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Association Schemes and Fusion Algebras (An Introduction)*

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Abstract. We introduce the concept of fusion algebras at algebraic level, as a purely algebraic concept for the fusion algebras which appear in conformal field theory in mathematical physics. We first discuss the connection between fusion algebras at algebraic level and character algebras, a purely algebraic concept for Bose-Mesner algebras of association schemes. Through this correspondence, we establish the condition when the matrix S of a fusion algebra at algebraic level is unitary or symmetric. We construct integral fusion algebras at algebraic level, from association schemes, in particular from group association schemes, whose matrix S is unitary and symmetric. Finally, we consider whether the modular invariance property is satisfied or not, namely whether there exists a diagonal matrix T satisfying the condition $(ST)^3 = S^2$. We prove that this property does not hold for some integral fusion algebras at algebraic level coming from the group association scheme of certain groups of order 64, and we also prove that the (nonintegral) fusion algebra at algebraic level obtained from the Hamming association scheme $H(d, q)$ has the modular invariance property.

Keywords: association scheme, fusion algebra, character algebra, Bose-Mesner algebra, modular invariance property

Introduction

This paper is based on my talk at the Workshop on Algebraic Combinatorics in Vladimir, USSR, August 7–16, 1991. This paper was originally prepared for the proceedings, but I missed the deadline for the submission. As was the intention of my talk at the workshop, the emphasis is on giving an overview of our current research, rather than giving the full technical details of the results. Some of the works presented here are based on some joint works of the author with Akihiro Munemasa and Etsuko Bannai. In this paper, I will present an introductory overview of these works, with my own viewpoint and responsibility. Further detailed discussions will be given in the subsequent papers.

The main purpose of this paper is to start with the study of fusion algebras from a purely algebraic viewpoint, by noticing and using the connection with association schemes. Here, fusion algebras are finite dimensional commutative and associative algebras (over the complex number field) which appear in conformal

*This paper is dedicated to Professor Katsumi Shiratani on the occasion of his 60th birthday.

field theory in mathematical physics. We consider a purely algebraic object, which we call fusion algebra at algebraic level, and find its connection with another purely algebraic object, called character algebra, which is closely connected with Bose-Mesner algebras (Hecke algebras) of an association scheme. There is the matrix S in fusion algebra which plays an important role in conformal field theory. We show, contrary to the theory of fusion algebras in conformal field theory, that the matrix S of a fusion algebra at algebraic level is not necessarily unitary nor symmetric. We give exact characterizations of when the matrix S is unitary or symmetric in terms of the corresponding character algebra.

By fully utilizing this connection between fusion algebras and character algebras, we try to find integral fusion algebras at algebraic level, in particular those whose matrix S is unitary and symmetric. We particularly discuss such examples coming from the group association schemes of certain groups of order 64.

Finally, we discuss the modular invariance of a fusion algebra. This property is very important in conformal field theory, because the fusion algebras of nice known conformal theories have this property; that is to say, there exists a diagonal matrix T satisfying $(ST)^3 = S^2$. We are interested in finding integral fusion algebras at algebraic level that satisfy the modular invariance property, as the chance of the existence of conformal field theory attached to them will increase if they satisfy the modular invariance property. We conclude this paper by mentioning that the (nonintegral) fusion algebra at algebraic level obtained from the Hamming association scheme $H(d, q)$ has the modular invariance, together with some conjectures on how this result will be generalized for other association schemes.

1. Fusion algebras at algebraic level

Fusion algebras appear in conformal field theory in mathematical physics. They are related to the representations of so-called Virasoro algebras or chiral algebras, and are finite dimensional commutative and associative algebras over the complex number field **C**. The reader is referred to [6, 7, 10, 12, 13, etc.] for a general understanding of fusion algebras which appear in conformal field theory.

When we look at the properties of fusion algebras, and when we forget (or ignore) the meanings about physics, they have very distinctive algebraic properties. By postulating only the algebraic properties of fusion algebras, we define the concept of "fusion algebras at algebraic level" as follows. The following set of axioms is just one attempt for this, and there will be room for further discussions of what is the best definition of fusion algebras at algebraic level. Also, I think that similar attempts should have been considered by many other authors independently, as this is a very natural thing to do.

Definition 1.1. (Fusion algebras at algebraic level). Let $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ be an algebra over C with basis $x_0, x_1, \ldots, x_d$ and with multiplication defined by

$$x_i x_j = \sum_{k=0}^{d} N_{ij}^k x_k.$$

Let us consider the following conditions.

