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Shellable complexes from multicomplexes

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Abstract Suppose a group G acts properly on a simplicial complex Γ. Let l be the number of G-invariant vertices, and $p_1, p_2, \dots, p_m$ be the sizes of the G-orbits having size greater than 1. Then Γ must be a subcomplex of $\Lambda = \Delta^{l-1} * \partial\Delta^{p_1-1} * \cdots * \partial\Delta^{p_m-1}$. A result of Novik gives necessary conditions on the face numbers of Cohen–Macaulay subcomplexes of Λ. We show that these conditions are also sufficient, and thus provide a complete characterization of the face numbers of these complexes.

Keywords Simplicial complex · f-vector · Multicomplex

1 Introduction

One of the central problems in geometric combinatorics is that of characterizing the face numbers of various classes of simplicial complexes. The Kruskal–Katona theorem [5, 6] characterized the f-vectors of all simplicial complexes, while a result of Stanley characterized the face numbers of all Cohen-Macaulay complexes [9]. One fruitful line of inquiry since then has been in determining additional conditions on the face numbers of complexes with certain types of symmetry.

In particular, let Γ be a simplicial complex on n vertices, and suppose G is a group which acts on Γ. We say the action of G is *proper* if whenever F is a face of Γ and $gF = F$ for some $g \in G$, then $gv = v$ for each vertex $v \in F$, i.e., whenever an element of G fixes a face of Γ, it fixes that face pointwise. Let V' be the set of G-invariant vertices of Γ, and let $V_1, V_2, \dots, V_m$ be the G-orbits on the vertex set of Γ with size greater than 1. If the action of G is proper, no face of Γ can contain any V_i, so Γ must be a subcomplex of $\Lambda(l; p_1, p_2, \dots, p_m) = \Delta^{l-1} * \partial\Delta^{p_1-1} * \cdots * \partial\Delta^{p_m-1}$,

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where $l = |V'|$, $p_i = |V_i|$, $*$ denotes the join of complexes, Δ^k is the k-simplex, and $\partial\Delta^k$ is the boundary complex of Δ^k. (Note also that as each face of Γ must miss at least one element of each V_i, the dimension of Γ is at most $n-m-1$.)

Let $S(a_1, a_2, \ldots, a_k)$ (for $0 \le a_i \le \infty$) denote the set of all monomials $x_1^{c_1}x_2^{c_2}\cdots x_k^{c_k}$ with $c_i \le a_i$. For short, we will write $S(\infty^r, a_{r+1}, \ldots, a_k)$ for $S(a_1, a_2, \ldots, a_k)$ when $a_i = \infty$ for $1 \le i \le r$. A nonempty subset M of $S(a_1, a_2, \ldots, a_k)$ is called a *multicomplex* if it is closed under divisibility; that is, if whenever $\mu | \mu'$ and $\mu' \in M$, then $\mu \in M$. For M finite, let $\deg(M) = \max\{\deg(\mu) : \mu \in M\}$. The F-vector of a multicomplex M is $F(M) = (F_0, F_1, F_2, \ldots)$, where F_i is the number of elements in M of total degree i.

Recall that the h-vector of a $(d-1)$-dimensional simplicial complex Γ is $h(\Gamma) = (h_0, h_1, \ldots, h_d)$ defined by $\sum_{i=0}^d h_i x^i = \sum_{i=0}^d f_{i-1}x^i(1-x)^{d-i}$, where f_i is the number of i-dimensional faces of Γ. In particular, the h-vector of Γ completely determines the face numbers of Γ.

The following result is essentially due to Novik [8]. (In fact Novik considered the case $p_i = p_j$ for all i, j, but with slight modifications her proof gives the general case, as we will address in Sect. 5.)

Theorem 1 *Let Γ be a $(d-1)$-dimensional Cohen–Macaulay complex having $n = l + \sum_{i=1}^m p_i$ vertices, where $p_1, p_2, \ldots, p_m \ge 2$, $m, l \ge 0$, are arbitrary integers. If Γ is a subcomplex of $\Lambda(l; p_1, p_2, \ldots, p_m)$, then there is a multicomplex $M \subseteq S(\infty^{n-d-m}, p_1-1, p_2-1, \ldots, p_m-1)$ such that the h-vector of Γ is equal to the F-vector of M.*

The goal of this paper is to show the converse to this theorem. In fact, we establish a slightly stronger result.

Theorem 2 *Let $l, m \ge 0$, $p_1, p_2, \ldots, p_m \ge 2$ be arbitrary integers. Let $n = l + \sum_{i=1}^m p_i$ and suppose $d \le n-m$. If $M \subseteq S(\infty^{n-d-m}, p_1-1, p_2-1, \ldots, p_m-1)$ is a multicomplex such that* $\deg(M) \le d$, *then there is a $(d-1)$-dimensional shellable subcomplex Γ of $\Lambda(l; p_1, p_2, \ldots, p_m)$ such that $h(\Gamma) = F(M)$.*

Combined with Theorem 1, this gives a generalization of a theorem of Stanley [9], which asserts that $h = (h_0, h_1, \ldots, h_d)$ is the h-vector of a Cohen–Macaulay complex of dimension $d-1$ if and only if h is the F-vector of some multicomplex $M \subseteq S(\infty^{n-d})$.

Corollary 1 *Let $p_1, p_2, \ldots, p_m \ge 2, m, l \ge 0$ be arbitrary integers, $n = l + \sum_{i=1}^m p_i$, and $d \le n-m$. Suppose $F = (F_0, F_1, \ldots, F_d)$. Then the following are equivalent*:

1. *F is the h-vector of a shellable subcomplex of $\Lambda(l; p_1, p_2, \ldots, p_m)$.*
2. *F is the h-vector of a Cohen–Macaulay subcomplex of $\Lambda(l; p_1, p_2, \ldots, p_m)$.*
3. *F is the F-vector of a multicomplex in $S(\infty^{n-d-m}, p_1-1, p_2-1, \ldots, p_m-1)$.*

A different generalization of Stanley's theorem was given by Björner, Frankl, and Stanley in [2] for the case of balanced complexes. Our proof of Theorem 2 shares a similar structure to the sufficiency portion of their proof.

