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Schur positivity and the q-log-convexity of the Narayana polynomials

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Abstract We prove two recent conjectures of Liu and Wang by establishing the strong q-log-convexity of the Narayana polynomials, and showing that the Narayana transformation preserves log-convexity. We begin with a formula of Brändén expressing the q-Narayana numbers as a specialization of Schur functions and, by deriving several symmetric function identities, we obtain the necessary Schur-positivity results. In addition, we prove the strong q-log-concavity of the q-Narayana numbers. The q-log-concavity of the q-Narayana numbers $N_q(n,k)$ for fixed k is a special case of a conjecture of McNamara and Sagan on the infinite q-log-concavity of the Gaussian coefficients.

Keywords q-Log-concavity · q-Log-convexity · q-Narayana number · Narayana polynomial · Lattice permutation · Schur positivity · Littlewood–Richardson rule

1 Introduction

The main objective of this paper is to prove two recent conjectures of Liu and Wang [19] on the q-log-convexity of the Narayana polynomials by using Schur positivity derived from the Littlewood–Richardson rule. Moreover, we prove that the Narayana polynomials are strongly q-log-convex. We also study the q-log-concavity of the q-Narayana numbers, and show that for fixed n or k the q-Narayana numbers $N_q(n,k)$ are strongly q-log-concave. It should be noticed that McNamara and

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Sagan [22] have proposed a conjecture on the infinite q-log-concavity of the Gaussian coefficients for fixed k. It turns out that the q-log-concavity of the q-Narayana numbers is equivalent to the 2-fold q-log-concavity of the Gaussian coefficients.

Unimodal and log-concave sequences and polynomials often arise in combinatorics, algebra and geometry; see, for example, Brenti [4, 5], Stanley [29], and Stembridge [32]. A sequence $(a_n)_{n\geq 0}$ of real numbers is said to be unimodal if there exists an integer $m \geq 0$ such that

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq a_{m+2} \geq \cdots.$$

It is said to be log-concave if

$$a_m^2 \geq a_{m+1}a_{m-1}, \quad m \geq 1,$$

and is said to be log-convex if

$$a_{m+1}a_{m-1} \geq a_m^2, \quad m \geq 1.$$

For polynomials, Stanley introduced the notion of q-log-concavity, which has been studied, for example, by Butler [6], Krattenthaler [16], Leroux [20], and Sagan [26]. A sequence of polynomials $(f_n(q))_{n\geq 0}$ over the field of real numbers is called q-log-concave if the difference

$$f_m(q)^2 - f_{m+1}(q)f_{m-1}(q)$$

as a polynomial in q has all nonnegative coefficients for any $m \geq 1$. Furthermore, Sagan [27] introduced the notion of strong q-log-concavity. We say that a sequence of polynomials $(f_n(q))_{n\geq 0}$ is strongly q-log-concave if

$$f_m(q)f_n(q) - f_{m+1}(q)f_{n-1}(q)$$

as a polynomial in q has all nonnegative coefficients for any $m \geq n \geq 1$.

Based on q-log-concavity, it is natural to define q-log-convexity. A polynomial sequence $(f_n(q))_{n\geq 0}$ is said to be q-log-convex if the difference

$$f_{m+1}(q)f_{m-1}(q) - f_m(q)^2$$

as a polynomial in q has all nonnegative coefficients for any $m \geq 1$. The notion of strong q-log-convexity is a natural counterpart of that of strong q-log-concavity. A sequence of polynomials $(f_n(q))_{n\geq 0}$ is called strongly q-log-convex if

$$f_{m+1}(q)f_{n-1}(q) - f_m(q)f_n(q)$$

as a polynomial in q has all nonnegative coefficients for any $m \geq n \geq 1$.

As noticed by Sagan [27], strong q-log-concavity is not equivalent to q-log-concavity, although it is the case for a sequence of positive numbers. Analogously, strong q-log-convexity is not equivalent to q-log-convexity. For example, the sequence

$$2q + q^2 + 3q^3, \qquad q + 2q^2 + 2q^3, \qquad q + 2q^2 + 2q^3, \qquad 2q + q^2 + 3q^3$$

is q-log-convex, but not strongly q-log-convex.

Liu and Wang [19] have shown that some well-known polynomials such as the Bell polynomials and the Eulerian polynomials are q-log-convex. They also proposed two conjectures on the q-log-convexity of the Narayana polynomials. To describe their conjectures, we begin with the classical Catalan numbers, as given by

$$C_n = \frac{1}{n+1}\binom{2n}{n},$$

which count the number of Dyck paths from $(0,0)$ to $(2n,0)$ with up steps $(1,1)$ and down steps $(1,-1)$ but never going below the x-axis; see, Stanley [30]. It is known that the Catalan numbers C_n form a log-convex sequence. Recall that a peak of a Dyck path is defined as a point where an up step is immediately followed by a down step. In this combinatorial setting, the Narayana number

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}, \quad n \geq 1 \tag{1.1}$$

equals the number of Dyck paths of length $2n$ with exactly k peaks. For many other statistics that satisfy the Narayana distribution, see [3, 10, 33, 34]. By setting $N(0,0)=1$, the Narayana polynomials are given by

$$N_n(q) = \sum_{k=0}^{n} N(n,k)q^k.$$

Using the Schur–Szegö composition map, Kostov, Martínez-Finkelshtein and Shapiro [15] presented a new interpretation of Narayana polynomials. The first few Narayana polynomials are listed below:

$$\begin{aligned}
N_1(q) &= q,\\
N_2(q) &= q + q^2,\\
N_3(q) &= q + 3q^2 + q^3,\\
N_4(q) &= q + 6q^2 + 6q^3 + q^4,\\
N_5(q) &= q + 10q^2 + 20q^3 + 10q^4 + q^5,\\
N_6(q) &= q + 15q^2 + 50q^3 + 50q^4 + 15q^5 + q^6.
\end{aligned}$$

Liu and Wang [19] have shown that, for a given positive number q, the log-convexity of the sequence $(N_n(q))_{n\geq 0}$ can be proved by using a criterion [19, Theorem 3.10] along with the following recurrence relation [19, (5.1)]

$$(n+1)N_n(q) = (2n-1)(1+q)N_{n-1}(q) - (n-2)(1-q)^2 N_{n-2}(q).$$

The first conjecture of Liu and Wang is as follows.

Conjecture 1.1 *The Narayana polynomials $N_n(q)$ form a q-log-convex sequence.*

We shall prove this conjecture by establishing the Schur positivity of certain sums of symmetric functions. Our proof heavily relies on the Littlewood–Richardson rule for the product of Schur functions of certain shapes with only two columns. A formula of Brändén [3] enables us to represent the Narayana polynomials in terms of Schur functions.

The second conjecture of Liu and Wang [19] is concerned with the Narayana transformation on sequences of positive real numbers. The Davenport–Pólya theorem [9] states that if $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are log-convex then their binomial convolution

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}, \quad n \geq 0$$

is also log-convex. It is known that the binomial convolution also preserves log-concavity [36]. Nevertheless, there exist log-convexity preserving transformations that do not preserve log-concavity, such as the componentwise sum [19]. There are also log-concavity preserving transformations that do not preserve log-convexity, such as the ordinary convolution [36].

Given an array of combinatorial numbers $(t(n,k))_{0\leq k\leq n}$ such as the binomial coefficients, one can define a linear operator which transforms a sequence $(a_n)_{n\geq 0}$ into another sequence $(b_n)_{n\geq 0}$ given by

$$b_n = \sum_{k=0}^{n} t(n,k) a_k, \quad n \geq 0.$$

Liu and Wang [19] have shown that log-convexity is preserved by linear transformations associated with the binomial coefficients, the Stirling numbers of the first kind, and the Stirling numbers of the second kind. The following conjecture is proposed by Liu and Wang [19].

Conjecture 1.2 *The Narayana transformation $b_n = \sum_{k=0}^{n} N(n,k) a_k$ preserves log-convexity.*

We shall give a proof of this conjecture based on the monotonicity of certain quartic polynomials and the q-log-convexity of the Narayana polynomials.

In addition, we shall prove the strong q-log-concavity of the q-Narayana numbers. The q-Narayana numbers, as a natural q-analogue of the Narayana numbers $N(n,k)$, arise in the study of q-Catalan numbers [12]. The q-Narayana number $N_q(n,k)$ is given by

$$N_q(n,k) = \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{k^2-k},$$

where we have adopted the common notation

$$[k] := \left(1-q^k\right)/(1-q), \qquad [k]! = [1][2]\cdots[k], \qquad \begin{bmatrix} n \\ j \end{bmatrix} := \frac{[n]!}{[j]![n-j]!}$$

for the q-analogues of the integer k, the q-factorial, and the q-binomial coefficient, respectively.

Recall that the q-Narayana number $N_q(n,k)$ is a natural refinement of the q-Catalan number $c_n(q) = \frac{1}{[n+1]}\left[\begin{smallmatrix} 2n \\ n \end{smallmatrix}\right]$ as defined in [12]. Brändén [3] studied several Narayana statistics and bi-statistics on Dyck paths, and noticed that the q-Narayana number $N_q(n,k)$ has a Schur function expression by a specialization of the variables. The following relation plays a key role in establishing the connection between q-log-convexity and Schur positivity.

Theorem 1.3 [3, Theorem 6] *For all* $n, k \in \mathbb{N}$, *we have*

$$N_q(n,k) = s_{(2^{k-1})}\big(q, q^2, \ldots, q^{n-1}\big). \tag{1.2}$$

It is known that the q-analogues of many classical combinatorial numbers are strongly q-log-concave. Bulter [6] and Krattenthaler [16] have proved the q-log-concavity of the q-binomial coefficients. Leroux [20] and Sagan [26] have proved the q-log-concavity of the q-Stirling numbers of the first kind and the second kind. It is also known that the Narayana numbers $N(n,k)$ are log-concave for fixed n or k. By establishing some symmetric function identities, it will be shown that $N_q(n,k)$ are strongly q-log-concave for fixed n or fixed k.

This paper is organized as follows. In Sect. 2, we give an overview of relevant background on symmetric functions. In Sect. 3, we prove the strong q-log-convexity of Narayana polynomials by using Schur positivity. In Sect. 4, we show that the Narayana transformation preserves log-convexity. In Sect. 5, we prove the strong q-log-concavity of the q-Narayana numbers. In the last section, we give several identities involving Schur functions indexed by two-column shapes, and prove the Schur positivity results required in the proofs in Sect. 3.

2 Background on symmetric functions

In this section, we give an overview of relevant background on symmetric functions and present several recurrence formulas for computing the principal specializations of Schur functions indexed by two-column shapes. These formulas are useful in the proofs of the main theorems. To be more specific, the hook-content formula plays an important role in reducing the log-convexity preserving property of the Narayana transformation to the monotonicity of certain polynomials, and the recurrence formulas enable us to reduce the q-log-convexity of the Narayana polynomials to the Schur positivity of certain sums of symmetric functions.

Throughout this paper, we will adopt the notation and terminology on partitions and symmetric functions in Stanley [30]. Given a nonnegative integer n, a partition λ of n is a weakly decreasing nonnegative integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^k$ such that $\sum_{i=1}^k \lambda_i = n$. The number of nonzero components λ_i is called the length of λ, denoted $\ell(\lambda)$. We also denote the partition λ by $(\ldots, 2^{m_2}, 1^{m_1})$ if i appears m_i times in λ. For example, $\lambda = (4,2,2,1,1,1)$ can be written as $(4^1, 2^2, 1^3)$, where we omit i^{m_i} if $m_i = 0$.

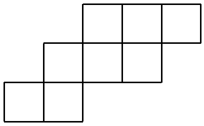

Fig. 1 The Young diagram of $(5,4,2)/(2,1)$

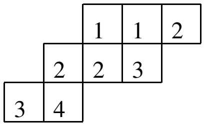

Fig. 2 A semistandard Young tableau

Let $\mathrm{Par}(n)$ denote the set of partitions of n. The Young diagram of λ is an array of squares in the plane justified from the top left corner with $\ell(\lambda)$ rows and λ_i squares in row i. By transposing the diagram of λ, we get the conjugate partition of λ, denoted λ'. A square labeled by (i,j) in the diagram of λ is meant to be the square in row i from the top and column j from the left. The hook length of (i,j), denoted $h(i,j)$, is given by $\lambda_i+\lambda'_j-i-j+1$. The content of (i,j), denoted $c(i,j)$, is given by $j-i$. Given two partitions λ and μ, we say that λ contains μ, denoted $\mu\subseteq\lambda$, if $\lambda_i\ge\mu_i$ holds for each i. When $\mu\subseteq\lambda$, we can define a skew partition λ/μ as the diagram obtained from the diagram of λ by removing the squares at the top left corner corresponding to the diagram of μ. Figure 1 illustrates the diagram of the skew partition $(5,4,2)/(2,1)$.