(0) The algebra $\mathfrak{A}$ is associative and commutative,
(i) $N_{ij}^k \in \boldsymbol{R}$,
[we also consider the following conditions which are stricter than (i):
(i′) $N_{ij}^k \in \boldsymbol{R}$ and $N_{ij}^k \geq 0$,
(i″) $N_{ij}^k \in \boldsymbol{N} = \{0, 1, 2, \ldots\}$],
(ii) There exists a bijection $\wedge : i \mapsto \hat{\imath}$ from $\{0, 1, \ldots, d\}$ to $\{0, 1, \ldots, d\}$ satisfying
(a) $\hat{\hat{\imath}} = i$,
(b) $N_{\hat{\imath}\hat{\jmath}}^{\hat{k}} = N_{ij}^k$,
and
(c) if we define $N_{ijk} = N_{ij}^{\hat{k}}$, then N_{ijk} is symmetric in i, j, k,
(iii) $N_{0j}^k = \delta_{jk}$ (i.e., x_0 is the identity element of $\mathfrak{A}$),
(iv) There exists a linear representation of $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ with $x_i \mapsto \sqrt{k_i}$ with $k_i > 0$ for all $i (0 \leq i \leq d)$.

We call $\mathfrak{A}$ a *fusion algebra at algebraic level* if $\mathfrak{A}$ satisfies the conditions (0), (i), (ii), (iii), and (iv). If (i′) is satisfied in addition, we call $\mathfrak{A}$ a fusion algebra at algebraic level *of nonnegative type.* If $\mathfrak{A}$ satisfies the condition (i″) furthermore, we call $\mathfrak{A}$ an *integral* fusion algebra at algebraic level.

Remark. In fusion algebras appearing in mathematical physics, the condition (i″) $N_{ij}^k \in \boldsymbol{N}$ is very fundamental. For those who are already familiar with the concept of fusion algebras in mathematical physics, we remark that the last condition (iv) is equivalent to the condition $S_0^i > 0$ for all i, and this is a very natural condition.

Here we give a well-known example of fusion algebras at algebraic level.

Example 1.1. Let G be any finite group, and let $\chi_0, \chi_1, \ldots, \chi_d$ be the (distinct) irreducible characters of G with $\chi_0 = 1_G$ being the identity character. Then we have the decomposition of the tensor product

$$\chi_i \otimes \chi_j = \sum_{k=0}^{d} N_{ij}^k \chi_k \quad \text{with} \quad N_{ij}^k \in \boldsymbol{N}.$$

Now, let x_i correspond to χ_i and let us consider the algebra defined by

$$x_i x_j = \sum_{k=0}^{d} N_{ij}^{k} x_k.$$

Then the algebra $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ is an integral fusion algebra at algebraic level. Note that here the map $\wedge$ is defined by the complex conjugation, i.e., $\chi_{\hat{i}} = \overline{\chi_i}$. Also note that N_{ijk} is symmetric in i, j, k because

$$N_{ijk} = N_{ij}^{\hat{k}} = (\chi_i \otimes \chi_j, \chi_{\hat{k}}) = (\chi_i \otimes \chi_j \otimes \chi_k, \chi_0).$$

Also note that $\sqrt{k_i} = \chi_i(1)$ (the degree of the irreducible character χ_i).

Of course, many other examples of integral fusion algebras at algebraic level are obtained from the fusion algebras of (known) conformal field theories in mathematical physics. Among them, very interesting examples are obtained from any finite group G with a basis, in which each element of the basis consists of a pair of a conjugacy class C of G and an irreducible representation of the centralizer $C_G(x)$ of an element x in the conjugacy class C, and by defining appropriate multiplications of these elements of the basis, (cf. Lusztig [11] and [7]. They are examples of integral fusion algebras (at algebraic level). We note that the Fourier transformations of these integral fusion algebras at algebraic level for certain finite groups are crucially used in the last step of the determination of irreducible characters of finite Chevalley groups, in the work of Lusztig, Asai, Kawanaka, and others. Also, we remark that the concept of based ring in Lusztig [11] is a proper setting to consider a noncommutative version of fusion algebras at algebraic level.

2. Association schemes, Bose-Mesner algebras, and character algebras

First we recall the definition of association schemes. The reader is referred to [3, 4, 5] for further information on association schemes.

Definition 2.1. (Association schemes). Let X be a finite set, and let $R_i (i = 0, 1, \ldots, d)$ be nonempty relations on X (i.e., subset of $X \times X$). Let the following conditions (1), (2), (3), and (4) be satisfied, then the pair $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ consisting of a set X and a set of relations $\{R_i\}_{0 \le i \le d}$ is called an association scheme.

(1) $R_0 = \{(x, x) \mid x \in X\}$,
(2) $X \times X = R_0 \cup R_1 \cup \ldots \cup R_d$, with $R_i \cap R_j = \emptyset$ if $i \neq j$,
(3) For $i \in \{0, 1, \ldots, d\}$, let ${}^tR_i$ be defined by

$${}^tR_i = \{(y, x) \mid (x, y) \in R_i\}.$$

Then ${}^tR_i = R_j$ for some $j \in \{0, 1, \ldots, d\}$. (We write ${}^tR_i = R_{\hat{i}}$).

(4) For $i, j, k \in \{0, 1, \ldots, d\}$,

$$\#\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\} \text{ (with } (x, y) \in R_k)$$

depends only on i, j, k and not on the choice of $(x, y) \in R_k$. (We write this number p_{ij}^k.)

Remark. We say that an association scheme $\mathfrak{X}$ is commutative if $p_{ij}^k = p_{ji}^k$ for all i, j, k.

