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A plactic algebra of extremal weight crystals and the Cauchy identity for Schur operators

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Abstract We give a new bijective interpretation of the Cauchy identity for Schur operators which is a commutation relation between two formal power series with operator coefficients. We introduce a plactic algebra associated with the Kashiwara's extremal weight crystals over the Kac–Moody algebra of type $A_{+\infty}$, and construct a Knuth type correspondence preserving the plactic relations. This bijection yields the Cauchy identity for Schur operators as a homomorphic image of its associated identity for plactic characters of extremal weight crystals, and also recovers Sagan and Stanley's correspondence for skew tableaux as its restriction.

Keywords Plactic algebra · Crystal · Schur operator

1 Introduction

Let $\Lambda = \Lambda_{\mathbf{x}}$ be the algebra of symmetric functions in a set of formal commuting variables $\mathbf{x} = \{x_1, x_2, \ldots\}$ over $\mathbb{Q}$. We denote by $\mathscr{P}$ the set of partitions and let $s_\lambda(\mathbf{x})$ be the Schur function in $\mathbf{x}$ corresponding to $\lambda \in \mathscr{P}$. Let

$$\mathscr{P}(\mathbf{x}) = \sum_{\lambda \in \mathscr{P}} s_\lambda s_\lambda(\mathbf{x}), \qquad \mathscr{Q}(\mathbf{x}) = \sum_{\lambda \in \mathscr{P}} s_\lambda^\perp s_\lambda(\mathbf{x}) \ \in \mathrm{End}_{\mathbb{Q}}(\Lambda)[[\mathbf{x}]],$$

where s_λ and $s_\lambda^\perp$ are linear operators on Λ induced from the left multiplication by $s_\lambda(\mathbf{x})$ and its adjoint with respect to the Hall inner product on Λ, respectively. One may regard s_λ and $s_\lambda^\perp$ as operators on $\mathbb{Q}\mathscr{P} = \bigoplus_{\lambda \in \mathscr{P}} \mathbb{Q}\lambda$, where λ is identified with

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$s_\lambda(\mathbf{x})$. Moreover s_λ and $s_\lambda^\perp$ can be given as Schur functions in certain locally non-commutative operators on $\mathbb{Q}\mathscr{P}$ called *Schur operators* by Fomin, while $\mathscr{P}(\mathbf{x})$ and $\mathscr{Q}(\mathbf{x})$ can be written as Cauchy products in Schur operators and $\mathbf{x}$ [3, 4].

Let $\mathbf{y} = \{y_1, y_2, \ldots\}$ be another set of formal commuting variables. It is well known that the following commutation relation holds:

$$\mathscr{Q}(\mathbf{y})\mathscr{P}(\mathbf{x}) = \frac{1}{\prod_{i,j}(1 - x_i y_j)}\mathscr{P}(\mathbf{x})\mathscr{Q}(\mathbf{y}) \tag{1.1}$$

called *generalized Cauchy identity* or *Cauchy identity for Schur operators*. Considering both sides as operators with coefficients in $\Lambda_{\mathbf{x}} \otimes \Lambda_{\mathbf{y}}$ and then equating each entry of their matrix forms, we obtain a Cauchy identity for skew Schur functions [16],

$$\sum_\lambda s_{\lambda/\alpha}(\mathbf{x}) s_{\lambda/\beta}(\mathbf{y}) = \frac{1}{\prod_{i,j}(1 - x_i y_j)} \sum_\eta s_{\beta/\eta}(\mathbf{x}) s_{\alpha/\eta}(\mathbf{y}),$$

where α, β are given partitions. A bijective interpretation of the Cauchy identity for skew Schur functions was given by Sagan and Stanley [17], and it was extended to a bijection in a more general framework by Fomin [3] including various analogues of Knuth correspondence.

Recently, a new representation theoretic interpretation of the Cauchy identity for Schur operators was given by the author [11] using the notion of Kashiwara's extremal weight crystals [8] over the quantized enveloping algebra associated with the Kac–Moody algebra of type $A_{+\infty}$, say $\mathfrak{gl}_{>0}$. It is proved that a Schur operator can be realized as a functor of tensoring by an extremal weight crystal element and (1.1) can be understood as a non-commutative character identity corresponding to the decomposition of the crystal graph of the Fock space with infinite positive level, which is an infinite analogue of the level n fermionic Fock space decomposition due to Frenkel [5].

Motivated by a categorification of Schur operators in [11], we give a new combinatorial way to explain both the Cauchy identities for Schur operators and skew Schur functions in terms of a single bijection. More precisely, the main result in this paper is to construct a Knuth type correspondence, which gives a bijective interpretation of the identity (1.1) or its dual form, as the usual Knuth correspondence does for the Cauchy product, and also recovers the Sagan and Stanley's correspondence as its restriction.

Our approach is to define a t-analogue of the plactic algebra $\mathcal{U}(t)$ for $\mathfrak{gl}_{>0}$ generated by u_i and $u_{i^\vee}$ for $i \in \mathbb{N}$ with t an indeterminate, where the subalgebra generated by u_i (resp. $u_{i^\vee}$) ($i \in \mathbb{N}$) is isomorphic to the usual plactic algebra introduced by Lascoux and Schützenberger [14]. We show that $\mathcal{U}(1)$ is isomorphic to the plactic algebra defined by using the notion of crystal equivalence (cf. [15]). Note that each monomial in $\mathcal{U}(1)$ corresponds in general to an element of an extremal weight crystal, which may not be either highest weight or lowest weight crystal.

Now, let $\mathcal{M}_{\mathbb{A},\mathbb{B}}$ be the set of $\mathbb{A} \times \mathbb{B}$ matrices $A = (a_{ij})$ with entries in $\mathbb{Z}_{\geq 0}$ such that $\sum_{i\in\mathbb{A}}\sum_{j\in\mathbb{B}} a_{ij} < \infty$ and $a_{ij} \leq 1$ for $|i| \neq |j|$, where $\mathbb{A}$ and $\mathbb{B}$ are arbitrary $\mathbb{Z}_2$-graded sets and $|\cdot|$ denotes the degree of an element in $\mathbb{A}$ or $\mathbb{B}$. We assume that all the elements in $\mathbb{N}$ and $\mathbb{N}^\vee = \{i^\vee | i \in \mathbb{N}\}$ are of degree 0. By using the usual Knuth map and non-commutative Littlewood–Richardson rule of extremal weight crystals for $\mathfrak{gl}_{>0}$ [11, 12], we construct an explicit bijection (Theorem 5.1);

$$\mathcal{M}_{\mathbb{A},\mathbb{N}} \times \mathcal{M}_{\mathbb{B},\mathbb{N}^\vee} \longrightarrow \mathcal{M}_{\mathbb{A},\mathbb{B}} \times \mathcal{M}_{\mathbb{B},\mathbb{N}^\vee} \times \mathcal{M}_{\mathbb{A},\mathbb{N}},$$

which preserves the weights with respect to $\mathbb{A}$ and $\mathbb{B}$, and the plactic relations of $\mathcal{U}(t)$ for the column words with entries in $\mathbb{N} \cup \mathbb{N}^\vee$ on both sides. As a corollary, we obtain a character identity in locally non-commuting variables $\mathbf{u} = \{u_i, u_{i^\vee} | i \in \mathbb{N}\}$ and commuting variables $\mathbf{x}_\mathbb{A} = \{x_a | a \in \mathbb{A}\}$, $\mathbf{x}_\mathbb{B} = \{x_b | b \in \mathbb{B}\}$ (Corollary 5.2). In particular, when $\mathbb{A} = \mathbb{B} = \mathbb{N}$, this identity recovers (1.1) under a homomorphism sending u_i and $u_{i^\vee}$ to Schur operators on $\mathbb{Q}\mathscr{P}$ and specializing $t = 1$. Moreover, the Knuth correspondence for skew tableaux by Sagan and Stanley can be recovered by restricting the above bijection to the pairs of matrices on the left-hand side whose column words are Littlewood–Richardson words of shape (α, β) with $\alpha, \beta \in \mathscr{P}$ (see Sect. 5.3 for a definition).

The paper is organized as follows. In Sect. 2, we briefly recall necessary background for semistandard tableaux and the Knuth correspondence. In Sect. 3, we recall the notion of rational semistandard tableaux for $\mathfrak{gl}_{>0}$ and their insertion algorithm. In Sect. 4, we introduce a plactic algebra for $\mathfrak{gl}_{>0}$ associated with rational semistandard tableaux. Finally, in Sect. 5, we construct a Knuth type correspondence and its associated non-commutative character identity.

2 Preliminaries

2.1 Semistandard tableaux

Throughout this paper, we assume that $\mathbb{A}$ (or $\mathbb{B}$) is a linearly ordered $\mathbb{Z}_2$-graded set, that is, $\mathbb{A} = \mathbb{A}_0 \sqcup \mathbb{A}_1$, which is at most countable. We usually denote by $<$ a linear ordering on a given linearly ordered $\mathbb{Z}_2$-graded set. For $a \in \mathbb{A}_\epsilon$ ($\epsilon \in \mathbb{Z}_2$), we put $|a| = \epsilon$. By convention, we let $\mathbb{N} = \{1 < 2 < \cdots\}$, and $[n] = \{1 < \cdots < n\}$ for $n \geq 1$, where all the elements are of degree 0.

Let $\mathscr{P}$ denote the set of partitions. We identify a partition $\lambda = (\lambda_i)_{i\geq 1}$ with a Young diagram or a subset $\{(i, j) | 1 \leq j \leq \lambda_i\}$ of $\mathbb{N} \times \mathbb{N}$ following [16]. Let $\ell(\lambda) = |\{i | \lambda_i \neq 0\}|$. We denote by $\lambda' = (\lambda'_i)_{i\geq 1}$ the conjugate partition of λ whose Young diagram is $\{(i, j) | (j, i) \in \lambda\}$. For $\mu \in \mathscr{P}$ with $\lambda \supset \mu$, λ/μ denotes the skew Young diagram.

For a skew Young diagram λ/μ, a tableau T obtained by filling λ/μ with entries in $\mathbb{A}$ is called $\mathbb{A}$-semistandard if (1) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (2) the entries in $\mathbb{A}_0$ (resp. $\mathbb{A}_1$) are strictly increasing in each column (resp. row). We say that λ/μ is the shape of T, and write $\mathrm{sh}(T) = \lambda/\mu$. We denote by $T(i, j)$ the entry of T at $(i, j) \in \lambda/\mu$.

We denote by $SST_\mathbb{A}(\lambda/\mu)$ the set of all $\mathbb{A}$-semistandard tableaux of shape λ/μ. We set $\mathscr{P}_\mathbb{A} = \{\lambda \in \mathscr{P} | SST_\mathbb{A}(\lambda) \neq \emptyset\}$. For example, $\mathscr{P}_\mathbb{A} = \mathscr{P}$ when $\mathbb{A}$ is an infinite set, and $\mathscr{P}_{[n]} = \{\lambda | \ell(\lambda) \leq n\}$.

Let $\mathcal{W}_\mathbb{A}$ be the set of finite words with letters in $\mathbb{A}$. For $T \in SST_\mathbb{A}(\lambda/\mu)$, we denote by $w(T) = w_{\mathrm{col}}(T) \in \mathcal{W}_\mathbb{A}$ the word obtained by reading the entries of T column by column from right to left, and from top to bottom in each column.

Let $P_{\mathbb{A}} = \bigoplus_{a\in\mathbb{A}} \mathbb{Z}\epsilon_a$ be the free abelian group with a basis $\{\epsilon_a | a \in \mathbb{A}\}$ and let $\mathbf{x}_{\mathbb{A}} = \{x_a | a \in \mathbb{A}\}$ be a set of formal commuting variables. For $\lambda = \sum_{a\in\mathbb{A}} \lambda_a \epsilon_a \in P_{\mathbb{A}}$, let $\mathbf{x}_{\mathbb{A}}^{\lambda} = \prod_{a\in\mathbb{A}} x_a^{\lambda_a}$. For $w = w_1 \dots w_r \in \mathcal{W}_{\mathbb{A}}$, we define $\mathrm{wt}_{\mathbb{A}}(w) = \sum_{1\le i\le r} \epsilon_{w_i} \in P_{\mathbb{A}}$. For $T \in SST_{\mathbb{A}}(\lambda/\mu)$, we define $\mathrm{wt}_{\mathbb{A}}(T) = \sum_{(i,j)\in\lambda/\mu} \epsilon_{T(i,j)}$. Let $s_{\lambda/\mu}(\mathbf{x}_{\mathbb{A}}) = \sum_{T\in SST_{\mathbb{A}}(\lambda/\mu)} \mathbf{x}_{\mathbb{A}}^{\mathrm{wt}_{\mathbb{A}}(T)}$, which is the character of $SST_{\mathbb{A}}(\lambda/\mu)$. Note that $s_{\lambda/\mu}(\mathbf{x}_{\mathbb{A}})$ is a usual (skew) Schur function when $\mathbb{A} = \mathbb{N}$.