2 Idea of the proof

For τ a face of some simplicial complex, denote by $\overline{\tau}$ the set of all subsets of τ. Recall that a $(d-1)$-dimensional simplicial complex Γ is *shellable* if it is pure (i.e., all of its facets have dimension $d-1$) and there is an ordering of its facets $(\tau_1, \tau_2, \ldots, \tau_r)$ such that for $1 < i \leq r$, the complex $\overline{\tau}_i \cap (\cup_{j<i}\overline{\tau}_j)$ is pure of dimension $d-2$. Such an ordering is then called a *shelling* of Γ. For $L = (\tau_1, \tau_2, \ldots, \tau_r)$ any ordering of the facets of Γ, let $T_L(\tau_i)$ denote the set of facets of $\overline{\tau}_i \cap \left(\cup_{j<i}\overline{\tau}_j\right)$ (which will be some set of subsets of τ_i of size $d-1$ if L is a shelling) for $i > 1$, and set $T_L(\tau_1) = \emptyset$. We then have the following nice characterization of the h-vector of Γ:

Proposition 1 [7] *Let $(h_0, h_1, \ldots, h_d)$ be the h-vector of Γ. If L is a shelling of Γ, then $h_i = |\{\tau_j : |T_L(\tau_j)| = i\}|$.*

Now, suppose Γ is a simplicial complex with shelling L, and suppose K is a subset of the set of facets of Γ. Let $L' = (\tau'_1, \tau'_2, \ldots, \tau'_{r'})$ be the ordering of K inherited from L. Suppose that $T_{L'}(\tau) = T_L(\tau)$ for each $\tau \in K$. Then it follows immediately that $\Gamma' = \bigcup_{i=1}^{r'} \overline{\tau'_i}$ is a shellable subcomplex of Γ with h-vector $(h'_1, h'_2, \ldots, h'_d)$, where $h'_i = |\{\tau \in K : |T_L(\tau)| = i\}|$.

For Γ a simplicial complex, let $\mathrm{skel}_d(\Gamma)$ denote its $(d-1)$-skeleton, that is, the set of faces of Γ of dimension no greater than $d-1$. For M a multicomplex, let M^d denote the set of monomials in M with degree no greater than d. Throughout this section let $\Lambda = \Lambda(l; p_1, p_2, \ldots, p_m)$ and $S = S(\infty^{n-d-m}, p_1 - 1, p_2 - 1, \ldots, p_m - 1)$.

To prove Theorem 2, we will construct a shelling L of $\mathrm{skel}_d(\Lambda)$ and show that for $(F_0, F_1, \ldots, F_d)$ the F-vector of some multicomplex M in S^d, there is a subsequence $L' = (\tau'_1, \ldots, \tau'_r)$ of L such that each $T_{L'}(\tau'_i) = T_L(\tau'_i)$, and the number of τ'_i with $|T_L(\tau'_i)| = j$ is F_j. We then have a shellable subcomplex of $\mathrm{skel}_d(\Lambda)$ with h-vector equal to the F-vector of M.

To do this, we will establish a bijection σ between the set of facets of $\mathrm{skel}_d(\Lambda)$ and S^d with the property that $|T_L(\tau)| = \deg(\sigma(\tau))$. For $M \subseteq S^d$ a multicomplex, let L^M be the restriction of L to $\sigma^{-1}(M)$. Then if $T_{L^M}(\tau) = T_L(\tau)$ for each $\tau \in \sigma^{-1}(M)$, L^M gives a shelling of a subcomplex of $\mathrm{skel}_d(\Lambda)$ (namely, the pure complex generated by the elements of L^M) with h-vector equal to the F-vector of M.

We will need to restrict our attention to a special class of multicomplexes. Define a partial order on our monomials as follows. For $\mu = x_1^{c_1} x_2^{c_2} \cdots x_k^{c_k}$ and $\mu' = x_1^{d_1} x_2^{d_2} \cdots x_k^{d_k}$ elements of $S(a_1, a_2, \ldots, a_k)$ with $\deg(\mu) = \deg(\mu')$, say $\mu < \mu'$ if for some i, $c_i < d_i$ and $c_j = d_j$ for all $j > i$ (reverse lexicographical order within degrees). For $F = (F_0, F_1, \ldots)$ the F-vector of some multicomplex $M \subseteq S(a_1, a_2, \ldots, a_k)$, let S_{i,F_i} be the set of the first F_i degree i elements of $S(a_1, \ldots, a_k)$ in the reverse lex order, and set $I_M = \bigcup_{i \geq 0} S_{i,F_i}$. A result of Clements and Lindström will allow us to replace M with I_M:

Theorem 3 [4] *Suppose M is a multicomplex in $S(a_1, a_2, \ldots, a_k)$, where $a_1 \geq a_2 \geq \cdots \geq a_k$. Then I_M is a multicomplex.*

In particular, we may from now on assume that our multicomplex M has the property that if $\deg(\mu) = \deg(\mu')$, $\mu < \mu'$, and $\mu' \in M$, then $\mu \in M$ (as I_M clearly has this property and $F(I_M) = F(M)$). Thus, it will suffice to construct L and σ such that whenever $\gamma \in T_L(\tau_i)$, there exist $j < i$ and divisor μ of $\sigma(\tau_i)$ such that $\gamma \subseteq \tau_j$, $\deg(\mu) = \deg(\sigma(\tau_j))$, and $\sigma(\tau_j) \le \mu$. Then if $\tau_i \in \sigma^{-1}(M)$, the properties of M require that $\sigma(\tau_j) \in M$, so $\tau_j \in \sigma^{-1}(M)$, and then as $\gamma \subseteq \tau_j$, $T_{L^M}(\tau_i) = T_L(\tau_i)$.