In this paper, we use the term association scheme to mean commutative association schemes, unless otherwise stated.

Definition 2.2. (Adjacency matrices and Bose-Mesner algebras). Let $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ be an association scheme. Let A_i be the adjacency matrix with respect to the relation R_i. Namely, let

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{A} = \langle A_0, A_1, \ldots, A_d \rangle$ be the subalgebra (in the complete matrix algebra of size $|X|$ over $\mathbf{C}$) generated by $A_0, A_1, \ldots, A_d$. The algebra $\mathfrak{A}$ is of dimension $d+1$ (by Definition 2.1), and is called the Bose-Mesner algebra of the association scheme $\mathfrak{X}$.

Remark. It is known that the Bose-Mesner algebra $\mathfrak{A}$ of $\mathfrak{X}$ is a semisimple algebra over $\mathbf{C}$ (even if $\mathfrak{X}$ is noncommutative), and that the association scheme $\mathfrak{X}$ is commutative if and only the Bose-Mesner algebra $\mathfrak{X}$ is commutative.

We now give two well-known examples of association schemes which are obtained from finite groups.

Example 2.1. Let G be a group acting on a finite set X transitively. Then G acts on the set $X \times X$ naturally. Let $Q_0 = \{(x, x) \mid x \in X\}, Q_1, Q_2, \ldots, Q_d$ be all the orbits of the action on G on $X \times X$. For $x, y \in X$ by defining

$$(x, y) \in R_i \Leftrightarrow (x, y) \in Q_i \quad (i = 0, 1, \ldots, d),$$

we have an association scheme $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ (which is not necessarily commutative). The association scheme $\mathfrak{X}$ is commutative if and only if the permutation representation π of G on X is multiplicity-free, i.e., π is decomposed into a direct sum of nonequivalent irreducible representations of G.

Example 2.2. [Group association scheme $\mathfrak{X}(G)$]. Let G be any finite group. Let $C_0 = \{1\}, C_1, C_2, \ldots, C_d$ be all the conjugacy classes of G. For $x, y \in G$, define

$$(x, y) \in R_i \Leftrightarrow yx^{-1} \in C_i \quad (i = 0, 1, \ldots, d),$$

then $\mathfrak{X}(G) = (G, \{R_i\}_{0 \le i \le d})$ becomes a commutative association scheme.

Remark. Example 2.2 is regarded as a special case of Example 2.1, by letting the group $G \times G$ act on G by $(x, y) \in G \times G$ act on $z \in G$ by $z \mapsto x^{-1}zy$. Note that the Bose-Mesner algebra of $\mathfrak{A}$ of Example 2.2 is isomorphic to $Z(\mathbf{C}G)$, the center of the group algebra of G over $\mathbf{C}$.

In what follows, we always assume that $\mathfrak{X} = (X, \{R_i\}_{0 \le i \le d})$ is a commutative association scheme. It is well known that the Bose-Mesner algebra $\mathfrak{A}$ of $\mathfrak{X}$ is not only closed by ordinary matrix multiplication but also by Hadamard product $\circ$ (i.e., the entrywise product of matrices). Also it is known that $\mathfrak{A} = \langle A_0, A_1, \ldots, A_d \rangle$ is commutative and semisimple, and that $\mathfrak{A}$ has a unique set of primitive idempotents $E_0, E_1, \ldots, E_d$. Now, let

$$(|X|E_i) \circ (|X|E_j) = \sum_{k=0}^{d} q_{ij}^k (|X|E_k) \quad \text{with} \quad q_{ij}^k \in \mathbf{C}.$$

Then it is known (as Krein condition) that the q_{ij}^k are nonnegative real numbers. The matrices P and Q which give transformations between two bases $A_0, A_1, \ldots, A_d$ and $E_0, E_1, \ldots, E_d$ of $\mathfrak{A}$ with the following normalizations are called the first and the second eigenmatrices of $\mathfrak{X}$ respectively:

$$(A_0, A_1, \ldots, A_d) = (E_0, E_1, \ldots, E_d)P$$

and

$$(|X|E_0, |X|E_1, \ldots, |X|E_d) = (A_0, A_1, \ldots, A_d)Q$$

(hence $PQ = QP = |X|I$). The matrix P is also called the character table of $\mathfrak{X}$. The reader is referred to [1] for a survey of recent research on commutative association schemes and their character tables.

The Bose-Mesner algebra was defined under the existence of a combinatorial object, namely association scheme. Here we define a purely algebraic concept which was obtained by extracting only the algebraic properties of the Bose-Mesner algebra.

Definition 2.3. (Character algebras, or Bose-Mesner algebras at algebraic level). Let $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ be an algebra over $\mathbf{C}$ with basis $x_0, x_1, \ldots, x_d$ and with multiplication defined by

$$x_i x_j = \sum_{k=0}^{d} p_{ij}^k x_k.$$

Let us consider the following conditions.

