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ERRATUM

Erratum to: Newton polygons and curve gonalities

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Erratum

The following statements involving 'the metric graph $\Gamma(\Delta_1, \ldots, \Delta_r)$' are false:

- [2, Theorem 10]
- [2, Conjecture 2]
- [2, Conjecture 3].

The erratum is remedied by replacing $\Gamma(\Delta_1, \ldots, \Delta_r)$ by another metric graph $\Gamma(v)$, which depends on an upper-convex piece-wise linear function $v : \Delta \to \mathbb{R}$ realizing the given subdivision $\Delta_1, \ldots, \Delta_r$ and satisfying $v(\Delta \cap \mathbb{Z}^2) \subset \mathbb{Z}$. The construction of $\Gamma(v)$ is discussed in Sect. 1 below.

Background to the erratum

We have made a conceptual error in the construction of our regular strongly semi-stable arithmetic surface $\mathfrak{X}$ over $\mathbb{C}[[t]]$, as explained in [2, Sect. 7]. The error lies in the last part, involving toric resolutions of singularities. Namely, it has been overlooked that the exceptional curves that are introduced during the resolution may appear with non-trivial multiplicities, turning $\mathfrak{X}$ non-stable. Whereas our construction

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suggested that one can keep blowing up to an arbitrary extent, one should be much more careful and blow up just the ‘right’ number of times:

- all singularities should become resolved (enough blow-ups),
- no non-trivial multiplicities should appear (not too many blow-ups).

Luckily, this ‘right’ number always exists and can be controlled in a purely combinatorial way.

1 The graph $\Gamma(v)$

Let $\Delta \subset \mathbb{R}^2$ be a two-dimensional lattice polygon. Let $\Delta_1, \dots, \Delta_r \subset \Delta$ be a regular subdivision and let $v : \Delta \to \mathbb{R}$ be an upper-convex piece-wise linear function realizing this subdivision. Assume that $v(\Delta \cap \mathbb{Z}^2) \subset \mathbb{Z}$. Let $G(\Delta_1, \dots, \Delta_r)$ be the graph with vertex set $\{v_1, \dots, v_r\}$ such that for all ℓ, m the number of edges between v_ℓ and v_m is equal to the integral length $L(\ell, m)$ of $\Delta_\ell \cap \Delta_m$ (i.e. the number of lattice points minus one).

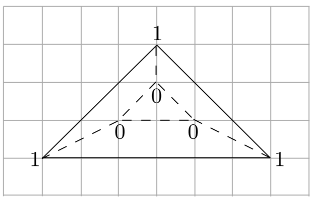
Let $G(v)$ be obtained from $G(\Delta_1, \dots, \Delta_r)$ by replacing each such edge with a linear graph of length $d(\ell, m)$. Here, $d(\ell, m)$ is the greatest common divisor of the (2×2)-minors of

$$\begin{pmatrix} a_{\ell 1} & a_{\ell 2} & 1 \\ a_{m1} & a_{m2} & 1 \end{pmatrix},$$

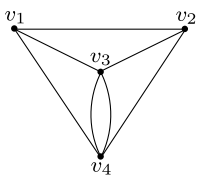
where $(a_{\ell 1}, a_{\ell 2}, 1)$ and $(a_{m1}, a_{m2}, 1)$ are primitive normal vectors to the graphs of v restricted to Δ_ℓ and Δ_m, respectively. The third coordinate can be taken 1 because $v(\Delta \cap \mathbb{Z}^2) \subset \mathbb{Z}$.

Finally, let $\Gamma(v)$ be the metric graph associated to $G(v)$, obtained by identifying each edge with the unit interval.

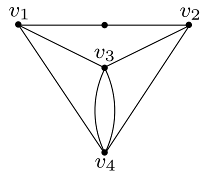
Example Consider $\Delta = \mathrm{Conv}\{(-3, 0), (3, 0), (0, 3)\}$. Let $v : \Delta \to \mathbb{R}$ be the piece-wise linear function whose graph is the lower convex hull of $\{(-1, 1, 0), (1, 1, 0), (0, 2, 0), (-3, 0, 1), (3, 0, 1), (0, 3, 1)\}$.