We will also use the following operations on tableaux.

(1) *dual*: Let $\mathbb{A}^\vee = \{a^\vee | a \in \mathbb{A}\}$ be the linearly ordered $\mathbb{Z}_2$-graded set with $|a^\vee| = |a|$ and $a_1^\vee < a_2^\vee$ for $a_1 > a_2$. For $T \in SST_{\mathbb{A}}(\lambda/\mu)$, we define $T^\vee$ to be the tableau obtained by applying 180°-rotation to T and replacing each entry a in T with $a^\vee$. Then $T^\vee \in SST_{\mathbb{A}^\vee}((\lambda/\mu)^\vee)$, where $(\lambda/\mu)^\vee$ denotes the shape of $T^\vee$. We use the convention that $(a^\vee)^\vee = a$ for $a \in \mathbb{A}$ and hence $(T^\vee)^\vee = T$.
(2) *gluing*: Let $\mathbb{A} * \mathbb{B}$ be the $\mathbb{Z}_2$-graded set $\mathbb{A} \sqcup \mathbb{B}$ with the extended linear ordering given by $a < b$ for $a \in \mathbb{A}$ and $b \in \mathbb{B}$. For $S \in SST_{\mathbb{A}}(\mu)$ and $T \in SST_{\mathbb{B}}(\lambda/\mu)$, we define $S * T \in SST_{\mathbb{A}*\mathbb{B}}(\lambda)$ by $S * T(i,j) = S(i,j)$ for $(i,j) \in \mu$ and $T(i,j)$ for $(i,j) \in \lambda/\mu$.

2.2 Littlewood–Richardson rule

For $a \in \mathbb{A}$ and $T \in SST_{\mathbb{A}}(\lambda)$ with $\lambda \in \mathscr{P}_{\mathbb{A}}$, $a \to T$ (resp. $T \leftarrow a$) denotes the tableau obtained by the Schensted column (resp. row) insertion (see for example, [6, Appendix A.2] and [2] for its super-analogue). For $w = w_1 \dots w_r \in \mathcal{W}_{\mathbb{A}}$, we let $(w \to T) = (w_r \to (\cdots (w_1 \to T) \cdots))$. For $S \in SST_{\mathbb{A}}(\mu)$ and $T \in SST_{\mathbb{A}}(\nu)$ with $\mu, \nu \in \mathscr{P}_{\mathbb{A}}$, we define $(T \to S) = (w(T) \to S)$.

For $\lambda, \mu, \nu \in \mathscr{P}$ with $|\lambda| = |\mu| + |\nu|$, let $\mathbf{LR}^{\lambda}_{\mu\nu}$ be the set of tableaux U in $SST_{\mathbb{N}}(\lambda/\mu)$ such that

(1) $\mathrm{wt}_{\mathbb{N}}(U) = \sum_{i\ge 1} \nu_i \epsilon_i$,
(2) for $1 \le k \le |\nu|$, the number of occurrences of each $i \ge 1$ in $w_1 \dots w_k$ is no less than that of $i+1$ in $w_1 \dots w_k$, where $w(U) = w_1 \dots w_{|\nu|}$.

We call $\mathbf{LR}^{\lambda}_{\mu\nu}$ the set of Littlewood–Richardson tableaux of shape λ/μ with content ν and put $c^{\lambda}_{\mu\nu} = |\mathbf{LR}^{\lambda}_{\mu\nu}|$ [16]. Let us introduce a variation of $\mathbf{LR}^{\lambda}_{\mu\nu}$, which is necessary for our later arguments. We define $\overline{\mathbf{LR}}^{\lambda}_{\mu\nu}$ to be the set of tableaux U in $SST_{-\mathbb{N}}(\lambda/\mu)$ such that

(1) $\mathrm{wt}_{-\mathbb{N}}(U) = \sum_{i\ge 1} \nu_i \epsilon_{-i}$,
(2) for $1 \le k \le |\nu|$, the number of occurrences of each $-i \le -1$ in $w_k \dots w_{|\nu|}$ is no less than that of $-(i+1)$ in $w_k \dots w_{|\nu|}$, where $w(U) = w_1 \dots w_{|\nu|}$.

Note that for $U \in SST_{\mathbb{N}}(\lambda/\mu)$, $U \in \mathbf{LR}^{\lambda}_{\mu\nu}$ if and only if U is Knuth equivalent to $H_\nu \in SST_{\mathbb{N}}(\nu)$, where $H_\nu(i,j) = i$ for $(i,j) \in \nu$ (cf. [6]). Similarly, we have for $U \in SST_{-\mathbb{N}}(\lambda/\mu)$, $U \in \overline{\mathbf{LR}}^{\lambda}_{\mu\nu}$ if and only if U is Knuth equivalent to $L_\nu \in SST_{-\mathbb{N}}(\nu)$, where $L_\nu(i,j) = -\nu'_j + i - 1$ for $(i,j) \in \nu$.

There is also a one-to-one correspondence from the set of $V \in SST_{\mathbb{N}}(\nu)$ such that $(V \to H_\mu) = H_\lambda$ to $\mathbf{LR}^{\lambda}_{\mu\nu}$. Indeed, V corresponds to $\imath(V) \in \mathbf{LR}^{\lambda}_{\mu\nu}$ where the number of k's in the ith row of V is equal to the number of i's in the kth row of $\imath(V)$ for $i, k \ge 1$.

Example 2.1

$$SST_{\mathbb{N}}\big((3,3,2)\big) \ni \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 4 & \end{array} \quad \stackrel{\imath}{\longrightarrow} \quad \begin{array}{ccccc} \bullet & \bullet & \bullet & 1 & 1 \\ \bullet & 1 & 2 & 2 & \\ 2 & 3 & & & \\ 3 & & & & \end{array} \quad \in \mathbf{LR}^{(5,4,2,1)}_{(3,1)\ (3,3,2)}$$

For $S \in SST_{\mathbb{A}}(\mu)$ and $T \in SST_{\mathbb{A}}(\nu)$ with $\mu, \nu \in \mathscr{P}_{\mathbb{A}}$, suppose that $\lambda = \mathrm{sh}(T \to S)$ and $w(T) = w_1 \dots w_r$. Define $(T \to S)_R$ to be the tableau of shape λ/μ such that $\mathrm{sh}(w_1 \dots w_k \to S)/\mathrm{sh}(w_1 \dots w_{k-1} \to S)$ is filled with i when w_k appears in the ith row of T for $1 \le k \le r$. Then the map $(S, T) \mapsto ((T \to S), (T \to S)_R)$ gives a bijection [20]

$$SST_{\mathbb{A}}(\mu) \times SST_{\mathbb{A}}(\nu) \longrightarrow \bigsqcup_{\lambda \in \mathscr{P}_{\mathbb{A}}} SST_{\mathbb{A}}(\lambda) \times \mathbf{LR}^{\lambda}_{\mu\nu}, \tag{2.1}$$

which also implies $s_\mu(\mathbf{x}_{\mathbb{A}}) s_\nu(\mathbf{x}_{\mathbb{A}}) = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda(\mathbf{x}_{\mathbb{A}})$.

2.3 Skew Littlewood–Richardson rule

Let λ/μ be a skew Young diagram. Let U be a tableau of shape λ/μ with entries in $\mathbb{A} \sqcup \mathbb{B}$, satisfying the following conditions;

(S1) $U(i,j) \le U(i',j')$ whenever $U(i,j), U(i',j') \in \mathbb{X}$ for $(i,j), (i',j') \in \lambda/\mu$ with $i \le i'$ and $j \le j'$,
(S2) in each column of U, entries in $\mathbb{X}_0$ increase strictly from top to bottom,
(S3) in each row of U, entries in $\mathbb{X}_1$ increase strictly from left to right,

where $\mathbb{X} = \mathbb{A}$ or $\mathbb{B}$. Suppose that $a \in \mathbb{A}$ and $b \in \mathbb{B}$ are two adjacent entries in U such that b is placed above or to the left of a. Interchanging a and b is called a switching if the resulting tableau still satisfies the conditions (S1), (S2) and (S3).

For $T \in SST_{\mathbb{A}}(\lambda/\mu)$, let U be a tableau obtained from $H_\mu * T$ by applying switching procedures as far as possible (in this case, $\mathbb{B} = \mathbb{N}$). Then $U = \jmath(T) * \jmath(T)_R$ for some $\jmath(T) \in SST_{\mathbb{A}}(\nu)$ and $\jmath(T)_R \in SST_{\mathbb{N}}(\lambda/\nu)$ with $\nu \in \mathscr{P}_{\mathbb{A}}$. Then by [1, Theorem 3.1], the map sending T to $(\jmath(T), \jmath(T)_R)$ gives a bijection

$$SST_{\mathbb{A}}(\lambda/\mu) \longrightarrow \bigsqcup_{\nu \in \mathscr{P}_{\mathbb{A}}} SST_{\mathbb{A}}(\nu) \times \mathbf{LR}^{\lambda}_{\nu\mu}. \tag{2.2}$$

In particular, the map $Q \mapsto \jmath(Q)_R$ restricts to a bijection from $\mathbf{LR}^{\lambda}_{\mu\nu}$ to $\mathbf{LR}^{\lambda}_{\nu\mu}$, and from $\overline{\mathbf{LR}}^{\lambda}_{\mu\nu}$ to $\mathbf{LR}^{\lambda}_{\nu\mu}$ when $\mathbb{A} = \pm\mathbb{N}$, respectively.

2.4 Knuth correspondence

Let $\mathcal{M}_{\mathbb{A},\mathbb{B}}$ be the set of $\mathbb{A} \times \mathbb{B}$ matrices $A = (a_{ij})$ with entries in $\mathbb{Z}_{\ge 0}$ such that $\sum_{i\in\mathbb{A}} \sum_{j\in\mathbb{B}} a_{ij} < \infty$ and $a_{ij} \le 1$ for $|i| \ne |j|$. Let $\Omega_{\mathbb{A},\mathbb{B}}$ be the set of biwords $(\mathbf{i}, \mathbf{j}) \in \mathcal{W}_{\mathbb{A}} \times \mathcal{W}_{\mathbb{B}}$ such that

(1) $\mathbf{i} = i_1 \cdots i_r$ and $\mathbf{j} = j_1 \cdots j_r$ for some $r \geq 0$,
(2) $(i_1, j_1) \leq \cdots \leq (i_r, j_r)$,
(3) $(i_s, j_s) < (i_{s+1}, j_{s+1})$ if $|i_s| \neq |j_s|$ for $1 \leq s < r$,

where for (i, j) and $(k, l) \in \mathbb{A} \times \mathbb{B}$,

$$(i, j) < (k, l) \iff \begin{cases} (i < k) & \text{or,} \\ (i = k,\ |i| = 0, \text{ and } j > l) & \text{or,} \\ (i = k,\ |i| = 1, \text{ and } j < l). \end{cases}$$

There is a bijection from $\Omega_{\mathbb{A},\mathbb{B}}$ to $\mathcal{M}_{\mathbb{A},\mathbb{B}}$, where $(\mathbf{i}, \mathbf{j})$ is mapped to $A(\mathbf{i}, \mathbf{j}) = (a_{ij})$ with $a_{ij} = |\{k | (i_k, j_k) = (i, j)\}|$. Note that the pair of empty words $(\emptyset, \emptyset)$ corresponds to zero matrix.

For $A = A(\mathbf{i}, \mathbf{j}) \in \mathcal{M}_{\mathbb{A},\mathbb{B}}$, we let $P(A) = (\mathbf{j} \to \emptyset)$, where $\emptyset$ is the empty tableau, and let $Q(A)$ be the tableau of the same shape as $P(A)$ such that $\mathrm{sh}(j_1 \ldots j_k \to \emptyset)/\mathrm{sh}(j_1 \ldots j_{k-1} \to \emptyset)$ is filled with i_k for $k \geq 1$. Then the map sending A to $(P(A), Q(A))$ gives a bijection

$$\mathcal{M}_{\mathbb{A},\mathbb{B}} \longrightarrow \bigsqcup_{\lambda \in \mathscr{P}_{\mathbb{A}} \cap \mathscr{P}_{\mathbb{B}}} SST_{\mathbb{B}}(\lambda) \times SST_{\mathbb{A}}(\lambda), \tag{2.3}$$

which is the (super-analogue of) Knuth (or RSK) correspondence [10]. If we define $\mathrm{wt}_{\mathbb{B}}(A) = \mathrm{wt}_{\mathbb{B}}(\mathbf{j})$ and $\mathrm{wt}_{\mathbb{A}}(A) = \mathrm{wt}_{\mathbb{A}}(\mathbf{i})$, then the bijection preserves $\mathrm{wt}_{\mathbb{A}}$ and $\mathrm{wt}_{\mathbb{B}}$. In terms of characters, we obtain the following Cauchy identity:

$$\frac{\prod_{|a| \neq |b|}(1 + x_a x_b)}{\prod_{|a| = |b|}(1 - x_a x_b)} = \sum_{\lambda \in \mathscr{P}_{\mathbb{A}} \cap \mathscr{P}_{\mathbb{B}}} s_\lambda(\mathbf{x}_{\mathbb{B}}) s_\lambda(\mathbf{x}_{\mathbb{A}}),$$

where $a \in \mathbb{A}$ and $b \in \mathbb{B}$.

