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An unpublished theorem of Manfred Schocker and the Patras-Reutenauer algebra

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Abstract Patras, Reutenauer (J. Algebr. Comb. 16:301–314, 2002) describe a subalgebra $\mathfrak{A}$ of the Malvenuto-Reutenauer algebra $\mathcal{P}$. Their paper contains several characteristic properties of this subalgebra. In an unpublished manuscript Manfred Schocker states without proof a theorem, providing two further characterizations of the Patras-Reutenauer algebra. In this paper we establish a slightly generalized version of Schocker's theorem, and give some applications. Finally we describe a derivation of the convolution algebra $\mathfrak{A}$, which is a homomorphism for the inner product.

Keywords Symmetric group algebras · Reciprocity laws · Lie idempotents · Solomon's descent algebra

1 Introduction

In this section we explain some different characterizations of the Patras-Reutenauer algebra, contained in [9], and the theorem of Schocker.

As a vector space the Malvenuto-Reutenauer algebra $\mathcal{P}$ is the direct sum of all group algebras $K\mathcal{S}_n$, where $\mathcal{S}_n$ denotes the group of all permutations of the set $[n] := \{1, \cdots, n\} \subseteq \mathbb{N}$. The set $\mathcal{S} := \bigcup_{n\geq 0} \mathcal{S}_n$ is a basis of $\mathcal{P}$. The field of coefficients K is assumed to be of characteristic 0. Via *Polya action* $\mathcal{P}$ acts on every free associative algebra $\mathcal{A} = \mathcal{A}\langle X\rangle$, freely generated by a set X. The multiplicative monoid X^* generated by X is a basis of $\mathcal{A}$. For all $\sigma \in \mathcal{S}$ and all *words* $x_1 \cdots x_n \in X^*$ of *length* n we put

$$\sigma x_1 \cdots x_n := \begin{cases} x_{1\sigma} \cdots x_{n\sigma} & \text{if } \sigma \in \mathcal{S}_n , \\ 0_{\mathcal{A}} & \text{if } \sigma \notin \mathcal{S}_n . \end{cases}$$

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Linear extension yields the Polya action of $\mathcal{P}$ on $\mathcal{A}$, which is a left action. For every subset Y of X we define a (uniquely determined) algebra endomorphism of $\mathcal{A}$ by

$$x \mapsto x_Y := \begin{cases} x & \text{for } x \in Y, \\ 1_{\mathcal{A}} & \text{for } x \in X \setminus Y. \end{cases}$$

We write a_Y for the image of a under this mapping, for all $a \in \mathcal{A}$. The coproduct $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the uniquely determined algebra homomorphism, such that

$$x\delta = x \otimes 1_{\mathcal{A}} + 1_{\mathcal{A}} \otimes x,$$

for all $x \in X$. The Lie subalgebra $\mathcal{L} := \mathcal{L}\langle X \rangle$ generated by X of the Lie algebra $\mathcal{A}_{\mathsf{Lie}}$ associated to $\mathcal{A}$ is the set of all $a \in \mathcal{A}$ with the following property:

$$a\delta = a \otimes 1_{\mathcal{A}} + 1_{\mathcal{A}} \otimes a,$$

by the theorem of Friedrichs. There is a coproduct $\downarrow$ on the vector space $\mathcal{P}$, too: for $\sigma \in \mathcal{S}_n$,

$$\sigma \downarrow := \sum_{j=0}^{n} \tau_j \otimes \rho_{n-j},$$

$\tau_j \in \mathcal{S}_j$ and $\rho_{n-j} \in \mathcal{S}_{n-j}$, where, viewed as a word, the permutation τ_j is obtained by lining up the elements $1, \cdots, j$ from σ in their present order. Similarly, the permutation ρ_{n-j} is obtained by reading out the numbers $j+1, \cdots, n$ from σ from left to right and subtracting j from each of them. A more conceptual description is given in (3). For example, $312 \downarrow = 312 \otimes \emptyset + 1 \otimes 21 + 12 \otimes 1 + \emptyset \otimes 312$. Another example is $1_{\mathcal{S}_n} \downarrow = \sum 1_{\mathcal{S}_j} \otimes 1_{\mathcal{S}_{n-j}}$. We embed $\mathcal{P}$ in $\mathcal{A}\langle \mathbb{N} \rangle$ simply by reading permutations as words over the alphabet $\mathbb{N}$. To be more precise, we define a linear mapping w of $\mathcal{P}$ into $\mathcal{A}\langle \mathbb{N} \rangle$ by