A semistandard Young tableau of shape λ/μ is an array $T=(T_{ij})$ of positive integers of shape λ/μ that is weakly increasing in every row and strictly increasing down every column. The type of T is defined as the composition $\alpha=(\alpha_1,\alpha_2,\ldots)$, where α_i is the number of i's in T. For example, the semistandard Young tableaux in Fig. 2 is of shape $(5,4,2)/(2,1)$ and type $(2,3,2,1,0,0,\ldots)$. Let x denote the set of variables $\{x_1,x_2,\ldots\}$. If T has type α, then we write

$$x^T=x_1^{\alpha_1}x_2^{\alpha_2}\cdots.$$

The skew Schur function $s_{\lambda/\mu}(x)$ of shape λ/μ is defined as the generating function

$$s_{\lambda/\mu}(x)=\sum_T x^T,$$

summed over all semistandard Young tableaux T of shape λ/μ filled with positive integers. When μ is the empty partition $\emptyset$, we call $s_\lambda(x)$ the Schur function of shape λ. In particular, we set $s_\emptyset(x)=1$. It is well known that the Schur functions s_λ form a basis of the ring of symmetric functions.

Let $y=\{y_1,y_2,\ldots\}$ be another set of variables, and let $s_{\lambda/\mu}(x,y)$ denote the Schur function in $x\cup y$. The following basic property will be used later:

$$s_{\lambda/\mu}(x,y)=\sum_\nu s_{\lambda/\nu}(x)s_{\nu/\mu}(y), \tag{2.1}$$

where the sum ranges over all partitions ν satisfying $\mu\subseteq\nu\subseteq\lambda$; see [21, 30].

For a symmetric function $f(x)$, its principle specialization $\mathrm{ps}_n(f)$ and specialization $\mathrm{ps}_n^1(f)$ of order n are defined by

$$\mathrm{ps}_n(f) = f\left(1, q, \ldots, q^{n-1}\right),$$
$$\mathrm{ps}_n^1(f) = \mathrm{ps}_n(f)|_{q=1} = f\left(1^n\right).$$

For notational convenience, we often omit the variable set x and simply write s_λ for the Schur function $s_\lambda(x)$ if no confusion arises. The following formula (2.2) due to Stanley [28] is called the hook-content formula.

Lemma 2.1 [30, Theorem 7.21.2, Corollary 7.21.4] *For any partition λ and $n \geq 1$, we have*

$$\mathrm{ps}_n(s_\lambda) = q^{\sum_{k\geq 1}(k-1)\lambda_k} \prod_{(i,j)\in\lambda} \frac{[n + c(i,j)]}{[h(i,j)]} \tag{2.2}$$

and

$$\mathrm{ps}_n^1(s_\lambda) = \prod_{(i,j)\in\lambda} \frac{n + c(i,j)}{h(i,j)}.$$

On the other hand, in view of (2.1), we deduce the following formulas for the principle specializations of the Schur functions s_λ indexed by two-column shapes.

Lemma 2.2 *Let k be a positive integer and $n > 1$. For any $a < 0$ or $b < 0$, set $s_{(2^a,1^b)} = 0$ by convention. Then we have*

$$\mathrm{ps}_n(s_{(2^k)}) = \mathrm{ps}_{n-1}(s_{(2^k)}) + q^{n-1}\mathrm{ps}_{n-1}(s_{(2^{k-1},1)}) + q^{2(n-1)}\mathrm{ps}_{n-1}(s_{(2^{k-1})}) \tag{2.3}$$

and

$$\begin{aligned}\mathrm{ps}_n(s_{(2^k,1)}) = {} & \mathrm{ps}_{n-1}(s_{(2^k,1)}) + q^{n-1}\mathrm{ps}_{n-1}(s_{(2^k)} + s_{(2^{k-1},1^2)}) \\ & + q^{2(n-1)}\mathrm{ps}_{n-1}(s_{(2^{k-1},1)}).\end{aligned} \tag{2.4}$$

Furthermore,

$$\mathrm{ps}_n^1(s_{(2^k)}) = \mathrm{ps}_{n-1}^1(s_{(2^k)} + s_{(2^{k-1},1)} + s_{(2^{k-1})}), \tag{2.5}$$

$$\mathrm{ps}_n^1(s_{(2^k,1)}) = \mathrm{ps}_{n-1}^1(s_{(2^k,1)} + s_{(2^k)} + s_{(2^{k-1},1^2)} + s_{(2^{k-1},1)}). \tag{2.6}$$

Lemma 2.3 *For any $m \geq n \geq 1$ and $k \geq 0$, we have*

$$\mathrm{ps}_m^1(s_{(2^k)}) = \sum_{0\leq a\leq b\leq m-n} \mathrm{ps}_n^1(s_{(2^{k-b},1^{b-a})})\mathrm{ps}_{m-n}^1(s_{(2^a,1^{b-a})}).$$

The Littlewood–Richardson rule in terms of lattice permutations enables us to expand a product of Schur functions in terms of Schur functions. Recall that a lattice permutation of length n is a sequence $w_1 w_2 \cdots w_n$ such that for any i and j, in the

$*$	$*$	$*$	$*$	1	1	1	1	1
$*$	$*$	1	1	2				
$*$	2	2						
2	3	3						
3								

$*$	$*$	$*$	$*$	1	1	1	1	1
$*$	$*$	1	2	2				
$*$	2	2						
1	3	3						
3								

$*$	$*$	$*$	$*$	1	1	1	1	1
$*$	$*$	1	2	2				
$*$	1	2						
2	3	3						
3								

Fig. 3 Skew Littlewood–Richardson tableaux

subsequence $w_1 w_2 \cdots w_j$, the number of i's is greater than or equal to the number of $i+1$'s. Let T be a semistandard Young tableau. The reverse reading word T^{rev} is a sequence of entries of T obtained by first reading each row from right to left and then concatenating the rows from top to bottom. If the reverse reading word T^{rev} is a lattice permutation, we call T a Littlewood–Richardson tableau. Given two Schur functions s_μ and s_ν, the Littlewood–Richardson coefficients $c^\lambda_{\mu\nu}$ can be defined by the relation

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda, \tag{2.7}$$

and can be determined using the following result, known as the Littlewood–Richardson rule.

Theorem 2.4 [30, Theorem A1.3.3] *The Littlewood–Richardson coefficient $c^\lambda_{\mu\nu}$ is equal to the number of Littlewood–Richardson tableaux of shape λ/μ and type ν.*

Let $\lambda = (9,5,3,3,1)$, $\mu = (4,2,1)$, $\nu = (7,4,3)$. By using the Maple package for symmetric functions [35], we find that $c^\lambda_{\mu\nu} = 3$. Indeed, there are three Littlewood–Richardson tableaux of shape λ/μ and type ν as shown in Fig. 3.

When taking $\nu = (n)$ or $\nu = (1^n)$ in (2.7), the Littlewood–Richardson rule has a simpler description, known as Pieri's rule. We need the notion of horizontal and vertical strips. A skew partition λ/μ is called a horizontal (or vertical) strip of size n if there are n squares in total with no two squares lying in the same column (resp. in the same row).

Theorem 2.5 [30, Theorem 7.15.7, Corollary 7.15.9] *We have*

$$s_\mu s_{(n)} = \sum_\lambda s_\lambda$$

summed over all partitions λ *such that* λ/μ *is a horizontal strip of size* n, *and*

$$s_\mu s_{(1^n)} = \sum_\lambda s_\lambda$$

summed over all partitions λ *such that* λ/μ *is a vertical strip of size* n.

3 q-Log-convexity

The main objective of this section is to show that the Narayana polynomials form a strongly q-log-convex sequence. This is a stronger version of the above Conjecture 1.1 of Liu and Wang.

We first consider certain products of Schur functions. Given $a, b, m \in \mathbb{N}$ and $0 \leq i \leq m$, let

$$D_1(m,i,a,b) = s_{(2^{i-b},1^{b-a})} s_{(2^{m-i-1})},$$
$$D_2(m,i,a,b) = s_{(2^{i-b-1},1^{b+2-a})} s_{(2^{m-i-1})},$$
$$D_3(m,i,a,b) = s_{(2^{i-b-1},1^{b+1-a})} s_{(2^{m-i-1},1)}$$

and let

$$D(m,i,a,b) = D_1(m,i,a,b) + D_2(m,i,a,b) - D_3(m,i,a,b),$$

where $s_{(2^k,1^l)} = 0$ for $k < 0$ or $l < 0$. It is easily checked that $D(m,m,a,b) = 0$. For $k = 1, 2, 3$, it is also clear that

$$D_k(m,i,a,b) = D_k(m-1,i-1,a-1,b-1),$$

and hence

$$D(m,i,a,b) = D(m-1,i-1,a-1,b-1).$$

Given a symmetric function f, recall that f is called Schur positive (or Schur negative) if the coefficients a_λ in the expansion $f = \sum_\lambda a_\lambda s_\lambda$ of f in terms of Schur functions are all nonnegative (resp., nonpositive). The following Schur positivity result can be employed to derive the q-log-convexity of the Narayana polynomials.

Theorem 3.1 *For any* $b \geq a \geq 0$ *and* $m \geq 0$, *the symmetric function* $\sum_{i=0}^m D(m,i,a,b)$ *is Schur positive.*

The proof of Theorem 3.1 will be postponed to Sect. 6. It plays a key role in the proof of the first main result of this paper.

Theorem 3.2 *The Narayana polynomials* $N_n(q)$ *form a strongly* q*-log-convex sequence.*

Proof By the definition of strong q-log-convexity, we need to prove that for any $m \geq n \geq 1$, the difference $N_{m+1}(q)N_{n-1}(q) - N_m(q)N_n(q)$ as a polynomial in q has all coefficients nonnegative.

First, we consider the case of $n = 1$. Using (1.1), it is routine to check that for $2 \leq k \leq m+1$,

$$\frac{N(m+1,k)}{N(m,k-1)} = \frac{m(m+1)}{k(k-1)} \geq 1.$$

Thus, for any $m \geq 1$, the difference

$$N_{m+1}(q)N_0(q) - N_m(q)N_1(q) = q + \sum_{k=2}^{m+1}\big(N(m+1,k) - N(m,k-1)\big)q^k$$

as a polynomial in q has all nonnegative coefficients.

Now it remains to consider the case $n \geq 2$. Evidently, for $1 \leq k \leq n$,

$$N(n,k) = N_q(n,k)|_{q=1} = s_{(2^{k-1})}\big(1^{n-1}\big) = \mathrm{ps}^1_{n-1}(s_{(2^{k-1})}).$$

On the other hand, for $k > n$, we have $N(n,k) = 0 = \mathrm{ps}^1_{n-1}(s_{(2^{k-1})})$.

For any $m \geq n \geq 2$ and $r \geq 0$, the coefficient of q^r in $N_{m+1}(q)N_{n-1}(q)$ equals

$$C_1 = \sum_{k=1}^{r-1} \mathrm{ps}^1_m(s_{(2^{k-1})})\mathrm{ps}^1_{n-2}(s_{(2^{r-k-1})}) = \sum_{k=0}^{r-2} \mathrm{ps}^1_m(s_{(2^k)})\mathrm{ps}^1_{n-2}(s_{(2^{r-2-k})}),$$

and the coefficient of q^r in $N_m(q)N_n(q)$ equals

$$C_2 = \sum_{k=1}^{r-1} \mathrm{ps}^1_{m-1}(s_{(2^{k-1})})\mathrm{ps}^1_{n-1}(s_{(2^{r-k-1})}) = \sum_{k=0}^{r-2} \mathrm{ps}^1_{m-1}(s_{(2^k)})\mathrm{ps}^1_{n-1}(s_{(2^{r-2-k})}).$$

From Lemma 2.2 and Lemma 2.3, it follows that

$$\mathrm{ps}^1_m(s_{(2^k)}) = \sum_{0\leq a\leq b\leq m-n+2} \mathrm{ps}^1_{n-2}(s_{(2^{k-b},1^{b-a})})\mathrm{ps}^1_{m-n+2}(s_{(2^a,1^{b-a})}),$$

$$\mathrm{ps}^1_{m-1}(s_{(2^k)}) = \sum_{0\leq a\leq b\leq m-n+1} \mathrm{ps}^1_{n-2}(s_{(2^{k-b},1^{b-a})})\mathrm{ps}^1_{m-n+1}(s_{(2^a,1^{b-a})}),$$

$$\mathrm{ps}^1_{n-1}(s_{(2^{r-2-k})}) = \mathrm{ps}^1_{n-2}(s_{(2^{r-2-k})} + s_{(2^{r-3-k},1)} + s_{(2^{r-3-k})}),$$

where for $k = r-2$ we set $s_{(2^{r-3-k},1)} = 0$ and $s_{(2^{r-3-k})} = 0$. Consequently,

$$\begin{aligned}C_1 - C_2 = &\sum_{k=0}^{r-2}\sum_{0\leq a\leq b\leq m-n+2} \mathrm{ps}^1_{m-n+2}(s_{(2^a,1^{b-a})})\mathrm{ps}^1_{n-2}(s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})})\\ &-\sum_{k=0}^{r-2}\sum_{0\leq a\leq b\leq m-n+1} \mathrm{ps}^1_{m-n+1}(s_{(2^a,1^{b-a})})\mathrm{ps}^1_{n-2}(s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})})\end{aligned}$$

$$-\sum_{k=0}^{r-2}\sum_{0\le a\le b\le m-n+1}\mathrm{ps}^1_{m-n+1}(s_{(2^a,1^{b-a})})\mathrm{ps}^1_{n-2}(s_{(2^{k-b},1^{b-a})}s_{(2^{r-3-k},1)})$$
$$-\sum_{k=0}^{r-2}\sum_{0\le a\le b\le m-n+1}\mathrm{ps}^1_{m-n+1}(s_{(2^a,1^{b-a})})\mathrm{ps}^1_{n-2}(s_{(2^{k-b},1^{b-a})}s_{(2^{r-3-k})}).$$