Similarly, for $A = (a_{ij}) \in \mathcal{M}_{\mathbb{A},\mathbb{B}}$, let $A' = (a'_{ij^\vee})$ be the unique matrix in $\mathcal{M}_{\mathbb{A},\mathbb{B}^\vee}$ such that $a_{ij} = a'_{ij^\vee}$ for $(i, j) \in \mathbb{A} \times \mathbb{B}$. Then the map sending A to $(P(A')^\vee, Q(A'))$ gives a bijection

$$\mathcal{M}_{\mathbb{A},\mathbb{B}} \longrightarrow \bigsqcup_{\lambda \in \mathscr{P}_{\mathbb{A}} \cap \mathscr{P}_{\mathbb{B}}} SST_{\mathbb{B}}(\lambda^\vee) \times SST_{\mathbb{A}}(\lambda). \tag{2.4}$$

Finally, for $\mu \in \mathscr{P}_{\mathbb{A}}$, we have

$$\begin{aligned} & SST_{\mathbb{A}}(\mu) \times \mathcal{M}_{\mathbb{A},\mathbb{B}} \\ & \overset{1-1}{\longleftrightarrow} \bigsqcup_{\nu \in \mathscr{P}_{\mathbb{A}} \cap \mathscr{P}_{\mathbb{B}}} SST_{\mathbb{A}}(\mu) \times SST_{\mathbb{B}}(\nu) \times SST_{\mathbb{A}}(\nu) \qquad \text{by (2.3)} \\ & \overset{1-1}{\longleftrightarrow} \bigsqcup_{\substack{\lambda \in \mathscr{P}_{\mathbb{A}} \\ \mu \subset \lambda}} SST_{\mathbb{A}}(\lambda) \times \Bigl(\bigsqcup_{\nu \in \mathscr{P}_{\mathbb{B}}} SST_{\mathbb{B}}(\nu) \times \mathbf{LR}^{\lambda}_{\nu\mu} \Bigr) \qquad \text{by (2.1)} \\ & \overset{1-1}{\longleftrightarrow} \bigsqcup_{\substack{\lambda \in \mathscr{P}_{\mathbb{A}} \\ \mu \subset \lambda}} SST_{\mathbb{A}}(\lambda) \times SST_{\mathbb{B}}(\lambda/\mu) \qquad \text{by (2.2).} \end{aligned}$$

Hence we obtain a bijection

$$SST_{\mathbb{A}}(\mu) \times \mathcal{M}_{\mathbb{A},\mathbb{B}} \longrightarrow \bigsqcup_{\substack{\lambda\in\mathscr{P}_{\mathbb{A}}\\ \mu\subset\lambda}} SST_{\mathbb{A}}(\lambda) \times SST_{\mathbb{B}}(\lambda/\mu). \tag{2.5}$$

Also by using (2.4), we have a bijection

$$SST_{\mathbb{A}}(\mu) \times \mathcal{M}_{\mathbb{A},\mathbb{B}} \longrightarrow \bigsqcup_{\substack{\lambda\in\mathscr{P}_{\mathbb{A}}\\ \mu\subset\lambda}} SST_{\mathbb{A}}(\lambda) \times SST_{\mathbb{B}}\big((\lambda/\mu)^{\vee}\big). \tag{2.6}$$

3 Rational semistandard tableaux

3.1 Rational semistandard tableaux for $\mathfrak{gl}_{>0}$

For convenience, we let for a skew Young diagram λ/μ,

$$\mathcal{B}_{\lambda/\mu} = SST_{\mathbb{N}}(\lambda/\mu), \qquad \mathcal{B}^{\vee}_{\lambda/\mu} = SST_{\mathbb{N}^{\vee}}\big((\lambda/\mu)^{\vee}\big).$$

Definition 3.1 For $\mu, \nu \in \mathscr{P}$, we define $\mathcal{B}_{\mu,\nu}$ to be the set of bitableaux (S, T) such that

(1) $(S, T) \in \mathcal{B}_{\mu} \times \mathcal{B}^{\vee}_{\nu}$,
(2) $|\{i|S(i, 1) \le k\}| + |\{i|T^{\vee}(i, 1) \le k\}| \le k$ for $k \ge 1$.

For convenience, we identify $\mathcal{B}_{\mu,\emptyset}$ and $\mathcal{B}_{\emptyset,\nu}$ with $\mathcal{B}_{\mu}$ and $\mathcal{B}^{\vee}_{\nu}$, respectively.

Example 3.2

$$\left(\begin{matrix} 1 & 1 & 3 \\ 2 & 3 & \\ 4 & & \end{matrix}\ ,\ \begin{matrix} & 7^{\vee} \\ 5^{\vee} & 5^{\vee} \\ 4^{\vee} & 3^{\vee} \end{matrix}\right) \in \mathcal{B}_{(3,2,1),(2,2,1)}.$$

Remark 3.3 Let $\mathfrak{gl}_{>0}$ be the general linear Lie algebra spanned by $\mathbb{N} \times \mathbb{N}$ complex matrices of finite support. Then $\mathcal{B}_{\mu,\nu}$ parameterizes a basis of an extremal weight module over the quantized universal enveloping algebra $U_q(\mathfrak{gl}_{>0})$ [11]. Recall that $\mathcal{B}_{\mu,\nu} \cap (SST_{[n]}(\mu) \times SST_{[n]^{\vee}}(\nu^{\vee}))$ $(n \ge 2)$ is a set of rational tableaux for $\mathfrak{gl}_n$ introduced by Stembridge, which parameterizes a basis of a finite dimensional complex irreducible representation of $\mathfrak{gl}_n$ [18].

Let us review an insertion algorithm for $\mathcal{B}_{\mu,\nu}$ [11], which is an infinite analogue of those for rational semistandard tableaux for $\mathfrak{gl}_n$ [18, 19]. For $a \in \mathbb{N}$ and $(S, T) \in \mathcal{B}_{\mu,\nu}$, we define $a \to (S, T)$ in the following way.

Suppose first that S is the empty tableau and T is a single column tableau. Let (T', a') be the pair obtained as follows.

(1) If T contains $a^\vee, (a+1)^\vee, \dots, (b-1)^\vee$ as its entries but not $b^\vee$, then T' is the tableau obtained from T by replacing $a^\vee, (a+1)^\vee, \dots, (b-1)^\vee$ with $(a+1)^\vee, (a+2)^\vee, \dots, b^\vee$, and put $a' = b$.
(2) If T does not contain $a^\vee$, then leave T unchanged and put $a' = a$.

Now, we suppose that S and T are arbitrary.

(1) Apply the above process to the leftmost column of T with a.
(2) Repeat (1) with a' and the next column to the right.
(3) Continue this process to the rightmost column of T to get a tableau T' and a''.
(4) Define

$$\bigl(a \to (S, T)\bigr) = \bigl((a'' \to S), T'\bigr).$$

Then $a \to (S, T) \in \mathcal{B}_{\sigma,\nu}$ for some $\sigma \in \mathscr{P}$ with $|\sigma/\mu| = 1$. For $w = w_1 \dots w_r \in \mathcal{W}_{\mathbb{N}}$, we let $(w \to (S, T)) = (w_r \to (\cdots (w_1 \to (S, T)) \cdots))$.

Next, we define $(S, T) \leftarrow a^\vee$ to be the pair (S', T') obtained in the following way:

(1) If the pair $(S, (T^\vee \leftarrow a)^\vee)$ satisfies the condition (2) in Definition 3.1, then put $S' = S$ and $T' = (T^\vee \leftarrow a)^\vee$.
(2) Otherwise, choose the smallest k such that a_k is bumped out of the kth row in the row insertion of a into $T^\vee$ and the insertion of a_k into the $(k+1)$-st row violates the condition (2) in Definition 3.1.
(2-a) Stop the row insertion of a into $T^\vee$ when a_k is bumped out, and let T' be the resulting tableau after taking $\vee$.
(2-b) Remove a_k in the leftmost column of S, which necessarily exists, and then play the jeu de taquin (see for example [6, Sect. 1.2]) to obtain a tableau S'.

In this case, $(S, T) \leftarrow a^\vee \in \mathcal{B}_{\sigma,\tau}$, where either (1) $|\mu/\sigma| = 1$ and $\tau = \nu$, or (2) $\sigma = \mu$ and $|\tau/\nu| = 1$. For $w = w_1 \dots w_r \in \mathcal{W}_{\mathbb{N}^\vee}$, we let $((S, T) \leftarrow w) = ((\cdots((S, T) \leftarrow w_1) \cdots) \leftarrow w_r)$.

3.2 Non-commutative Littlewood–Richardson rule

Let us recall the Littlewood–Richardson rule for $\mathcal{B}_{\mu,\nu}$ (see [11, Proposition 4.9] for more details).

Let $\mu, \nu \in \mathscr{P}$ be given. For $(S, T) \in \mathcal{B}_\nu^\vee \times \mathcal{B}_\mu$, consider $(w(T) \to (\emptyset, S))$. Suppose that $w(T) = w_1 \dots w_r$ and $(w_1 \dots w_k \to (\emptyset, S)) \in \mathcal{B}_{\mu^{(k)},\nu}$ for $1 \le k \le r$. Let $(i_k, j_k) \in \mu$ correspond to w_k in T $(1 \le k \le r)$. Then $\mu^{(k)} = \mu^{(k-1)} \cup \{(i_k, \mu^{(k-1)}_{i_k} + 1)\}$ (adding a box or dot in the i_kth row of the Young diagram $\mu^{(k-1)}$), where $\mu^{(0)} = \emptyset$. In particular, $\mu^{(r)} = \mu$. Hence, the map sending (S, T) to $(w(T) \to (\emptyset, S))$ gives a bijection [11, Corollary 4.11]

$$\mathcal{B}_\nu^\vee \times \mathcal{B}_\mu \longrightarrow \mathcal{B}_{\mu,\nu}. \tag{3.1}$$

Next, for $(S, T) \in \mathcal{B}_\mu \times \mathcal{B}_\nu^\vee$, consider $((S, \emptyset) \leftarrow w(T))$. Suppose that $w(T) = w_1 \dots w_r$ and $((S, \emptyset) \leftarrow w_1 \dots w_k) \in \mathcal{B}_{\mu^{(k)},\nu^{(k)}}$ for $1 \le k \le r$. Let $(i_k, j_k) \in \nu$ correspond to w_k in T $(1 \le k \le r)$. Define U to be the tableau of shape ν such that for

$1 \le k \le r$

$$U(i_k, j_k) = \begin{cases} i, & \text{if } \mu^{(k)} = \mu^{(k-1)} \setminus \{(i, \mu_i^{(k-1)})\}, \\ -i, & \text{if } \nu^{(k)} = \nu^{(k-1)} \cup \{(i, \nu_i^{(k-1)} + 1)\}, \end{cases}$$

where $\mu^{(0)} = \mu$ and $\nu^{(0)} = \emptyset$. Then $U = U_+ * U_-$, where $U_+ \in SST_{\mathbb{N}}(\lambda)$ and $U_- \in SST_{-\mathbb{N}}(\nu/\lambda)$ for some $\lambda \subset \nu$. Let $\sigma = \mu^{(r)}$ and $\tau = \nu^{(r)}$. We have $\imath(U_+) \in \mathbf{LR}^{\mu}_{\sigma\lambda}$ and $U_- \in \overline{\mathbf{LR}}^{\nu}_{\lambda\tau}$, hence $\jmath(U_-)_R \in \mathbf{LR}^{\nu}_{\tau\lambda}$ (see Sects. 2.2 and 2.3). Therefore, we have a bijection [12, Proposition 4.3]

$$\mathcal{B}_\mu \times \mathcal{B}^\vee_\nu \longrightarrow \bigsqcup_{\lambda,\sigma,\tau} \mathcal{B}_{\sigma,\tau} \times \mathbf{LR}^{\mu}_{\sigma\lambda} \times \mathbf{LR}^{\nu}_{\tau\lambda}, \tag{3.2}$$

where (S, T) is mapped to $(((S, \emptyset) \leftarrow w(T)), \imath(U_+), \jmath(U_-)_R)$. Now, we have

$$\begin{aligned} &\bigsqcup_{\lambda,\sigma,\tau} \mathcal{B}_{\sigma,\tau} \times \mathbf{LR}^{\mu}_{\sigma\lambda} \times \mathbf{LR}^{\nu}_{\tau\lambda} \\ &\quad \overset{1-1}{\longleftrightarrow} \bigsqcup_{\lambda,\sigma,\tau} \mathcal{B}^\vee_\tau \times \mathcal{B}_\sigma \times \mathbf{LR}^{\mu}_{\sigma\lambda} \times \mathbf{LR}^{\nu}_{\tau\lambda} \quad \text{by (3.1)} \\ &\quad \overset{1-1}{\longleftrightarrow} \bigsqcup_{\lambda,\sigma,\tau} \mathcal{B}^\vee_\tau \times \mathbf{LR}^{\nu}_{\tau\lambda} \times \mathcal{B}_\sigma \times \mathbf{LR}^{\mu}_{\sigma\lambda} \\ &\quad \overset{1-1}{\longleftrightarrow} \bigsqcup_{\lambda\subset\mu,\nu} \mathcal{B}^\vee_{\nu/\lambda} \times \mathcal{B}_{\mu/\lambda} \quad \text{by (2.2).} \end{aligned}$$