$$\mathsf{w} : \sigma \mapsto \sigma\mathsf{w} := 1\sigma. \cdots .n\sigma = \sigma(1.2. \cdots .n) \in \mathbb{N}^*,$$

for all $\sigma \in \mathcal{S}_n$, and linear extension.[1] By $*$ we denote the *inner product* on $\mathcal{P}$, inherited from the algebra structure of all $K\mathcal{S}_n$. Then,

$$(x * y)\mathsf{w} = x(y\mathsf{w}),$$

for all $x, y \in \mathcal{P}$. Finally put

$$\mathcal{O} := \{x \in \mathcal{P} \mid x\mathsf{w} \in \mathcal{L}\langle \mathbb{N} \rangle\}.$$

The elements of $\mathcal{O}$ are often called *multilinear Lie elements*. Obviously, $\mathcal{O} = \bigoplus_{n \geq 1} \mathcal{O}_n$ where $\mathcal{O}_n := \mathcal{O} \cap K\mathcal{S}_n$. In addition to the inner product the *convolution product* $\star$ on $\mathcal{P}$ is defined as follows: For all $\sigma \in \mathcal{S}_n$ and $\tau \in \mathcal{S}_m$ we define $\sigma\#\tau \in \mathcal{S}_{n+m}$ by

$$i(\sigma\#\tau) := \begin{cases} i\sigma & \text{for} \quad 1 \leq i \leq n, \\ (i-n)\tau + n & \text{for} \quad n+1 \leq i \leq n+m, \end{cases}$$

[1] We denote the *concatenation* of $x, y \in \mathcal{A}\langle \mathbb{N} \rangle$ by $x.y$.

then

$$\sigma \star \tau := \sum_{\rho} (\sigma \# \tau)\rho, \tag{1}$$

where the summation is extended over all permutations $\rho \in \mathcal{S}_{n+m}$, which are increasing on $[n]$ and on $n + [m] = \{n+1, \cdots, n+m\}$. For example, $1 \star 21 = 132 + 231 + 321$. This turns $\mathcal{P}$ into an associative algebra with neutral element $\emptyset$, the only element of $\mathcal{S}_0$. By $\mathfrak{A}$ we denote the unitary subalgebra of $(\mathcal{P}, \star)$ generated by $\mathcal{O}$. Observe $\mathfrak{A} = \bigoplus_{n \geq 0} \mathfrak{A}_n$, where $\mathfrak{A}_n := \mathfrak{A} \cap K\mathcal{S}_n$, i.e. $\mathfrak{A}$ is a homogeneous subspace of $\mathcal{P}$, just as $\mathcal{O}$. The first characterization of $\mathfrak{A}$ is a consequence of several statements in [9] (Proposition-Definition 3, Theorem 4 and the assertions about primitive elements):

Theorem 1.1 *Let $x \in \mathcal{P}$. Then $x \in \mathfrak{A}$ if and only if for all sets X and all $a \in \mathcal{A} := \mathcal{A}\langle X \rangle$:*

$$(xa)\delta = (x\downarrow)(a\delta). \tag{2}$$

The left hand side refers to Polya action of $\mathcal{P}$ on $\mathcal{A}$ and the right hand side to Polya action of $\mathcal{P} \otimes \mathcal{P}$ on $\mathcal{A} \otimes \mathcal{A}$. In particular, $1_{\mathcal{S}_n} \in \mathfrak{A}$ for all $n \in \mathbb{N}$.

