 2.3.6 and 2.4.1. Since there is only one path in a tree that connects any vertex with the root, we will actually obtain bijection(s) between paths in *BinWord* and permutations.

We restrict ourselves to the "vertex versions" (i.e., specializations of Algorithms 3.7.9–3.7.10) which are valid as both graphs have no multiple edges.

There are many simple ways to define an r-correspondence for this dual pair. The following one seems to be the most natural.

Lemma 4.6.1 *Let* G_1 *be the lifted binary tree and* G_2 *the dual graph BinWord. Define the functions* ΦV, ΨV *as follows*:

$\boxed{\Phi V}$: replace the dots in Definition 3.7.11 by
case
$x \neq y$ **or** $z = w0$ for some $w \Rightarrow t := x$ with the last symbol removed; $\alpha := 0$;
$x = y$ **and** $z = w1$ for some $w \Rightarrow t := x$; $\alpha := 1$
endcase
$\boxed{\Psi V}$: replace the dots in Definition 3.7.12 by
case
$x \neq y \Rightarrow z := y\omega$; {here ω is the last symbol of x; so $x = t\omega$}
$x = y \neq t \Rightarrow z := y0$;
$x = y = t,\ \alpha = 1 \Rightarrow z := y1$ {if $y = 0$ then $y1$ means 1}
endcase

Then the corresponding maps Φ and Ψ are inverse to each other, and Φ is an r-correspondence in G.

In the case under consideration Algorithms 3.7.9–3.7.10 (with the above choice of an r-correspondence) define a bijection between permutations and paths in *BinWord*. The corresponding map v_Φ assigns binary words (vertices of *BinWord*) to permutations. This map can be directly defined as follows.

Proposition 4.6.2 *Let σ be an $(n \times n)$-permutation. Then the binary word $v_\Phi(\sigma)$ is a descent word of σ. Formally, if $\sigma = \sigma_1 \cdots \sigma_n$, then $v_\Phi(\sigma) = w_1 \cdots w_n$ where*

$$w_1 = 1; \quad w_k = \begin{cases} 0, & if\ \sigma_{k-1} > \sigma_k \\ 1, & if\ \sigma_{k-1} < \sigma_k \end{cases}, \quad k = 2, \ldots, n;$$

in addition, $v_\Phi(\sigma) = 0$ when $n = 0$.

Proof: Induction on the size of σ based on Algorithm 3.7.10. □

This proposition implies that our Schensted correspondence for the subword order essentially coincides with the bijection of X. G. Viennot [28].

4.7. Pascal graphs

The Pascal graph $\mathbb{N}^r$ is a lattice of r-dimensional points with nonnegative integer coordinates. The r-dual graph is $\mathbb{N}^r$ with appropriate weights assigned to its edges (see Examples 2.2.2 and 2.3.3). Since this dual pair is, in a sense, the rth cartesian power of the dual pair for the infinite chain (see Example 2.2.1 and Lemma 2.2.3), we start with the latter.

Definition 4.7.1 Let G_1 be the infinite chain $\mathbb{N}$. Then G_2 has the same vertices but multiple edges; namely, n edges connect $n-1$ and n for $n = 1, 2, \ldots$. Let us label these edges with *marks* $1, 2, \ldots, n$ in order to distinguish among them. The most natural way of defining an r-correspondence seems to be the following:

$\boxed{\Phi}$: replace the dots in Definition 3.4.3 by
{it suffices to determine whether a_1 and a_2 are degenerate or not, and in the latter case, also the mark of a_2}
case
mark$(b_2) \neq 1 \Rightarrow$
a_1 and a_2 are not degenerate; mark$(a_2) :=$ mark$(b_2) - 1$; $\alpha := 0$;
mark$(b_2) = 1 \Rightarrow a_1$ and a_2 are degenerate; $\alpha := 1$
endcase

$\boxed{\Psi}$: replace the dots in Definition 3.4.4 by
{we should only determine the mark of b_2}
case
a_1 and a_2 are not degenerate $\Rightarrow$ mark$(b_2) :=$ mark$(a_2) + 1$;
a_1 and a_2 are degenerate, $\alpha = 1 \Rightarrow$ mark$(b_2) := 1$;
endcase