Hence, we obtain the following bijection [12, Proposition 5.1]:

$$\mathcal{B}_\mu \times \mathcal{B}^\vee_\nu \longrightarrow \bigsqcup_{\lambda\subset\mu,\nu} \mathcal{B}^\vee_{\nu/\lambda} \times \mathcal{B}_{\mu/\lambda}. \tag{3.3}$$

4 Plactic algebra

4.1 A plactic algebra for $\mathfrak{gl}_{>0}$

Let t be an indeterminate. Define $\mathcal{U}(t)$ to be an associative $\mathbb{Q}[t, t^{-1}]$-algebra with unity generated by u_i and $u_{i^\vee}$ ($i \in \mathbb{N}$) subject to the following relations:

$$\begin{aligned} &u_i u_j u_k = u_i u_k u_j, \qquad u_{k^\vee} u_{j^\vee} u_{i^\vee} = u_{j^\vee} u_{k^\vee} u_{i^\vee} \quad (j \le i < k), \\ &u_i u_j u_k = u_j u_i u_k, \qquad u_{k^\vee} u_{j^\vee} u_{i^\vee} = u_{k^\vee} u_{i^\vee} u_{j^\vee} \quad (j < k \le i), \\ &u_{i+1} u_{(i+1)^\vee} = u_{i^\vee} u_i \qquad (i \ge 1), \qquad u_1 u_{1^\vee} = t, \\ &u_i u_{j^\vee} = u_{j^\vee} u_i \quad (i \ne j). \end{aligned} \tag{4.1}$$

Let $\mathcal{U}(t)_+$ (resp. $\mathcal{U}(t)_-$) be the subalgebra of $\mathcal{U}(t)$ generated by u_i (resp. $u_{i^\vee}$) for $i \in \mathbb{N}$. Then $\mathcal{U}(t)_\pm$ is isomorphic to the usual plactic algebra for $\mathfrak{gl}_{>0}$ over $\mathbb{Q}[t, t^{-1}]$ [14], where the first two relations in (4.1) are Knuth relations.

Let $\mathcal{W}$ be the set of finite words with letters in $\mathbb{N}\cup\mathbb{N}^\vee$. For $w=w_1\cdots w_r\in\mathcal{W}$, put $u_w=u_{w_1}\cdots u_{w_r}\in\mathcal{U}(t)$, where we assume that $u_\emptyset=1$ for the empty word $\emptyset$. It is well known that if $w\in\mathcal{W}_\mathbb{N}$ (resp. $\mathcal{W}_{\mathbb{N}^\vee}$), then there exists a unique $T\in\mathcal{B}_\mu$ (resp. $\mathcal{B}_\mu^\vee$) such that $u_w=u_{w(T)}$.

For a skew Young diagram λ/μ and $T\in\mathcal{B}_{\lambda/\mu}$ or $\mathcal{B}^\vee_{\lambda/\mu}$, we let $u_T=u_{w(T)}$, and for $\mu,\nu\in\mathscr{P}$ and $(S,T)\in\mathcal{B}_{\mu,\nu}$, we let $u_{(S,T)}=u_Su_T$.

Lemma 4.1 *For $p,q\geq 1$, let $S\in\mathcal{B}_{(1^p)}$ and $T\in\mathcal{B}^\vee_{(1^q)}$ be given and let $(S',T')=(w(S)\to(\emptyset,T))\in\mathcal{B}_{(1^p),(1^q)}$. Then $u_Tu_S=u_{(S',T')}$.*

Proof It is straightforward to check it from (3.1) and (4.1). □

Lemma 4.2 *For $p,q\geq 1$, let $S\in\mathcal{B}_{(1^p)}$ and $T\in\mathcal{B}^\vee_{(1^q)}$ be given with $w(S)=w_1^+\ldots w_p^+$ and $w(T)=w_q^-\ldots w_1^-$. Suppose that there exists $k\geq 1$ such that $|\{i|w_i^+\leq k\}|+|\{j|w_j^-\geq k^\vee\}|>k$. If $w_i^+=k$ and $w_j^-=k^\vee$ for some i and j, and $(S,T')\in\mathcal{B}_{(1^p),(1^{q-1})}$, where T' is obtained from T by removing $k^\vee$, then*

$$u_Su_T=tu_{w_1^+\ldots\widehat{w_i^+}\ldots w_p^+}u_{w_q^-\ldots\widehat{w_j^-}\ldots w_1^-}.$$

Proof We use induction on $p+q$. If $p+q=2$, then $k=1$ and $u_{w_1^+}u_{w_1^-}=u_1u_{1^\vee}=t$.

Suppose that $p+q>2$. First we assume that $i<p$ or $j<q$. Then

$$\begin{aligned}w_{i+1}^+\ldots w_p^+&=(k+a_1)\ldots(k+a_{p-i}),\\ w_q^-\ldots w_{j+1}^-&=(k+b_{q-j})^\vee\ldots(k+b_1)^\vee,\end{aligned}$$

for some $1\leq a_1<\cdots<a_{p-i}$ and $1\leq b_1<\cdots<b_{q-j}$. Also it follows from our hypothesis that $i+j=k+1$, and hence we can choose $(\overline{S},\overline{T})\in\mathcal{B}_{(1^{p-i}),(1^{q-j})}$ such that $w(\overline{S})=a_1\ldots a_{p-i}$ and $w(\overline{T})=b_{q-j}^\vee\ldots b_1^\vee$. By Lemma 4.1, there exists $(\overline{T}',\overline{S}')\in\mathcal{B}^\vee_{(1^{q-j})}\times\mathcal{B}_{(1^{p-i})}$ with $w(\overline{S}')=a_1'\ldots a_{p-i}'$ and $w(\overline{T}')=(b_{q-j}')^\vee\ldots(b_1')^\vee$ such that $u_{\overline{T}'}u_{\overline{S}'}=u_{(\overline{S},\overline{T})}=u_{\overline{S}}u_{\overline{T}}$. This implies that

$$u_{w_{i+1}^+\ldots w_p^+}u_{w_q^-\ldots w_{j+1}^-}=u_{(k+b_{q-j}')^\vee}\ldots u_{(k+b_1')^\vee}u_{(k+a_1')}\ldots u_{(k+a_{p-i}')}.$$

Since $w_i^+=k<k+b_1'$ and $w_j^-=k^\vee>(k+a_1')^\vee$, we have

$$u_Su_T=u_{(k+b_{q-j}')^\vee\ldots(k+b_1')^\vee}(u_{w_1^+}\ldots u_{w_i^+}u_{w_j^-}\ldots u_{w_1^-})u_{(k+a_1')\ldots(k+a_{p-i}')},$$

and by induction hypothesis,

$$\begin{aligned}u_Su_T&=tu_{(k+b_{q-j}')^\vee\ldots(k+b_1')^\vee}\big(u_{w_1^+}\ldots\widehat{u_{w_i^+}}\widehat{u_{w_j^-}}\ldots u_{w_1^-}\big)u_{(k+a_1')\ldots(k+a_{p-i}')}\\ &=tu_{w_1^+}\ldots\widehat{u_{w_i^+}}\ldots u_{w_p^+}u_{w_q^-}\ldots\widehat{u_{w_j^-}}\ldots u_{w_1^-}.\end{aligned}$$

Next, we assume that $i=p$ and $j=q$, that is, $w_p^+=k$ and $w_q^-=k^\vee$. Note that $p+q=k+1$. If $w_{p-1}^+\neq k-1$ and $w_{q-1}^-\neq (k-1)^\vee$, then

$$\left|\left\{i|w_i^+\le k-2\right\}\right|+\left|\left\{j|w_j^-\ge (k-2)^\vee\right\}\right|=p+q-2=k-1>k-2,$$

which contradicts the fact that $(S,T')\in\mathcal{B}_{(1^p),(1^{q-1})}$. So we also assume that either $w_{p-1}^+=k-1$ or $w_{q-1}^-=(k-1)^\vee$.

Case 1. Suppose that $w_{p-1}^+\neq k-1$ and $w_{q-1}^-=(k-1)^\vee$. We have

$$\begin{aligned}
u_Su_T&=u_{w_1^+}\dots u_{w_{p-1}^+}u_ku_{k^\vee}u_{w_{q-1}^-}\dots u_{w_1^-}\\
&=u_{w_1^+}\dots u_{w_{p-1}^+}u_{(k-1)^\vee}u_{k-1}u_{w_{q-1}^-}\dots u_{w_1^-}\\
&=u_{(k-1)^\vee}u_{w_1^+}\dots u_{w_{p-1}^+}u_{k-1}u_{(k-1)^\vee}u_{w_{q-2}^-}\dots u_{w_1^-}\\
&=tu_{(k-1)^\vee}u_{w_1^+}\dots u_{w_{p-1}^+}u_{w_{q-2}^-}\dots u_{w_1^-}\\
&=tu_{w_1^+}\dots u_{w_{p-1}^+}u_{(k-1)^\vee}u_{w_{q-2}^-}\dots u_{w_1^-}\\
&=tu_{w_1^+}\dots u_{w_{p-1}^+}u_{w_{q-1}^-}\dots u_{w_1^-},
\end{aligned}$$

where we use induction hypothesis in the third line.

Case 2. Suppose that $w_{p-1}^+=k-1$ and $w_{q-1}^-\neq(k-1)^\vee$. By almost the same argument as in Case 1, we have

$$u_Su_T=tu_{w_1^+}\dots u_{w_{p-1}^+}u_{w_{q-1}^-}\dots u_{w_1^-}.$$

Case 3. Suppose that $w_{p-1}^+=k-1$ and $w_{q-1}^-=(k-1)^\vee$. We have

$$\begin{aligned}
u_Su_T&=u_{w_1^+}\dots u_{w_{p-1}^+}u_ku_{k^\vee}u_{w_{q-1}^-}\dots u_{w_1^-}\\
&=u_{w_1^+}\dots u_{w_{p-1}^+}u_{(k-1)^\vee}u_{k-1}u_{w_{q-1}^-}\dots u_{w_1^-}\\
&=u_{(k-a)^\vee}u_{v_1}\dots u_{v_{p-2}}u_{k-2}u_{k-1}u_{w_{q-1}^-}\dots u_{w_1^-},
\end{aligned}$$

for some $1\le a<k$ and $1\le v_1<\dots<v_{p-2}<k-2$. So by induction hypothesis,

$$\begin{aligned}
u_Su_T&=u_{(k-a)^\vee}u_{v_1}\dots u_{v_{p-2}}u_{k-2}u_{k-1}u_{(k-1)^\vee}u_{w_{q-2}^-}\dots u_{w_1^-}\\
&=tu_{(k-a)^\vee}u_{v_1}\dots u_{v_{p-2}}u_{k-2}u_{w_{q-2}^-}\dots u_{w_1^-}\\
&=tu_{w_1^+}\dots u_{w_{p-2}^+}u_{k-1}u_{(k-1)^\vee}u_{w_{q-2}^-}\dots u_{w_1^-}\\
&=tu_{w_1^+}\dots u_{w_{p-1}^+}u_{w_{q-1}^-}\dots u_{w_1^-}.
\end{aligned}$$

This completes the induction. □

Lemma 4.3 *Let $\mu, \nu \in \mathscr{P}$ be given. For $a \in \mathbb{N}$ and $(S,T) \in \mathcal{B}_{\mu,\nu}$,*

(1) $u_{(S,T)} u_a = u_{(a \to (S,T))}$,
(2) $u_{(S,T)} u_{a^\vee} = t^\epsilon u_{((S,T) \leftarrow a^\vee)}$, *where* $\epsilon = 0, 1$.