The second characterization needs some more preparation. Let X be an alphabet and $<$ a total ordering on X. We define the standard permutation $\pi_w \in \mathcal{S}_n$ belonging to a word $w = x_1 \cdots x_n \in X^*$ as follows:

$$i\pi_w < j\pi_w \quad \Longleftrightarrow \quad \begin{cases} x_i < x_j, \\ \text{or } x_i = x_j \text{ and } i < j. \end{cases}$$

For example, let $a < b < \cdots < z$ and $w = rccacd$, then $\pi_w = 623145 \in \mathcal{S}_6$. By linear extension we get a mapping

$$\mathsf{st} : \mathcal{A}\langle X \rangle \to \mathcal{P}, \quad w \mapsto w\,\mathsf{st} := \pi_w.$$

Then the coproduct $\downarrow$ can be described as follows:

$$\sigma\downarrow = \sum_{j=0}^{n} (\sigma \mathsf{w})_{[j]}\,\mathsf{st} \otimes (\sigma \mathsf{w})_{[n]\setminus[j]}\,\mathsf{st}, \tag{3}$$

for all $\sigma \in \mathcal{S}_n$. A slight modification of 1.1 follows:

Proposition 1.2 *Let $x = \sum_{\sigma \in \mathcal{S}_n} k_\sigma \sigma \in K\mathcal{S}_n$. Then $x \in \mathfrak{A}$ if and only if*

$$(x\mathsf{w})\delta = (x\downarrow)((1.2.\cdots.n)\delta). \tag{4}$$

Proof By 1.1 the condition (4) is necessary since $x\mathsf{w} = x(1.2.\cdots.n)$. On the other hand, if X is a set and $a = x_1 \cdots x_n \in X^n$ define the algebra homomorphism $\varphi : \mathcal{A}\langle \mathbb{N} \rangle \to \mathcal{A}\langle X \rangle$ by

$$i\varphi := \begin{cases} x_i & \text{if } 1 \leq i \leq n, \\ 1_{\mathcal{A}\langle X \rangle} & \text{if } n < i. \end{cases}$$

Then φ is permutable with Polya action, i.e. $(\sigma i_1.\cdots.i_n)\varphi = \sigma(i_1\varphi\cdots i_n\varphi)$, for all $\sigma \in \mathcal{S}_n$, further $xa = (x(1.2.\cdots.n))\varphi$ and

$$\begin{aligned}(xa)\delta &= (x(1.2.\cdots.n))\varphi\delta\\ &= (x(1.2.\cdots.n))\delta(\varphi\otimes\varphi)\\ &= \big((x\downarrow)(1.2.\cdots.n)\delta\big)(\varphi\otimes\varphi)\\ &= (x\downarrow)\big((1.2.\cdots.n)\delta(\varphi\otimes\varphi)\big)\\ &= (x\downarrow)(a\delta),\end{aligned}$$

therefore $x \in \mathfrak{A}$ by 1.1. □

Condition (4) is equivalent to

$$\begin{aligned}&\sum_{\substack{J\subseteq[n]\\|J|=j}}\sum_{\sigma\in\mathcal{S}_n} k_\sigma\big((\sigma\mathsf{w})_J\otimes(\sigma\mathsf{w})_{[n]\backslash J}\big)\\ &\quad=\sum_{\substack{J\subseteq[n]\\|J|=j}}\sum_{\sigma\in\mathcal{S}_n} k_\sigma\big((\sigma\mathsf{w})_{[j]}\,\mathsf{st}(1.2.\cdots.n)_J\otimes(\sigma\mathsf{w})_{[n]\backslash[j]}\,\mathsf{st}(1.2.\cdots.n)_{[n]\backslash J}\big),\end{aligned}$$

for $0\le j\le n$. Applying $\mathsf{st}\otimes\mathsf{st}$ yields

$$\begin{aligned}&\sum_{\substack{J\subseteq[n]\\|J|=j}}\sum_{\sigma\in\mathcal{S}_n} k_\sigma\big((\sigma\mathsf{w})_J\,\mathsf{st}\otimes(\sigma\mathsf{w})_{[n]\backslash J}\,\mathsf{st}\big)\\ &\qquad=\binom{n}{j}\sum_{\sigma\in\mathcal{S}_n} k_\sigma\big((\sigma\mathsf{w})_{[j]}\,\mathsf{st}\otimes(\sigma\mathsf{w})_{[n]\backslash[j]}\,\mathsf{st}\big).\end{aligned}$$

The following combinatorial characterization of $\mathfrak{A}$ ([9], Theorem 4) is much stronger than this last assertion.

