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Reconstructing a Generalized Quadrangle from its Distance Two Association Scheme

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Abstract. Payne [4] constructed an association scheme from a generalized quadrangle with a quasiregular point. We show that an association scheme with appropriate parameters and satisfying an assumption about maximal cliques must be one of these schemes arising from a generalized quadrangle.

Keywords: association scheme, generalized quadrangle, quasiregular point

1. Introduction

It is well known that the points of a generalized quadrangle (GQ) under the relation of collinearity form a strongly regular graph. Recently there has been interest in the structure of the induced subgraph of points at distance 2 from a fixed point. In a translation generalized quadrangle (see [5, chapter 8]) with parameters $(s, t) = (q^a, q^{a+1})$ there is a quasiregular point, i.e., a point p for which each triad (p, x, y) of pairwise noncollinear points has $k = |\{p, x, y\}^\perp| = 0$ or $q + 1$. With fixed x and $a > 1$ (necessarily a is odd), the numbers of points y with $k = 0$ and with $k = q + 1$ are positive and easily computed. Payne observed in [4] that the points at distance 2 from such a quasiregular point p form an association scheme $\mathcal{A}(\mathcal{S}, p)$ with three classes with one of the relations being collinearity.

A GQ with parameters (s, s^2) may be considered a special case of this where the association scheme has two classes and the induced subgraph is strongly regular. Ivanov and Shpectorov [3] showed that a strongly regular graph which has the same parameters and such that all maximal cliques have s points is the distance 2 subgraph of a $GQ(s, s^2)$; a shorter proof was later given by Brouwer and Haemers [1].

In this note, we consider the analogous question for the quadrangle association schemes of [4]. The techniques used are mostly eigenvalue techniques, and [2] is a general reference for these.

Throughout, we will assume that $(X, \{f_1, f_2, f_3\})$ is an association scheme with parameters as in [4] such that all maximal cliques in $\Gamma_1 = (X, f_1)$ have size s. We will call such maximal cliques *lines*, the elements of X *points*, and refer to x and y as *adjacent* if $(x, y) \in f_1$. Let A_i be the adjacency matrix and M_i be the intersection matrix for the relation f_i; so A_i is the matrix with rows and columns indexed by elements of X with (x, y) entry equal to 1 if $(x, y) \in f_i$ and 0 otherwise, and M_i is the 4×4 matrix with (j, k) entry p^k_{ji}. We will also use the notation $\Gamma_i = (X, f_i)$ and $v_i = |\{y : (x, y) \in f_i\}|$ where $x \in X$ is fixed; note v_i does not depend on the choice of x.

The relevant parameters from [4] are as follows.

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ (qs+1)(s-1) & s-2 & qs-q & qs+1 \\ 0 & \frac{qs(qs-1)}{q+1} & \frac{(s-1)(qs^2-1)}{q+1} & \frac{q^2s^2-1}{q+1} \\ 0 & \frac{qs(s-q)}{q+1} & \frac{(qs-q)(s-q)}{q+1} & \frac{(qs+1)(s-2q-1)}{q+1} \end{pmatrix}$$

$$v_1 = (qs+1)(s-1)$$
$$v_2 = \frac{s(q^2s^2-1)}{q+1}$$
$$v_3 = \frac{qs(s-1)(s-q)}{q+1}$$

The adjacency matrices A_i have eigenvalues and multiplicities:

A_1	A_2	A_3	multiplicity
$(qs+1)(s-1)$	$\frac{s(q^2s^2-1)}{q+1}$	$\frac{qs(s-1)(s-q)}{q+1}$	1
$s-qs-1$	$\frac{s(qs-1)}{q+1}$	$\frac{-qs(s-q)}{q+1}$	$(qs+1)(s-1)$
$s-1$	$-s$	0	$\frac{s(q^2s^2-1)}{q+1}$
$-(qs+1)$	0	qs	$\frac{qs(s-1)(s-q)}{q+1}$

Our notation differs from [4] in that we use only the parameters q and s; the remaining parameter t satisfies $t = qs$.

Our main result is:

THEOREM. *There exists a GQ* $\mathcal{S}$ *such that* $(X, \{f_1, f_2, f_3\}) = \mathcal{A}(\mathcal{S}, p)$.

2. Reconstructing the generalized quadrangle

If L is a line, we will say a point x is adjacent to L or has distance 1 from L if x is adjacent to some point of L; x is at distance 2 from L if x is not adjacent to L but, there exists a point y adjacent to L such that x is adjacent to y. Note that any point is at distance at most 2 from L since p^2_{11} and p^3_{11} are not zero.