Proof We keep the notations in Sect. 3.1. Consider $(a \to (S,T)) = (S', T')$. Let (T', a') be the pair obtained by the first step in the definition of $a \to (S,T)$. It is straightforward to check that $u_T u_a = u_{a'} u_{T'}$. Since $S' = (a' \to S)$, which is a usual column insertion, we have $u_{S'} = u_S u_{a'}$. Hence

$$u_{(S,T)} u_a = u_S u_T u_a = u_S u_{a'} u_{T'} = u_{S'} u_{T'} = u_{(a \to (S,T))}.$$

Next, consider $((S,T) \leftarrow a^\vee) = (S', T')$. If the pair $(S, (T^\vee \leftarrow a)^\vee)$ satisfies the condition (2) in Definition 3.1, then $(S', T') = (S, (T^\vee \leftarrow a)^\vee)$, which implies that $u_S = u_{S'}$ and $u_T u_{a^\vee} = u_{T'}$. Hence, $u_{(S,T)} u_{a^\vee} = u_{((S,T) \leftarrow a^\vee)}$.

Suppose that there exists j such that $a_j = k$ is bumped out of the $(j-1)$-st row in the row insertion of a into $T^\vee$ and the insertion of a_j into the jth row violates the condition (2) in Definition 3.1.

Let $T'' = (T^\vee \leftarrow a)^\vee$. Suppose that $w(S) = \widetilde{w}^+ w^+$ and $w(T'') = w^- \widetilde{w}^-$, where $w^+ = w_1^+ \dots w_p^+$ is the subword corresponding to the leftmost column of S and $w^- = w_q^- \dots w_1^-$ is the subword corresponding to the rightmost column of T'' reading from top to bottom. Note that $w_j^- = k^\vee$. Suppose that $w_i^+ = k$. By Lemma 4.2, we have

$$u_{w^+} u_{w^-} = t u_{w_1^+ \dots \widehat{w_i^+} \dots w_p^+} u_{w_q^- \dots \widehat{w_j^-} \dots w_1^-}.$$

Note that $w(T') = w_q^- \dots \widehat{w_j^-} \dots w_1^- \widetilde{w}^-$. Recalling that S' is obtained by playing the jeu de taquin after removing k in the first column of S, it follows that $u_{S'} = u_{\widetilde{w}^+} u_{w_1^+ \dots \widehat{w_i^+} \dots w_p^+}$. Therefore,

$$\begin{aligned} u_{(S,T)} u_{a^\vee} &= u_S u_T u_{a^\vee} = u_S u_{T''} \\ &= u_{\widetilde{w}^+} u_{w^+} u_{w^-} u_{\widetilde{w}^-} \\ &= t u_{\widetilde{w}^+} u_{w_1^+ \dots \widehat{w_i^+} \dots w_p^+} u_{w_q^- \dots \widehat{w_j^-} \dots w_1^-} u_{\widetilde{w}^-} \\ &= t u_{S'} u_{T'} = t u_{((S,T) \leftarrow a^\vee)}. \end{aligned}$$ □

Now, we obtain the following immediately.

Proposition 4.4 *For $w = w_1 \dots w_r \in \mathcal{W}$, there exists $(S,T) \in \mathcal{B}_{\mu,\nu}$ such that $u_w = t^\epsilon u_{(S,T)}$ where $\epsilon = r - |\mu| - |\nu|$.*

Corollary 4.5 *The set $\{u_{(S,T)} \,|\, (S,T) \in \mathcal{B}_{\mu,\nu},\ \mu, \nu \in \mathscr{P}\}$ spans $\mathcal{U}(t)$ over $\mathbb{Q}[t, t^{-1}]$.*

The uniqueness of (S,T) in Proposition 4.4 and the linear independence of the spanning set in Corollary 4.5 will be proved in Sect. 4.2.

4.2 Crystal equivalence

Let $P = P_{\mathbb{N}} = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}\epsilon_i$ be the weight lattice of $\mathfrak{gl}_{>0}$ with a symmetric bilinear form $(\cdot,\cdot)$ given by $(\epsilon_i, \epsilon_j) = \delta_{ij}$. Let $\{\alpha_i = \epsilon_i - \epsilon_{i+1} | i \in \mathbb{N}\}$ be the set of simple roots of $\mathfrak{gl}_{>0}$. A (normal) $\mathfrak{gl}_{>0}$-crystal is a set B together with the maps $\mathrm{wt}: B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z}_{\geq 0}$ and $\widetilde{e}_i, \widetilde{f}_i : B \to B \cup \{\mathbf{0}\}$ $(i \in \mathbb{N})$ such that for $b, b' \in B$

(1) $\varphi_i(b) = (\mathrm{wt}(b), \alpha_i) + \varepsilon_i(b)$,
(2) $\varepsilon_i(b) = \max\{k | \widetilde{e}_i^k b \neq \mathbf{0}\}$ and $\varphi_i(b) = \max\{k | \widetilde{f}_i^k b \neq \mathbf{0}\}$,
(3) $\mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $\widetilde{e}_i b \neq \mathbf{0}$, and $\mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $\widetilde{f}_i b \neq \mathbf{0}$,
(4) $\widetilde{f}_i b = b'$ if and only if $b = \widetilde{e}_i b'$,

where $\mathbf{0}$ is a formal symbol (cf. [7]). Note that B is equipped with a colored oriented graph structure, where $b \stackrel{i}{\to} b'$ if and only if $b' = \widetilde{f}_i b$ for $b, b' \in B$ and $i \in \mathbb{N}$. The dual crystal $B^\vee$ of B is defined to be the set $\{b^\vee | b \in B\}$ with $\mathrm{wt}(b^\vee) = -\mathrm{wt}(b)$, $\widetilde{e}_i(b^\vee) = (\widetilde{f}_i b)^\vee$ and $\widetilde{f}_i(b^\vee) = (\widetilde{e}_i b)^\vee$ for $b \in B$ and $i \in \mathbb{N}$. We assume that $\mathbf{0}^\vee = \mathbf{0}$. Note that $\mathbb{N}$ is naturally equipped with a $\mathfrak{gl}_{>0}$-crystal structure;

$$1 \stackrel{1}{\longrightarrow} 2 \stackrel{2}{\longrightarrow} 3 \stackrel{3}{\longrightarrow} \cdots$$

with $\mathrm{wt}(i) = \epsilon_i$ $(i \in \mathbb{N})$, while $\mathbb{N}^\vee$ is its dual.

For $\mathfrak{gl}_{>0}$-crystals B_1 and B_2, a tensor product $B_1 \otimes B_2$ is defined to be $B_1 \times B_2$ as a set with elements denoted by $b_1 \otimes b_2$, where

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2),$$

$$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \widetilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$

for $i \in \mathbb{N}$ and $b_1 \otimes b_2 \in B_1 \otimes B_2$. Here we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$. For example, $\mathcal{W}$ is a $\mathfrak{gl}_{>0}$-crystal, where each word $w_1 \ldots w_r$ is identified with $w_1 \otimes \cdots \otimes w_r$ in a mixed r-tensor product of $\mathbb{N}$ and $\mathbb{N}^\vee$.

For $b_i \in B_i$ $(i = 1, 2)$, we say that b_1 is equivalent to b_2, and write $b_1 \equiv b_2$ if $\mathrm{wt}(b_1) = \mathrm{wt}(b_2)$ and they generate the same $\mathbb{N}$-colored graph with respect to $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in \mathbb{N})$. We usually call $\equiv$ the crystal equivalence.

For a skew Young diagram λ/μ, $\mathcal{B}_{\lambda/\mu}$ has a well-defined $\mathfrak{gl}_{>0}$-crystal structure such that $\widetilde{x}_i(S) = S'$ if $\widetilde{x}_i w(S) \neq \mathbf{0}$ $(i \in \mathbb{N}, x = e, f)$, where S' is the unique tableau in $\mathcal{B}_{\lambda/\mu}$ with $w(S') = \widetilde{x}_i w(S)$, and $\widetilde{x}_i(S) = \mathbf{0}$ otherwise [9]. We regard $\mathcal{B}^\vee_{\lambda/\mu}$ as the dual of $\mathcal{B}_{\lambda/\mu}$. Moreover, for $\mu, \nu \in \mathscr{P}$, $\mathcal{B}_{\mu,\nu} \cup \{\mathbf{0}\} \subset (\mathcal{B}_\mu \otimes \mathcal{B}^\vee_\nu) \cup \{\mathbf{0}\}$ is invariant under $\widetilde{e}_i$, $\widetilde{f}_i$ $(i \in \mathbb{N})$, and hence a $\mathfrak{gl}_{>0}$-crystal, which is connected as a graph [11, Proposition 3.4]. It is shown in [11, Theorem 3.5] that $\mathcal{B}_{\mu,\nu}$ is an extremal weight crystal which was introduced by Kashiwara [8].

Let $\mathcal{W}$ be an associative $\mathbb{Q}$-algebra with unity generated by the symbol $[w]$ ($w \in \mathcal{W}$) subject to the relations;

$$[w][w'] = [ww'],$$
$$[w] = [w'], \quad \text{if } w \equiv w',$$

for $w, w' \in \mathcal{W}$. Note that $[\emptyset] = 1$ is the unity in $\mathcal{W}$, where $\emptyset$ is the empty word.

Lemma 4.6 *The set*

$$\mathscr{B} = \left\{ \left[w(S)w(T) \right] \middle| (S,T) \in \mathcal{B}_{\mu,\nu},\ \mu,\nu \in \mathscr{P} \right\}$$

is a $\mathbb{Q}$-basis of $\mathcal{W}$.

Proof For $a \in \mathbb{N}$ and $(S,T) \in \mathcal{B}_{\mu,\nu}$, it is shown in [11, Lemma 4.4] that

$$\big(a \to (S,T)\big) \equiv (S,T) \otimes a, \qquad \big((S,T) \leftarrow a^\vee\big) \equiv (S,T) \otimes a^\vee. \tag{4.2}$$

This implies that for $w \in \mathcal{W}$, $[w] = [w(S)w(T)]$ for some $(S,T) \in \mathcal{B}_{\mu,\nu}$, and hence $\mathcal{W}$ is spanned by $\mathscr{B}$.

Now, suppose that

$$\sum_{i=1}^n c_i \big[w\big(S^{(i)}\big) w\big(T^{(i)}\big) \big] = 0 \tag{4.3}$$

for some $c_i \in \mathbb{Q}$ and $(S^{(i)}, T^{(i)}) \in \mathcal{B}_{\mu^{(i)},\nu^{(i)}}$ $(1 \le i \le n)$. Since $(S,T) \equiv (S',T')$ implies $(S,T) = (S',T')$ for $(S,T) \in \mathcal{B}_{\mu,\nu}$ and $(S',T') \in \mathcal{B}_{\sigma,\tau}$ [11, Lemma 5.1], we assume that $(S^{(i)}, T^{(i)})$'s are mutually different.

We use induction on n to show that $c_i = 0$ for $1 \le i \le n$. It is clear when $n = 1$. Suppose that $n \ge 2$.

We claim that there exist $j_1, \ldots, j_r$ such that $\widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r}(S^{(1)}, T^{(1)}) = \mathbf{0}$ but $\widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r}(S^{(i)}, T^{(i)}) \neq \mathbf{0}$ for some $2 \le i \le n$, where x denotes e or f for each j_k.

Consider $(S^{(i)}, T^{(i)})$ $(i = 1,2)$ and suppose that $\widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r}(S^{(1)}, T^{(1)}) \neq \mathbf{0}$ if and only if $\widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r}(S^{(2)}, T^{(2)}) \neq \mathbf{0}$ for all $j_1, \ldots, j_r$. Then by applying suitable $\widetilde{e}_k$'s, we may assume that $S^{(i)} = H_{\mu^{(i)}}$ and $(T^{(i)})^\vee(k,l) \ge p$ for $(k,l) \in \nu^{(i)}$, where $p \gg \ell(\mu^{(i)})$ $(i = 1,2)$. Now, $\widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r} H_{\mu^{(1)}} \neq \mathbf{0}$ if and only if $\widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r} H_{\mu^{(2)}} \neq \mathbf{0}$ for all $1 \le j_1, \ldots, j_r \le p-2$ since $\widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r} T^{(i)} = \mathbf{0}$ $(i = 1,2)$. This implies that $H_{\mu^{(1)}} = H_{\mu^{(2)}}$. Also, we regard $T^{(i)}$ $(i = 1,2)$ as elements in $\mathfrak{gl}_{>0}$-crystals (whose weight lattice is $\bigoplus_{i \ge p} \mathbb{Z}\epsilon_i$) with respect to $\widetilde{e}_k$ and $\widetilde{f}_k$ $(k \ge p)$. Then $\widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r} T^{(1)} \neq \mathbf{0}$ if and only if $\widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r} T^{(2)} \neq \mathbf{0}$ for all $j_1, \ldots, j_r \ge p$ since $\widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r} H_{\mu^{(i)}} = \mathbf{0}$ $(i = 1,2)$. This implies that $T^{(1)} = T^{(2)}$. Therefore, $(S^{(1)}, T^{(1)}) = (S^{(2)}, T^{(2)})$, which is a contradiction. This proves our claim.