This is certainly an r-correspondence, and the corresponding Greene analogue is simply a function v_Φ that assigns n to a permutation of n elements. Since $\mathbb{N}$ is a tree, the Schensted analogue is a bijection between permutations and paths in G_2; the latters are determined by sequences of marks. The only condition such a sequence should satisfy is

$$1 \leq m_i \leq i \quad \text{for} \quad i = 1, 2, \ldots, n.$$

The total number of such sequences is certainly $n!$. The meaning of the sequence $\{m_i\}$ assigned to a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is the following:

$$\text{for each } i \text{ let } \sigma_k = i; \text{ then } m_i = \#\{j\colon j \geq k,\ \sigma_j \leq i\}. \tag{4.7.1}$$

This is a modification of the so-called *code* of a permutation.

Now we turn to the general case of an arbitrary r. To save space, we do not describe the r-correspondence in detail; just note that it is, in a natural sense, a cartesian r-power of the r-correspondence for the case $r = 1$, i.e., for the infinite chain (see above).

The resulting Greene analogue is a function v_Φ that assigns to an r-colored permutation σ the point $(\gamma_1, \cdots, \gamma_r) \in \mathbb{N}^r$ where γ_i is the number of elements of σ which have color i. The Schensted analogue is a bijection between r-colored permutations and pairs

$$(\text{ path in } \mathbb{N}^r, \text{ path in the dual graph}) \tag{4.7.2}$$

that can be alternatively described as follows. Let $\sigma = \sigma_1 \ldots \sigma_n$ where σ_i has color c_i. Define a path in $\mathbb{N}^r$ that starts at $\hat{0} = (0, \ldots, 0)$ and at step i adds 1 to the c_ith coordinate. This is the first path of (4.7.2), or the "P-tableau". The second path (the "Q-tableau") is defined in a similar way; namely, for each i find k such that $\sigma_k = i$ (cf. (4.7.1)) and add 1 to the c_kth coordinate. We should also fix a *mark* of every edge, since the path is to be in the dual graph (do not confuse marks and colors!). These marks are defined similarly to (4.7.1), taking the colors into account:

$$\text{for each } i \text{ let } \sigma_k = i; \quad \text{then} \quad m_i = \#\{j\colon j \geq k,\ \sigma_j \leq i,\ c_j = c_k\}.$$

Thus the marks determine how the elements of each color are permuted; the distribution of colors among rows and columns is governed by the first path and the "unmarked" second path, respectively.

One can also easily describe the corresponding algorithms in insertion-replacement terms.

4.8. *Insertion graphs and path trees*

Let $G_1 = (P, \rho, E_1)$ be a graded graph with zero. The paths in G_1 starting at zero form a *path tree* $\mathbb{T}(G_1)$ (see Definition 2.5.1). In other terminology, vertices of $\mathbb{T}(G_1)$ are "standard tableaux"; so $\mathbb{T}$ abbreviates both *tree* and *tableau*. It turns out that, once a dual graph $G_2 = (P, \rho, E_2)$ is found, one can construct a dual to $\mathbb{T}(G_1)$ as well.

Assume Φ is an r-correspondence in $G = (P, \rho, E_1, E_2)$. Then an analogue of the Schensted insertion appears (see Comments 3.7.6.1). This allows us to define a graph $Ins(G_1, G_2, \Phi)$ called an insertion graph that proves to be dual to $\mathbb{T}(G_1)$.

Definition 4.8.1 The definition of an *insertion graph* $Ins(G_1, G_2, \Phi)$ generalizes Definition 2.5.5 of the Schensted graph. The set of vertices is the same as that of $\mathbb{T}(G_1)$, namely, the paths in G_1 starting at zero. The rank function is the same, too. Each vertex w of rank $n-1$ is covered in $Ins(G_1, G_2, \Phi)$ by exactly n vertices of rank n (cf. Corollary 2.9.2(2)). These vertices can be obtained by inserting a number into w, i.e., by applying the procedure of Comments 3.7.6.1. Note that we use an r-correspondence Φ, fixed in advance, while making these insertions.