(0) The algebra $\mathfrak{A}$ is associative and commutative,

(i) $p_{ij}^k \in \mathbf{R}$,
[we also consider the following condition which is stricter than (i):
(i′) $p_{ij}^k \in \mathbf{R}$ and $p_{ij}^k \geq 0$],
(ii) There exists a bijection $\wedge : i \mapsto \hat{i}$ from $\{0, 1, \ldots, d\}$ to $\{0, 1, \ldots, d\}$ satisfying
(a) $\hat{\hat{i}} = i$, and
(b) $p_{\hat{i}\hat{j}}^{\hat{k}} = p_{ij}^k$,
(iii) $p_{0j}^k = \delta_{jk}$ (i.e., x_0 is the identity element of $\mathfrak{A}$),
(iv) $p_{ij}^0 = \delta_{\hat{i}j} k_i$ with $k_i > 0$ for all i, and the map $x_i \mapsto k_i (i = 0, 1, \ldots, d)$ gives a linear representation of $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$.

Then we call an algebra $\mathfrak{A}$ a *character algebra,* if the above conditions (0), (i), (ii), (iii), and (iv) are satisfied. Furthermore, if $\mathfrak{A}$ satisfies the condition (i′) in addition, then it is called a character algebra of *nonnegative type* (or Bose-Mesner algebra at algebraic level).

Remark. The concept of character algebra was defined by Y. Kawada [9] in 1942 (50 years ago!). The reader is referred to [3, §2.5] for further details of character algebras.

Now we mention that character algebras (of nonnegative type) are obtained from a (commutative) association scheme in two different ways.

Example 2.3. Let $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a (commutative) association scheme, and let $A_i (i = 0, 1, \ldots, d)$ be the adjacency matrices and $E_i (i = 0, 1, \ldots, d)$ be the primitive idempotents. Suppose that

$$A_i \cdot A_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

and

$$(|X|E_i) \circ (|X|E_j) = \sum_{k=0}^{d} q_{ij}^k (|X|E_k)$$

Then we have the following assertions.

(a) Defining

$$x_i \cdot x_j = \sum_{k=0}^{d} p_{ij}^k x_k,$$

the algebra $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ becomes a character algebra of nonnegative type.

(b) Defining

$$x_i \cdot x_j = \sum_{k=0}^{d} q_{ij}^k x_k,$$

the algebra $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ becomes a character algebra of nonnegative type. (In this example, $m_i(:= \mathit{rank}\ E_i)$ corresponds to the k_i in the definition of character algebra (cf. Definition 2.3).

3. Connection between fusion algebras at algebraic level and character algebras

In this section we show that there is a natural one-to-one correspondence between fusion algebras at algebraic level and character algebras. Namely, we prove the following.

THEOREM 3.1. *There exists a natural one-to-one correspondence between fusion algebras at algebraic level and character algebras. Moreover this gives a one-to-one correspondence between fusion algebras at algebraic level of nonnegative type and character algebras of nonnegative type.*

Proof. Let $\tilde{\mathfrak{A}} = \langle y_0, y_1, \ldots, y_d \rangle$ be a character algebra with basis $y_0, y_1, \ldots, y_d$ and multiplication

$$y_i y_j = \sum_{k=0}^{d} p_{ij}^k y_k.$$

Let us define

$$N_{ij}^k = \sqrt{\frac{k_k}{k_i k_j}} p_{ij}^k,$$

and let $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ be the algebra with basis $x_0, x_1, \ldots, x_d$ and multiplication

$$x_i \cdot x_j = \sum_{k=0}^{d} N_{ij}^k x_k.$$

Then $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ becomes a fusion algebra at algebraic level. (The details of the proof are left to the reader. For example, the fact that N_{ijk} is symmetric in i, j, k comes from the relation in character algebra such that $k_\gamma p_{\alpha\beta}^\gamma = k_\beta p_{\hat{\alpha}\gamma}^\beta$, cf. [3, §2.5, Proposition 5.1].)

Conversely, starting from a fusion algebra at algebraic level with structure constant N_{ij}^k, by defining

$$p_{ij}^k = \sqrt{\frac{k_i k_j}{k_k}} N_{ij}^k,$$

we get a character algebra. This establishes the one-to-one correspondence between fusion algebras at algebraic level and character algebras. Also, it is clear that nonnegative type corresponds to nonnegative type in this correspondence. [Note that the k_i in Definition 1.1 (fusion algebra at algebraic level) corresponds to the k_i in Definition 2.3 (character algebra).] □

Remark. By this correspondence integral fusion algebras at algebraic level do not necessarily correspond to integral (i.e., $p_{ij}^k \in N$) character algebras. It is an interesting question to know when integral fusion algebras at algebraic level come from integral character algebras or association schemes.

We conclude this section by giving an example of this correspondence.

Example 3.1. Let $\mathfrak{X}(G)$ be the group association scheme (see Example 2.2) for a finite group G. As mentioned in Example 2.3 (b), we get the character algebra of nonnegative type defined by

$$x_i x_j = \sum_{k=0}^{d} q_{ij}^k x_k,$$

where q_{ij}^k are the Krein parameters of the association scheme $\mathfrak{X}(G)$. By Theorem 3.1, there exists a fusion algebra at algebraic level corresponding to this character algebra. This fusion algebra at algebraic level is in fact the fusion algebra given in Definition 1.1. Here note that in $\mathfrak{X}(G)$, the primitive idempotent E_i corresponds to the irreducible character χ_i and that $m_i = \mathit{rank}\ E_i = \chi_i(1)^2$.