Note that $\widetilde{x}_i$ $(x = e, f, i \in \mathbb{N})$ acts on $\mathcal{W}$ by $\widetilde{x}_i[w] = [\widetilde{x}_i w]$, where we assume that $[\mathbf{0}] = 0$. Hence by applying $X = \widetilde{x}_{j_1} \cdots \widetilde{x}_{j_r}$ to (4.3), we get

$$\sum_{i=2}^n c_i \big[Xw\big(S^{(i)}\big) w\big(T^{(i)}\big) \big] = \sum_{i=2}^n c_i \big[w\big(\overline{S}^{(i)}\big) w\big(\overline{T}^{(i)}\big) \big] = 0$$

for some $[w(\overline{S}^{(i)})w(\overline{T}^{(i)})] \in \mathcal{B}$. Here, we assume that $c_i = 0$ if $X(w(S^{(i)})w(T^{(i)})) = \mathbf{0}$. By induction hypothesis, we have $c_2 = \cdots = c_n = 0$, and hence $c_1 = 0$. Therefore, $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathcal{W}$. □

Theorem 4.7 *Let $\mathcal{U}(1)$ be the $\mathbb{Q}$-algebra obtained from $\mathcal{U}(t)$ by specializing $t = 1$. Then the assignment $u_a \mapsto [a]$ for $a \in \mathbb{N} \cup \mathbb{N}^\vee$ gives a $\mathbb{Q}$-algebra isomorphism*

$$\mathcal{U}(1) \simeq \mathcal{W}.$$

Proof By (4.2), the relations in (4.1) when $t = 1$ are preserved in $\mathcal{W}$ under the correspondence $u_a \mapsto [a]$. Hence there exists a $\mathbb{Q}$-algebra homomorphism $\psi : \mathcal{U}(1) \to \mathcal{W}$ sending u_a to $[a]$ for $a \in \mathbb{N} \cup \mathbb{N}^\vee$. Since $\{u_{(S,T)} | (S,T) \in \mathcal{B}_{\mu,\nu},\ \mu,\nu \in \mathcal{P}\}$ spans $\mathcal{U}(1)$ and $\psi(u_{(S,T)}) = [w(S)w(T)]$, it follows from Lemma 4.6 that ψ is an isomorphism. □

Corollary 4.8 *The set*

$$\left\{u_{(S,T)} \,\middle|\, (S,T) \in \mathcal{B}_{\mu,\nu},\ \mu,\nu \in \mathcal{P}\right\}$$

is a $\mathbb{Q}[t,t^{-1}]$-basis of $\mathcal{U}(t)$.

Proof Note that $\{u_{(S,T)} | (S,T) \in \mathcal{B}_{\mu,\nu},\ \mu,\nu \in \mathcal{P}\} \subset \mathcal{U}(1)$ is a $\mathbb{Q}$-basis of $\mathcal{U}(1)$ since it is mapped to $\mathcal{B}$ by Theorem 4.7. Then it is not difficult to check that $\{u_{(S,T)} | (S,T) \in \mathcal{B}_{\mu,\nu},\ \mu,\nu \in \mathcal{P}\} \subset \mathcal{U}(t)$ is linearly independent over $\mathbb{Q}[t,t^{-1}]$ and hence a $\mathbb{Q}[t,t^{-1}]$-basis of $\mathcal{U}(t)$ since $\mathcal{U}(t)$ is a $\mathbb{Q}[t,t^{-1}]$-submodule of a $\mathbb{Q}(t)$-vector space $\mathbb{Q}(t) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathcal{U}(t)$. □

Corollary 4.9 *For $w \in \mathcal{W}$, there exist unique $(S,T) \in \mathcal{B}_{\mu,\nu}$ and $\epsilon \in \mathbb{Z}_{\geq 0}$ such that $u_w = t^\epsilon u_{(S,T)}$.*

4.3 Non-commutative Schur functions

Let $\widehat{\mathcal{U}(t)} = \bigoplus_{n\geq 0} \widehat{\mathcal{U}(t)}_n$, where $\widehat{\mathcal{U}(t)}_n$ is the completion of $\mathbb{Q}[t,t^{-1}]$-submodule of $\mathcal{U}(t)$ spanned by $\{(S,T) | (S,T) \in \mathcal{B}_{\mu,\nu}, |\mu| + |\nu| = n\}$. For a skew Young diagram λ/μ, let

$$s_{\lambda/\mu}(\mathbf{u}) = \sum_{S \in \mathcal{B}_{\lambda/\mu}} u_S, \qquad s^\vee_{\lambda/\mu}(\mathbf{u}) = \sum_{S \in \mathcal{B}^\vee_{\lambda/\mu}} u_S \in \widehat{\mathcal{U}(t)},$$

which are plactic skew Schur functions in u_i's and $u_{i^\vee}$'s, respectively.

Let $\Lambda(t)$ be the algebra of symmetric functions in $\mathbf{x} = \mathbf{x}_\mathbb{N}$ over $\mathbb{Q}[t,t^{-1}]$. Then $\{s_{(k)}(\mathbf{u}) | k \geq 0\}$ (resp. $\{s^\vee_{(k)}(\mathbf{u}) | k \geq 0\}$) generates the subalgebra $\mathcal{S}(t)_\pm$ of $\widehat{\mathcal{U}(t)}$ isomorphic to $\Lambda(t)$ [14], where $s_{(k)}(\mathbf{u})$ (resp. $s^\vee_{(k)}(\mathbf{u})$) corresponds to the kth complete symmetric function $h_k(\mathbf{x}) = s_{(k)}(\mathbf{x})$, and $\{s_\lambda(\mathbf{u}) | \lambda \in \mathcal{P}\}$ (resp. $\{s^\vee_\lambda(\mathbf{u}) | \lambda \in \mathcal{P}\}$) is a $\mathbb{Q}[t,t^{-1}]$-basis of $\mathcal{S}(t)_+$ (resp. $\mathcal{S}(t)_-$).

We define

$$s_{\mu,\nu}(\mathbf{u}) = \sum_{(S,T)\in\mathcal{B}_{\mu,\nu}} u_{(S,T)}$$

for $\mu, \nu \in \mathcal{P}$ and let

$$\mathcal{S}(t) = \sum_{\mu,\nu\in\mathcal{P}} \mathbb{Q}\big[t, t^{-1}\big] s_{\mu,\nu}(\mathbf{u}) \subset \widehat{\mathcal{U}(t)}.$$

Lemma 4.10 *For $\mu, \nu \in \mathcal{P}$, we have*

$$s_\mu(\mathbf{u}) s^\vee_\nu(\mathbf{u}) = \sum_{\lambda\subset\mu,\nu} t^{|\lambda|} s^\vee_{\nu/\lambda}(\mathbf{u}) s_{\mu/\lambda}(\mathbf{u}) = \sum_{\lambda,\sigma,\tau} t^{|\lambda|} c^\mu_{\lambda\sigma} c^\nu_{\lambda\tau} s_{\sigma,\tau}(\mathbf{u}).$$

Proof By (3.1) and Lemma 4.3(1), we have $s_{\mu,\nu}(\mathbf{u}) = s^\vee_\nu(\mathbf{u}) s_\mu(\mathbf{u})$. The identity follows from (3.3) and Lemma 4.3(2). □

Proposition 4.11 *$\mathcal{S}(t)$ is a $\mathbb{Q}[t, t^{-1}]$-algebra with a basis $\{s_{\mu,\nu}(\mathbf{u}) | \mu, \nu \in \mathcal{P}\}$, where*

$$s_{\mu,\nu}(\mathbf{u}) s_{\sigma,\tau}(\mathbf{u}) = \sum_{\zeta,\eta} \Bigg(\sum_{\alpha,\beta,\gamma} t^{|\beta|} c^\zeta_{\sigma\alpha} c^\mu_{\alpha\beta} c^\tau_{\beta\gamma} c^\eta_{\gamma\nu} \Bigg) s_{\zeta,\eta}(\mathbf{u})$$

for $\mu, \nu, \sigma, \tau \in \mathcal{P}$.

Proof In fact, $\{s_{\mu,\nu}(\mathbf{u}) | \mu, \nu \in \mathcal{P}\}$ is linearly independent by Lemma 4.8, and hence a basis of $\mathcal{S}(t)$. Combining Lemma 4.10 with the usual Littlewood–Richardson rule (2.1) for $s_\mu(\mathbf{u})$'s and $s^\vee_\nu(\mathbf{u})$'s, we obtain the above identity. Since the sum on the right hand side is finite, $\mathcal{S}(t)$ has a well-defined multiplication and hence is a $\mathbb{Q}[t, t^{-1}]$-algebra. □

4.4 Heisenberg algebra

Let $\mathcal{H}(t)$ be an associative $\mathbb{Q}[t, t^{-1}]$-algebra with unity generated by B_n ($n \in \mathbb{Z} \setminus \{0\}$) subject to the relations

$$B_k B_l - B_l B_k = k t^k \delta_{k+l,0}.$$

For $k \geq 1$, let $p_k(\mathbf{u}) \in \mathcal{S}(t)_+$ (resp. $p^\vee_k(\mathbf{u}) \in \mathcal{S}(t)_-$) correspond to the kth power sum symmetric function $p_k(\mathbf{x}) \in \Lambda(t)$.

Proposition 4.12 *The assignment $p_k(\mathbf{u}) \mapsto B_k$ $p^\vee_k(\mathbf{u}) \mapsto B_{-k}$ for $k \geq 1$ gives a $\mathbb{Q}[t, t^{-1}]$-algebra isomorphism*

$$\mathcal{S}(t) \simeq \mathcal{H}(t).$$

Proof Put $h_k(\mathbf{u}) = s_{(k)}(\mathbf{u})$ and $h_k^\vee(\mathbf{u}) = s_{(k)}^\vee(\mathbf{u})$ for $k \geq 0$ (note that $h_0(\mathbf{u}) = h_0^\vee(\mathbf{u}) = 1$). By Lemma 4.10, we have

$$h_s(\mathbf{u})h_r^\vee(\mathbf{u}) = \sum_{i=0}^{m} t^i h_{r-i}^\vee(\mathbf{u})h_{s-i}(\mathbf{u}) \tag{4.4}$$

for $r, s \geq 0$, where $m = \min\{r, s\}$. We may view $\mathcal{S}(t)$ as an algebra generated by $\{h_k^\vee(\mathbf{u}), h_k(\mathbf{u}) | k \geq 0\}$ with the defining relations (4.4). Since

$$h_r(\mathbf{u}) = \sum_{|\lambda|=r} \frac{1}{z_\lambda} p_\lambda(\mathbf{u}),$$

where $z_\lambda = \prod_{i\geq 1} i^{m_i(\lambda)} m_i(\lambda)!$ and $m_i(\lambda) = |\{k | \lambda_k = i\}|$, we obtain

$$p_k(\mathbf{u})p_l^\vee(\mathbf{u}) - p_l^\vee(\mathbf{u})p_k(\mathbf{u}) = kt^k \delta_{k,l}$$

for $k, l \geq 1$ by using the same argument as in [13, Corollary 8]. This implies that there exists an isomorphism $\psi : \mathcal{H}(t) \to \mathcal{S}(t)$ sending B_{-k} (resp. B_k) to $p_k^\vee(\mathbf{u})$ (resp. $p_k(\mathbf{u})$) for $k \geq 1$. □

Remark 4.13 Regarding $\mathcal{S}(0)$ and $\mathcal{S}(1)$ as $\mathbb{Q}$-algebras generated by $h_k(\mathbf{u})$ and $h_k^\vee(\mathbf{u})$ ($k \geq 0$), we have $\mathcal{S}(0) \simeq \Lambda \otimes \Lambda$, and $\mathcal{S}(1) \simeq \langle \frac{\partial}{\partial p_k}, p_k | k \geq 1 \rangle \subset \mathrm{End}_{\mathbb{Q}}(\Lambda)$, where Λ is the algebra of symmetric functions in $\mathbf{x}$ over $\mathbb{Q}$ and p_k is the operator on Λ induced from the multiplication by $p_k(\mathbf{x})$. Therefore, we may view $\mathcal{S}(t)$ as an algebra interpolating the algebra of double symmetric functions and the Weyl algebra of infinite rank.

5 Knuth correspondence and Cauchy identity

5.1 Main result

Let $\mathbb{A}$ and $\mathbb{B}$ be linearly ordered $\mathbb{Z}_2$-graded sets. For $A \in \mathcal{M}_{\mathbb{A},\mathbb{N}}$ (or $\mathcal{M}_{\mathbb{B},\mathbb{N}^\vee}$), we put $u_A = u_{\mathbf{j}}$ if $A = A(\mathbf{i}, \mathbf{j})$. Now we are in a position to state and prove our main theorem.