For notational convenience, let $d=m-n+1$. By (2.1), it is easily verified that

$$\mathrm{ps}^1_{d+1}(s_{(2^a,1^{b-a})})=\mathrm{ps}^1_d(s_{(2^a,1^{b-a})})+\mathrm{ps}^1_d(s_{(2^a,1^{b-a-1})})$$
$$+\mathrm{ps}^1_d(s_{(2^{a-1},1^{b-a})})+\mathrm{ps}^1_d(s_{(2^{a-1},1^{b-a+1})}).$$

Thus, the double summation

$$\sum_{k=0}^{r-2}\sum_{0\le a\le b\le d+1}\mathrm{ps}^1_{d+1}(s_{(2^a,1^{b-a})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})}$$

can be divided into four sums

$$A_1=\sum_{k=0}^{r-2}\sum_{0\le a\le b\le d+1}\mathrm{ps}^1_d(s_{(2^a,1^{b-a})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})},$$
$$A_2=\sum_{k=0}^{r-2}\sum_{0\le a\le b\le d+1}\mathrm{ps}^1_d(s_{(2^a,1^{b-a-1})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})},$$
$$A_3=\sum_{k=0}^{r-2}\sum_{0\le a\le b\le d+1}\mathrm{ps}^1_d(s_{(2^{a-1},1^{b-a})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})},$$
$$A_4=\sum_{k=0}^{r-2}\sum_{0\le a\le b\le d+1}\mathrm{ps}^1_d(s_{(2^{a-1},1^{b-a+1})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})}.$$

Let

$$B_1=\sum_{k=0}^{r-2}\sum_{0\le a\le b\le d}\mathrm{ps}^1_d(s_{(2^a,1^{b-a})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})},$$
$$B_2=\sum_{k=0}^{r-2}\sum_{0\le a\le b\le d}\mathrm{ps}^1_d(s_{(2^a,1^{b-a})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-3-k},1)},$$
$$B_3=\sum_{k=0}^{r-2}\sum_{0\le a\le b\le d}\mathrm{ps}^1_d(s_{(2^a,1^{b-a})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-3-k})}.$$

The equality $A_1 = B_1$ holds since

$$A_1 = B_1 + \sum_{k=0}^{r-2} \sum_{0\le a\le d+1} \mathrm{ps}_d^1(s_{(2^a,1^{d+1-a})}) s_{(2^{k-d-1},1^{d+1-a})} s_{(2^{r-2-k})},$$

but $\mathrm{ps}_d^1(s_{(2^a,1^{d+1-a})}) = 0$. We also have $A_3 = B_3$ since

$$\begin{aligned} A_3 &= \sum_{k=0}^{r-2} \sum_{0\le a\le b\le d+1} \mathrm{ps}_d^1(s_{(2^{a-1},1^{b-a})}) s_{(2^{k-b},1^{b-a})} s_{(2^{r-2-k})} \\ &= \sum_{k=0}^{r-2} \sum_{1\le a\le b\le d+1} \mathrm{ps}_d^1(s_{(2^{a-1},1^{b-a})}) s_{(2^{k-b},1^{b-a})} s_{(2^{r-2-k})} \\ &= \sum_{k=0}^{r-2} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1(s_{(2^{a},1^{b-a})}) s_{(2^{k-b-1},1^{b-a})} s_{(2^{r-2-k})} \\ &= \sum_{k=1}^{r-2} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1(s_{(2^{a},1^{b-a})}) s_{(2^{k-b-1},1^{b-a})} s_{(2^{r-2-k})} \\ &= \sum_{k=0}^{r-3} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1(s_{(2^{a},1^{b-a})}) s_{(2^{k-b},1^{b-a})} s_{(2^{r-3-k})}, \end{aligned}$$

which can be rewritten in the form of B_3.

Moreover, we have

$$\begin{aligned} A_2 &= \sum_{k=0}^{r-2} \sum_{0\le a\le b\le d+1} \mathrm{ps}_d^1(s_{(2^{a},1^{b-a-1})}) s_{(2^{k-b},1^{b-a})} s_{(2^{r-2-k})} \\ &= \sum_{k=0}^{r-2} \sum_{0\le a< b\le d+1} \mathrm{ps}_d^1(s_{(2^{a},1^{b-a-1})}) s_{(2^{k-b},1^{b-a})} s_{(2^{r-2-k})} \\ &= \sum_{k=0}^{r-2} \sum_{0\le a\le b\le d} \mathrm{ps}_d^1(s_{(2^{a},1^{b-a})}) s_{(2^{k-b-1},1^{b+1-a})} s_{(2^{r-2-k})} \\ &= \sum_{k=0}^{r-2} \sum_{0\le a< b\le d} \mathrm{ps}_d^1(s_{(2^{a},1^{b-a})}) s_{(2^{k-b-1},1^{b+1-a})} s_{(2^{r-2-k})} \\ &\quad + \sum_{k=0}^{r-2} \sum_{0\le a\le d} \mathrm{ps}_d^1(s_{(2^{a})}) s_{(2^{k-a-1},1)} s_{(2^{r-2-k})} \\ &= \sum_{k=1}^{r-2} \sum_{0\le a< b\le d} \mathrm{ps}_d^1(s_{(2^{a},1^{b-a})}) s_{(2^{k-b-1},1^{b+1-a})} s_{(2^{r-2-k})} \end{aligned}$$

$$+\sum_{k=0}^{r-2}\sum_{0\le a\le d}\mathrm{ps}_d^1(s_{(2^a)})s_{(2^{k-a-1},1)}s_{(2^{r-2-k})}$$

$$=\sum_{k=1}^{r-2}\sum_{0\le a\le b\le d-1}\mathrm{ps}_d^1(s_{(2^a,1^{b+1-a})})s_{(2^{k-b-2},1^{b+2-a})}s_{(2^{r-2-k})}$$

$$+\sum_{k=0}^{r-2}\sum_{0\le a\le d}\mathrm{ps}_d^1(s_{(2^a)})s_{(2^{k-a-1},1)}s_{(2^{r-2-k})}$$

$$=\sum_{k=0}^{r-3}\sum_{0\le a\le b\le d-1}\mathrm{ps}_d^1(s_{(2^a,1^{b+1-a})})s_{(2^{k-b-1},1^{b+2-a})}s_{(2^{r-3-k})}$$

$$+\sum_{k=0}^{r-2}\sum_{0\le a\le d}\mathrm{ps}_d^1(s_{(2^a)})s_{(2^{k-a-1},1)}s_{(2^{r-2-k})}$$

and

$$A_4=\sum_{k=0}^{r-2}\sum_{0\le a\le b\le d+1}\mathrm{ps}_d^1(s_{(2^{a-1},1^{b-a+1})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})}$$

$$=\sum_{k=0}^{r-2}\sum_{1\le a\le b\le d}\mathrm{ps}_d^1(s_{(2^{a-1},1^{b-a+1})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})}$$

$$=\sum_{k=1}^{r-2}\sum_{1\le a\le b\le d}\mathrm{ps}_d^1(s_{(2^{a-1},1^{b-a+1})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-2-k})}$$

$$=\sum_{k=1}^{r-2}\sum_{0\le a\le b\le d-1}\mathrm{ps}_d^1(s_{(2^a,1^{b+1-a})})s_{(2^{k-b-1},1^{b-a})}s_{(2^{r-2-k})}$$

$$=\sum_{k=0}^{r-3}\sum_{0\le a\le b\le d-1}\mathrm{ps}_d^1(s_{(2^a,1^{b+1-a})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-3-k})}.$$

On the other hand, we have

$$B_2=\sum_{k=0}^{r-2}\sum_{0\le a\le b\le d}\mathrm{ps}_d^1(s_{(2^a,1^{b-a})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-3-k},1)}$$

$$=\sum_{k=0}^{r-2}\sum_{0\le a<b\le d}\mathrm{ps}_d^1(s_{(2^a,1^{b-a})})s_{(2^{k-b},1^{b-a})}s_{(2^{r-3-k},1)}$$

$$+\sum_{k=0}^{r-2}\sum_{0\le a\le d}\mathrm{ps}_d^1(s_{(2^a)})s_{(2^{k-a})}s_{(2^{r-3-k},1)}$$

$$= \sum_{k=0}^{r-2} \sum_{0\le a<b\le d} \mathrm{ps}_d^1(s_{(2^a,1^{b-a})}) s_{(2^{k-b},1^{b-a})} s_{(2^{r-3-k},1)}$$
$$+ \sum_{k=0}^{r-2} \sum_{0\le a\le d} \mathrm{ps}_d^1(s_{(2^a)}) s_{(2^{k-a-1},1)} s_{(2^{r-2-k})}$$
$$= \sum_{k=0}^{r-3} \sum_{0\le a\le b\le d-1} \mathrm{ps}_d^1(s_{(2^a,1^{b+1-a})}) s_{(2^{k-b-1},1^{b+1-a})} s_{(2^{r-3-k},1)}$$
$$+ \sum_{k=0}^{r-2} \sum_{0\le a\le d} \mathrm{ps}_d^1(s_{(2^a)}) s_{(2^{k-a-1},1)} s_{(2^{r-2-k})}.$$

Therefore,

$$\begin{aligned} C_1 - C_2 &= \mathrm{ps}_{n-2}^1\big((A_1+A_2+A_3+A_4) - (B_1+B_2+B_3)\big) \\ &= \mathrm{ps}_{n-2}^1(A_2+A_4-B_2) \\ &= \mathrm{ps}_{n-2}^1\left(\sum_{0\le a\le b\le d-1} \mathrm{ps}_d^1(s_{(2^a,1^{b+1-a})}) \sum_{k=0}^{r-2} D(r-2,k,a,b)\right). \end{aligned}$$

From Theorem 3.1, we deduce that

$$\sum_{0\le a\le b\le d-1} \mathrm{ps}_d^1(s_{(2^a,1^{b+1-a})}) \sum_{k=0}^{r-2} D(r-2,k,a,b)$$

is Schur positive, hence $C_1 - C_2$ is nonnegative, as desired. □

As a corollary, we are led to an affirmative answer to Conjecture 1.1.

Corollary 3.3 *The Narayana polynomials $N_n(q)$ form a q-log-convex sequence.*

We remark that Butler and Flanigan [7] defined a different q-analogue of log-convexity. In their definition, a sequence of polynomials $(f_k(q))_{k\ge 0}$ is called q-log-convex if

$$f_{m-1}(q) f_{n+1}(q) - q^{n-m+1} f_m(q) f_n(q)$$

has nonnegative coefficients for $n \ge m \ge 1$. They have shown that the q-Catalan numbers of Carlitz and Riordan [8] form a q-log-convex sequence. However, the Narayana polynomial sequence $(N_n(q))_{n\ge 0}$ is not q-log-convex in the sense of Butler and Flanigan.

4 The Narayana transformation

In this section, we give a proof of the conjecture of Liu and Wang on the log-convexity preserving property of the Narayana transformation. We first give two lemmas.

For any $n \geq 1$ and $0 \leq r \leq 2n$, we define the following polynomials in x:

$$f_1(x) = (n+1)(n-r+x)(n-r+x+1)^2(n-r+x+2),$$
$$f_2(x) = (n+1)(n-x)(n-x+1)^2(n-x+2),$$
$$f_3(x) = (n-1)(n-x+1)(n-x+2)(n-r+x+1)(n-r+x+2).$$

Let

$$f(x) = f_1(x) + f_2(x) - 2f_3(x).$$

Lemma 4.1 *For fixed integers $n \geq 1$ and $0 \leq r \leq 2n$, the polynomial $f(x)$ is monotonically decreasing in x on the interval $(-\infty, \frac{r}{2}]$.*

Proof To prove the monotonicity of $f(x)$, we take the derivative $f'(x)$ of $f(x)$ with respect to x,

$$f'(x) = 2(2x-r)g(x),$$

where

$$g(x) = 4x^2 - 4rx + 16n - 3r + 2r^2 - 13nr - 8n^2r + 2nr^2 + 22n^2 + 8n^3.$$

In order to show that $f(x)$ is decreasing on $(-\infty, \frac{r}{2}]$, it suffices to show that $g(x) > 0$ for $x \leq \frac{r}{2}$. Since $g(x)$ is a quadratic polynomial with a positive leading coefficient, it suffices to verify that its discriminant, which equals

$$16\big(-r^2 - 16n + 3r + 13nr + 8n^2r - 2nr^2 - 22n^2 - 8n^3\big),$$

is negative. To this end, we consider the polynomial

$$g_1(y) = -y^2 - 16n + 3y + 13ny + 8n^2y - 2ny^2 - 22n^2 - 8n^3$$

in y, and shall show that it is increasing on the interval $(-\infty, 2n]$. Note that the derivative of $g_1(y)$ with respect to y equals

$$g_1'(y) = -2y + 3 + 13n + 8n^2 - 4ny = (4n+2)(2n-y) + 9n + 3.$$

Therefore, $g_1'(y) > 0$ for $y \in (-\infty, 2n]$ and $g_1(y)$ is increasing on $(-\infty, 2n]$. Consequently, for any $0 \leq r \leq 2n$ and $n \geq 1$, we have

$$g_1(r) \leq g_1(2n) = -10n < 0.$$

This implies that $g(x) > 0$ and $f'(x) = 2(2x-r)g(x) < 0$ for $x \in (-\infty, \frac{r}{2})$. Therefore, $f(x)$ is monotonically decreasing on $(-\infty, \frac{r}{2}]$. □

Lemma 4.2 *For any* $n \geq 1$, $0 \leq r \leq 2n$ *and* $0 \leq k \leq \lfloor \frac{r}{2} \rfloor$, *let*

$$\alpha(n,r,k) = N(n+1,k)N(n-1,r-k) + N(n+1,r-k)N(n-1,k) \\ - 2N(n,r-k)N(n,k).$$

Then, for given n *and* r, *there always exists an integer* $k' = k'(n,r)$ *such that* $\alpha(n,r,k) \geq 0$ *for* $k \leq k'$ *and* $\alpha(n,r,k) \leq 0$ *for* $k > k'$.