4. The symmetry of the matrix S and Verlinde's formula

The following result is well known for fusion algebras appearing in mathematical physics.

THEOREM (Verlinde [6, 13]). *Let $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ be the fusion algebra (at algebraic level) which appears in mathematical physics. Let us set*

$$N_i = (N_{ij}^k)_{0 \le j \le d, 0 \le k \le d}.$$

Then there exists a matrix $S = (S_i^j)_{0 \le i \le d, 0 \le j \le d}$ satisfying

$$S^{-1} N_i S = \begin{pmatrix} \lambda_i^{(0)} & & & 0 \\ & \lambda_i^{(1)} & & \\ & & \ddots & \\ 0 & & & \lambda_i^{(d)} \end{pmatrix}$$

with

$$\lambda_i^{(j)} = \frac{S_i^j}{S_0^j}, \quad S_0^i = S_i^0$$

for all $i (0 \le i \le d)$.

Also the following formula is known as Verlinde's formula:

$$\text{(I)} \qquad N_{ij}^k = \sum_{n=0}^{d} \frac{S_i^n S_j^n \overline{S}_n^k}{S_n^0}.$$

The following result is well known in the theory of association schemes, and in the theory of character algebras:

THEOREM (Kawada [9], see also Bannai [3, §2.5]). *Let* $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ *be a character algebra. Let us set*

$$B_i = (p_{ij}^k)_{0 \le j \le d,\ 0 \le k \le d},$$

and let $P = (P_{ij})_{0 \le i \le d,\ 0 \le j \le d}$ *be the matrix defined by*

$$x_i = \sum_{j=0}^{d} P_{ij} e_j,$$

where $e_0, e_1, \ldots, e_d$ *are the primitive idempotents of the character algebra* $\mathfrak{A}$. *(Note that* P *is the character table (the first eigenmatrix) of the association scheme when* $\mathfrak{A}$ *comes from an association scheme as in Example 2.3(a). Also note that* P_{ij} *is written as* $P_j(i)$ *in [3, §2.5]. Then we have*

$$P^{-1} B_i P = \begin{pmatrix} P_{0i} & & & 0 \\ & P_{1i} & & \\ & & \ddots & \\ 0 & & & P_{di} \end{pmatrix}$$

for all $i\,(0 \le i \le d)$. *Moreover, the following formula is well known.*

$$\text{(II)} \qquad p_{ij}^k = \frac{1}{|X| \cdot k_k} \sum_{\nu=0}^{d} P_{\nu i} P_{\nu j} \overline{P_{\nu k}} m_\nu,$$

where $|X| = k_0 + k_1 + \cdots + k_d$ *and the* m_ν*s are certain numbers defined for any character algebra (see [3, §2.5]).*

We prove the following

THEOREM 4.1. *(i) Let $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ be any fusion algebra at algebraic level. Let us set*

$$N_i = (N_{ij}^k)_{0 \le j \le d,\ 0 \le k \le d}.$$

Then there exists a matrix $S = (S_i^j)_{0 \le i \le d,\ 0 \le j \le d}$, satisfying

$$S^{-1} N_i S = \begin{pmatrix} \lambda_i^{(0)} & & & & 0 \\ & \lambda_i^{(1)} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \lambda_i^{(d)} \end{pmatrix}$$

with

$$\lambda_i^{(j)} = \frac{S_i^j}{S_0^j}, \quad S_0^i = S_i^0$$

for all $i (0 \le i \le d)$.

(ii) Let P be the matrix $P = (P_{ij})_{0 \le i \le d,\ 0 \le j \le d}$ (mentioned in the preceding theorem) for the character algebra $\mathfrak{A} = \langle y_0, y_1, \ldots, y_d \rangle$ which corresponds to $\mathfrak{A}$, the fusion algebra at algebraic level (by Theorem 3.1). Then the matrix S and the matrix P are related by the following relation.

$$S = \frac{1}{\sqrt{|X|}} \begin{pmatrix} \sqrt{k_0} & & & & 0 \\ & \sqrt{k_1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \sqrt{k_d} \end{pmatrix} P \begin{pmatrix} \frac{1}{\sqrt{k_0}} & & & & 0 \\ & \frac{1}{\sqrt{k_1}} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \frac{1}{\sqrt{k_d}} \end{pmatrix}$$

where $|X| = k_0 + k_1 + \cdots + k_d$ by definition.