PROPOSITION. *Let L be a line, and let Z be the set of points at distance 2 from L. Then $L \cup Z$ is a set of qs^2 points which induces a regular subgraph of degree $s-1$ in Γ_1. Further, if $x, y \in L \cup Z$ and x and y are not adjacent, then $(x,y) \in f_2$.*

Proof. Clearly any point not in L is adjacent to at most one point of L. Any point of L is adjacent to $v_1 = qs^2 - qs + s - 1$ points, of which $s-1$ are on L, hence to $qs^2 - qs$ points not on L.

Now, $|Z|$ = number of points adjacent to no point on L

$$= |X| - |L|(qs^2 - qs) - |L| = qs^2 - s.$$

It follows that $|L \cup Z| = qs^2$.

Fix $x \in L$, and count the number of points of Z having relation f_2 to x.

If $y \notin L \cup Z$, there is a unique point of L adjacent to y, and hence the points of $L - \{x\}$ partition the points not in $L \cup Z$ having relation f_2 to x into $s-1$ sets of size p^1_{21}. It follows that the number of $z \in Z$ with $(x,z) \in f_2$ is

$$v_2 - (s-1)p^1_{21} = \frac{s((qs)^2-1)}{q+1} - \frac{(s-1)(qs)(qs-1)}{q+1} = qs^2 - s = |Z|.$$

Therefore, if $z \in Z$ and $x \in L$, $(x,z) \in f_2$.

If $z \in Z$, the points not in Z adjacent to z are partitioned (as above) by the points of L into s sets of size p^2_{11}. Hence, the number of points in Z adjacent to z is

$$v_1 - sp^2_{11} = qs^2 - qs + s - 1 - s(qs - q) = s - 1.$$

Hence $L \cup Z$ is a regular subgraph of degree $s-1$.

Similarly, if $z \in Z$, the number of points not in Z having relation f_3 to z is $sp^2_{31} = v_3$, hence if z and w are distinct nonadjacent points of Z, then $(z,w) \in f_2$. □

We now consider the relationship in the graph between a set having the properties of $L \cup Z$ and a point outside the set. Let $\mathcal{L}$ be the set of all subsets of X which induce a regular subgraph on qs^2 points in Γ_1 with valency $s-1$ and form a coclique in Γ_3. By the previous proposition, each line is contained in a (clearly unique) element of $\mathcal{L}$.

PROPOSITION. *Suppose $T \in \mathcal{L}$, and $x \notin T$. Then x is adjacent to qs points of T, and has relation f_3 to $\frac{qs(s-q)}{q+1}$ points of T.*

Proof. Order X so that the points of T are first, and consider the corresponding partition of A_2, so

$$A_2 = \begin{pmatrix} A_{2T} & M \\ M^t & A_{2X\backslash T} \end{pmatrix},$$

where A_{2T} and $A_{2X\backslash T}$ are the adjacency matrices for f_2 restricted to T and $X\backslash T$ respectively. Let B_2 be the matrix of average row sums for this partition. Then

$$B_2 = \begin{pmatrix} s(qs-1) & \frac{qs(qs-1)(s-1)}{q+1} \\ \frac{qs(qs-1)}{q+1} & \frac{q^2s^3-q^2s^2+qs-s}{q+1} \end{pmatrix},$$

which has eigenvalues $v_2 = \frac{q^2s^3-s}{q+1}$ and $\frac{s(qs-1)}{q+1}$. These interlace the eigenvalues of A_2, and the interlacing is tight, hence, the row sums are constant. It follows that if $x \notin T$, then x has relation f_2 to $\frac{qs(qs-1)}{q+1}$ points to T.

Similarly partition A_3, and let B_3 be the matrix of average row sums. Then

$$B_3 = \begin{pmatrix} 0 & v_3 \\ v_3/(s-1) & v_3(s-2)/(s-1) \end{pmatrix}$$

with eigenvalues v_3 and $\frac{-qs(s-q)}{q+1}$. Again the interlacing is tight, so if $x \notin T, x$ has relation f_3 to $v_3/(s-1) = \frac{qs(s-q)}{q+1}$ points of T.