Proof By (1.1), it is straightforward to verify that

$$\alpha(1,0,0) = 0, \qquad \alpha(1,1,0) = 1, \qquad \alpha(1,2,0) = 1, \qquad \alpha(1,2,1) = -2.$$

Hence the lemma holds for $n = 1$.

We may now assume that $n \geq 2$. In this case, it is clear that the lemma holds for $r = 0$. So we may further assume that $r \geq 1$. Obviously, for given n and r, if $k \leq r - n - 2$, then $n \leq (r-k) - 2$ and $\alpha(n,r,k) = 0$. Since for $n \geq 2$ and $k = 0$ we always have $\alpha(n,r,k) = 0$, it remains to determine the sign of $\alpha(n,r,k)$ for $\max(0, r-n-2) < k \leq \lfloor \frac{r}{2} \rfloor$. Let

$$C = \frac{(n-2)!(n-1)!}{k!(k-1)!(n-k+1)!(n-k+2)!},$$

$$C' = \frac{(n!)^2}{(r-k)!(r-k-1)!(n-r+k+1)!(n-r+k+2)!}.$$

By (1.1), we have

$$\alpha(n,r,k) = C \cdot C' \cdot f(k).$$

By Lemma 4.1, we deduce that

$$f(1) \geq f(2) \geq \cdots \geq f\left(\left\lfloor \frac{r}{2} \right\rfloor\right).$$

Because $\alpha(n,r,0) = 0$ for $n \geq 2$, and C, C' are two nonnegative numbers, this completes the proof. □

Theorem 4.3 *If the sequence* $(a_k)_{k \geq 0}$ *of positive real numbers is log-convex, then the sequence*

$$b_n = \sum_{k=0}^{n} N(n,k) a_k, \quad n \geq 0$$

is log-convex.

In general, the Narayana transformation does not preserve log-convexity, and the condition that $(a_k)_{k\geq 0}$ is a positive sequence is necessary for the above theorem. For example, if we take $a_k = (-1)^k$ for $k \geq 0$, then it is easy to see that $(a_k)_{k\geq 0}$ is log-convex, but $(b_n)_{n\geq 0}$ is not log-convex.

Proof of Theorem 4.3 For any $n, r, k \geq 0$, let

$$\alpha'(n,r,k) = \begin{cases} \alpha(n,r,k)/2, & \text{if } r \text{ is even and } k = r/2, \\ \alpha(n,r,k), & \text{otherwise.} \end{cases}$$

For $n \geq 1$

$$b_{n-1}b_{n+1} - b_n^2 = \sum_{r=0}^{2n} \left(\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) a_k a_{r-k} \right)$$

and

$$N_{n-1}(q)N_{n+1}(q) - N_n(q)^2 = \sum_{r=0}^{2n} \left(\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) \right) q^r.$$

By Corollary 3.3, we see that, for any $r \geq 0$,

$$\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) \geq 0.$$

Since the sequence $(a_k)_{k\geq 0}$ is a log-convex sequence of positive real numbers, we obtain that

$$a_0 a_r \geq a_1 a_{r-1} \geq a_2 a_{r-2} \geq \cdots.$$

From Lemma 4.2, it follows that there exists an integer $k' = k'(n,r)$ such that

$$\sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) a_k a_{r-k} \geq \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \alpha'(n,r,k) a_{k'} a_{r-k'} \geq 0.$$

Thus $(b_n)_{n\geq 0}$ is log-convex. This completes the proof. □

5 q-Log-concavity

This section is concerned with the q-log-concavity of the q-Narayana numbers $N_q(n,k)$ for fixed n or k. First we apply Brändén's formula (1.2) to express the q-Narayana numbers in terms of specializations of Schur functions. This formulation allows us to reduce the q-log-concavity of the q-Narayana numbers to the Schur positivity of some differences of products of Schur functions indexed by two-column shapes. Notice that much work has been done on the Schur positivity of differences of products of Schur functions; see, for example, Bergeron, Biagioli and Rosas [2], Fomin, Fulton, Li and Poon [11] and Okounkov [23].

To prove the q-log-concavity of q-Narayana numbers $N_q(n,k)$ for fixed n, we will use the following result of Bergeron and McNamara [1].

Theorem 5.1 [1, Remark 7.2] *For $k \geq 1$ and $a \geq b$, the symmetric function $s_{(k^a)}s_{(k^b)} - s_{(k^{a+1})}s_{(k^{b-1})}$ is Schur positive.*

For $a=b$, the above result was proved earlier by Kirillov [13], and a different proof was given by Kleber [14].

Theorem 5.2 *Given an integer n, the sequence $(N_q(n,k))_{k\geq 1}$ of polynomials in q is strongly q-log-concave.*

Proof Using (1.2), for any $k\geq l\geq 2$, we get

$$N_q(n,k)N_q(n,l)-N_q(n,k+1)N_q(n,l-1)=s_{(2^{k-1})}s_{(2^{l-1})}-s_{(2^k)}s_{(2^{l-2})},$$

where the Schur functions are evaluated at the variable set $\{q,q^2,\ldots,q^{n-1}\}$. By Theorem 5.1, the difference $s_{(2^{k-1})}s_{(2^{l-1})}-s_{(2^k)}s_{(2^{l-2})}$ is Schur positive for $k\geq l$. In view of the variable set for symmetric functions, we see that the difference $N_q(n,k)N_q(n,l)-N_q(n,k+1)N_q(n,l-1)$ as a polynomial in q has nonnegative coefficients. This completes the proof. □

We now turn to the q-log-concavity of the q-Narayana numbers $N_q(n,k)$ for fixed k. To this end, we recall a result due to Lam, Postnikov and Pylyavaskyy [18], which was conjectured by Lam and Pylyavaskyy [17]. Given two partitions $\lambda=(\lambda_1,\lambda_2,\ldots)$ and $\mu=(\mu_1,\mu_2,\ldots)$, let

$$\lambda\vee\mu=\big(\max(\lambda_1,\mu_1),\max(\lambda_2,\mu_2),\ldots\big),$$
$$\lambda\wedge\mu=\big(\min(\lambda_1,\mu_1),\min(\lambda_2,\mu_2),\ldots\big).$$

For two skew partitions λ/μ and ν/ρ, we define

$$(\lambda/\mu)\vee(\nu/\rho)=(\lambda\vee\nu)/(\mu\vee\rho),$$
$$(\lambda/\mu)\wedge(\nu/\rho)=(\lambda\wedge\nu)/(\mu\wedge\rho).$$

Theorem 5.3 [18, Theorem 5] *For any two skew partitions λ/μ and ν/ρ, the difference*

$$s_{(\lambda/\mu)\vee(\nu/\rho)}s_{(\lambda/\mu)\wedge(\nu/\rho)}-s_{\lambda/\mu}s_{\nu/\rho}$$

is Schur positive.

In particular, we will need the following special cases.

Corollary 5.4 *Let k be an integer greater than 1. If I,J are partitions with $I\subseteq(2^{k-2})$ and $J\subseteq(2^{k-2},1)$, then both*

$$s_{(2^{k-2})}s_{(2^{k-1})/I}-s_{(2^{k-2})/I}s_{(2^{k-1})} \tag{5.1}$$

and

$$s_{(2^{k-2},1)}s_{(2^{k-1})/J}-s_{(2^{k-2},1)/J}s_{(2^{k-1})} \tag{5.2}$$

are Schur positive.

Proof For (5.1), take $\lambda=(2^{k-2})$, $\mu=I$, $\nu=(2^{k-1})$ and $\rho=\emptyset$ in Theorem 5.3. For (5.2), take $\lambda=(2^{k-2},1)$, $\mu=J$, $\nu=(2^{k-1})$ and $\rho=\emptyset$. □

For any $r\geq 1$, let

$$X_r=\{q,q^2,\ldots,q^{r-1}\},\qquad X_r^{-1}=\{q^{-1},q^{-2},\ldots,q^{-(r-1)}\}.$$

The following relations will be used in the proof of the q-log-concavity of the q-Narayana numbers $N_q(n,k)$ for given k.

Lemma 5.5 *For any $m\geq n\geq 1$ and $k\geq 1$, we have*

$$\begin{aligned}q^{n-1}&s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(X_m)-q^m s_{(2^{k-1},1)}(X_m)s_{(2^k)}(X_{n-1})\\&=q^{k-1}\big(s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(X_m)-s_{(2^{k-1},1)}(X_m)s_{(2^k)}(X_{n-1})\big)\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}q^{2(n-1)}&s_{(2^{k-1})}(X_{n-1})s_{(2^k)}(X_m)-q^{2m}s_{(2^{k-1})}(X_m)s_{(2^k)}(X_{n-1})\\&=q^{2k(m+n-1)}\big(s_{(2^{k-1})}\big(X_{n-1}^{-1}\big)s_{(2^k)}\big(X_m^{-1}\big)-s_{(2^{k-1})}\big(X_m^{-1}\big)s_{(2^k)}\big(X_{n-1}^{-1}\big)\big).\end{aligned}\tag{5.4}$$

Proof We shall adopt the following common notation. For indeterminates $a,a_1,\ldots,a_s$ and an integer $r\geq 0$, let

$$\begin{aligned}(a;q)_r&=(1-a)(1-aq)\cdots\big(1-aq^{r-1}\big),\\(a_1,a_2,\ldots,a_s;q)_r&=(a_1;q)_r(a_2;q)_r\cdots(a_s;q)_r.\end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}s_{(2^{k-1},1)}(X_{n-1})&=s_{(2^{k-1},1)}\big(q,q^2,\ldots,q^{n-2}\big)\\&=\frac{q^{k^2}(q^{n-k-1};q)_k(q^{n-k+1};q)_{k-1}}{(1-q)(q;q)_{k-1}(q^3;q)_{k-1}}\end{aligned}$$

and

$$\begin{aligned}s_{(2^k)}(X_n)&=s_{(2^k)}\big(q,q^2,\ldots,q^{n-1}\big)\\&=\frac{q^{k(k+1)}(q^{n-k};q)_k(q^{n-k+1};q)_k}{(q;q)_k(q^2;q)_k}.\end{aligned}$$

Therefore, the left hand side of (5.3) equals

$$\begin{aligned}&\frac{q^{2k^2+k+n-1}(q^{n-k+1};q)_{k-1}(q^{n-k-1},q^{m-k},q^{m-k+1};q)_k}{(1-q)(q,q^3;q)_{k-1}(q,q^2;q)_k}\\&\quad-\frac{q^{2k^2+k+m}(q^{m-k+2};q)_{k-1}(q^{m-k},q^{n-k-1},q^{n-k};q)_k}{(1-q)(q,q^3;q)_{k-1}(q,q^2;q)_k}\end{aligned}$$

$$= \frac{q^{2k^2+k+n-1}(1-q^{m-n+1})(q^{m-k+2}, q^{n-k+1}; q)_{k-1}(q^{m-k}, q^{n-k-1}; q)_k}{(1-q)(q, q^3; q)_{k-1}(q, q^2; q)_k}$$

and the difference $s_{(2^{k-1},1)}(X_{n-1})s_{(2^k)}(X_m) - s_{(2^{k-1},1)}(X_m)s_{(2^k)}(X_{n-1})$ equals

$$\frac{q^{2k^2+k}(q^{n-k+1}; q)_{k-1}(q^{n-k-1}, q^{m-k}, q^{m-k+1}; q)_k}{(1-q)(q, q^3; q)_{k-1}(q, q^2; q)_k} - \frac{q^{2k^2+k}(q^{m-k+2}; q)_{k-1}(q^{m-k}, q^{n-k-1}, q^{n-k}; q)_k}{(1-q)(q, q^3; q)_{k-1}(q, q^2; q)_k}$$
$$= \frac{q^{2k^2+n}(1-q^{m-n+1})(q^{m-k+2}, q^{n-k+1}; q)_{k-1}(q^{m-k}, q^{n-k-1}; q)_k}{(1-q)(q, q^3; q)_{k-1}(q, q^2; q)_k}.$$

Combining the above two relations, we arrive at (5.3).