Denoting the induced subdivision by $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, one finds that $G(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ equals



However, it is easily verified that $G(v)$ equals



Remark. By Khovanskii's theorem, a generic Laurent polynomial $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ for which $\Delta(f) \subset \Delta$, defines a non-hyperelliptic genus 4 curve $U(f)$ which canonically embeds into $\mathrm{Tor}(\Delta^{(1)}) \subset \mathbb{P}^3$. Since $\mathrm{Tor}(\Delta^{(1)})$ is a cone, by [3, Example 5.5.2] this curve carries a unique g^1_3 (computed by projection from the singular top of the cone). If $\Gamma(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ is the metric graph associated to $G(\Delta_1, \dots, \Delta_4)$, then $\Gamma(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ carries at least two distinct g^1_3's: it can be verified that the divisors $3v_1$ and $3v_2$ are non-equivalent. One concludes that $\Gamma(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ cannot be the 'correct' metric graph associated to this example, since the existence of two distinct g^1_3's would contradict M. Baker's specialization theory [1, Lemma 2.1 and Remark 2.3]. In the case of $\Gamma(v)$, the divisors $3v_1$ and $3v_2$ are easily seen to become equivalent.

2 Details of the toric resolution

We resume at [2, Sect. 7], right after the sentence 'For more details on resolving non-degenerate hypersurface singularities, …'.

Let $\Sigma(\tilde{\Delta})$ be the normal fan of $\tilde{\Delta}$. Any subdivision Σ' of $\Sigma(\tilde{\Delta})$ induces a birational morphism $\rho : \mathrm{Tor}(\Sigma') \to \mathrm{Tor}(\Sigma(\tilde{\Delta})) \cong \mathrm{Tor}(\tilde{\Delta})$. If one writes $Y' = \mathrm{Tor}(\Sigma')$ and let $X' \subset Y'$ be the strict transform of X under ρ, then the morphism $p' = p \circ \rho$ yields a fibration $Y' \to \mathbb{P}^1$, and one can redo the argument to obtain an arithmetic surface $\mathfrak{X}'$ over $\mathbb{C}[[t]]$. One can always choose Σ' such that $\mathfrak{X}'$ is a regular, strongly semi-stable arithmetic surface. Such a Σ' can be constructed as follows. Let $\sigma_1, \dots, \sigma_k \in \Sigma(\tilde{\Delta})$ be the two-dimensional cones that are strictly contained in the open upper half-space—these correspond to the edges of $\tilde{\Delta}$ that are not projected on the boundary of Δ. Let $\Sigma_0 = \Sigma(\tilde{\Delta})$ and repeat the following for $i = 1, \dots, k$:

Because $v(\Delta \cap \mathbb{Z}^2) \subset \mathbb{Z}$, the extremal rays of σ_i are generated by vectors

$$(\alpha, \beta, 1) \quad \text{and} \quad (\gamma, \delta, 1) \quad \text{with } \alpha, \beta, \gamma, \delta \in \mathbb{Z},$$

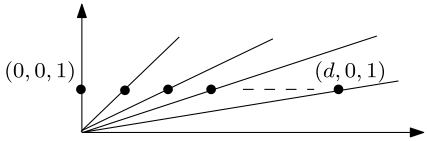
hence by applying a $\mathbb{Z}$-affine transformation if necessary, we may assume that σ_i is generated by

$$(0, 0, 1) \quad \text{and} \quad (d, 0, 1),$$

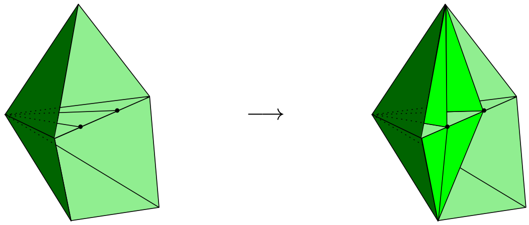
where d is the greatest common divisor of the (2×2)-minors of

$$\begin{pmatrix} \alpha & \beta & 1 \\ \gamma & \delta & 1 \end{pmatrix}.$$

Subdivide σ_i by introducing $d-1$ new rays, generated by $(1,0,1), (2,0,1), \ldots, (d-1,0,1)$,



and denote the resulting fan by σ_i^{sub}. Extend this to a subdivision of Σ_{i-1} by connecting each of the newly introduced rays with a fixed third extremal ray of each three-dimensional cone adjacent to σ_i. Let Σ_i be the resulting fan.



Then let $\Sigma' = \Sigma_k$. Note that the construction of Σ' is highly non-canonical: it depends on the way we ordered $\sigma_1, \ldots, \sigma_k$ and on the respective choices of the third extremal rays of the adjacent cones. But since by local non-degeneracy X_0 does not contain any of the zero-dimensional toric orbits of Y, these choices affect neither X' nor p'.