Proof. (i) Straightforward from the correspondence in Theorem 3.1 and from the fact that there are exactly $d + 1$ linear representations in the character algebra (cf. [9] or [3, §2.5]). (ii) This is proved by a straightforward calculation. The details are left to the reader. Here we note that

$$\text{(III)} \qquad N_i = \frac{1}{\sqrt{k_i}} \begin{pmatrix} \frac{1}{\sqrt{k_0}} & & & & 0 \\ & \frac{1}{\sqrt{k_1}} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \frac{1}{\sqrt{k_d}} \end{pmatrix} B_i \begin{pmatrix} \sqrt{k_0} & & & & 0 \\ & \sqrt{k_1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \sqrt{k_d} \end{pmatrix}$$

□

Remark. By Theorem 4.1(ii), we have the explicit connection between the two matrices S and P. Since the N_{ij}^k and the p_{ij}^k are related by (III), starting from

(II), we can get a formula for expressing N_{ij}^k in terms of S_i^j for any fusion algebra at algebraic level. It will be natural to expect that that is the formula (I). However, this is not the case. The formula we get is

$$(\mathrm{I}') \qquad N_{ij}^k = \sum_{n=0}^{d} \frac{S_i^n S_j^n \overline{S}_k^n}{S_0^n} \cdot \frac{m_n}{k_n}$$

Note that the formula (I′) is considerably different from Verlinde's formula (I) for the fusion algebras which appear in mathematical physics. [Verlinde's formula (I) does not hold for general fusion algebras at algebraic level.]

For the fusion algebras which appear in conformal field theory in mathematical physics, it is shown that the matrix S is always unitary and symmetric. However, it is shown that this is not always true for fusion algebras at algebraic level. To be more precise, we can characterize when the matrix S is unitary or is symmetric.

THEOREM 4.2. *Let* $\mathfrak{A} = \langle x_0, x_1, \ldots, x_d \rangle$ *be a fusion algebra at algebraic level, and let* S *be the matrix defined in Theorem 4.1, (i). Then we have the following two assertions.*

(i) The matrix S *is unitary (i.e.,* ${}^t\overline{S}S = S \cdot {}^t\overline{S} = I$*) if and only if* $k_i = m_i (i = 0, 1, \ldots, d)$ *for the corresponding (by Theorem 3.1) character algebra.*
(ii) The matrix S *is symmetric (i.e.,* ${}^tS = S$*) if and only if* $P = \overline{Q}$ *for the corresponding character algebra.*

(Note that Q is the matrix defined by $PQ = QP = |X|I$, where $|X| = k_0 + k_1 + \cdots + k_d$ by definition. For the definition of m_i, see [3, §2.5].)

Proof. For a character algebra, it is shown from the second orthogonality relation (cf. [3, p96, (5.12)]), that the matrix

$$\begin{pmatrix} \sqrt{m_0} & & & 0 \\ & \sqrt{m_1} & & \\ & & \ddots & \\ 0 & & & \sqrt{m_d} \end{pmatrix} P \begin{pmatrix} \frac{1}{\sqrt{k_0}} & & & 0 \\ & \frac{1}{\sqrt{k_1}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{k_d}} \end{pmatrix}$$

is a unitary matrix. Since the k_is and the m_is are all real numbers, we have that S is unitary if and only if $k_i = m_i$ for all $i (0 \le i \le d)$. The second assertion follows from the relation

$$Q = \begin{pmatrix} \frac{1}{k_0} & & & 0 \\ & \frac{1}{k_1} & & \\ & & \ddots & \\ 0 & & & \frac{1}{k_d} \end{pmatrix} {}^t\overline{P} \begin{pmatrix} m_0 & & & 0 \\ & m_1 & & \\ & & \ddots & \\ 0 & & & m_d \end{pmatrix}$$

(cf. [3, §2.5, (5.13)] or $PQ = QP = |X|I$) and the second orthogonality relation (cf. [3, §2.5, (5.12)]). □

Remark. Note that S is unitary if S is symmetric.

Remark. It seems that the proof of unitarity and the symmetry of the matrix S of a fusion algebra (which appears in mathematical physics) is a "physical" proof and is not a "mathematical" proof. Furthermore, it is difficult to see the validity of Verlinde's formula (I) at a mathematical level. Actually, there are plenty of counterexamples with the matrix S not unitary (and not symmetric) for fusion algebras at algebraic level.

Remark. If the matrix S is symmetric, then Verlinde's formula (I) is identical to the formula (I′). The fusion algebras studied in the original Verlinde's papers [6, 13] are self-dual, i.e., $P = \overline{Q}$, hence Verlinde's formula (I) for the original fusion algebras is true. Association schemes (or character algebras) with $P = \overline{Q}$ are called self-dual. (For examples of many self-dual and nonself-dual association schemes, cf. [3, Chapter III] or [4].) The condition $P = \overline{Q}$ implies that $p_{ij}^k = q_{ij}^k$ for all i, j, k. Hence in a self-dual association scheme, two fusion algebras described in Example 2.3(a) and (b) are identical. It is an open question whether $p_{ij}^k = q_{ij}^k$ implies $P = \overline{Q}$ for suitable arrangements of rows and columns. (This last question was first raised by Akihiro Munemasa.)