3 An illustrative example

At this point it will be helpful to look at a small but nontrivial example. Let $d = 4$ and $\Lambda = \Lambda(0; 3, 3) = \partial\Delta^2 * \partial\Delta^2$. The vertex set V of Λ decomposes into the vertex sets P_1 and P_2 of the two copies of $\partial\Delta^2$. The faces of $\mathrm{skel}_4(\Lambda)$ are precisely the subsets of V of size 4 that do not contain either P_1 or P_2. Label the vertices of Λ as shown:

P_1	P_2
∘ 1	∘ 2
∘ 3	∘ 4
∘ 5	∘ 6

We want to build a shelling of $\mathrm{skel}_4(\Lambda)$ and a correspondence σ between the facets of $\mathrm{skel}_4(\Lambda)$ and the elements of $S = S(2, 2)$ with the properties described at the end of the last section. Given our use of the reverse lexicographical order on the set of monomials, it is tempting to simply list the facets in reverse lex order L_R (which will indeed give a shelling) and for τ the ith facet of $\mathrm{skel}_4(\Lambda)$ having $|T_{L_R}(\tau)| = j$, let $\sigma(\tau)$ be the ith monomial in $S(2, 2)$ of degree j. In fact such an approach will work in some simple cases. Here, however, it fails:

τ	$\lvert T_{L_R}(\tau)\rvert$	$\sigma(\tau)$	τ	$\lvert T_{L_R}(\tau)\rvert$	$\sigma(\tau)$
1234	0	1	1256	2	x_2^2
1245	1	x_1	2356	3	$x_1^2x_2$
2345	2	x_1^2	1456	3	$x_1x_2^2$
1236	1	x_2	3456	4	$x_1^2x_2^2$
1346	2	x_1x_2			

In particular, consider the multicomplex $M = \{1, x_1, x_2, x_1^2, x_1x_2, x_1^2x_2\}$. Note that $M = I_M$, but $L^M = (1234, 1245, 2345, 1236, 1346, 2356)$. Then $T_{L^M}(2356) = \{235, 236\} \neq T_L(2356)$, and letting $\Gamma = \bigcup_{\tau\in\sigma^{-1}(M)} \overline{\tau}$ be the pure complex generated by the elements of L^M, $h(\Gamma) = (1, 2, 3, 0, 0) \neq F(M)$. The problem is that $T_L(2356) = \{235, 236, 256\}$, and these faces first appear in facets corresponding to x_1^2, x_2, and x_2^2. But $\sigma(2356) = x_1^2x_2$, the presence of which in M does not imply that of x_2^2.

Let us examine the problem more closely. Notice that our ordering on the vertex set has resulted in each facet ending in 5 corresponding to a monomial with greatest variable x_1, and any facet ending in 6 corresponding to a monomial with greatest variable x_2. This leads us to make the following definitions.

Suppose we have a total order on the set of vertices of simplicial complex Λ and label the elements of this set $y_1 < y_2 < \cdots$ accordingly. Then let

$$\Lambda_i := \Big\{\gamma \in \operatorname{link}_{\Lambda}(y_i) : \gamma \subseteq \{y_1, \ldots, y_{i-1}\}\Big\}.$$

Similarly, if the variables in a multicomplex M are ordered $x_1 < x_2 < \cdots$, define

$$M_i := \big\{\mu \in M : \operatorname{supp}(\mu) \subseteq \{x_1, \ldots, x_i\} \text{ and } \mu x_i \in M\big\}.$$

Now observe that any facet of $\operatorname{skel}_d(\Lambda)$ is, for some i, of the form $\gamma \cup y_i$, where $\gamma \in \operatorname{skel}_{d-1}(\Lambda_i)$, and any element of S^d (aside from 1) is, for some i, of the form μx_i, where $\mu \in S_i^{d-1}$.

Consider Λ_6. This is isomorphic to $\Lambda(0; 3, 2)$.

Note that our original ordering of facets gives a shelling of $\operatorname{skel}_3(\Lambda_6)$ and correspondence σ' to elements of $S(2, 1)$, by taking $\sigma'(\tau) = \frac{\sigma(\tau \cup \{6\})}{x_2}$.

τ	$\lvert T(\tau)\rvert$	$\sigma'(\tau)$
123	0	1
134	1	x_1
125	1	x_2
235	2	x_1^2
145	2	x_1x_2
345	3	$x_1^2x_2$

Here we see the same problem as before, occurring at 235. Naïvely we might note that here we no longer have the nice correspondence between last variable and last vertex we had in the larger ordering, but this deficiency is easily fixed by a simple reordering of the vertex set. In fact, consider the shelling and map obtained if we order our facets as if $4 > 5$, while retaining our ordering on the monomials:

τ	$\lvert T(\tau)\rvert$	$\sigma'(\tau)$
123	0	1
125	1	x_1
235	2	x_1^2
134	1	x_2
154	2	x_1x_2
354	3	$x_1^2x_2$

It is simple to check that this correspondence has the property described at the end of the previous section, and furthermore we can use this to fix our original attempt, by reordering the facets ending in 6 to match our new ordering on the facets of $\mathrm{skel}_3(\Lambda_6)$:

τ	$\|T(\tau)\|$	$\sigma(\tau)$	τ	$\|T(\tau)\|$	$\sigma(\tau)$
1234	0	1	2356	3	$x_1^2x_2$
1245	1	x_1	1346	2	x_2^2
2345	2	x_1^2	1456	3	$x_1x_2^2$
1236	1	x_2	3456	4	$x_1^2x_2^2$
1256	2	x_1x_2			

The example suggests that we should build our shelling and map σ inductively, at each step making sure the vertices are ordered so that the last m vertices are from $P_1, P_2, \dots, P_m$, respectively. This is how we shall proceed.

4 Construction of the shelling and bijection

Let $\Lambda = \Lambda(l; p_1, \dots, p_m)$ with $p_1 \geq p_2 \geq \cdots \geq p_m$. Let V' be the vertex set of the Δ^{l-1} in the construction of Λ and for $1 \leq i \leq m$, let P_i be the vertex set of $\partial\Delta^{p_i-1}$. For ease of a later induction argument, we will now allow $p_i = 1$, in which case we will take P_i to contain a solitary placeholder vertex which is contained in no face of Λ. Similarly, we will allow $S(a_1, \dots, a_k)$ where $a_i = 0$, in which case the variable x_i appears in no monomial. Let $V = V' \cup (\bigcup_i P_i)$, and let $n = l + \sum |P_i|$. Let $S = S(\infty^{n-d-m}, p_1 - 1, p_2 - 1, \dots, p_m - 1)$.

As we will be changing the ordering on the vertices at different steps of our induction, we will require some additional notation. For O denoting a total ordering $y_1 < y_2 < \cdots < y_n$ of V, let $\Lambda_{k,O}$ be Λ_k, as defined in the previous section, with respect to ordering O. (The ordering $x_1, x_2, \dots, x_{n-d}$ will remain fixed, so S_k may remain as above.)