Theorem 5.1 *There exists a bijection*

$$\mathcal{M}_{\mathbb{A},\mathbb{N}} \times \mathcal{M}_{\mathbb{B},\mathbb{N}^\vee} \longrightarrow \mathcal{M}_{\mathbb{A},\mathbb{B}} \times \mathcal{M}_{\mathbb{B},\mathbb{N}^\vee} \times \mathcal{M}_{\mathbb{A},\mathbb{N}}$$

sending (X, Y) *to* (Z, Y', X') *such that*

(1) $\mathrm{wt}_{\mathbb{A}}(X) = \mathrm{wt}_{\mathbb{A}}(X') + \mathrm{wt}_{\mathbb{A}}(Z)$ *and* $\mathrm{wt}_{\mathbb{B}}(Y) = \mathrm{wt}_{\mathbb{B}}(Y') + \mathrm{wt}_{\mathbb{B}}(Z)$,
(2) $u_X u_Y = t^{|Z|} u_{Y'} u_{X'}$ *where* $Z = (z_{ij})$ *and* $|Z| = \sum_{i,j} z_{ij}$.

Proof It is obtained by composing the following bijections, which preserve $\mathrm{wt}_{\mathbb{A}}$, $\mathrm{wt}_{\mathbb{B}}$ and (4.1):

$$
\begin{aligned}
&\mathcal{M}_{\mathbb{A},\mathbb{N}} \times \mathcal{M}_{\mathbb{B},\mathbb{N}^\vee} \\
&\longrightarrow \bigsqcup_{\mu\in\mathcal{P}_\mathbb{A},\nu\in\mathcal{P}_\mathbb{B}} \mathcal{B}_\mu \times SST_\mathbb{A}(\mu) \times \mathcal{B}_\nu^\vee \times SST_\mathbb{B}(\nu) \quad \text{by (2.3) and (2.4)} \\
&\longrightarrow \bigsqcup_{\mu\in\mathcal{P}_\mathbb{A},\nu\in\mathcal{P}_\mathbb{B}} SST_\mathbb{A}(\mu) \times SST_\mathbb{B}(\nu) \times \mathcal{B}_\mu \times \mathcal{B}_\nu^\vee \\
&\longrightarrow \bigsqcup_{\mu\in\mathcal{P}_\mathbb{A},\nu\in\mathcal{P}_\mathbb{B}} SST_\mathbb{A}(\mu) \times SST_\mathbb{B}(\nu) \times \Big(\bigsqcup_{\lambda\subset\mu,\nu} \mathcal{B}_{\nu/\lambda}^\vee \times \mathcal{B}_{\mu/\lambda}\Big) \quad \text{by (3.3)} \\
&\longrightarrow \bigsqcup_{\mu\in\mathcal{P}_\mathbb{A},\nu\in\mathcal{P}_\mathbb{B}} \bigsqcup_{\lambda\subset\mu,\nu} SST_\mathbb{B}(\nu) \times \mathcal{B}_{\nu/\lambda}^\vee \times SST_\mathbb{A}(\mu) \times \mathcal{B}_{\mu/\lambda} \\
&\longrightarrow \bigsqcup_{\lambda\in\mathcal{P}_\mathbb{A}\cap\mathcal{P}_\mathbb{B}} SST_\mathbb{B}(\lambda) \times \mathcal{M}_{\mathbb{B},\mathbb{N}^\vee} \times SST_\mathbb{A}(\lambda) \times \mathcal{M}_{\mathbb{A},\mathbb{N}} \quad \text{by (2.5) and (2.6)} \\
&\longrightarrow \Big(\bigsqcup_{\lambda\in\mathcal{P}_\mathbb{A}\cap\mathcal{P}_\mathbb{B}} SST_\mathbb{B}(\lambda) \times SST_\mathbb{A}(\lambda)\Big) \times \mathcal{M}_{\mathbb{B},\mathbb{N}^\vee} \times \mathcal{M}_{\mathbb{A},\mathbb{N}} \\
&\longrightarrow \mathcal{M}_{\mathbb{A},\mathbb{B}} \times \mathcal{M}_{\mathbb{B},\mathbb{N}^\vee} \times \mathcal{M}_{\mathbb{A},\mathbb{N}} \quad \text{by (2.3).}
\end{aligned}
$$

□

Now, let us consider the non-commutative character identity associated with Theorem 5.1. We first define the plactic Cauchy products

$$
\mathcal{Q}(\mathbf{x}_\mathbb{A}) = \overrightarrow{\prod_{a\in\mathbb{A}}} \mathcal{Q}(x_a), \qquad \mathcal{P}(\mathbf{x}_\mathbb{B}) = \overrightarrow{\prod_{b\in\mathbb{B}}} \mathcal{P}(x_b),
$$

where the products are given with respect to the linear ordering on $\mathbb{A}$ or $\mathbb{B}$ so that smaller terms are to the left, and

$$
\mathcal{Q}(x_a) = \begin{cases} \frac{1}{\cdots(1-u_2x_a)(1-u_1x_a)}, & \text{if } |a|=0, \\ (1+u_1x_a)(1+u_2x_a)\cdots, & \text{if } |a|=1, \end{cases}
$$

$$
\mathcal{P}(x_b) = \begin{cases} \frac{1}{\cdots(1-u_{2^\vee}x_b)(1-u_{1^\vee}x_b)}, & \text{if } |b|=0, \\ (1+u_{1^\vee}x_b)(1+u_{2^\vee}x_b)\cdots, & \text{if } |b|=1. \end{cases}
$$

We assume that $\mathbf{x}_\mathbb{A}$ and $\mathbf{x}_\mathbb{B}$ commute with $\mathbf{u}$. Note that

$$
\mathcal{Q}(\mathbf{x}_\mathbb{A}) = \sum_{\lambda\in\mathcal{P}_\mathbb{A}} s_\lambda(\mathbf{u}) s_\lambda(\mathbf{x}_\mathbb{A}), \qquad \mathcal{P}(\mathbf{x}_\mathbb{B}) = \sum_{\lambda\in\mathcal{P}_\mathbb{B}} s_\lambda^\vee(\mathbf{u}) s_\lambda(\mathbf{x}_\mathbb{B}),
$$

by (2.3) and (2.4).

Corollary 5.2

$$
\mathcal{Q}(\mathbf{x}_\mathbb{A})\mathcal{P}(\mathbf{x}_\mathbb{B}) = \frac{\prod_{|a|\neq|b|}(1+tx_ax_b)}{\prod_{|a|=|b|}(1-tx_ax_b)} \mathcal{P}(\mathbf{x}_\mathbb{B})\mathcal{Q}(\mathbf{x}_\mathbb{A}).
$$

Proof By definition, we have

$$\mathcal{Q}(\mathbf{x}_{\mathbb{A}}) = \sum_{X\in\mathcal{M}_{\mathbb{A},\mathbb{N}}} u_X \mathbf{x}_{\mathbb{A}}^{\mathrm{wt}_{\mathbb{A}}(X)}, \qquad \mathcal{P}(\mathbf{x}_{\mathbb{B}}) = \sum_{Y\in\mathcal{M}_{\mathbb{B},\mathbb{N}^\vee}} u_Y \mathbf{x}_{\mathbb{B}}^{\mathrm{wt}_{\mathbb{B}}(Y)}.$$

Since the bijections in the proof of Theorem 5.1 preserve the plactic relations (4.1), $\mathrm{wt}_{\mathbb{A}}$ and $\mathrm{wt}_{\mathbb{B}}$, we obtain the identity. □

5.2 Cauchy identity for Schur operators

For $i\in\mathbb{N}$, we define operators $\overline{u}_i, \overline{u}_{i^\vee} \in \mathrm{End}_{\mathbb{Q}[t,t^{-1}]}(\Lambda(t))$ by

$$\overline{u}_{i^\vee}\bigl(s_\mu(\mathbf{x})\bigr) = \begin{cases} s_{\mu\cup\{(i,\mu_i+1)\}}(\mathbf{x}), & \text{if } \mu\cup\{(i,\mu_i+1)\}\in\mathcal{P},\\ 0, & \text{if } \mu\cup\{(i,\mu_i+1)\}\notin\mathcal{P},\end{cases}$$

$$\overline{u}_i\bigl(s_\mu(\mathbf{x})\bigr) = \begin{cases} t s_{\mu\setminus\{(i,\mu_i)\}}(\mathbf{x}), & \text{if } \mu\setminus\{(i,\mu_i)\}\in\mathcal{P},\\ 0, & \text{if } \mu\setminus\{(i,\mu_i)\}\notin\mathcal{P}.\end{cases}$$

These operators are called Schur operators [3]. Let $\overline{\mathcal{U}}(t)$ be the subalgebra of $\mathrm{End}_{\mathbb{Q}[t,t^{-1}]}(\Lambda(t))$ generated by $\overline{u}_i, \overline{u}_{i^\vee}$ $(i\in\mathbb{N})$. It is easy to see that there exists a surjective $\mathbb{Q}[t,t^{-1}]$-algebra homomorphism $\psi:\mathcal{U}(t)\to\overline{\mathcal{U}}(t)$ such that $\psi(u_i)=\overline{u}_i$ and $\psi(u_{i^\vee})=\overline{u}_{i^\vee}$ for $i\in\mathbb{N}$.

For $\lambda\in\mathcal{P}$, let

$$s_\lambda(\overline{\mathbf{u}}) = \sum_{S\in\mathcal{B}_\lambda}\overline{u}_S, \qquad s_\lambda^\vee(\overline{\mathbf{u}}) = \sum_{S\in\mathcal{B}_\lambda^\vee}\overline{u}_S,$$

where $\overline{u}_S=\psi(u_S)$ for $S\in\mathcal{B}_\lambda$ or $\mathcal{B}_\lambda^\vee$. For $\lambda,\mu\in\mathcal{P}$, we have

$$s_\mu^\vee(\overline{\mathbf{u}})\bigl(s_\lambda(\mathbf{x})\bigr) = s_\lambda(\mathbf{x})s_\mu(\mathbf{x}), \qquad s_\mu(\overline{\mathbf{u}})\bigl(s_\lambda(\mathbf{x})\bigr) = t^{|\mu|}s_{\lambda/\mu}(\mathbf{x})$$

(see [3]). We also have

$$\overline{\mathcal{Q}}(\mathbf{x}_{\mathbb{A}}) = \overrightarrow{\prod_{a\in\mathbb{A}}}\overline{\mathcal{Q}}(x_a) = \sum_{\lambda\in\mathcal{P}_{\mathbb{A}}} s_\lambda(\overline{\mathbf{u}})s_\lambda(\mathbf{x}_{\mathbb{A}}),$$

$$\overline{\mathcal{P}}(\mathbf{x}_{\mathbb{B}}) = \overrightarrow{\prod_{b\in\mathbb{B}}}\overline{\mathcal{P}}(x_b) = \sum_{\lambda\in\mathcal{P}_{\mathbb{B}}} s_\lambda^\vee(\overline{\mathbf{u}})s_\lambda(\mathbf{x}_{\mathbb{B}}),$$

where $\overline{\mathcal{P}}(x_a)$ and $\overline{\mathcal{Q}}(x_b)$ are obtained from $\mathcal{P}(x_a)$ and $\mathcal{Q}(x_b)$ by replacing $u_i, u_{i^\vee}$ with $\overline{u}_i, \overline{u}_{i^\vee}$, respectively. Therefore, the products $\overline{\mathcal{Q}}(\mathbf{x}_{\mathbb{A}})\overline{\mathcal{P}}(\mathbf{x}_{\mathbb{B}})$ and $\overline{\mathcal{P}}(\mathbf{x}_{\mathbb{B}})\overline{\mathcal{Q}}(\mathbf{x}_{\mathbb{A}})$ are well defined, and the identity in Corollary 5.2 gives the following, which recovers the generalized Cauchy identity for Schur operators [3] when $t=1$:

$$\overline{\mathcal{Q}}(\mathbf{x}_{\mathbb{A}})\overline{\mathcal{P}}(\mathbf{x}_{\mathbb{B}}) = \frac{\prod_{|a|\neq|b|}(1+tx_ax_b)}{\prod_{|a|=|b|}(1-tx_ax_b)}\overline{\mathcal{P}}(\mathbf{x}_{\mathbb{B}})\overline{\mathcal{Q}}(\mathbf{x}_{\mathbb{A}}).$$

5.3 Knuth correspondence for skew tableaux

Fix $\alpha, \beta \in \mathscr{P}$. For $w = w_1 \ldots w_r \in \mathcal{W}$, we define $(S^{(k)}, T^{(k)}) \in \mathcal{B}_{\sigma^{(k)},\tau^{(k)}}$ ($1 \le k \le r$) inductively as follows: (1) $(S^{(0)}, T^{(0)}) = (H_\alpha, \emptyset)$, (2) $(S^{(k)}, T^{(k)}) = (w_k \to (S^{(k-1)}, T^{(k-1)}))$ if $w_k \in \mathbb{N}$ and $(S^{(k)}, T^{(k)}) = ((S^{(k-1)}, T^{(k-1)}) \leftarrow w_k)$ if $w_k \in \mathbb{N}^\vee$ for $1 \le k \le r$.