We now proceed to prove (5.4). The left hand side can be written as

$$\frac{q^{2(n+k^2-1)}(q^{n-k}, q^{n-k+1}; q)_{k-1}(q^{m-k}, q^{m-k+1}; q)_k}{(q, q^2; q)_{k-1}(q, q^2; q)_k} - \frac{q^{2(m+k^2)}(q^{m-k+1}, q^{m-k+2}; q)_{k-1}(q^{n-k-1}, q^{n-k}; q)_k}{(q, q^2; q)_{k-1}(q, q^2; q)_k}$$
$$= \frac{f(q)(q^{n-k}, q^{n-k+1}, q^{m-k+1}, q^{m-k+2}; q)_{k-1}}{(q, q^2; q)_{k-1}(q, q^2; q)_k},$$

where

$$f(q) = q^{2k^2-k-2}\big(q^{m+1} - q^n\big)\big(q^{m+n+1} + q^{m+n} - q^{m+k+1} - q^{n+k}\big).$$

The difference $s_{(2^{k-1})}(X_{n-1})s_{(2^k)}(X_m) - s_{(2^{k-1})}(X_m)s_{(2^k)}(X_{n-1})$ equals

$$\frac{q^{2k^2}(q^{n-k}, q^{n-k+1}; q)_{k-1}(q^{m-k}, q^{m-k+1}; q)_k}{(q, q^2; q)_{k-1}(q, q^2; q)_k} - \frac{q^{2k^2}(q^{m-k+1}, q^{m-k+2}; q)_{k-1}(q^{n-k-1}, q^{n-k}; q)_k}{(q, q^2; q)_{k-1}(q, q^2; q)_k}$$
$$= \frac{g(q)(q^{n-k}, q^{n-k+1}, q^{m-k+1}, q^{m-k+2}; q)_{k-1}}{(q, q^2; q)_{k-1}(q, q^2; q)_k},$$

where

$$g(q) = q^{2k^2-2k-1}\big(q^{m+1} - q^n\big)\big(q^{m+1} + q^n - q^{k+1} - q^k\big).$$

It is easily checked that $g(q^{-1}) = q^{2k+1-4k^2-2m-2n} f(q)$. Since $(1 - q^{-r}) = -q^{-r}(1 - q^r)$ for any r, we arrive at (5.4). □

Now we are ready to prove the q-log-concavity of the q-Narayana numbers $(N_q(n,k))_{n\geq k}$ for given k.

Theorem 5.6 *Given an integer k, the sequence $(N_q(n,k))_{n\geq k}$ is strongly q-log-concave.*

Proof Clearly, the theorem is valid for $k=1$. So we may assume that $k\geq 2$. For any $m\geq n\geq k$, let

$$A_{m,n}(q)=N_q(m,k)N_q(n,k)-N_q(m+1,k)N_q(n-1,k).$$

By (1.2), we have

$$A_{m,n}(q)=s_{(2^{k-1})}(X_m)s_{(2^{k-1})}(X_n)-s_{(2^{k-1})}(X_{m+1})s_{(2^{k-1})}(X_{n-1}).$$

Applying (2.3) to $s_{(2^{k-1})}(X_n)$ and $s_{(2^{k-1})}(X_{m+1})$, $A_{m,n}(q)$ equals

$$\begin{aligned}
&s_{(2^{k-1})}(X_m)\big(s_{(2^{k-1})}(X_{n-1})+q^{n-1}s_{(2^{k-2},1)}(X_{n-1})+q^{2(n-1)}s_{(2^{k-2})}(X_{n-1})\big)\\
&\quad-\big(s_{(2^{k-1})}(X_m)+q^{m}s_{(2^{k-2},1)}(X_m)+q^{2m}s_{(2^{k-2})}(X_m)\big)s_{(2^{k-1})}(X_{n-1})\\
&=\big(q^{n-1}s_{(2^{k-2},1)}(X_{n-1})s_{(2^{k-1})}(X_m)-q^{m}s_{(2^{k-2},1)}(X_m)s_{(2^{k-1})}(X_{n-1})\big)\\
&\quad+\big(q^{2(n-1)}s_{(2^{k-2})}(X_{n-1})s_{(2^{k-1})}(X_m)-q^{2m}s_{(2^{k-2})}(X_m)s_{(2^{k-1})}(X_{n-1})\big).
\end{aligned}$$

By Lemma 5.5 and (2.1), we find that $A_{m,n}(q)$ equals

$$\begin{aligned}
&q^{k-2}\big(s_{(2^{k-2},1)}(X_{n-1})s_{(2^{k-1})}(X_m)-s_{(2^{k-2},1)}(X_m)s_{(2^{k-1})}(X_{n-1})\big)\\
&\quad+q^{2(k-1)(m+n-1)}\big(s_{(2^{k-2})}\big(X_{n-1}^{-1}\big)s_{(2^{k-1})}\big(X_m^{-1}\big)-s_{(2^{k-2})}\big(X_m^{-1}\big)s_{(2^{k-1})}\big(X_{n-1}^{-1}\big)\big)\\
&=q^{k-2}s_{(2^{k-2},1)}(X_{n-1})s_{(2^{k-1})}(Z)\\
&\quad+q^{k-2}\sum_{J\subseteq(2^{k-2},1)}s_J(Z)(s_{(2^{k-2},1)}s_{(2^{k-1})/J}-s_{(2^{k-2},1)/J}s_{(2^{k-1})})(X_{n-1})\\
&\quad+q^{2(k-1)(m+n-1)}s_{(2^{k-2})}\big(X_{n-1}^{-1}\big)s_{(2^{k-1})}\big(Z^{-1}\big)\\
&\quad+q^{2(k-1)(m+n-1)}s_{(2^{k-2})}\big(X_{n-1}^{-1}\big)s_{(2^{k-2},1)}\big(Z^{-1}\big)s_{(1)}\big(X_{n-1}^{-1}\big)\\
&\quad+q^{2(k-1)(m+n-1)}\sum_{I\subseteq(2^{k-2})}s_I\big(Z^{-1}\big)(s_{(2^{k-2})}s_{(2^{k-1})/I}-s_{(2^{k-2})/I}s_{(2^{k-1})})\big(X_{n-1}^{-1}\big),
\end{aligned}$$

where $Z=\{q^{n-1},\ldots,q^{m-1}\}$ and $Z^{-1}=\{q^{1-n},\ldots,q^{1-m}\}$. Applying Corollary 5.4, the proof is complete. □

6 Schur positivity

The main goal of this section is to give a proof of Theorem 3.1. We shall establish several symmetric function identities which will be proved by induction based on the Littlewood–Richardson rule. These identities involve products of Schur functions indexed by partitions with only two-columns. Such Schur functions are of particular interest for their own sake; see, for example, Rosas [25], and Remmel and Whitehead [24].

We should note that the Littlewood–Richardson coefficients in the context of this section are either one or two. It is also worth mentioning that the Schur expansion of the product of two Schur functions indexed by partitions with only two-columns turns out to be multiplicity-free if one is indexed by a rectangular shape; see Stembridge [31].

Let us first introduce certain classes of products of Schur functions that are the ingredients to establish the desired Schur positivity. Given $m \in \mathbb{N}$ and $0 \le i \le m$, let

$$D^{(1)}_{m,i} = s_{(2^i)} s_{(2^{m-i-1})},$$

$$D^{(2)}_{m,i} = s_{(2^{i-1},1^2)} s_{(2^{m-i-1})},$$

$$D^{(3)}_{m,i} = s_{(2^{i-1},1)} s_{(2^{m-i-1},1)},$$

and let

$$D_{m,i} = D^{(1)}_{m,i} + D^{(2)}_{m,i} - D^{(3)}_{m,i}, \tag{6.1}$$

where $s_{(2^i)} = s_{(2^i,1)} = s_{(2^i,1^2)} = 0$ for $i < 0$ by convention. It is clear that $D_{m,m} \equiv 0$.

For two partitions λ and μ, let $\lambda \cup \mu$ be the partition obtained by taking the union of all parts of λ and μ and then rearranging them in the weakly decreasing order. For $k \in \mathbb{N}$, we use λ^k to represent the union of k λ's, and in particular put $\lambda^k = \emptyset$ if $k = 0$. In this notation, we define an operator Δ^μ on the ring of symmetric functions defined by a partition μ. For a symmetric function f, if it has the expansion

$$f = \sum_\lambda a_\lambda s_\lambda,$$

then the action of Δ^μ on f is given by

$$\Delta^\mu(f) = \sum_\lambda a_\lambda s_{\lambda \cup \mu}.$$

For example, if

$$f = s_{(4,3,2)} + 3s_{(2,2,1)} + 2s_{(5)},$$

then

$$\Delta^{(3,1)} f = s_{(4,3,3,2,1)} + 3s_{(3,2,2,1,1)} + 2s_{(5,3,1)}.$$

Lemma 6.1 *For $n \ge k \ge 1$, we have*

$$s_{(2^k)} s_{(2^{n+1})} = \Delta^{(2)}(s_{(2^k)} s_{(2^n)}), \tag{6.2}$$

$$s_{(2^{k-1},1^2)} s_{(2^{n+1})} = \Delta^{(2)}(s_{(2^{k-1},1^2)} s_{(2^n)}), \tag{6.3}$$

$$s_{(2^k)} s_{(2^{n+1},1^2)} = \Delta^{(2)}(s_{(2^k)} s_{(2^n,1^2)}), \tag{6.4}$$

$$s_{(2^{k-1},1)} s_{(2^{n+1},1)} = \Delta^{(2)}(s_{(2^{k-1},1)} s_{(2^n,1)}). \tag{6.5}$$

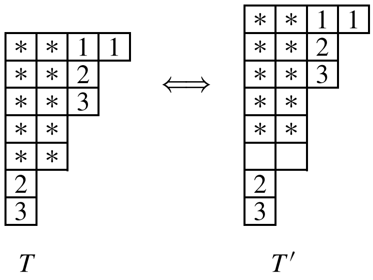

Fig. 4 The bijection between Littlewood–Richardson tableaux

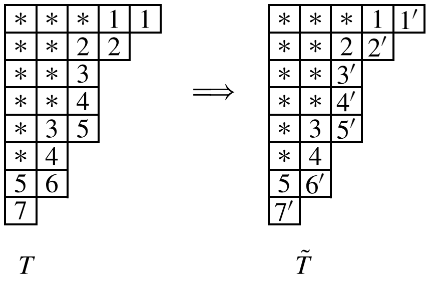

Fig. 5 Construct $\tilde{T}$ from T

Proof Let

$$s_{(2^k)}s_{(2^n)} = \sum_{\lambda} a_\lambda s_\lambda.$$

By Theorem 2.4, the coefficient a_λ is equal to the number of Littlewood–Richardson tableaux of shape $\lambda/(2^n)$ and type (2^k). We claim that $a_\lambda = 0$ if the diagram of λ contains the square $(n+1, 3)$; Otherwise, we get a contradiction to the assumption $n \geq k$ since the column strictness of Young tableaux requires that there should be at least $n+1$ distinct numbers in the tableau. Therefore, for a Littlewood–Richardson tableau T of shape $\lambda/(2^n)$ and type (2^k), we can construct a Littlewood–Richardson tableau T' of shape $\lambda \cup (2)/(2^{n+1})$ and of the same type by moving all rows of T to the next row except for the first n rows and inserting two empty squares in the $(n+1)$th row. Clearly, the above procedure to construct T' is reversible, as illustrated in Fig. 4. Thus we have verified (6.2). By similar reasoning, the three remaining identities can be justified. This completes the proof. □

Sometimes it is convenient to regard a tableau T of type $(2^k, 1^l)$ as a semistandard tableau $\tilde{T}$ filled with distinct numbers in the ordered set

$$\{1 < 1' < 2 < 2' < \cdots < n < n' < \cdots\}.$$

In this context, let $\tilde{T}$ be the tableau obtained from T by changing the first occurrence of i in the reverse reading word of T to i'. An example is given in Fig. 5. Given a partition μ, let

$$Q_\mu(n) = \left\{\lambda \in \mathrm{Par}(n) : \lambda = \mu \cup (4)^a \cup (3,1)^b \cup (2,2)^c \text{ for } a, b, c \in \mathbb{N}\right\}.$$

Lemma 6.2 *Let $m=2k+1$. The following statements hold.*

(i)

$$D^{(1)}_{m,k}=D^{(1)}_{2k+1,k}=s_{(2^k)}s_{(2^k)}=\sum_{\lambda\in Q_{\emptyset}(4k)}s_\lambda.$$

(ii)

$$D^{(1)}_{m,k+1}=D^{(1)}_{2k+1,k+1}=s_{(2^{k+1})}s_{(2^{k-1})}=\sum_{\lambda\in Q_{(2,2)}(4k)}s_\lambda.$$

(iii) *Let $Q_1(n)=Q_{(3,1)}(n)\cup Q_{(2,1,1)}(n)\cup Q_{(3,3,2)}(n)$. Then*

$$D^{(2)}_{m,k}=D^{(2)}_{2k+1,k}=s_{(2^{k-1},1^2)}s_{(2^k)}=\sum_{\lambda\in Q_1(4k)}s_\lambda.$$

(iv) *Let $Q_2(n)=Q_{(2,1,1)}(n)\cup Q_{(3,2,2,1)}(n)\cup Q_{(3,3,2,2,2)}(n)$. Then*

$$D^{(2)}_{m,k+1}=D^{(2)}_{2k+1,k+1}=s_{(2^k,1^2)}s_{(2^{k-1})}=\sum_{\lambda\in Q_2(4k)}s_\lambda.$$

(v) *Let $Q_3(n)=Q_{(3,1)}(n)\cup Q_{(2,2)}(n)\cup Q_{(2,1,1)}(n)\cup Q_{(3,3,2)}(n)$. Then*

$$D^{(3)}_{m,k}=D^{(3)}_{2k+1,k}=s_{(2^{k-1},1)}s_{(2^k,1)}=\sum_{\lambda\in Q_3(4k)}a_\lambda s_\lambda,$$

where $a_\lambda=2$ if $\lambda\in Q_{(3,2,2,1)}(4k)$, otherwise $a_\lambda=1$.