Recall that one characterization of a shelling $L = (\tau_1, \tau_2, \dots, \tau_r)$ is that for each i, there exists a face $R(\tau_i)$ of τ_i such that $\overline{\tau_i} - (\bigcup_{j<i} \overline{\tau_j}) = \{\gamma \subseteq \tau_i : R(\tau_i) \subseteq \gamma\}$. (Note in particular that $|T_L(\tau_i)| = |R_i(\tau)|$.) Examining the two shellings of $\mathrm{skel}_4(\Lambda(0; 3, 3))$ in our example in the last section, we see that both yield the same $R(\tau)$ for each facet τ of $\mathrm{skel}_4(\Lambda)$. It will be helpful to determine the exact structure of the $R(\tau)$ in the shelling obtained by listing the facets of $\mathrm{skel}_d(\Lambda)$ in the reverse lexicographical order.

Let τ be a facet of $\mathrm{skel}_d(\Lambda)$, and O an ordering of V. If the corresponding reverse lexicographical ordering of the facets of Λ is a shelling, the corresponding subset $R(\tau)$ of τ should be the unique minimal subset of τ contained in no earlier facet. In other words, any subset of τ which does not contain $R(\tau)$ should appear in an earlier facet; in particular, for any $v \in R(\tau)$, $\tau - v$ must appear in an earlier facet. Such facet would have the form $(\tau - v) \cup w$ where $w < v$ and $w \notin \tau$. There are two ways to find such a facet. First, there may be $w < v$ which is contained in a part of the partition P_j with $\tau \cap P_j \neq P_j - w$, in which case $(\tau - v) \cup w$ is a facet of $\mathrm{skel}_d(\Lambda)$. So let $\mathrm{full}(\tau) = \{i : |P_i \cap \tau| = |P_i| - 1\}$ (the notation is meant to suggest that $\mathrm{full}(\tau)$

collects the indices of the sets P_i such that $\tau \cap P_i$ is "full" in the sense that no further elements of P_i could be added without leaving Λ), and let $s_O(\tau)$ be the first element (with respect to order O) of $V - \tau$ not appearing in P_j with $j \in \text{full}(\tau)$ if such an element exists, otherwise set $s_O(\tau) = \infty$. Let $\tau_{>s_O(\tau)} = \{y \in \tau : y > s_O(\tau)\}$; this contains all v such that we may find a $w < v$, $w \notin \tau$ and w not in a "full" part of the partition.

On the other hand, suppose each $w < v$, $w \notin \tau$, is in some P_j such that $j \in \text{full}(\tau)$. For such a w, $(\tau - v) \cup w$ is a facet of $\text{skel}_d(\Lambda)$ if and only if $v \in P_j$. So for $i \in \text{full}(\tau)$, set $\text{miss}(\tau, i)$ be the element of P_i *not* in τ, and let $U_O(\tau) = \{y : y \in P_j$ and $y > \text{miss}(\tau, j)$ for some $j \in \text{full}(\tau)\}$.

Finally, let $R_O(\tau) = \tau_{>(s_O(\tau))} \cup U_O(\tau)$. From the discussion above it follows that any subset of τ not containing R_O occurs in an earlier facet; on the other hand, it is simple to check that $R_O(\tau)$ is not itself contained in any earlier facet. Thus, if $L = (\tau_1, \tau_2, \dots, \tau_r)$ is the reverse lex order on the facets of $\text{skel}_d(\Lambda)$, $\overline{\tau_i} - (\bigcup_{j<i} \overline{\tau_j}) = \{\gamma \subseteq \tau : R_O(\tau_i) \subseteq \gamma\}$. Our inductively built shelling will share this structure.

Example 1 Let $\Lambda = \Lambda(1; 5, 4, 3)$, with vertex ordering O as shown:

V'	P_1	P_2	P_3
y_1	y_2	y_3	y_4
	y_5	y_6	y_7
	y_8	y_9	y_{13}
	y_{10}	y_{12}	
	y_{11}		

Consider the face $\tau = \{y_1, y_2, y_4, y_5, y_6, y_9, y_{11}, y_{12}\}$ (Fig. 1). Then $\text{full}(\tau) = \{2\}$, $\text{miss}(\tau, 2) = y_3$, $U_O(\tau) = \{y_6, y_9, y_{12}\}$ (Fig. 2), $s_O(\tau) = y_7$, and, finally, $\tau_{>s_O(\tau)} = \{y_9, y_{11}, y_{12}\}$ (Fig. 3). So $R_O(\tau) = \{y_6, y_9, y_{11}, y_{12}\}$.

We are now ready to prove our central theorem. Recall that for $L = (\tau_1, \tau_2, \dots, \tau_t)$ a shelling, $T_L(\tau_i)$ denotes the set of facets of $\overline{\tau}_i \cap (\bigcup_{j<i} \overline{\tau}_j)$.

Fig. 1 τ

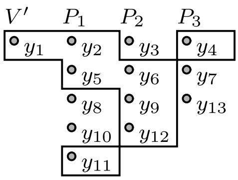


Fig. 2 $U_O(\tau)$

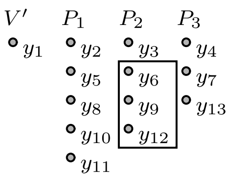
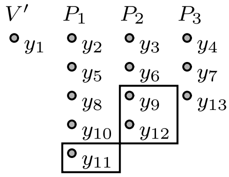


Fig. 3 $\tau_{>s_O(\tau)}$



Theorem 4 *Let Λ and S be as above, and let O be an ordering $y_1 < y_2 < \cdots < y_n$ of V such that for $1 \le i \le m$, $y_{n-m+i} \in P_i$. Let $1 \le d \le n-m$. Then there exist a shelling $L = (\tau_1, \tau_2, \ldots, \tau_t)$ of* $\mathrm{skel}_d(\Lambda)$ *and bijection σ from the set of facets of* $\mathrm{skel}_d(\Lambda)$ *to S^d such that*:

1. $\overline{\tau_i} - (\bigcup_{j<i} \overline{\tau_j}) = \{\gamma \subseteq \tau : R_O(\tau) \subseteq \gamma\}$.
2. $\deg(\sigma(\tau_i)) = |T_L(\tau_i)|$.
3. *If $\gamma \in T_L(\tau_i)$, then there exist $j < i$ and $\mu | \sigma(\tau_i)$ such that $\gamma \subseteq \tau_j$,* $\deg(\mu) = \deg(\sigma(\tau_j))$, *and $\sigma(\tau_j) \le \mu$.*

Again, condition (1) is sufficient to show that L is a shelling. Theorem 2 follows from (2) and (3), the proof of the latter requiring our precise definition of R_O.