Theorem 1.3 *Let* $x=\sum_{\sigma\in\mathcal{S}_n}k_\sigma\sigma\in K\mathcal{S}_n$. *Then* $x\in\mathfrak{A}$ *if and only if*

$$\begin{aligned}&\sum_{\sigma\in\mathcal{S}_n} k_\sigma\big((\sigma\mathsf{w})_J\,\mathsf{st}\otimes(\sigma\mathsf{w})_{[n]\backslash J}\,\mathsf{st}\big)\\ &\qquad=\sum_{\sigma\in\mathcal{S}_n} k_\sigma\big((\sigma\mathsf{w})_{[|J|]}\,\mathsf{st}\otimes(\sigma\mathsf{w})_{[n]\backslash[|J|]}\,\mathsf{st}\big),\end{aligned}$$

for all $J\subseteq[n]$.

The succeeding theorem of Manfred Schocker provides two further characterizations of the Patras-Reutenauer algebra. It is contained without proof in an unpublished manuscript.

Theorem 1.4 *For all* $x\in\mathcal{P}$ *the following statements are equivalent*:

(i) $x \in \mathfrak{A}$,
(ii) $(z_1 \star z_2) * x = ((z_1 \otimes z_2) *_{\otimes} x\downarrow)\,\mathsf{conv}$, *for all* $z_1, z_2 \in \mathcal{P}$,
(iii) $(x * z)\downarrow = x\downarrow *_{\otimes} z\downarrow$, *for all* $z \in \mathcal{P}$.

Here $\mathsf{conv} : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P},\ x \otimes y \mapsto x \star y$ *denotes the linearization of the convolution.*

The statement (ii) is called the *multiplicative reciprocity law*. It was proved in [5] for elements z_1, z_2, x in Solomon's algebra $\mathcal{D}$, which is the subalgebra of $(\mathcal{P}, \star)$ generated by all $1_{\mathcal{S}_n}$. In particular, $\mathcal{D} \subseteq \mathfrak{A}$. By [13] $\mathcal{D}$ is also a subalgebra of $(\mathcal{P}, *)$. In this paper we give a proof of Schocker's result in a slightly generalized version and some applications.

2 The Lie projector *R*

We present a useful instrument in this section, which is useful in the proof of Schocker's theorem and for applications.

Let p_n be the canonical projection of $\mathcal{P}$ onto $K\mathcal{S}_n$. The algebra $(\mathcal{P}, \star)$ is graded since $K\mathcal{S}_n \star K\mathcal{S}_m \subseteq K\mathcal{S}_{n+m}$. We denote by

$$\widehat{\mathcal{P}} := \prod_{n \geq 0} K\mathcal{S}_n$$

the completion of $\mathcal{P}$ with respect to the metric given by

$$d(\alpha, \beta) := \begin{cases} e^{-\min\{n \mid p_n(\alpha-\beta)\neq 0\}} & \text{if } \alpha \neq \beta, \\ 0 & \text{if } \alpha = \beta. \end{cases}$$