By normality, Y is non-singular except possibly at its one-dimensional and zero-dimensional toric orbits. If $\tilde{\tau}$ is the graph of v restricted to an edge τ of Δ, then Y is non-singular at $O(\tilde{\tau})$: the corresponding cone of $\Sigma(\tilde{\Delta})$ is generated by vectors of the form $(a,b,0), (\alpha,\beta,1)$ with $\gcd(a,b)=1$, hence it is smooth. By local non-degeneracy, we conclude that X cannot have any singularities at X_0, except possibly at the toric orbits $O(\tilde{\tau})$ associated to the lower edges $\tilde{\tau}$ of $\tilde{\Delta}$ that are not of the above form $\mathrm{graph}(v|_\tau)$. These edges exactly correspond to the cones $\sigma_1, \ldots, \sigma_k$.

To prove that $\mathfrak{X}'$ is a regular strongly semi-stable arithmetic surface, it suffices to make a local analysis around these toric orbits. That is, for $i = 1, \ldots, k$ we consider the strict transform of $X \cap \mathrm{Tor}(\sigma_i)$ under the restriction of ρ to $\mathrm{Tor}(\sigma_i^{\mathrm{sub}})$. Let $\tilde{\tau}_i$ be the lower edge of $\tilde{\Delta}$ corresponding to σ_i. As mentioned above, modulo a $\mathbb{Z}$-affine transformation we may assume that σ_i is generated by $(0,0,1)$ and $(d,0,1)$. In fact, we can make the slightly stronger assumption that $\tilde{\tau}_i$ is supported on the y-axis, that the supporting hyperplanes of the adjacent facets $\tilde{\Delta}_\ell$ and $\tilde{\Delta}_m$ contain $(1,0,0)$ resp. $(-1,0,d)$, and that the t-direction remains vertical. By local non-degeneracy, we can write

$$f_t = g_{\tilde{\tau}}\bigl(y^{\pm 1}\bigr) + u \cdot g_{\tilde{\Delta}_\ell}\bigl(y^{\pm 1}, u\bigr) + x \cdot g_{\tilde{\Delta}_m}\bigl(y^{\pm 1}, x\bigr) + t \cdot g_{\tilde{\Delta}}\bigl(y^{\pm 1}, u, x, t\bigr), \quad u = x^{-1}t^d,$$

where

- $g_{\tilde{\tau}} \in \mathbb{C}[y^{\pm 1}]$ is a square-free Laurent polynomial (having $L(\ell, m)$ zeroes in $O(\tilde{\tau})$),

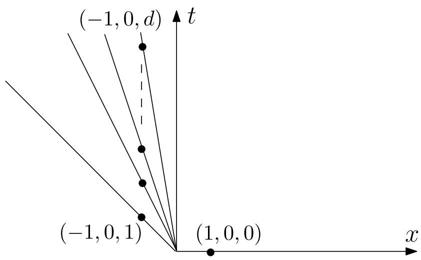
- $g_{\tilde{\tau}} + u \cdot g_{\tilde{\Delta}_\ell} \in \mathbb{C}[y^{\pm 1}, u]$ defines a smooth curve in $\mathbb{T}^2 = O(\tilde{\Delta}_\ell)$ (the completion inside $\mathrm{Tor}(\tilde{\Delta}_\ell)$ of which is exactly $X^{(\ell)}$),
- $g_{\tilde{\tau}} + x \cdot g_{\tilde{\Delta}_m} \in \mathbb{C}[y^{\pm 1}, x]$ defines a smooth curve in $\mathbb{T}^2 = O(\tilde{\Delta}_m)$ (the completion inside $\mathrm{Tor}(\tilde{\Delta}_m)$ of which is exactly $X^{(m)}$).