Proof of Theorem 4 We will proceed by induction on d. Suppose $d = 1$, and note that by assumption if we have any $p_i = 1$, the elements of these P_i must occur after all the other vertices in our order. Furthermore, not every vertex may be contained in such a P_i, as then we would have $n - m = m - m = 0 < d$. So let y_k be the last vertex not contained in such a P_i, and let $L = (y_1, y_2, \ldots, y_k)$, $\sigma(y_1) = 1$, and $\sigma(y_i) = x_{i-1}$ for $1 < i \le k$. Properties (1)–(3) immediately follow.

Now, suppose $1 < d \le n - m$. Set $\tau_1^0 = \{y_1, y_2, \ldots, y_d\}$. By the properties of our order on V, τ_1^0 does not contain any P_i and hence is a facet of $\mathrm{skel}_d(\Lambda)$. Set $\sigma(\tau_1^0) = 1$.

Any other facet τ of $\mathrm{skel}_d(\Lambda)$ has the form $\tau = \tau' \cup y_{d+k}$, where $\tau' \in \mathrm{skel}_{d-1}(\Lambda_{d+k,O})$ for some $k > 0$. Similarly, any element of S^d aside from 1 is of the form μx_k, where $\mu \in S_k^{d-1}$ for some $k \ge 1$.

Suppose $d + k \le n - m$. Then at least one vertex from each P_j occurs after y_{d+k}, so the union of y_{d+k} and any $(d-1)$-subset of the preceding vertices is in Λ, i.e., $\mathrm{skel}_{d-1}(\Lambda_{d+k,O})$ is the $(d-2)$-skeleton of the simplex on the first $d + k - 1$ vertices in V. Then the ordering O_k on these vertices inherited from the original order on V satisfies the conditions of our theorem, so by induction there exist a shelling of $\mathrm{skel}_{d-1}(\Lambda_{d+k,O_k})$ and map σ_k from its set of facets to $S^{d-1}(\infty^{d+k-1-(d-1)}) = S^{d-1}(\infty^k) = S_k^{d-1}$ satisfying (1)–(3). Call this shelling $L_k = (G_1^k, G_2^k, \ldots, G_{r_k}^k)$.

On the other hand, suppose $d + k = n - m + i$ for some $1 \le i \le m$. If $p_i = 1$, then $\mathrm{skel}_{d-1}(\Lambda_{d+k,O}) = \emptyset$, as y_{d+k} is the sole element of P_i and thus is not in any face of Λ. Otherwise, $\mathrm{skel}_{d-1}(\Lambda_{d+k,O}) = \mathrm{skel}_{d-1}(\Lambda(l + \sum_{j>i}(p_j - 1); p_1, p_2, \ldots, p_{i-1}, p_i - 1))$ with $p_i - 1 \ge 1$. In this case, the restriction of the order on V does not quite meet the conditions of the theorem. Let y^k be the largest element of $P_i - y_{d+k}$ with respect to O (this set is not empty as we are assuming $p_i \ge 2$). Define a new order O_k by $y_1 <_k y_2 <_k \cdots <_k \widehat{y^k} <_k \cdots <_k y_{d+k-1} <_k y^k$ (i.e., take

Fig. 4 G

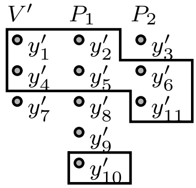
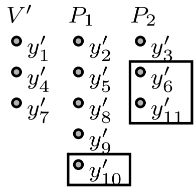


Fig. 5 $R_{O_{12}}(G)$



the original order but set $y_t <_k y^k$ for all $y_t \neq y^k$, as in the example in the previous section). This new order satisfies the conditions of our theorem, and so by induction we have a shelling $L_k = (G_1^k, G_2^k, \dots, G_{r_k}^k)$ of $\mathrm{skel}_{d-1}(\Lambda_{d+k,O_k})$ and map σ_k from the set of its facets to $S^{d-1}(\infty^{d+k-1-(d-1)-i}, p_1 - 1, \dots, p_{i-1} - 1, p_i - 2) = S^{d-1}(\infty^{n-m-d}, p_1 - 1, \dots, p_{i-1} - 1, p_i - 2) = S_k^{d-1}$ satisfying (1)–(3).

For $1 \le k \le n-d$ such that y_{d+k} is not in P_j with $p_j = 1$, set $\tau_i^k = G_i^k \cup y_{d+k}$ and $\sigma(\tau_i^k) = \sigma(G_i^k)x_k$ for each $1 \le i \le r_k$. We claim that the listing of facets $L = (\tau_1^0, \tau_1^1, \dots, \tau_{r_1}^1, \tau_1^2, \dots, \tau_{r_2}^2, \dots)$ and σ satisfy (1)–(3) for any τ_i^k.

(1) The $k = 0$ case is immediate. Now suppose $k > 0$. Set $R_i^k = R_{O_k}(G_i^k) \cup y_{d+k}$. We will first show that $\overline{\tau_i^k} - (\bigcup_{k'<k} \overline{\tau_j^{k'}} \cup (\bigcup_{j<i} \overline{\tau_j^k})) = \{\gamma \subseteq \tau_i^k : R_i^k \subseteq \gamma\}$ and then that $R_i^k = R_O(\tau_i^k)$.

Example 2 Let Λ, O, and τ be as in Example 1. Then $\tau = G \cup y_{12}$, where $G \in \mathrm{skel}_7(\Lambda_{12}) = \mathrm{skel}_7(\Lambda(3; 5, 3))$. In the new ordering O_{12}, y_9 becomes the last vertex, so labeling the ith vertex in this ordering y'_i, we have $y'_i = y_i$ for $i < 9$, $y'_9 = y_{10}$, $y'_{10} = y_{11}$, and $y'_{11} = y_9$. Observe that $R_O(\tau) = R_{O_{12}}(G) \cup y_{12}$ (see Figs. 4 and 5).