Consider $\mathcal{P}$ as a subspace of $\widehat{\mathcal{P}}$. The projection of $\widehat{\mathcal{P}}$ onto $K\mathcal{S}_n$ is again denoted by p_n. For every sequence $(\alpha_n)_{n\geq 0}$ converging to $0_{\widehat{\mathcal{P}}}$ the series $\sum \alpha_n$ itself is convergent. Cauchy multiplication turns $\widehat{\mathcal{P}}$ into an associative algebra with neutral element $\emptyset \in \mathcal{S}_0$. Obviously, $(\mathcal{P}, \star)$ is a subalgebra of $(\widehat{\mathcal{P}}, \star)$. The inner product $*$ and the Polya action of $\mathcal{P}$ on $\mathcal{A}$ can also be extended to $\widehat{\mathcal{P}}$. The latter is a homogeneous operation, i.e., the subspaces $\mathcal{A}_n$ are invariant, where $\mathcal{A}_n$ denotes the subspace of $\mathcal{A}$ generated by all elements of X^* of length n. In contrast with $(\mathcal{P}, *)$ the algebra $(\widehat{\mathcal{P}}, *)$ has a neutral element:

$$E := \sum_{n\geq 0} 1_{\mathcal{S}_n}.$$

Polya action defines a linear mapping pol of $\widehat{\mathcal{P}}$ into the algebra $\mathcal{E}$ of all linear endomorphisms of $\mathcal{A}$, which is injective if (and only if) X is infinite.With respect to the inner product pol is an anti-homomorphism. The following statement[2] is well known:

$$\mathsf{pol}\,(S \star T) = \delta(\mathsf{pol}\,(S) \otimes \mathsf{pol}\,(T))\,\mathsf{conc} \tag{5}$$

[2] In [7] this is used to *define* the convolution product on $\mathcal{P}$.

for all $S, T \in \widehat{\mathcal{P}}$. Here $\mathsf{conc}: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $a \otimes b \mapsto ab$ denotes the linearization of the concatenation in $\mathcal{A}$. Putting

$$\varphi \star \psi := \delta(\varphi \otimes \psi)\,\mathsf{conc}$$

for all $\varphi, \psi \in \mathcal{E}$ turns $\mathcal{E}$ into an associative algebra with neutral element ε, the canonical projection of $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ onto $\mathcal{A}_0$. Then pol is a unital homomorphism of $(\widehat{\mathcal{P}}, \star)$ in $(\mathcal{E}, \star)$. It is convenient (see [11]), to generalize conc and δ in the following way: for all $k \in \mathbb{N}$ define an algebra homomorphism

$$\delta^{(k)} : \mathcal{A} \to \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{k} = \mathcal{A}^{\otimes k}$$

by

$$\delta^{(k)} : x \mapsto \sum_{i=0}^{k-1} \underbrace{1_{\mathcal{A}} \otimes \cdots \otimes 1_{\mathcal{A}}}_{i} \otimes x \otimes \underbrace{1_{\mathcal{A}} \otimes \cdots \otimes 1_{\mathcal{A}}}_{k-i-1},$$

for all $x \in X$, further $(a_1 \otimes \cdots \otimes a_k)\,\mathsf{conc}^{(k)} := a_1 \cdots a_k$. Then for all $\varphi_1, \cdots, \varphi_k \in \mathcal{E}$

$$\varphi_1 \star \cdots \star \varphi_k = \delta^{(k)}(\varphi_1 \otimes \cdots \otimes \varphi_k)\,\mathsf{conc}^{(k)}. \tag{6}$$

As a consequence of (6) and (5), observe the *reciprocity law for Polya action*:

$$(S_1 \star \cdots \star S_k)a = ((S_1 \otimes \cdots \otimes S_k)(a\delta^{(k)}))\,\mathsf{conc}^{(k)}, \tag{7}$$

for all $S_1, \cdots, S_k \in \widehat{\mathcal{P}}$ and all $a \in \mathcal{A}$. An element P of $\widehat{\mathcal{P}}$ is a *Lie projector*, if $P * P = P$ and if $P\mathcal{A} = \mathcal{L}$ for all sets X. As a consequence of (7), we note

$$(P_1 \star \cdots \star P_k)\mathcal{A} \subseteq \underbrace{\mathcal{L} \cdots \mathcal{L}}_{k} =: \mathcal{L}^k, \tag{8}$$

for all Lie projectors $P_1, \cdots, P_k$. Furthermore, (cf. [11], 1.5.6.)