Let us say that w is a Littlewood–Richardson (simply LR) word of shape (α, β) if $(S^{(k)}, T^{(k)}) = (H_{\sigma^{(k)}}, \emptyset)$ for $1 \le k \le r$, and $\sigma^{(r)} = \beta$. Note that for $1 \le k \le r$, $|\sigma^{(k)}| = |\sigma^{(k-1)}| + 1$ if $w_k \in \mathbb{N}$ and $|\sigma^{(k)}| = |\sigma^{(k-1)}| - 1$ if $w_k \in \mathbb{N}^\vee$ (we assume that $\sigma^{(0)} = \alpha$). By definition, the subword $w_s w_{s+1} \ldots w_t$ of w is also an LR word of shape $(\sigma^{(s-1)}, \sigma^{(t)})$ for $1 \le s < t \le r$.

Lemma 5.3 *For $w \in \mathcal{W}$, w is an LR word of shape (α, β) if and only if $H_\alpha \otimes w \equiv H_\beta$. In particular, if w is an LR word of shape (α, β) and $w' \equiv w$ for $w' \in \mathcal{W}$, then w' is also an LR word of shape (α, β).*

Proof We keep the above notations. Suppose that w is an LR word of shape (α, β). Since $H_\alpha \otimes w_1 \ldots w_k \equiv (H_{\sigma^{(k)}}, \emptyset) \equiv H_{\sigma^{(k)}}$ for $1 \le k \le r$, we have $H_\alpha \otimes w \equiv H_\beta$.

Conversely, suppose that $H_\alpha \otimes w \equiv H_\beta \equiv (H_\beta, \emptyset)$. If $\tau^{(k)} \ne \emptyset$ (that is, $T^{(k)} \ne \emptyset$) for some k, then we have $\tau^{(r)} \ne \emptyset$ (that is, $T^{(r)} \ne \emptyset$) by definition of the insertions (Sect. 3.1), which contradicts the fact that $(S^{(r)}, T^{(r)}) \equiv H_\alpha \otimes w \equiv (H_\beta, \emptyset)$. Hence $\tau^{(k)} = \emptyset$ (that is, $T^{(k)} = \emptyset$) for $1 \le k \le r$ and $\sigma^{(r)} = \beta$.

Now suppose that $S^{(k)} \ne H_{\sigma^{(k)}}$ for some $1 \le k < r$, which is equivalent to saying that $\widetilde{e}_i S^{(k)} \ne \mathbf{0}$ for some $i \ge 1$. Then

$$\widetilde{e}_i H_\beta \equiv \widetilde{e}_i (H_\alpha \otimes w) \equiv \widetilde{e}_i \big(S^{(k)} \otimes w_{k+1} \ldots w_r\big) = \big(\widetilde{e}_i S^{(k)}\big) \otimes w_{k+1} \ldots w_r \ne \mathbf{0},$$

which is also a contradiction. Hence w is an LR word of shape (α, β). □

For $\lambda, \mu \in \mathscr{P}$ with $|\lambda| = |\alpha| + |\mu|$, we have by (2.1)

$$\big\{S \in \mathcal{B}_\mu \,|\, w(S) \text{ is an LR word of shape } (\alpha, \lambda)\big\} \stackrel{1-1}{\longleftrightarrow} \mathbf{LR}^\lambda_{\alpha\mu}. \tag{5.1}$$

For $\lambda, \nu \in \mathscr{P}$ with $|\lambda| = |\beta| + |\nu|$, we have by (3.2)

$$\big\{S \in \mathcal{B}^\vee_\nu \,|\, w(S) \text{ is an LR word of shape } (\lambda, \beta)\big\} \stackrel{1-1}{\longleftrightarrow} \mathbf{LR}^\lambda_{\beta\nu}. \tag{5.2}$$

Let $(\mathcal{B}_\mu \times \mathcal{B}^\vee_\nu)_{(\alpha,\beta)}$ be the set of $(S, T) \in \mathcal{B}_\mu \times \mathcal{B}^\vee_\nu$ such that $w(S)w(T)$ is an LR word of shape (α, β). Combining (5.1) and (5.2), we have

$$\big(\mathcal{B}_\mu \times \mathcal{B}^\vee_\nu\big)_{(\alpha,\beta)} \stackrel{1-1}{\longleftrightarrow} \bigsqcup_\lambda \mathbf{LR}^\lambda_{\alpha\mu} \times \mathbf{LR}^\lambda_{\beta\nu}. \tag{5.3}$$

Similarly, for $\sigma, \tau \in \mathscr{P}$, let $(\mathcal{B}^\vee_\tau \times \mathcal{B}_\sigma)_{(\alpha,\beta)}$ be the set of $(S, T) \in \mathcal{B}^\vee_\tau \times \mathcal{B}_\sigma$ such that $w(S)w(T)$ is an LR word of shape (α, β). As in (5.3), we have a bijection

$$\big(\mathcal{B}^\vee_\tau \times \mathcal{B}_\sigma\big)_{(\alpha,\beta)} \stackrel{1-1}{\longleftrightarrow} \bigsqcup_\lambda \mathbf{LR}^\alpha_{\lambda\tau} \times \mathbf{LR}^\beta_{\lambda\sigma}. \tag{5.4}$$

Corollary 5.4 *Let $\alpha,\beta,\mu,\nu\in\mathcal{P}$ be given. The bijection* (3.2) *when restricted to* $(\mathcal{B}_\mu\times\mathcal{B}_\nu^\vee)_{(\alpha,\beta)}$ *gives the following bijection*:

$$\bigsqcup_\lambda \mathbf{LR}^\lambda_{\alpha\mu}\times\mathbf{LR}^\lambda_{\beta\nu}\longrightarrow\bigsqcup_{\eta,\zeta,\sigma,\tau}\mathbf{LR}^\alpha_{\eta\tau}\times\mathbf{LR}^\beta_{\eta\sigma}\times\mathbf{LR}^\mu_{\sigma\zeta}\times\mathbf{LR}^\nu_{\tau\zeta}.$$

Proof Since the bijection (3.2) preserves the plactic relations or the crystal equivalence, we have by Lemma 5.3

$$\left(\mathcal{B}_\mu\times\mathcal{B}_\nu^\vee\right)_{(\alpha,\beta)}\longrightarrow\bigsqcup_{\zeta,\sigma,\tau}\left(\mathcal{B}_\tau^\vee\times\mathcal{B}_\sigma\right)_{(\alpha,\beta)}\times\mathbf{LR}^\mu_{\sigma\zeta}\times\mathbf{LR}^\nu_{\tau\zeta}.$$

Hence, it follows from (5.3) and (5.4). □

Let $(\mathcal{M}_{\mathbb{A},\mathbb{N}}\times\mathcal{M}_{\mathbb{B},\mathbb{N}^\vee})_{(\alpha,\beta)}$ be the set of (A,A') such that $\mathbf{j}\cdot\mathbf{j}'\in\mathcal{W}$ is an LR word of shape (α,β), where $A=A(\mathbf{i},\mathbf{j})$ and $A'=A(\mathbf{i}',\mathbf{j}')$, and let $(\mathcal{M}_{\mathbb{B},\mathbb{N}^\vee}\times\mathcal{M}_{\mathbb{A},\mathbb{N}})_{(\alpha,\beta)}$ be defined in the same way.

Now, we recover the Knuth type correspondence for skew tableaux by Sagan and Stanley [17] as a restriction of the bijection in Theorem 5.1 to the set of LR words of shape (α,β).

Theorem 5.5 *Let $\alpha,\beta\in\mathcal{P}$ be given. The bijection in Theorem* 5.1 *when restricted to* $(\mathcal{M}_{\mathbb{A},\mathbb{N}}\times\mathcal{M}_{\mathbb{B},\mathbb{N}^\vee})_{(\alpha,\beta)}$ *gives a bijection*

$$\bigsqcup_\lambda SST_{\mathbb{A}}(\lambda/\alpha)\times SST_{\mathbb{B}}(\lambda/\beta)\longrightarrow\bigsqcup_\eta\mathcal{M}_{\mathbb{A},\mathbb{B}}\times SST_{\mathbb{A}}(\beta/\eta)\times SST_{\mathbb{B}}(\alpha/\eta).$$

Proof Since the bijection in Theorem 5.1 preserves the plactic relations, we have a bijection by Lemma 5.3

$$(\mathcal{M}_{\mathbb{A},\mathbb{N}}\times\mathcal{M}_{\mathbb{B},\mathbb{N}^\vee})_{(\alpha,\beta)}\longrightarrow\mathcal{M}_{\mathbb{A},\mathbb{B}}\times(\mathcal{M}_{\mathbb{B},\mathbb{N}^\vee}\times\mathcal{M}_{\mathbb{A},\mathbb{N}})_{(\alpha,\beta)}. \tag{5.5}$$

On the other hand, we have

$$\begin{aligned}
&(\mathcal{M}_{\mathbb{A},\mathbb{N}}\times\mathcal{M}_{\mathbb{B},\mathbb{N}^\vee})_{(\alpha,\beta)}\\
&\stackrel{1-1}{\longleftrightarrow}\bigsqcup_{\mu\in\mathcal{P}_{\mathbb{A}},\nu\in\mathcal{P}_{\mathbb{B}}}\left(\mathcal{B}_\mu\times\mathcal{B}_\nu^\vee\right)_{(\alpha,\beta)}\times SST_{\mathbb{A}}(\mu)\times SST_{\mathbb{B}}(\nu)\\
&\stackrel{1-1}{\longleftrightarrow}\bigsqcup_{\substack{\mu\in\mathcal{P}_{\mathbb{A}},\nu\in\mathcal{P}_{\mathbb{B}}\\ \mu,\nu\subset\lambda}}\mathbf{LR}^\lambda_{\alpha\mu}\times\mathbf{LR}^\lambda_{\beta\nu}\times SST_{\mathbb{A}}(\mu)\times SST_{\mathbb{B}}(\nu)\quad\text{by (5.3)}\\
&\stackrel{1-1}{\longleftrightarrow}\bigsqcup_{\substack{\mu\in\mathcal{P}_{\mathbb{A}},\nu\in\mathcal{P}_{\mathbb{B}}\\ \mu,\nu\subset\lambda}}SST_{\mathbb{A}}(\mu)\times\mathbf{LR}^\lambda_{\mu\alpha}\times SST_{\mathbb{B}}(\nu)\times\mathbf{LR}^\lambda_{\nu\beta}\\
&\stackrel{1-1}{\longleftrightarrow}\bigsqcup_{\alpha,\beta\subset\lambda}SST_{\mathbb{A}}(\lambda/\alpha)\times SST_{\mathbb{B}}(\lambda/\beta)\quad\text{by (2.2)}.
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&(\mathcal{M}_{\mathbb{B},\mathbb{N}^\vee} \times \mathcal{M}_{\mathbb{A},\mathbb{N}})_{(\alpha,\beta)} \\
&\overset{1-1}{\longleftrightarrow} \bigsqcup_{\sigma\in\mathcal{P}_{\mathbb{A}},\tau\in\mathcal{P}_{\mathbb{B}}} \big(\mathcal{B}_\tau^\vee \times \mathcal{B}_\sigma\big)_{(\alpha,\beta)} \times SST_{\mathbb{A}}(\sigma) \times SST_{\mathbb{B}}(\tau) \\
&\overset{1-1}{\longleftrightarrow} \bigsqcup_{\substack{\sigma\in\mathcal{P}_{\mathbb{A}},\tau\in\mathcal{P}_{\mathbb{B}}\\ \eta\subset\alpha,\beta}} \mathbf{LR}^{\alpha}_{\eta\tau} \times \mathbf{LR}^{\beta}_{\eta\sigma} \times SST_{\mathbb{A}}(\sigma) \times SST_{\mathbb{B}}(\tau) \quad \text{by (5.4)} \\
&\overset{1-1}{\longleftrightarrow} \bigsqcup_{\substack{\sigma\in\mathcal{P}_{\mathbb{A}},\tau\in\mathcal{P}_{\mathbb{B}}\\ \eta\subset\alpha,\beta}} SST_{\mathbb{A}}(\sigma) \times \mathbf{LR}^{\beta}_{\sigma\eta} \times SST_{\mathbb{B}}(\tau) \times \mathbf{LR}^{\alpha}_{\tau\eta} \\
&\overset{1-1}{\longleftrightarrow} \bigsqcup_{\eta\subset\alpha,\beta} SST_{\mathbb{A}}(\beta/\eta) \times SST_{\mathbb{B}}(\alpha/\eta) \quad \text{by (2.2)}.
\end{aligned}
$$

Combining with (5.5), we obtain the result. □

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