(vi) *We have*

$$D^{(3)}_{m,k+1}=D^{(3)}_{2k+1,k+1}=s_{(2^k,1)}s_{(2^{k-1},1)}=\sum_{\lambda\in Q_3(4k)}a_\lambda s_\lambda,$$

where $a_\lambda=2$ if $\lambda\in Q_{(3,2,2,1)}(4k)$, otherwise $a_\lambda=1$.

Proof We use induction on k to prove (i). Clearly, the assertion holds for $k=0$ since $s_\emptyset=1$, and it also holds for $k=1$ because of Pieri's rule; see Theorem 2.5. Using the Littlewood–Richardson rule, we see that if s_λ appears in the Schur expansion of $s_{(2^k)}s_{(2^k)}$, then λ does not contain any part greater than 4. We claim that for each Littlewood–Richardson tableau T of shape $\mu/(2^k)$ and type (2^k), subject to the conditions on the shapes and types, there are three Littlewood–Richardson tableaux of type (2^{k+1}), which are T_1 of shape $\mu\cup(4)/(2^{k+1})$, T_2 of shape $\mu\cup(3,1)/(2^{k+1})$ and T_3 of shape $\mu\cup(2,2)/(2^{k+1})$.

Let T_1 be the tableau obtained from T by increasing all numbers by 1 and then inserting a four-square row on top of T such that the rightmost two squares are filled with 1's and the leftmost two squares are empty.

Next, suppose that T has r rows of length greater than 2, and that the largest number in the first r rows is j, where we set $j=0$ if $r=0$. Consider the relabeled tableau $\tilde{T}$ corresponding to T. Let T^* be the tableau obtained from $\tilde{T}$ by increasing all numbers below the r-th row by 1 (i.e., changing i to i' and i' to $i+1$), inserting a three-square row in the $(r+1)$th row such that the rightmost square is filled with $(j+1)'$, and appending a single square row at the bottom filled with $k+1$. Let T_2 be the tableau obtained from T^* by replacing each i' with i.

We continue to construct a tableau T_3. Note that the tableau T does not contain the square $(k+1,3)$. Consider the numbers in the first k rows. Let j_1 and j_2 be the smallest and the largest numbers which appear only once in the first k rows of T. Starting with the tableau $\tilde{T}$, let T^* be the tableau obtained from $\tilde{T}$ by increasing all numbers below the kth row by 2 (i.e., changing i to $i+1$ and i' to $(i+1)'$), inserting a row of two empty squares below the kth row, and then inserting a two-square row filled with $(j_1,(j_2+1)')$ immediately below the row that has been inserted. If no number appears only once in the first k rows, then we consider the largest number j which appears twice in these rows (taking $j=0$ if no such number exists). Let T^* be the tableau obtained from $\tilde{T}$ by increasing all numbers below the kth row by 2, inserting a row of two empty squares below the kth row, and then inserting a two-square row filled with $(j+1,(j+1)')$ immediately below the row just inserted. Now we obtain the tableau T_3 by replacing each i' with i in T^*.

Note that if T is a Littlewood–Richardson tableau of shape $\mu/(2^k)$ and type (2^k), then there exist some nonnegative integers r,s,t such that the reverse reading word $\tilde{T}^{\mathrm{rev}}$ is of the form (w_a,w_b,w_c,w_d), where

$$w_a = 1',1,\ldots,r',r,$$
$$w_b = (r+1)',\ldots,(r+s)',$$
$$w_c = (r+s+1)',(r+1),\ldots,(r+s+t)',(r+s),$$
$$w_d = (r+s+1),\ldots,(r+s+t),$$

and $r+s+t=k$. It is easily seen that ${T_1}^{\mathrm{rev}}, {T_2}^{\mathrm{rev}}, {T_3}^{\mathrm{rev}}$ can be recovered from $\tilde{T}^{\mathrm{rev}}$. It is also easy to verify that they are lattice permutations. Figure 6 is an illustration of the constructions of T_1,T_2,T_3.

On the other hand, it is necessary to show that for each Littlewood–Richardson tableau T' of shape $\lambda/(2^{k+1})$ and type (2^{k+1}), we can find a Littlewood–Richardson tableau T of shape $\mu/(2^k)$ and of type 2^k such that $\lambda=\mu\cup(4)$, $\lambda=\mu\cup(3,1)$ or $\lambda=\mu\cup(2,2)$. Evidently, if λ contains at least one row of length 4, then T can be obtained from T' by reversing the construction of T_1. If T' has a two-square row fully filled with numbers and all rows of T' contain at most three squares, then T' has a two-square row with no numbers since it is a Littlewood–Richardson tableau of type (2^{k+1}), and hence T can be obtained by reversing the construction of T_3. Otherwise, T' contains at least one row of length 1 and at least one row of length 3 because of the type of T'. In this case, we can reverse the construction of T_2 to recover T. However, we should note that T is not uniquely determined by T'. By Stembridge's characterization of multiplicity-free products of Schur functions [31, Theorem 3.1], there exists a unique Littlewood–Richardson tableau of shape $\lambda/(2^k)$ and type (2^k) if s_λ appears in the expansion of $s_{(2^k)}s_{(2^k)}$. This completes the proof of (i).

We now give a sketch of the proof of (ii) which is similar to that of (i). Clearly, the assertion holds for $k=0,1$, and $D^1_{3,2}=s_{(2,2)}$. For $k\geq 2$, we may consider the Littlewood–Richardson tableau of shape $\lambda/(2^{k+1})$ and type (2^{k-1}) if s_λ appears in $s_{(2^{k+1})}s_{(2^{k-1})}$.

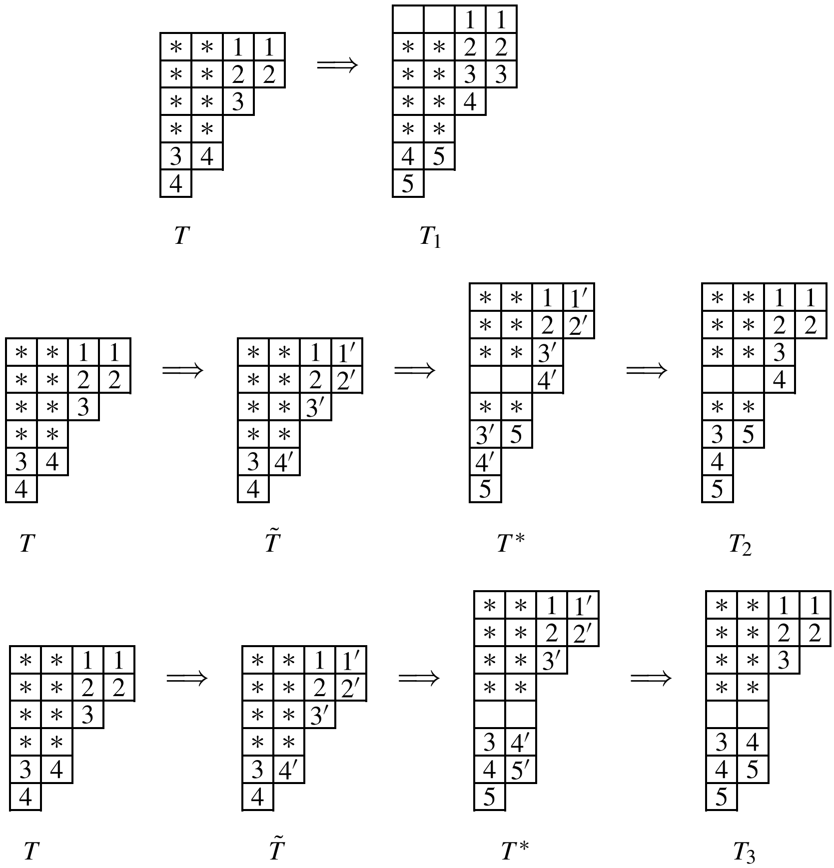

Fig. 6 Construction of T_1, T_2, T_3

To prove (iii), notice that $D^{(2)}_{2k+1,k}=0$ for $k=0$, and $D^{(2)}_{2k+1,k}=s_{(3,1)}+s_{(2,1,1)}$ for $k=1$. For $k=2$, we have

$$D^{(2)}_{2k+1,k}=s_{(4,3,1)}+s_{(4,2,1^2)}+s_{(3^2,2)}+s_{(3^2,1^2)}\\+s_{(3,2^2,1)}+s_{(3,2,1^3)}+s_{(2^3,1^2)}.$$

Again, we use induction on k. If s_λ appears in the expansion of $D^{(2)}_{2k+1,k}$, then λ does not contain the square $(k+1,3)$, because there exists no Littlewood–Richardson tableau of shape $\lambda/(2^k)$ and type $(2^{k-1},1,1)$, or equivalently, there is no filling of the $(k+1)$th row satisfying the lattice permutation condition. Thus we can proceed as in the proof of (i).

In the same manner we can prove (iv). For $k=0$, it is clear that $D^{(2)}_{2k+1,k+1}=0$. For $k=1$, we have $D^{(2)}_{2k+1,k+1}=s_{2,1,1}$. For $k=2$, we find

$$D^{(2)}_{2k+1,k+1}=s_{(4,2,1^2)}+s_{(3,2^2,1)}+s_{(3,2,1^3)}+s_{(2^3,1^2)}.$$

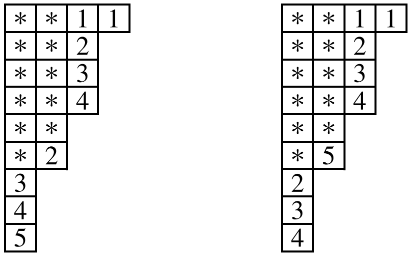

Fig. 7 The Littlewood–Richardson tableaux

For $k = 3$, we have

$$D^{(2)}_{2k+1,k+1} = s_{(4^2,2,1^2)} + s_{(4,3,2^2,1)} + s_{(4,3,2,1^3)} + s_{(4,2^3,1^2)} + s_{(3^2,2^3)} \\ + s_{(3^2,2^2,1^2)} + s_{(3,2^4,1)} + s_{(3^2,2,1^4)} + s_{(3,2^3,1^3)} + s_{(2^5,1^2)}.$$

Now we can use induction on k and consider Littlewood–Richardson tableaux of shape $\lambda/(2^k, 1^2)$ and type (2^{k-1}).

Finally, we come to (v) and (vi) which are concerned with the Schur positivity of the same product $s_{(2^{k-1},1)}s_{(2^k,1)}$. For $k = 0$, $D^{(3)}_{2k+1,k} = 0$. For $k = 1$, we get

$$D^{(3)}_{2k+1,k} = s_{(3,1)} + s_{(2^2)} + s_{(2,1^2)}.$$

For $k = 2$, we have

$$D^{(3)}_{2k+1,k} = s_{(4,3,1)} + s_{(4,2^2)} + s_{(4,2,1^2)} + s_{(3^2,2)} \\ + 2s_{(3,2^2,1)} + s_{(3^2,1^2)} + s_{(3,2,1^3)} + s_{(2^4)} + s_{(2^3,1^2)}.$$

To use induction on k, we consider Littlewood–Richardson tableaux of shape $\lambda/(2^k, 1)$ and type $(2^{k-1}, 1)$. If $\lambda \in Q_{(3,2,2,1)}(4k)$, there are exactly two such Littlewood–Richardson tableaux, see Fig. 7 for the case of $\lambda = (4, 3^3, 2^2, 1^3)$. The rest of the proof is similar to that of (i). Thus the proof of the lemma is complete. □

Theorem 6.3 *Let* $m = 2k + 1$. *We have*

$$D_{m,k} = s_{(3^k)}s_{(1^k)}, \tag{6.6}$$

$$D_{m,k+1} = s_{(4^k)} - s_{(3^k)}s_{(1^k)} - \Delta^{(2)}(s_{(3^k)}s_{1^{(k-2)}}); \tag{6.7}$$

and for $0 \le i \le k - 1$, *we have*

$$D_{m,i} = \Delta^{(2)}(D_{m-1,i}), \tag{6.8}$$

$$D_{m,m-i} = \Delta^{(2)}(D_{m-1,m-1-i}). \tag{6.9}$$

Proof To prove (6.6), we need (i), (iii) and (v) of Lemma 6.2. If $\lambda \in Q_{(3,2,2,1)}(4k)$, then s_λ appears in the expansion of both $D^{(1)}_{m,k}$ and $D^{(2)}_{m,k}$, and hence it vanishes in

$D_{m,k}$. If $\lambda \in Q_{(3,3,2)}(4k) \cup Q_{(2,1,1)}(4k)$, then s_λ appears in both $D^{(2)}_{m,k}$ and $D^{(3)}_{m,k}$, so it vanishes in $D_{m,k}$. If $\lambda \in Q_{(2,2)}(4k)$ but $\lambda \notin Q_{(3,1)}(4k)$, then s_λ appears in both $D^{(1)}_{m,k}$ and $D^{(3)}_{m,k}$. So we deduce that s_λ vanishes in $D_{m,k}$. Therefore, for a term s_λ which does not vanish in $D_{m,k}$, the index partition λ belongs to the set $Q_\emptyset(4k)$ and 2 does not appear as a part. By virtue of Pieri's rule, the Schur functions not vanishing in $D_{m,k}$ coincide with the terms in the Schur expansion of $s_{(3^k)}s_{(1^k)}$.