Returning now to the proof, suppose that $\gamma \subseteq \tau_i^k$ and $R_i^k \subseteq \gamma$. Then $y_{d+k} \in \gamma$, so γ cannot be in any $\tau_i^{k'}$ for $k' < k$. On the other hand, as $\gamma - y_{d+k}$ contains $R_{O_k}(G_i^k)$, there is no $j < i$ such that G_j^k contains $\gamma - y_{d+k}$. Hence γ is not in any τ_j^k for $j < i$, so γ can occur in no facet appearing before τ_i^k.

We next show that any subset of τ_i^k not containing R_i^k is contained in an earlier facet. Suppose $R_i^k \nsubseteq \gamma \subseteq \tau_i^k$. Then there is at least one element of R_i^k not in γ. If some such element is in $R_{O_k}(G_i^k)$, then $\gamma - y_{d+k}$ is a face of G_i^k not containing $R_{O_k}(G_i^k)$, so there is $j < i$ such that $\gamma - y_{d+k} \subseteq G_j^k$, and then $\gamma \subseteq \tau_j^k$. Otherwise, $\gamma = \tau_i^k - y_{d+k} = G_i^k$. Now, there is clearly some $r < d+k$ such that $y_r \notin G_i^k$. Suppose, in order to obtain a contradiction, that for each such r, $G_i^k \cup y_r$ is not a facet of $\mathrm{skel}_d(\Lambda)$, i.e., $G_i^k \cup y_r$ contains some P_s. Then $d+k = n-m+j$ for some $j > 1$ (otherwise, G_i^k cannot contain any element of the form y_{n-m+s} for $1 \le s \le m$, and adding any vertex before y_{d+k} cannot complete P_s). But then there are at least $|P_{j'}| - 1$ elements of each $P_{j'}$ occurring before y_{d+k} in our ordering for each $1 \le j' \le m$, so G_i^k must contain at least $|P_{j'}| - 1$ elements of each $P_{j'}$, in addition to all of V'. Hence $d = |G_i^k| + 1 \ge l + \sum(|P_{j'}| - 1) + 1 \ge n - m + 1$, a contradiction. Hence, there is some $r < d + k$ such that $G_i^k \cup y_r$ is a facet of Λ^d. This facet occurs before τ_i^k and contains γ.

It remains to show that $R_i^k = R_O(\tau_i^k)$ for $k > 0$. We first confirm that $y_{d+k} \in R_O(\tau_i^k)$. If $y_{d+k} > s_O(\tau_i^k)$, then y_{d+k} is in $R_O(\tau_i^k)$, so suppose $y_{d+k} < s_O(\tau_i^k)$. Then as y_{d+k} is the greatest element of τ_i^k, τ_i^k must consist of all elements of $V - \bigcup_{j \in \mathrm{full}(\tau_j^k)} \{\mathrm{miss}(\tau_i^k, j)\}$ less than y_{d+k}. Suppose $d + k \le n - m$. Then τ_i^k cannot contain the largest element of any P_i, so in particular $\mathrm{miss}(\tau_i^k, j) > y_{d+k}$ for any $j \in \mathrm{full}(\tau_i^k)$. Thus, τ_i^k is just the first d elements of V, i.e., τ_1^0. But $k > 0$, so we must have $d + k = n - m + j$ for some j, and in particular y_{d+k} is the largest element of P_j. But then $j \in \mathrm{full}(\tau_i^k)$ and $y_{d+k} > \mathrm{miss}(\tau_i^k, j)$, so $y_{d+k} \in U_O(\tau_i^k) \subseteq R_O(\tau_i^k)$.

We next show that $R_O(\tau_i^k) - y_{d+k} = R_{O_k}(G_i^k)$. Suppose $d + k \le n - m$. In this case our orderings O and O_k are the same, so $s_{O_k}(G_i^k) = s_O(\tau_i^k)$. Furthermore, as τ_i^k cannot contain the largest element of any P_j, $U_O(\tau_i^k) = \emptyset$. Thus $R_O(\tau_i^k) - y_{d+k} = R_{O_k}(G_i^k)$.

On the other hand, suppose $d + k = n - m + j$. Observe that the vertices corresponding to the indices in $\mathrm{full}(G_i^k)$ are the same as those corresponding to the indices in $\mathrm{full}(\tau_i^k)$, and as y^k is the largest element of $P_j - y_{d+k}$ (with respect to both orders), with O_k matching O on all the other vertices, $U_O(\tau_i^k)$ and $U_{O_k}(G_i^k)$ agree, except for the possible presence of y_{d+k} in the former.

Suppose $s_{O_k}(G_i^k) < y^k$. Then $s_O(\tau_i^k) = s_{O_k}(G_i^k)$. Furthermore, for $y \in G_i^k$, $y > s_O(\tau_i^k)$ if and only if $y >_k s_O(\tau_i^k)$. Thus $R_O(\tau_i^k) - y_{d+k} = R_{O_k}(G_i^k)$.

On the other hand, suppose $s_{O_k}(G_i^k) \ge y^k$. As y^k is the greatest element of $P_j - y_{d+k}$, any other element y of $P_j - y_{d+k}$ is less than $s_{O_k}(G_i^k)$ in both orders. Thus, either (i) all of these elements are in G_i^k, or (ii) exactly one is missing, and every other element of $P_j - y_{d+k}$ is in G_i^k.

Case (i): Every element of $P_j - y_{d+k}$ less than y^k is in G_i^k. Then y^k cannot be in G_i^k, and in particular $y^k = \mathrm{miss}(G_i^k, j) = \mathrm{miss}(\tau_i^k, j)$. Thus $s_O(\tau_i^k) = s_{O_k}(G_i^k)$ (as the changing of the position of y^k in the order will have no effect on s). Furthermore, $s_O(\tau_i^k) \ne y^k$, and $y^k \notin \tau_i^k$. Then for $y \in G_i^k$, $y >_k s_{O_k}(G_i^k)$ if and only if $y > s_O(\tau_i^k)$. Thus $R_O(\tau_i^k) - y_{d+k} = R_{O_k}(G_i^k)$.

Case (ii): G_i^k contains every element of P_j except some $y < y^k$. Then in particular $y^k \in G_i^k$, so $s_O(\tau_i^k) = s_{O_k}(G_i^k) > y^k$. It now only remains to check the membership of y^k in R_i^k and $R_O(\tau_i^k)$. But $y^k \in U_{O_k}(G_i^k) = U_O(\tau_i^k)$ and is thus in both R_i^k and $R_O(\tau_i^k)$. Hence, (1) is proved.