$$a\delta^{(k)} = \sum_{i=0}^{k-1} \underbrace{1_{\mathcal{A}} \otimes \cdots \otimes 1_{\mathcal{A}}}_{i} \otimes a \otimes \underbrace{1_{\mathcal{A}} \otimes \cdots \otimes 1_{\mathcal{A}}}_{k-i-1}, \tag{9}$$

for all $a \in \mathcal{L}$. From (6) and (9) we deduce:

$$a(\varphi_1 \star \cdots \star \varphi_k) = 0_{\mathcal{A}} \tag{10}$$

for all $a \in \mathcal{L}$, $k > 1$ and $\varphi_1, \cdots, \varphi_k \in \mathcal{E}$ with the property $1_{\mathcal{A}}\varphi_j = 0_{\mathcal{A}}$, $1 \leq j \leq k$. Recall $E = \sum_{n \geq 0} 1_{\mathcal{S}_n} = 1_{(\mathcal{P}, *)}$. Put $I := \sum_{n \geq 1} 1_{\mathcal{S}_n}$, that means $E = \emptyset + I$, then $I^{\star n}a = 0_{\mathcal{A}}$ for all $a \in \mathcal{L}$ and $n = 0$ or $n > 1$, by (10).[3] Define

$$R := \log E = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} I^{\star n},$$

[3] $I^{\star n}$ denotes the n-fold convolution product of I with itself.

then $Ra = a$ for all $a \in \mathcal{L}$. Define the mapping

$$\widehat{\downarrow} : \widehat{\mathcal{P}} \to \prod_{n=0}^{\infty} \big(\bigoplus_{k=0}^{n} K\mathcal{S}_k \otimes K\mathcal{S}_{n-k}\big) = \widehat{\mathcal{P} \otimes \mathcal{P}}$$

as the continuous extension of $\downarrow$. For any sequence $(\alpha_n)_{n\geq 0}$ with $\alpha_n \in K\mathcal{S}_n$ the series $\alpha := \sum_{n\geq 0} \alpha_n$ is convergent and

$$\alpha \widehat{\downarrow} = \sum_{n\geq 0} (\alpha_n \downarrow).$$

From $1_{\mathcal{S}_n} \downarrow = \sum_{k=0}^{n} 1_{\mathcal{S}_k} \otimes 1_{\mathcal{S}_{n-k}}$ we deduce

$$E \widehat{\downarrow} = E \otimes E = \sum_{n=0}^{\infty} \sum_{k=0}^{n} 1_{\mathcal{S}_k} \otimes 1_{\mathcal{S}_{n-k}} \in \widehat{\mathcal{P} \otimes \mathcal{P}}.$$

It is well known that $R = \log E$ is primitive with respect to $\widehat{\downarrow}$, i.e.

$$R \widehat{\downarrow} = R \otimes \emptyset + \emptyset \otimes R. \tag{11}$$

Putting $\rho_n := p_n(R)$, in particular $R = \sum_{n\geq 1} \rho_n$, we have now:

$$\rho_n \downarrow = \rho_n \otimes \emptyset + \emptyset \otimes \rho_n.$$

The elements ρ_n are contained in the subalgebra of $(\mathcal{P}, \star)$ generated by all $1_{\mathcal{S}_n}$, i.e. in Solomons Algebra $\mathcal{D} = \bigoplus_{n\geq 0} \mathcal{D}_n$, which is contained in $\mathfrak{A}$ by 1.1. Again by 1.1, we conclude for $n \geq 1$ and all $y \in \mathcal{A}_n$

$$\begin{aligned}(Ry)\delta = (\rho_n y)\delta = (\rho_n \downarrow)(y\delta) = \\ = \rho_n y \otimes 1_{\mathcal{A}} + 1_{\mathcal{A}} \otimes \rho_n y = Ry \otimes 1_{\mathcal{A}} + 1_{\mathcal{A}} \otimes Ry,\end{aligned}$$

therefore $R\mathcal{A} \subseteq \mathcal{L}$ by the theorem of Friedrichs. Since $Ra = a$ for all $a \in \mathcal{L}$ we get