Therefore if $x \notin T, x$ is adjacent to $|T| - \frac{qs(qs-1)}{q+1} - \frac{qs(s-q)}{q+1} = qs$ points of T. □

PROPOSITION. *Suppose $T \in \mathcal{L}, x \notin T$. Then there exists $T_x \in \mathcal{L}$ such that $x \in T_x$ and $T_x \cap T = \phi$.*

Proof. Since x is adjacent to qs points of T, and x lies on $qs+1$ lines, there exists a line L_x with $x \in L_x, L_x$ disjoint from T. Let Z_x be the set of points at distance two from L_x. Each point of L_x is adjacent to qs points of T, and these sets of points are disjoint. Thus, each point of T is adjacent to L_x, and it follows that $T_x = L_x \cup Z_x$ is the desired set. □

THEOREM. *Let $T \in \mathcal{L}$. Then T is the disjoint union of qs lines, and T is uniquely determined by each of its lines.*

Proof. We wish to show that (T, f_1) is the graph $qs \cdot K_s$; this is equivalent to showing that its adjacency matrix A_{1T} has eigenvalue $s-1$ with multiplicity at least qs.

Let $E = -J + qA_1 + (q+1)A_3$. Then E has eigenvalues $qs-q$ and $qs-q-qs^2$ with multiplicities $qs^3 - qs^2 + qs - s$ and $qs^2 - qs + s$, respectively. Partition E according to the partition of X into T and $X\backslash T$, so

$$E = \begin{pmatrix} E_T & M \\ M^t & E_{X\backslash T} \end{pmatrix}.$$

Note E_T is $qs^2 \times qs^2$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{qs^2}$ be the eigenvalues of E_T, and $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{qs^3-qs^2}$ be the eigenvalues of $E_{X\backslash T}$. Then by [2, Theorem 1. 3. 3], $\lambda_i = qs - q$ for $i = 1, \ldots, qs - s$ and $\lambda_i = 2qs - 2q - qs^2 - \mu_{qs^3-qs^2+qs-s+1-i}$ for $qs - s + 1 \leq i \leq qs^2$.

Let $U \in \mathcal{L}$ with $U \cap T = \phi$, and χ_U be the characteristic vector of U in the set $X\backslash T$. If $x \in X\backslash T, x$ is adjacent to qs points of U and has relation f_3 to $\frac{qs(s-q)}{q+1}$, hence $E_{X\backslash T}\mathcal{X}_U = (qs - q - qs^2)\mathcal{X}_U$. There are at least $\frac{qs^3-qs^2}{qs^2} = s - 1$ linearly independent such vectors, since each point of $X\backslash T$ is in such a U. By Cauchy interlacing $\mu_i \geq qs - q - qs^2$, therefore $\lambda_i = qs - q$ for $qs - s + 1 \leq i \leq qs - 1$.

We now relate this to the eigenvalues of A_{1T}. Note that $E_T = -J_T + qA_{1T}$, and $E_T\mathbf{j} = (qs - q - qs^2)\mathbf{j}$, where $\mathbf{j}$ is the vector of all ones. The eigenvectors of E_T with eigenvalue $qs - q$ are orthogonal to $\mathbf{j}$, hence they are eigenvectors for A_{1T} with eigenvalue $s - 1$. Since $\mathbf{j}$ is an eigenvector for A_{1T} with eigenvalue $s - 1$, A_{1T} has eigenvalue $s - 1$ with multiplicity qs. It follows that the subgraph induced by T is $qs \cdot K_s$. □

If $T \in \mathcal{L}$, each point not in T is on a unique line disjoint from T and this determines a partition of X into a disjoint union of elements of $\mathcal{L}$. Let Π be the set of such partitions. Proceeding as in [1], we construct a GQ as follows:

The point set is $X \cup \mathcal{L} \cup \{\infty\}$.

The lines are of two types:

(1) $L \cup \{T\}$ where L is an s-clique in $\Gamma_1, T \in \mathcal{L}, L \subset T$.
(2) $\pi \cup \{\infty\}$ where $\pi \in \Pi$.

Incidence is inclusion.

It is easy to check that this gives a GQ with ∞ a quasiregular point, and that $(X, \{f_1, f_2, f_3\})$ is the coherent configuration of points at distance 2 from ∞ derived as in [4].

That T is uniquely determined by each of its lines follows from the first proposition.

3. Remarks

If $s = 3$, $p^1_{11} = 1$, which implies the condition on maximal cliques is satisfied. Then $q = 1$ is the only value giving nonnegative parameters, and the constructed GQ has parameters (3, 3) and a quasiregular point, hence is unique. This implies that the corresponding association scheme is unique.

The case considered by Brouwer and Haemers [1] of a GQ(s, s^2) may be considered to be a special case of our result, where the relation f_3 is empty and $A_3 = 0$. Our proof works also in this case, and essentially reduces to theirs.

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