5. Construction of integral fusion algebras at algebraic level from association schemes

In Section 3, we have seen that a fusion algebra at algebraic level is constructed from a character algebra, and that two character algebras are attached to an association scheme. So, in this section, we consider the problem of constructing integral fusion algebras at algebraic level, from association schemes. We are particularly interested in such fusion algebras at algebraic level satisfying the condition that the matrix S is unitary and symmetric. (Here note that the symmetry of S implies the unitarity of S.)

First we consider this problem for the group association scheme $\mathfrak{X}(G)$ (cf.

Example 2.2(a)). If G is finite abelian group, it is easy to see that

$$S = \frac{1}{\sqrt{|G|}} P,$$

and that S is unitary and symmetric (with respect to a suitable ordering of irreducible characters, namely of primitive idempotents). It was not obvious at the beginning whether there exists at all any group association scheme $\mathfrak{X}(G)$ for a nonabelian group G such that the matrix S is unitary and symmetric. Since $k_i = |C_i|$ (the size of the conjugacy class C_i), and since $m_i = \chi_i(1)^2$ (where $\chi_i(1)$ is the degree of the irreducible character χ_i), in order that the matrix S is unitary, the set of numbers $k_0, k_1, \ldots, k_d$ must coincide with the set $m_0, m_1, \ldots, m_d$ including the repetition. Using this condition, first I was able to see that nonabelian such G do not exist for small orders, say for $|G| < 64$. Then, I tried to find such G of order 64, by looking at the complete list of groups of order $64 = 2^6$ in Hall-Senior [8]. (It is known that there are 257 nonisomorphic groups of order 64, including abelian groups.) By checking the list of Hall-Senior [8], we can easily see that such G are exactly those in the class ${}^3_3B$ in [8], namely G must be the 10 groups numbered from 144 to 153 in [8]. It is known (cf. [3, §2.7] that the matrix P of $\mathfrak{X}(G)$ is calculated from the character table T of the group G by the formula

$$P = \begin{pmatrix} \frac{1}{\sqrt{m_0}} & & & 0 \\ & \frac{1}{\sqrt{m_1}} & & \\ & & \ddots & \\ 0 & & & \frac{1}{\sqrt{m_d}} \end{pmatrix} T \begin{pmatrix} k_0 & & & 0 \\ & k_1 & & \\ & & \ddots & \\ 0 & & & k_d \end{pmatrix}$$

and that the matrix S is calculated from P by Theorem 4.1(ii). So, at this stage, I asked Akihiro Munemasa to calculate the character tables of these 10 groups by using "Cayley," and to show that these 10 fusion algebras at algebraic level are all integral (i.e., $N_{ij}^k \in \boldsymbol{N} = \{0, 1, \ldots\}$ for all i, j, k) and that the matrix S are all symmetric (with respect to suitable orderings of primitive idempotents). Actually, all the calculations were carried out by Munemasa rather quickly using a computer. Thus we get:

THEOREM 5.1. *Let G be one of the groups from 144 to 153 in the list of [8] of order 64. Then the fusion algebra at algebraic level corresponding to the Bose-Mesner algebra of $\mathfrak{X}(G)$ are all integral, and the matrix S of the fusion algebra at algebraic level becomes symmetric (with respect to a suitable ordering of primitive idempotents).*

Proof. Here we give a description of the matrix S for the group 153. This group is a Sylow 2-subgroup of Suzuki simple group $Sz(8)$ of order $64 \cdot 63 \cdot 7 = 29{,}120$.

$$S = \frac{1}{8} \begin{pmatrix} J_8 & A \\ {}^tA & B \end{pmatrix}$$

with J_8 being the matrix of size 8 by 8 whose entries are all 1,

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\ 2 & 2 & -2 & -2 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 \\ 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & 2 & 2 \\ -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & -2 & -2 \\ -2 & -2 & 2 & 2 & -2 & -2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 \\ -2 & -2 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 & -2 & -2 & 2 & 2 \\ -2 & -2 & -2 & -2 & 2 & 2 & -2 & -2 & 2 & 2 & 2 & 2 & -2 & -2 \end{pmatrix}$$

(size 8 by 14), and

$$B = \begin{pmatrix} a & b & & & & 0 \\ b & a & & & & \\ & & a & b & & \\ & & b & a & & \\ & & & & \ddots & \\ 0 & & & & & \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \otimes I_7$$

(size 14 by 14), where $a = 4i$ and $b = -4i$. We also note that the character tables of other groups are similar to the above matrix. That is, change some of blocks

$$\begin{pmatrix} 4i & -4i \\ -4i & 4i \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix},$$

and change some signs in the entries 2 or -2. We also note that the character table of the group (hence the character table P of $\mathfrak{X}(G)$) and hence the matrix S of the corresponding fusion algebra at algebraic level is the same for the following groups.

No. 144 ≅ No. 145,
No. 146 ≅ No. 147 ≅ No. 148 ≅ No. 149,
No. 151 ≅ No. 152. □

There are many other constructions of integral fusion algebras at algebraic level. Many such examples with S being symmetric were constructed from association schemes by Akihiro Munemasa. Also, there are some other constructions without using association schemes, though it is not generally easy to find those with S symmetric. This topic will be discussed in subsequent papers.