Proposition 2.1 *R is a Lie projector, i.e. R is an idempotent in $(\widehat{\mathcal{P}}, *)$ and $R\mathcal{A}\langle X\rangle = \mathcal{L}\langle X\rangle$ for all sets X. Especially, ρ_n is an idempotent in $\mathfrak{A}_n$.*[4]

We can easily deduce from 2.1:

$$\mathcal{O} = R * \mathcal{P} = \bigoplus_{n\geq 1} \rho_n * K\mathcal{S}_n \quad \text{and} \quad \mathcal{O}_n = \rho_n * K\mathcal{S}_n. \tag{12}$$

In particular, $\mathcal{O}$ is a right ideal in $(\mathcal{P}, *)$. Concerning the proof, we remark that

$$x * y = (x(y\mathsf{w}))\,\mathsf{st} = y\mathsf{w}\,(\mathsf{pol}\, x)\,\mathsf{st}$$

[4] The elements ρ_n first appeared in [12], by communication of a referee also in [2], [8]. Further, $\mathsf{pol}\, R$ is the canonical projection π_1 from [10].

for all $x \in \widehat{\mathcal{P}}$ and $y \in \mathcal{P}$, therefore

$$R * y = (R(y\mathsf{w}))\,\mathsf{st} \in (\mathcal{P}\mathsf{w} \cap \mathcal{L})\,\mathsf{st} = \mathcal{O},$$

that means $R * \mathcal{P} \subseteq \mathcal{O}$. On the other hand, if $y \in \mathcal{O}$ then $y\mathsf{w} \in \mathcal{L}$ and $R * y = (R(y\mathsf{w}))\,\mathsf{st} = y\mathsf{w}\,\mathsf{st} = y$.

3 Multiplicative reciprocity

We prove Schocker's theorem in a rather generalized version, which is more convenient for applications.

In analogy to $\delta^{(k)}$ we define recursively

$$\downarrow^{(k)} := \downarrow (\downarrow^{(k-1)} \otimes \mathrm{id}).$$

Then

$$\downarrow^{(k)} : \mathcal{P} \to \mathcal{P}^{\otimes k} = \bigoplus_{n=0}^{\infty} \left(\bigoplus_{j_1+\cdots+j_k=n} K\mathcal{S}_{j_1} \otimes \cdots \otimes K\mathcal{S}_{j_k} \right)$$

is a homogeneous algebra homomorphism. Further,

$$\widehat{\downarrow}^{(k)} : \widehat{\mathcal{P}} \to \prod_{n=0}^{\infty} \left(\bigoplus_{j_1+\cdots+j_k=n} K\mathcal{S}_{j_1} \otimes \cdots \otimes K\mathcal{S}_{j_k} \right) = \widehat{\mathcal{P}^{\otimes k}}$$

denotes the continuous extension of $\downarrow^{(k)}$, that means $x\widehat{\downarrow}^{(k)} = \sum_{n\geq 0} x_n \downarrow^{(k)}$ for $x = \sum_{n\geq 0} x_n$, $x_n \in K\mathcal{S}_n$. Similarly $\widehat{\mathsf{conv}}^{(k)}$ is the continuous extension of $\mathsf{conv}^{(k)} : \mathcal{P}^{\otimes k} \to \mathcal{P}$, $z_1 \otimes \cdots \otimes z_k \mapsto z_1 \star \cdots \star z_k$. Put $\mathcal{O}_n := \mathcal{O} \cap K\mathcal{S}_n$ and $\mathfrak{A}_n := \mathfrak{A} \cap K\mathcal{S}_n$, then $\mathcal{O} = \bigoplus_{n\geq 1} \mathcal{O}_n$ and $\mathfrak{A} = \bigoplus_{n\geq 0} \mathfrak{A}_n$. We put $\widehat{\mathfrak{A}} := \prod_{n\geq 0} \mathfrak{A}_n$ and $\widehat{\mathcal{O}} := \prod_{n\geq 1} \mathcal{O}_n$. For the proof of Schocker's theorem we need a remarkable relationship between the convolution product on $\mathcal{P}$ and the coproduct $\downarrow$, the *reciprocity law*. We define a symmetric, non-degenerate bilinear form $(\ ,\ )_{\mathcal{P}}$ on $\mathcal{P}$ by