To prove (6.7), we shall use (ii), (iv) and (vi) of Lemma 6.2. If $\lambda \in Q_{(3,2,2,1)}(4k)$, then s_λ appears in the expansion of both $D^{(1)}_{m,k+1}$ and $D^{(2)}_{m,k+1}$, which implies that it vanishes in $D_{m,k+1}$. If $\lambda \in Q_{(3,3,2,2,2)}(4k) \cup Q_{(2,1,1)}(4k)$, then s_λ appears in both $D^{(2)}_{m,k+1}$ and $D^{(3)}_{m,k+1}$. But since it disappears in $D^{(1)}_{m,k+1}$, it follows that s_λ vanishes in $D_{m,k+1}$. If $\lambda \in Q_{(2,2)}(4k)$ but $\lambda \notin Q_{(3,2,2,1)}(4k)$, then s_λ appears in both $D^{(1)}_{m,k+1}$ and $D^{(3)}_{m,k+1}$, but disappears in $D^{(2)}_{m,k+1}$, and hence it vanishes in $D_{m,k+1}$. Therefore,

$$D_{m,k+1} = -\sum_{\substack{\lambda \in Q_{(3,3,2)},\\ \lambda \notin Q_{(3,3,2,2,2)}}} s_\lambda - \sum_{\substack{\lambda \in Q_{(3,1)},\\ \lambda \notin Q_{(3,2,2,1)}}} s_\lambda.$$

So (6.7) can be verified by applying Pieri's rule to $s_{(3^k)}s_{(1^k)}$ and $s_{(3^k)}s_{1^{(k-2)}}$.

The remaining two identities (6.8) and (6.9) are direct consequences of Lemma 6.1. This completes the proof of the theorem. □

When m is even, we can deduce the following expansion formulas. The proof is similar to that of Lemma 6.2 and is omitted.

Lemma 6.4 *Let $m = 2k$. The following statements hold.*

(i)

$$D^{(1)}_{m,k} = D^{(1)}_{2k,k} = s_{(2^k)}s_{(2^{k-1})} = \sum_{\lambda \in Q_{(2)}(4k-2)} s_\lambda.$$

(ii)

$$D^{(1)}_{m,k-1} = D^{(1)}_{2k,k-1} = s_{(2^{k-1})}s_{(2^k)} = \sum_{\lambda \in Q_{(2)}(4k-2)} s_\lambda.$$

(iii) *Let $R_1(n) = Q_{(1,1)}(n) \cup Q_{(3,3,2,2)}(n) \cup Q_{(3,2,1)}(n)$. Then*

$$D^{(2)}_{m,k} = D^{(2)}_{2k,k} = s_{(2^{k-1},1^2)}s_{(2^{k-1})} = \sum_{\lambda \in R_1(4k-2)} s_\lambda.$$

(iv) *Let $R_2(n) = Q_{(3,3)}(n) \cup Q_{(3,2,1)}(n) \cup Q_{(2,2,1,1)}(n)$. Then*

$$D^{(2)}_{m,k-1} = D^{(2)}_{2k,k-1} = s_{(2^{k-2},1^2)}s_{(2^k)} = \sum_{\lambda \in R_2(4k-2)} s_\lambda.$$

(v) *Let* $R_3(n) = Q_{(3,3)}(n) \cup Q_{(2)}(n) \cup Q_{(1,1)}(n)$. *Then*

$$D^{(3)}_{m,k} = D^{(3)}_{2k,k} = s_{(2^{k-1},1)}s_{(2^{k-1},1)} = \sum_{\lambda\in R_3(4k-2)} a_\lambda s_\lambda,$$

where $a_\lambda = 2$ *if* $\lambda \in Q_{(3,2,1)}(4k-2)$, *otherwise* $a_\lambda = 1$.

(vi) *Let* $R_4(n) = Q_{(3,3,2,2)}(n) \cup Q_{(3,2,1)}(n) \cup Q_{(2,2,2)}(n) \cup Q_{(2,2,1,1)}(n)$. *Then*

$$D^{(3)}_{m,k-1} = s_{(2^{k-2},1)}s_{(2^k,1)} = \sum_{\lambda\in R_4(4k-2)} a_\lambda s_\lambda,$$

where $a_\lambda = 2$ *if* $\lambda \in Q_{(3,2,2,2,1)}(4k)$, *otherwise* $a_\lambda = 1$.

With the aid of Lemmas 6.1 and 6.4, we obtain the following theorem for even m. The proof is similar to that of Theorem 6.3 and is omitted.

Theorem 6.5 *Let* $m = 2k$. *We have*

$$D_{m,k-1} = s_{(3^k)}s_{(1^{k-2})} + \Delta^{(2)}(s_{(3^{k-1})}s_{(1^{k-1})}), \tag{6.10}$$

$$D_{m,k} = -s_{(3^k)}s_{(1^{k-2})}, \tag{6.11}$$

$$D_{m,k+1} = \Delta^{(2)}(D_{m-1,k}), \tag{6.12}$$

and for $0 \le i \le k-2$, *we have*

$$D_{m,i} = \Delta^{(2)}(D_{m-1,i}), \tag{6.13}$$

$$D_{m,m-i} = \Delta^{(2)}(D_{m-1,m-1-i}). \tag{6.14}$$

Theorems 6.3 and 6.5 lead to a construction for the underlying partitions corresponding to the Schur expansion of $D_{m,i}$. Table 1 gives an illustration. The proof of Schur positivity in Theorem 6.7 reflects the following observation.

Corollary 6.6 *Assume that* $k \ge 1$.

(i) *If* $m = 2k+1$, *then* $D_{m,i}$ *is Schur positive for* $0 \le i \le k$, *and* $D_{m,i}$ *is Schur negative for* $k+1 \le i \le m-1$.
(ii) *If* $m = 2k$, *then* $D_{m,i}$ *is Schur positive for* $0 \le i \le k-1$, *and* $D_{m,i}$ *is Schur negative for* $k \le i \le m-1$.

Proof We conduct induction on m. It is easy to check that the result holds for $m = 2$. For $m \ge 3$, assume that the corollary holds for $m-1$. We aim to show that it holds for m.

If $m = 2k+1$, then $D_{m,k}$ is Schur positive and $D_{m,k+1}$ is Schur negative according to (6.6) and (6.7) of Theorem 6.3. For $0 \le i \le k-1$, using (6.8) of Theorem 6.3 we see that $D_{m,i} = \Delta^{(2)}(D_{2k,i})$ is Schur positive by the inductive hypothesis. Similarly, for $k+2 \le i \le 2k$, we find that $D_{m,i} = \Delta^{(2)}(D_{2k,i-1})$ is Schur negative by (6.9) and the inductive hypothesis.

Table 1 Schur function expansions of $D_{m,k}$ for $m=8,9$

$m=8$	
$D_{8,0}$	$s_{(2^7)}$
$D_{8,1}$	$s_{(4,2^5)}+s_{(3^2,2^4)}+s_{(3,2^5,1)}$
$D_{8,2}$	$s_{(3^2,2^3,1^2)}+s_{(4,3^2,2^2)}+s_{(4^2,2^3)}+s_{(3^3,2^2,1)}+s_{(4,3,2^3,1)}$
$D_{8,3}$	$s_{(4,3^2,2,1^2)}+s_{(3^3,2,1^3)}+s_{(4^2,3,2,1)}+s_{(4^3,2)}$ $+s_{(3^4,1^2)}+s_{(4^2,3^2)}+s_{(4,3^3,1)}$
$D_{8,4}$	$-s_{(3^4,1^2)}-s_{(4^2,3^2)}-s_{(4,3^3,1)}$
$D_{8,5}$	$-s_{(4^2,3,2,1)}-s_{(3^3,2^2,1)}-s_{(3^3,2,1^3)}-s_{(4,3^2,2,1^2)}$ $-s_{(4,3^2,2^2)}$
$D_{8,6}$	$-s_{(3^2,2^4)}-s_{(3^2,2^3,1^2)}-s_{(4,3,2^3,1)}$
$D_{8,7}$	$-s_{(3,2^5,1)}$
$D_{8,8}$	0
$m=9$	
$D_{9,0}$	$s_{(2^8)}$
$D_{9,1}$	$s_{(4,2^6)}+s_{(3^2,2^5)}+s_{(3,2^6,1)}$
$D_{9,2}$	$s_{(3^2,2^4,1^2)}+s_{(4,3^2,2^3)}+s_{(4^2,2^4)}+s_{(3^3,2^3,1)}+s_{(4,3,2^4,1)}$
$D_{9,3}$	$s_{(4,3^2,2^2,1^2)}+s_{(3^3,2^2,1^3)}+s_{(4^2,3,2^2,1)}+s_{(4^3,2^2)}$ $+s_{(3^4,2,1^2)}+s_{(4^2,3^2,2)}+s_{(4,3^3,2,1)}$
$D_{9,4}$	$s_{(4,3^3,1^3)}+s_{(4^2,3^2,1^2)}+s_{(4^4)}+s_{(4^3,3,1)}+s_{(3^4,1^4)}$
$D_{9,5}$	$-s_{(4,3^3,1^3)}-s_{(4^2,3^2,1^2)}-s_{(4^3,3,1)}-s_{(3^4,1^4)}$ $-s_{(3^4,2,1^2)}-s_{(4^2,3^2,2)}-s_{(4,3^3,2,1)}$
$D_{9,6}$	$-s_{(4^2,3,2^2,1)}-s_{(3^3,2^3,1)}-s_{(3^3,2^2,1^3)}-s_{(4,3^2,2^2,1^2)}$ $-s_{(4,3^2,2^3)}$
$D_{9,7}$	$-s_{(3^2,2^5)}-s_{(3^2,2^4,1^2)}-s_{(4,3,2^4,1)}$
$D_{9,8}$	$-s_{(3,2^6,1)}$
$D_{9,9}$	0

If $m=2k$, from (6.10) and (6.11) of Theorem 6.5 it follows that $D_{m,k-1}$ is Schur positive and $D_{m,k}$ is Schur negative. For $0\le i\le k-2$, by (6.13) of Theorem 6.5, together with the inductive hypothesis, we obtain that $D_{m,i}=\Delta^{(2)}(D_{2k-1,i})$ is Schur positive. Similarly, for $k+1\le i\le 2k-1$, by virtue of (6.12) and (6.14), together with the inductive hypothesis, we find that $D_{m,i}=\Delta^{(2)}(D_{2k-1,i-1})$ is Schur negative. This completes the proof. □

Given a set S of positive integers, let $\mathrm{Par}_S(n)$ denote the set of partitions of n whose parts belong to S. We have the following Schur positivity result.