Then locally, X is defined by $f_t(y^{\pm 1}, u, t, x)$ inside

$$\mathrm{Tor}(\sigma_i) = \mathrm{Spec}\frac{\mathbb{C}[y^{\pm 1}, u, t, x]}{(t^d - ux)} \subset \mathrm{Tor}\big(\Sigma(\tilde{\Delta})\big).$$

We will restrict our analysis of its strict transform in $\mathrm{Tor}(\sigma_i^{\mathrm{sub}})$ to the patch $\mathrm{Tor}(\sigma_{i,1}^{\mathrm{sub}})$, where $\sigma_{i,1}^{\mathrm{sub}}$ is the cone spanned by $(0,0,1)$ and $(1,0,1)$. The dual cone is generated by $(-1,0,1)$ and $(1,0,0)$, hence

$$\mathrm{Tor}\big(\sigma_{i,1}^{\mathrm{sub}}\big) = \mathrm{Spec}\ \mathbb{C}\big[y^{\pm 1}, v, x\big], \quad v = x^{-1}t.$$



Since this dual cone contains the dual cone of σ_i, we have a natural inclusion map which exactly describes our toric resolution ρ:

$$\mathrm{Tor}\big(\sigma_{i,1}^{\mathrm{sub}}\big) \to \mathrm{Tor}(\sigma_i) : (y, v, x) \mapsto \big(y, v^d x^{d-1}, vx, x\big). \tag{1}$$

The strict transform of X under this map is described by

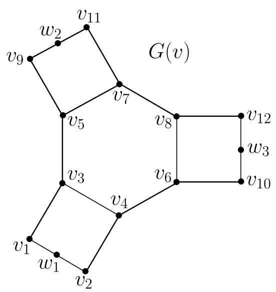
$$f_t'\big(y^{\pm 1}, v, x\big) = g_{\tilde{\tau}}\big(y^{\pm 1}\big) + v^d x^{d-1} \cdot g_{\tilde{\Delta}_\ell}\big(y^{\pm 1}, v^d x^{d-1}\big) + x \cdot g_{\tilde{\Delta}_m}\big(y^{\pm 1}, x\big)$$
$$+ vx \cdot g_{\tilde{\Delta}}\big(y^{\pm 1}, v^d x^{d-1}, x, vx\big).$$

The fiber above $t = 0$ corresponds to taking $v = 0$, in which case we find $g_{\tilde{\tau}}(y^{\pm 1}) + x \cdot g_{\tilde{\Delta}_m}(y^{\pm 1}, x) = 0$ (the curve $X^{(m)}$), and taking $x = 0$, in which case we find $g_{\tilde{\tau}}(y^{\pm 1}) = 0$ ($L(\ell, m)$ exceptional lines). These are easily checked to be non-singular points of the strict transform, and all components intersect each other transversally. By making a similar analysis of the other patches, one concludes that $\mathfrak{X}'$ is indeed a regular, strongly semi-stable arithmetic surface.

Now the generic fibers of $\mathfrak{X}$ and $\mathfrak{X}'$ are isomorphic, because $\rho|_{X'}$ is an isomorphism on $p'^{-1}(V)$ for an open subset V of $\mathbb{P}^1$. On the other hand, the special fiber of $\mathfrak{X}'$ differs from the special fiber of $\mathfrak{X}$. To see how the latter modifies under the above toric resolution, it suffices to have a second look at the above analysis. Suppose that τ_i corresponds to adjacent lower facets $\tilde{\Delta}_\ell$ and $\tilde{\Delta}_m$ of $\tilde{\Delta}$. Then $d(\ell, m) - 1$ new rays are introduced. The introduction of a first new ray separates the curves $X^{(\ell)}$ and $X^{(m)}$, and each intersection point becomes replaced by an exceptional curve intersecting $X^{(\ell)}$ and $X^{(m)}$ transversally. This exceptional curve is contained in the strict transform of X and hence belongs to the special fiber of our new arithmetic surface. All intersections remain transversal. More generally, if $d(\ell, m) - 1$ rays are added, then each intersection point becomes replaced by a chain of $d(\ell, m) - 1$ transversally intersecting exceptional curves. Hence in the dual graph of $\mathfrak{X}$, if an edge corresponds to Δ_ℓ and Δ_m, then it becomes replaced by a linear graph of length $d(\ell, m)$.

3 Further remarks and adjustments

- In the third paragraph of the proof of [2, Theorem 10], one should let $\mathfrak{X}'$ be the regular strongly semi-stable arithmetic surface constructed in Sect. 2 above.
- The graph associated to the example following [2, Conjecture 2] should be replaced, e.g. by



  (we leave the determination of v as an easy exercise)—the according conclusion remains unaffected.
- In the proof of [2, Theorem 11], it is easy to find an upper-convex piece-wise linear function v realizing the given subdivision, such that $v(\Delta \cap \mathbb{Z}^2) \subset \mathbb{Z}$ and all $d(\ell, m)$'s equal 1. Therefore, the proof remains valid.

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