It would be interesting to know whether more such examples are constructed from group association schemes $\mathfrak{X}(G)$. Recently, Masao Kiyota proved (by using the classification of finite simple groups) that the condition $k_i = m_i$ (for all i) cannot hold for $\mathfrak{X}(G)$ for a nonabelian finite simple group G. He conjectures that G must be nilpotent if the condition is satisfied. Also, we remark that Hironobu Okuyama (a student of S. Koshitani at Chiba University) in his master's degree thesis checked when this condition is satisfied for groups of order 2^7 by using "Cayley." (We do not know at present whether these examples obtained by Okuyama actually give integral fusion algebras at algebraic level with the matrix S being symmetric.)

6. Modular invariance property

For fusion algebras attached to known conformal field theory, the matrix S satisfies the condition that there exists a diagonal matrix T with

$$(ST)^3 = S^2 \quad \text{and} \quad S^4 = I.$$

(cf. [6, 7, 10, 12, 13] etc.) We say that the matrix S (or the fusion algebra) has the modular invariance property if this condition is satisfied. The name modular invariance comes from the fact that S and T correspond to the elements.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(respectively) of the modular group $SL(2, \mathbb{Z})$.

It is generally an open difficult question to know, for a given fusion algebra at algebraic level, whether there is attached a conformal theory on it. Since the fusion algebra attached to a conformal field theory is likely to have the modular invariance property for S, we are interested in finding fusion algebras at algebraic level with the modular invariance property. As mentioned in §5, there are many known integral fusion algebras at algebraic level. However, not many of them satisfy the modular invariance property. The following result was obtained by Etsuko Bannai.

PROPOSITION 6.1. *The integral fusion algebra at algebraic level obtained from the group association scheme $\mathfrak{X}(G)$ for G one of the groups 144–153 of order 64 does not have any diagonal matrix T satisfying*

$$(ST)^3 = S^2.$$

Proof. Straightforward but very involved calculation shows the assertion of Proposition 6.1. Here, there are several possibilities to make the matrix S for each case (by rearranging rows and columns), but the nonexistence of the desired matrix T is proved for each of the possible matrix S. □

On the other hand, Akihiro Munemasa has shown that there are integral fusion algebras at algebraic level obtained from certain Schur rings attached to the elementary abelian group of order 64 (and of order 2^{2m}, $m \geq 3$), in which the matrix S is symmetric and moreover satisfies the modular invariance property: $(ST)^3 = S^2$. In these examples, we have $(ST)^3 = S^2 = T^2 = I$, hence the group $\langle S, T \rangle$ is isomorphic to the symmetric group of degree 3. Further details of these and related topics will be discussed by A. Munemasa elsewhere.

Now, we are interested in the following question. Let S be the matrix of (a not necessarily integral) fusion algebra at algebraic level. Then whether S satisfies the modular invariance property or not. We are interested in this problem for a particularly important class of association schemes, namely, for P-and Q-polynomial association schemes with $P = Q(= \overline{Q})$, i.e., self-dual P- and Q-polynomial association schemes. (See [3] for the definition and the examples of P-and Q-polynomial association schemes.)

THEOREM 6.2. (Bannai [2]) *Let P be the character table of the Hamming association scheme $H(d, q)$. Then we have*

$$(PT)^3 = q^{\frac{3d}{2}} I$$

for the matrix

$$T = \alpha_0 \begin{pmatrix} 1 & & & & 0 \\ & \alpha^1 & & & \\ & & \alpha^2 & & \\ & & & \ddots & \\ 0 & & & & \alpha^d \end{pmatrix},$$

where α and α_0 are the numbers defined by the following relations:

$$\alpha^2 + (q-2)\alpha + 1 = 0 \quad and \quad \alpha_0^3 = \frac{q^{\frac{d}{2}}}{(1 + (q-1)\alpha)^d}.$$

(So, there are 6 choices for T.)
Here we note that $P = (P_j(i))_{0 \leq i \leq d, 0 \leq j \leq d} = (K_j(i))_{0 \leq i \leq d, 0 \leq j \leq d}$ where $K_j(\theta)$ is the Krawtchouk polynomial defined by

$$K_j(\theta) = \sum_{u=0}^{j} (-q)^u (q-1)^{j-u} \binom{d-u}{j-u} \binom{\theta}{u}.$$

COROLLARY 6.1. (Bannai [2]) *Let S be the matrix of the fusion algebra at algebraic level obtained from the Hamming association scheme $H(d, q)$. (This fusion algebra is not integral in general.) Then S satisfies the modular invariance property, namely $(ST)^3 = S^2 = I$ for each of the six diagonal matrices T given in Theorem 6.2.*

We conjecture that similar results may be obtained for any self-dual P- and Q- polynomial (symmetric) association scheme.

For nonself-dual P- and Q-polynomial association schemes, say for Johnson association schemes, $J(v, d)$, we cannot expect the relation $S^4 = I$, hence we cannot expect the modular invariance property in the original sense. Then is there any natural diagonal matrix T where S and T satisfy some nice relation?

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