(2) Note that $|T_L(\tau_i^k)| = |R_O(\tau_i^k)| = |R_{O_k}(G_i^k)| + 1$. Then as $|R_{O_k}(G_i^k)| = \deg(\sigma_k(G_i^k))$ by induction, (2) follows from the definition of σ.

(3) The $k = 0$ case is trivial. Suppose $k > 0$ and $\gamma \in T_L(\tau_i^k)$. Then γ is obtained from τ_i^k by removing some element of $R_O(\tau_i^k)$. Suppose that the element is not y_{d+k}.

Then $\gamma - y_{d+k} \in T_{L_k}(G_i^k)$. Thus, there exist $j < i$ and divisor μ of $\sigma_k(G_i^k)$ such that $\gamma - y_{d+k} \in G_j^k$, $\deg(\mu) = \deg(\sigma_k(G_j^k))$, and $\sigma_k(G_j^k) \le \mu$. Then $\gamma \in \tau_j^k$, μx_k is a divisor of $\sigma(\tau_i^k)$, $\deg(\mu x_k) = \deg(\sigma(\tau_j^k))$, and $\sigma(\tau_j^k) \le \mu x_k$.

On the other hand, suppose $\gamma = \tau_i^k - y_{d+k}$. We claim that there exists a facet τ_t^r for some $r < k$ such that $\deg(\sigma(\tau_t^r)) \le \deg(\sigma(\tau_i^k))$ and $\gamma \subset \tau_t^r$. From this it will follow that (3) is satisfied with τ_t^r playing the role of τ_j in the theorem's statement.

Case 1: $y_{d+k} \in U_O(\tau_i^k)$, where $y_{d+k} \in P_j$. Then let $\tau' = \gamma \cup \text{miss}(\tau_i^k, j)$. Note that $s_O(\tau') = s_O(\tau_i^k)$ and $\text{full}(\tau_i^k) = \text{full}(\tau')$. Suppose $y \in R_O(\tau')$ and $y \neq \text{miss}(\tau_i^k, j)$. We will show that $y \in R_O(\tau_i^k)$. Note that $y \in \tau_i^k$, and if $y > s_O(\tau')$, then $y > s_O(\tau_i^k)$, so $y \in R_O(\tau_i^k)$. On the other hand, if $y \in U_O(\tau')$, then $y \in U_O(\tau_i^k)$, as $\text{miss}(\tau', q) \ge \text{miss}(\tau_i^k, q)$ for all $q \in \text{full}(\tau') = \text{full}(\tau_i^k)$. Thus $y \in R_O(\tau_i^k)$. So every element of $R_O(\tau') - \text{miss}(\tau_i^k, j)$ is in $R_O(\tau_i^k)$, and as y_{d+k} is in $R_O(\tau_i^k)$ but not τ', $R_O(\tau') - \text{miss}(\tau_i^k, j) \subseteq R_O(\tau_i^k) - y_{d+k}$. Hence $|R_O(\tau')| \le |R_O(\tau_i^k)|$.

Example 3 Again take Λ, O, and τ as in Example 1, and consider $\gamma = \tau - y_{12}$. Then $R_O(\tau') = \{y_9, y_{11}\}$, see Figs. 6, 7 and 8.

Case 2: $y_{d+k} \notin U_O(\tau_i^k)$. Let $\tau' = \gamma \cup s_O(\tau_i^k)$ (and recall that since $k > 0$, we have seen that $y_{d+k} \in R_O(\tau_i^k)$, so we must have $s_O(\tau_i^k) < y_{d+k}$). Then $s_O(\tau') > s_O(\tau_i^k)$. Suppose $y \in R_O(\tau')$ and $y \neq s_O(\tau_i^k)$. Again, $y \in \tau_i^k$; we will show that $y \in R_O(\tau_i^k)$. If $y > s_O(\tau')$, then $y > s_O(\tau_i^k)$, so $y \in R_O(\tau_i^k)$. On the other hand, suppose $y \in U_O(\tau')$. If $y \in U_O(\tau_i^k)$, then $y \in R_O(\tau_i^k)$. So finally suppose y is in $U_O(\tau')$ but not

Fig. 6 G

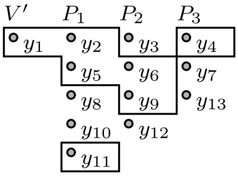


Fig. 7 τ'

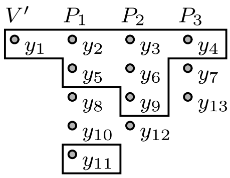
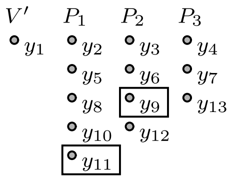


Fig. 8 $R_O(\tau')$



in $U_O(\tau_i^k)$. Then for some q, τ' contains all but one element, b, of P_q, $y \in P_q$, and $y > b$, but τ_i^k is missing at least 2 elements of P_q. The only element of τ_i^k which is not in τ' is $y_{d+k} > y$, so $b \notin \tau_i^k$. Then as $q \notin \mathrm{full}(\tau_i^k)$, $b \geq s_O(\tau_i^k)$. Thus $y > s_O(\tau_i^k)$, and so $y \in R_O(\tau_i^k)$. Thus, every element of $R_O(\tau') - s_O(\tau_i^k)$ is in $R_O(\tau_i^k)$, and as before we see that $R_O(\tau') - s_O(\tau_i^k) \subseteq R_O(\tau_i^k) - y_{d+k}$. Hence $|R_O(\tau')| \leq |R_O(\tau_i^k)|$.

In either case, τ' is a facet of $\mathrm{skel}_d(\Lambda)$ containing γ and by construction must be equal to τ_t^r for some $r < k$. Since $|R_O(\tau')| \leq |R_O(\tau_i^k)|$, $\deg(\sigma(\tau')) \leq \deg(\sigma(\tau_i^k))$. If $r = 0$, $\sigma(\tau') = 1$, a divisor of $\sigma(\tau_i^k)$. Otherwise, $\sigma(\tau')$ is some monomial in $x_1, \dots, x_r$. Let μ be the reverse lexicographically largest divisor of $\sigma(\tau_i^k)$ whose degree is the same as that of $\sigma(\tau')$. Then x_k divides μ, and as the support of $\sigma(\tau')$ is in variables less than x_k, $\sigma(\tau') < \mu$. Thus (3) is proved. □

5 Theorem 1

The proof of Theorem 1 is essentially that given by Novik in [8] for the $p_i = p_j$ case, so we here give an abbreviated account with the necessary modifications, referring the reader to [8] for full details.