The construction of Definition 4.8.1 leads to the Schensted graph and the graph *InsYF* (see Definitions 2.5.4–2.5.5 and 2.5.10–2.5.11) when specialized to the Young and Young-Fibonacci graphs, respectively.

Theorem 4.8.2 *A path tree* $\mathbb{T}(G_1)$ *and an insertion graph* $Ins(G_1, G_2, \Phi)$ *are dual to each other.*

Proof: Let U be the up operator in $Ins(G_1, G_2, \Phi)$ and D the down operator in $\mathbb{T}(G_1)$. Applying UD to a vertex, i.e., a path $(a_1, \ldots, a_n)$ in G_1, can be described as removing a_n and then inserting each possible k, namely, $k = 1, \ldots, n$. On the other hand, DU inserts all the k (thus resulting in a formal sum of respective vertices = paths) and then removes the last edge from each path. Note that in the latter procedure k can be either of $1, \ldots, n+1$. Moreover, corresponding operations (i.e., inserting k and deleting the last edge, or the maximal entry of a tableau) *commute* for each fixed $k \leq n$. Finally, inserting $n+1$ and deleting last edge results in the initial path (vertex). Hence $DU = UD + I$. □

This construction can be extended,—say, to the case of commutation relations

$$D_{n+1}U_n = U_{n-1}D_n + r_nI_n \tag{4.8.1}$$

where $r_n \geq 0$ for all n (cf. (1.3.4)). The statement of Theorem 4.8.2 remains valid and essentially the same proof works.

The following particular case leads us to the *permutation trees* of Section 2.6.

Theorem 4.8.3 *Assume G_1 and G_2 are dual and G_2 is a tree. Then $\mathbb{T}(G_1)$ and $Ins(G_1, G_2, \Phi)$ are dual permutation trees.*

A **proof** of this theorem is omitted; it is technical and not very hard.

Strictly speaking, we should not say that $\mathbb{T}(G_1)$ and $Ins(G_1, G_2, \Phi)$ *are* permutation trees but rather that there exists a pair of isomorphisms between them and permutation trees that respects the whole construction.

Corollary 4.8.4 *As $\mathbb{T}(G_1)$ and $Ins(G_1, G_2, \Phi)$ are dual (Theorem 4.8.2), let Ξ be an r-correspondence for this pair of graphs. Make one additional step to obtain dual graphs*

$$\mathbb{T}(Ins(G_1, G_2, \Phi)) \quad \text{and} \quad Ins(Ins(G_1, G_2, \Phi), \mathbb{T}(G_1), \Xi) .$$

These graphs are isomorphic to dual permutation trees.

In other words, a repeated use of the main construction of this section results (for *any* pair of dual graded graphs!) in the canonical Example 2.6.8.

Now we list the main applications of this construction.

4.8.5 Theorem *Appropriate choices of r-correspondences Φ produce the following series of examples (consult Index 2.8 for definitions):*

G_1	G_2	$\mathbb{T}(G_1)$	$Ins(G_1, G_2, \Phi)$
Young graph	*Young graph*	*SYTTree*	*Schensted graph*
Y.-Fibonacci graph	*Y.-Fibonacci graph*	*SYFTTree*	*InsYF*
Pascal graph $\mathbb{N}^r$	*Weighted Pascal graph*	*Infinite r-nary tree*	*See Ex. 2.3.5*
Schensted graph	*SYTTree*	*Permutation tree*	*Perm. tree (dual)*
InsYF	*SYFTTree*	*Permutation tree*	*Perm. tree (dual)*
See Ex. 2.3.5	*Infinite r-nary tree*	*Permutation tree*	*Perm. tree (dual)*
BinWord	*Lifted binary tree*	*Permutation tree*	*Perm. tree (dual)*

Proof: The first two lines are valid since these are particular cases of the general construction. Lines 4–7 follow from Theorem 4.8.3. It only remains to check the third line. Clearly, $\mathbb{T}(\mathbb{N}^r) = \mathbb{T}_r$. A proof of the fact that in this case the insertion graph is [isomorphic to] the graph related to derivation (2.3.7) is left to the reader. □

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