Theorem 6.7 *For any $m \geq 0$, we have*

$$\sum_{i=0}^{m} D_{m,i} = \sum_{\lambda \in \mathrm{Par}_{\{2,4\}}(2m-2)} s_\lambda. \tag{6.15}$$

Before proving the above theorem, let us give some examples. For $1 \leq m \leq 5$, using the Maple package ACE [35], we obtain

$$\sum_{i=0}^{1} D_{1,i} = s_\emptyset = 1,$$

$$\sum_{i=0}^{2} D_{2,i} = s_{(2)},$$

$$\sum_{i=0}^{3} D_{3,i} = s_{(4)} + s_{(2,2)},$$

$$\sum_{i=0}^{4} D_{4,i} = s_{(4,2)} + s_{(2,2,2)},$$

$$\sum_{i=0}^{5} D_{5,i} = s_{(4,4)} + s_{(4,2,2)} + s_{(2,2,2,2)}.$$

Proof of Theorem 6.7 We use induction on m. It is readily seen that the theorem holds for $m = 0, 1$. We assume that it is true for $m-1$. For $m \geq 2$, it suffices to show that

$$\sum_{i=0}^{m} D_{m,i} = \begin{cases} \Delta^{(2)}(\sum_{i=0}^{m-1} D_{m-1,i}), & \text{if } m = 2k, \\ s_{(4^k)} + \Delta^{(2)}(\sum_{i=0}^{m-1} D_{m-1,i}), & \text{if } m = 2k+1. \end{cases} \tag{6.16}$$

If $m = 2k$, we have

$$\begin{aligned}
\sum_{i=0}^{m} D_{m,i} &= \sum_{i=0}^{2k} D_{2k,i} \\
&= \sum_{i=0}^{k-2} D_{2k,i} + D_{2k,k-1} + D_{2k,k} + D_{2k,k+1} + \sum_{i=0}^{k-2} D_{2k,2k-i} \\
&= \sum_{i=0}^{k-2} \Delta^{(2)}(D_{2k-1,i}) + \big(s_{(3^k)} s_{(1^{k-2})} + \Delta^{(2)}(s_{(3^{k-1})} s_{(1^{k-1})})\big) \\
&\quad + (-s_{(3^k)} s_{(1^{k-2})}) + \Delta^{(2)}(D_{2k-1,k}) \\
&\quad + \sum_{i=0}^{k-2} \Delta^{(2)}(D_{2k-1,2k-1-i}) \quad \text{(by Theorem 6.5)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{k-2} \Delta^{(2)}(D_{2k-1,i}) + \Delta^{(2)}(D_{2k-1,k-1}) + \Delta^{(2)}(D_{2k-1,k}) \\
&\quad + \sum_{i=0}^{k-2} \Delta^{(2)}(D_{2k-1,2k-1-i}) \quad \text{(by (6.6))} \\
&= \sum_{i=0}^{2k-1} \Delta^{(2)}(D_{2k-1,i}) \\
&= \Delta^{(2)}\left(\sum_{i=0}^{m-1} D_{m-1,i}\right).
\end{aligned}
$$

If $m = 2k + 1$, we have

$$
\begin{aligned}
\sum_{i=0}^{m} D_{m,i} &= \sum_{i=0}^{2k+1} D_{2k+1,i} \\
&= \sum_{i=0}^{k-1} D_{2k+1,i} + D_{2k+1,k} + D_{2k+1,k+1} + \sum_{i=0}^{k-1} D_{2k+1,2k+1-i} \\
&= \sum_{i=0}^{k-1} \Delta^{(2)}(D_{2k,i}) + s_{(3^k)} s_{(1^k)} \\
&\quad + \left(s_{(4^k)} - s_{(3^k)} s_{(1^k)} - \Delta^{(2)}(s_{(3^k)} s_{1^{(k-2)}})\right) \\
&\quad + \sum_{i=0}^{k-1} \Delta^{(2)}(D_{2k,2k-i}) \quad \text{(by Theorem 6.3)} \\
&= s_{(4^k)} + \sum_{i=0}^{2k} \Delta^{(2)}(D_{2k,i}) \quad \text{(by (6.11))} \\
&= s_{(4^k)} + \Delta^{(2)}\left(\sum_{i=0}^{m-1} D_{m-1,i}\right).
\end{aligned}
$$

Using (6.16), together with the inductive hypothesis, we complete the proof. □

We are now ready to prove Theorem 3.1, that is, the Schur positivity of $\sum_{i=0}^{m} D(m,i,a,b)$. Note that for $a = b = 0$, $D(m,i,a,b)$ reduces to $D_{m,i}$. Some values of $D(m,i,a,b)$ are given in Table 2 for $m = 10, a = 0, b = 2$.

For the sake of presentation, we introduce the following notation. Given a pair (λ, μ) of partitions and a pair (f_1, f_2) of symmetric functions, let

$$
\Delta^{\lambda}(f_1) = \sum_{\nu} a_{\nu} s_{\nu}, \qquad \Delta^{\mu}(f_2) = \sum_{\nu} b_{\nu} s_{\nu}.
$$

Table 2 Schur function expansion of $D(10,i,0,2)$

$D(10,1,0,2)$	0
$D(10,2,0,2)$	$s_{(3^2,2^5)}+s_{(3,2^6,1)}+s_{(2^7,1^2)}$
$D(10,3,0,2)$	$s_{(3^4,2^2)}+s_{(4,3^2,2^3)}+s_{(4,2^5,1^2)}$ $+s_{(3^3,2^3,1)}+s_{(3^2,2^4,1^2)}+s_{(3,2^5,1^3)}+s_{(4,3,2^4,1)}$
$D(10,4,0,2)$	$s_{(3^5,1)}+s_{(4,3^4)}+s_{(4^2,3^2,2)}+s_{(4,3^3,2,1)}+s_{(3^4,2,1^2)}+s_{(4^2,2^3,1^2)}$ $+s_{(4,3,2^3,1^3)}+s_{(3^2,2^3,1^4)}+s_{(4^2,3,2^2,1)}+s_{(4,3^2,2^2,1^2)}+s_{(3^3,2^2,1^3)}$
$D(10,5,0,2)$	$-s_{(3^5,1)}-s_{(4,3^4)}+s_{(4^3,3,1)}+s_{(4^2,3^2,1^2)}+s_{(4,3^3,1^3)}+s_{(3^4,1^4)}$ $+s_{(4^3,2,1^2)}+s_{(4^2,3,2,1^3)}+s_{(4,3^2,2,1^4)}+s_{(3^3,2,1^5)}$
$D(10,6,0,2)$	$-s_{(4^2,3^2,1^2)}-s_{(4,3^3,1^3)}-s_{(3^4,1^4)}-s_{(4,3^3,2,1)}-s_{(3^4,2,1^2)}-s_{(3^4,2^2)}$
$D(10,7,0,2)$	$-s_{(4,3^2,2^2,1^2)}-s_{(3^3,2^2,1^3)}-s_{(3^3,2^3,1)}-s_{(3^3,2,1^5)}$ $-s_{(4,3^2,2,1^4)}-s_{(4^2,3,2,1^3)}$
$D(10,8,0,2)$	$-s_{(4,3,2^3,1^3)}-s_{(3^2,2^4,1^2)}-s_{(3^2,2^3,1^4)}$
$D(10,9,0,2)$	$-s_{(3,2^5,1^3)}$

Then we define

$$\tilde{\Delta}^{\lambda,\mu}(f_1,f_2)=\sum_{\nu}\max(a_\nu,b_\nu)s_\nu.$$

The following lemma gives a recurrence relation for $D_k(m,i,a,b)$.

Lemma 6.8 *For $m\ge i\ge b>a\ge 0$ and $k=1,2,3$, we have*

$$D_k(m,i,a,b)=\tilde{\Delta}^{(1),(3)}\big(D_k(m-1,i-1,a,b-1),D_k(m-2,i-1,a,b-1)\big).$$

Proof We shall consider only the case $k=2$, that is,

$$s_{(2^{i-b-1},1^{b+2-a})}s_{(2^{m-i-1})}=\tilde{\Delta}^{(1),(3)}(s_{(2^{i-b-1},1^{b+1-a})}s_{(2^{m-i-1})},s_{(2^{i-b-1},1^{b+1-a})}s_{(2^{m-i-2})}).$$

The cases $k=1$ and $k=3$ can be dealt with by the same argument.

First, we need to show that if s_λ appears in the Schur expansion of $s_{(2^{i-b-1},1^{b+1-a})}s_{(2^{m-i-1})}$ with multiplicity n, then the multiplicity of $s_{\lambda\cup(1)}$ in $s_{(2^{i-b-1},1^{b+2-a})}s_{(2^{m-i-1})}$ is at least n. To this end, we shall construct an injective map φ from the set of Littlewood–Richardson tableaux T of shape $\lambda/(2^{m-i-1})$ and type $(2^{i-b-1},1^{b+1-a})$ to the set of Littlewood–Richardson tableaux T' of shape $\lambda\cup(1)/(2^{m-i-1})$ and type $(2^{i-b-1},1^{b+2-a})$. For a given T, let T' be the tableau obtained from T by appending one row consisting of a single square filled with $i+1-a$. Clearly, T' is a Littlewood–Richardson tableau of $\lambda\cup(1)/(2^{m-i-1})$ and type $(2^{i-b-1},1^{b+2-a})$. See the first two tableaux in Fig. 8.

Moreover, we proceed to show that if s_λ appears in the Schur expansion of $s_{(2^{i-b-1},1^{b+1-a})}s_{(2^{m-i-2})}$ with multiplicity n, then the multiplicity of $s_{\lambda\cup(3)}$ in $s_{(2^{i-b-1},1^{b+2-a})}s_{(2^{m-i-1})}$ is at least n. To accomplish this task, we shall construct

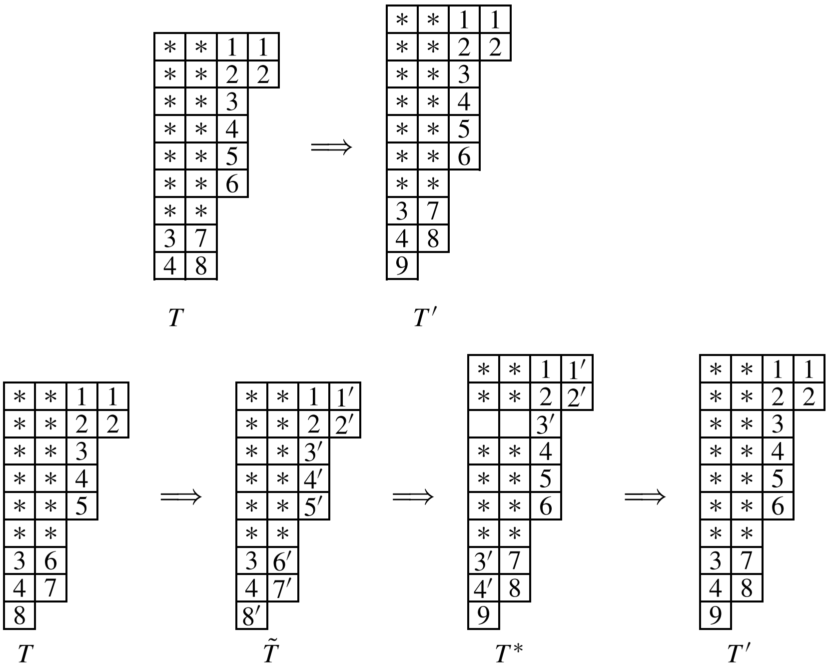

Fig. 8 Two ways to construct T'

an injective map ψ from the set of Littlewood–Richardson tableaux T of shape $\lambda/(2^{m-i-2})$ and type $(2^{i-b-1}, 1^{b+1-a})$ to the set of Littlewood–Richardson tableaux T' of shape $\lambda \cup (3)/(2^{m-i-1})$ and type $(2^{i-b-1}, 1^{b+2-a})$. For a given T, let us consider the corresponding tableau $\tilde{T}$, defined just before Lemma 6.2. Suppose that T has n rows of length 4 (setting $n = 0$ if no such a row exists). Let T^* be the tableau obtained from $\tilde{T}$ by inserting one row of three squares in the $(n+1)$th row in which the rightmost square is filled with $(n+1)'$, and then increasing all numbers below the $(n+1)$th row by 1, namely, changing j to j' and j' to $j+1$. Let T' be the tableau obtained from T^* by replacing j' with j for each j. It is straightforward to verify that T' is a Littlewood–Richardson tableau of shape $\lambda \cup (3)/(2^{m-i-1})$ and type $(2^{i-b-1}, 1^{b+2-a})$, as desired. Clearly, the map ψ is injective. See the last four tableaux in Fig. 8.

Finally, it remains to show that if s_λ appears in the Schur expansion of $s_{(2^{i-b-1},1^{b+2-a})}s_{(2^{m-i-1})}$ with multiplicity n, then one of the following two cases must occur: (i) there exists a partition μ such that $\lambda = \mu \cup (3)$ and the multiplicity of s_μ in $s_{(2^{i-b-1},1^{b+1-a})}s_{(2^{m-i-2})}$ is at least n; (ii) there exists a partition ν such that $\lambda = \nu \cup (1)$ and the multiplicity of s_ν in $s_{(2^{i-b-1},1^{b+1-a})}s_{(2^{m-i-1})}$ is at least n. We claim that if 3 is a part of λ, then case (i) must occur. This is readily seen in view of the reverse map of ψ, since for a given Littlewood–Richardson tableau T' of shape $\lambda/(2^{m-i-1})$ and type $(2^{i-b-1}, 1^{b+2-a})$, we can uniquely determine a Littlewood–Richardson tableau T of shape $\mu/(2^{m-i-2})$ and type $(2^{i-b-1}, 1^{b+1-a})$. If 3 does not appear as a part of λ, then we need to show that case (ii) must occur. This is true, because we can obtain a Littlewood–Richardson tableau T of shape $\nu/(2^{m-i-1})$ and type $(2^{i-b-1}, 1^{b+1-a})$ from T' by using the reverse map of φ. To be more specific, it is possible to apply the reverse map of φ to T' since the lattice permutation property requires that λ should contain a part of size 1 and the bottom square should be filled with $i+1-a$. This completes the proof. □

Now we are ready to finish the proof of Theorem 3.1.

Proof of Theorem 3.1 We use induction on the difference $b-a$. For $a=b$,

$$\sum_{i=0}^{m} D(m,i,a,b) = \sum_{i=a}^{m} D(m-a,i-a,0,0) = \sum_{i=0}^{m-a} D_{m-a,i},$$

which, according to Theorem 6.7, is Schur positive. Suppose $b-a \geq 1$. By Lemma 6.8, the negative terms of $D(m,i,a,b)$ come from either $\Delta^{(1)}(D(m-1, i-1,a,b-1))$ or $\Delta^{(3)}(D(m-2,i-1,a,b-1))$. They always vanish in $\sum_{i=0}^{m} D(m,i,a,b)$, since by induction both $\sum_{i=0}^{m} D(m-1,i-1,a,b-1)$ and $\sum_{i=0}^{m} D(m-2,i-1,a,b-1)$ are Schur positive. This completes the proof. □

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