Let $\Lambda = \Lambda(l; p_1, p_2, \dots, p_m)$, and let Γ be a $(d-1)$-dimensional Cohen–Macaulay subcomplex of Λ. Let P_i and V' be as defined in the previous section and label the vertices of Λ with variables $x_1, x_2, \dots, x_n$, ordered so that $x_i \in P_i$ for $1 \leq i \leq m$, and $x_i \in V'$ for $n-l+1 \leq i \leq n$. Let $\mathbf{k}$ be a field and $\mathbf{k}[\mathbf{x}] = \mathbf{k}[x_1, x_2, \dots, x_n]$. Recall that the Stanley–Reisner ideal of Γ, I_Γ, is the ideal generated by squarefree monomials $x_{i_1}x_{i_2}\cdots x_{i_s}$ such that $\{x_{i_1}, x_{i_2}, \dots, x_{i_s}\}$ is *not* a face of Γ.

For $g \in GL_n(\mathbf{k})$, g defines an automorphism of $\mathbf{k}[\mathbf{x}]$ by $g(x_j) = \sum_{i=1}^n g_{ij}x_i$. We say g possesses the Kind–Kleinschmidt condition if for every facet of Γ, $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}\}$, the submatrix of g^{-1} obtained by taking the intersection of the rows numbered $i_1, i_2, \dots, i_r$ with the last d columns has rank r. For such a g, let $J(g, \Gamma) = gI_\Gamma + \langle x_{n-d+1}, \dots, x_n \rangle$. Such a g exists as long as $\mathbf{k}$ is infinite.

Finally, for I an ideal in $\mathbf{k}[\mathbf{x}]$, let $\mathrm{Bs}(I) = \{\mu \in S(\infty^n) : \mu \notin \mathrm{span}_{\mathbf{k}}(\{\mu' : \mu \prec \mu'\} \cup I)\}$, where $\prec$ is the order given by $\mu \prec \mu'$ if either $\deg(\mu) < \deg(\mu')$ or $\deg(\mu) = \deg(\mu')$ and $\mu' < \mu$ in our original order on monomials (notice the reversal). The crux of the proof lies in the fact that $\mathrm{Bs}(J(g, \Gamma))$ is a multicomplex and that $F(\mathrm{Bs}(J(g, \Gamma))) = h(\Gamma)$. We additionally make use of the fact $\mathrm{Bs}(J(g, \Gamma)) = \mathrm{Bs}(gI_\Gamma) \cap S(\infty^{n-d})$. It thus suffices to construct a matrix g satisfying the Kind–Kleinschmidt condition such that $\mathrm{Bs}(gI_\Gamma)$ does not contain $x_i^{p_i}$ for $1 \leq i \leq m$.

To do this we first pass to a larger field. Let $\mathbf{K} = \mathbf{k}(y_{ij}, w_{ij}, z_{ij})$ be the field of rational functions in $\sum_i (p_i - 1)^2 + l^2 + l(\sum_i p_i)$ variables, where $Y = (y_{ij})$, $W = (w_{ij})$, and $Z = (z_{ij})$ are $(\sum_i (p_i - 1)) \times (\sum_i (p_i - 1))$, $l \times l$, and $(\sum_i p_i) \times l$ matrices, respectively. Let $E = (E_{ij})$ be the $m \times (\sum_i (p_i - 1))$ matrix, where

$$E_{ij} = \begin{cases} 1 & \text{if } x_{j-m} \in P_i, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Define

$$g^{-1}=\begin{bmatrix}\begin{bmatrix}I_m & -EY\\ 0 & Y\end{bmatrix} & Z\\ 0 & W\end{bmatrix}\quad\text{so that}\quad g=\begin{bmatrix}\begin{bmatrix}I_m & E\\ 0 & Y^{-1}\end{bmatrix} & *\\ 0 & W^{-1}\end{bmatrix}.$$

Now, for each i, $P_i \notin \Gamma$, so I_Γ contains $\prod_{x_j\in P_i} x_j$. Then gI_Γ contains

$$\prod_{x_j\in P_i} g(x_j)=\prod_{x_j\in P_i}\left(x_i+\sum_{k>m} g_{kj}x_k\right)=x_i^{p_i}+\sum\{\alpha_\mu\mu : \mu\prec x_i^{p_i}\}.$$

Thus $x_i^{p_i}\notin \mathrm{Bs}(gI_\Gamma)$, so $\mathrm{Bs}(J)\subseteq S(p_1-1, p_2-1,\dots,p_m-1,\infty^{n-d-m})$.

It remains only to show that g satisfies the Kind–Kleinschmidt condition. Note that the ith row of EY is equal to the sum of the rows of Y indexed (in the larger matrix g) by $j>m$ such that $x_j\in P_i$. Since no facet of Λ contains P_i, and the entries of Y, W, and Z are algebraically independent, it then follows that for $\{x_{i_1}, x_{i_2},\dots,x_{i_d}\}$ a facet of Γ, the determinant of the submatrix of g^{-1} defined by the intersection of the last d columns and the rows numbered $i_1,\dots,i_d$ is nonzero, so the Kind–Kleinschmidt condition holds.

6 Remarks

Note that the class of Cohen–Macaulay subcomplexes of $\Lambda(l; p_1, p_2,\dots,p_m)$ is larger than that of Cohen–Macaulay complexes having proper G action with corresponding orbit structure. Thus, one does not expect our conditions to be sufficient for face numbers of Cohen–Macaulay complexes with proper group action. Indeed, in [10] Stanley showed necessary conditions on the h-vectors of centrally symmetric Cohen–Macaulay complexes not implied by our conditions, which were later generalized by Adin in [1] to the case of Cohen–Macaulay complexes with proper $\mathbb{Z}_p$-action. It would be of interest to determine sufficient conditions in this more restricted case.

As mentioned in the introduction, Corollary 1 is similar in structure to a result of Björner, Frankl, and Stanley in [2], which characterizes the f-vectors of balanced complexes. In fact, since the first draft of this paper a common generalization of both results has been shown, see [3].

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