$$(\sigma, \tau)_{\mathcal{P}} := \begin{cases} 1 & \text{if } \sigma = \tau^{-1} \\ 0 & \text{otherwise} \end{cases}$$

for all permutations σ and τ and bilinear extension. A moment's reflection reveals

$$(\alpha * \beta, \gamma)_{\mathcal{P}} = (\alpha, \beta * \gamma)_{\mathcal{P}} = (\beta, \gamma * \alpha)_{\mathcal{P}},$$

for all $\alpha, \beta, \gamma \in \mathcal{P}$. There exists exactly one non-degenerate and symmetric bilinear form $(\ ,\ )_{\mathcal{P}\otimes\mathcal{P}}$ on $\mathcal{P} \otimes \mathcal{P}$ with the property

$$(\alpha \otimes \beta, \gamma \otimes \delta)_{\mathcal{P}\otimes\mathcal{P}} := (\alpha, \gamma)_{\mathcal{P}} (\beta, \delta)_{\mathcal{P}},$$

for all $\alpha, \beta, \gamma, \delta \in \mathcal{P}$.

Reciprocity Law 3.1 (cf. [3]) *For all* $\alpha_1, \alpha_2, \beta \in \mathcal{P}$:

$$(\alpha_1 \star \alpha_2, \beta)_{\mathcal{P}} = (\alpha_1 \otimes \alpha_2, \beta\downarrow)_{\mathcal{P}\otimes\mathcal{P}}.$$

The following assertion is a little bit changed and more convenient formulation of Schocker's theorem 1.4.

Main Lemma 3.2 *For all* $x \in \widehat{\mathcal{P}}$ *the following statements are equivalent*:

(i) $x \in \widehat{\mathfrak{A}}$,

(ii) $(z_1 \star \cdots \star z_k) * x = ((z_1 \otimes \cdots \otimes z_k) *_{\otimes} x\widehat{\downarrow}^{(k)})\widehat{\mathsf{conv}}^{(k)}$ *for all* $z_1, \cdots, z_k \in \widehat{\mathcal{P}}$ *and for all* $k \in \mathbb{N}$,

(iii) $(z_1 \star z_2) * x = ((z_1 \otimes z_2) *_{\otimes} x\widehat{\downarrow})\,\mathsf{conv}$ *for all* $z_1, z_2 \in \widehat{\mathcal{P}}$,

(iv) $(z_1 \star \cdots \star z_k) * x = ((z_1 \otimes \cdots \otimes z_k) *_{\otimes} x\widehat{\downarrow}^{(k)})\,\mathsf{conv}^{(k)}$ *for all* $z_1, \cdots, z_k \in \mathcal{P}$ *and for all* $k \in \mathbb{N}$,

(v) $(z_1 \star z_2) * x = ((z_1 \otimes z_2) *_{\otimes} x\widehat{\downarrow})\,\mathsf{conv}$ *for all* $z_1, z_2 \in \mathcal{P}$,

(vi) $(x * z)\widehat{\downarrow} = x\widehat{\downarrow} *_{\otimes} z\widehat{\downarrow}$ *for all* $z \in \widehat{\mathcal{P}}$,

(vii) $(x * z)\widehat{\downarrow} = x\widehat{\downarrow} *_{\otimes} z\downarrow$ *for all* $z \in \mathcal{P}$.[5]

Proof Let $x = \sum_{n\geq 0} x_n \in \widehat{\mathcal{P}}$, $x_n \in K\mathcal{S}_n$. Every statement (i) up to (vii) is true for x if and only if it is true for all x_n. Therefore, we can assume $x \in K\mathcal{S}_n$ for some n. First we show the equivalence of (ii) up to (vii). Furthermore, we can assume $z_1 \star \cdots \star z_k, z_1 \star z_2, z \in K\mathcal{S}_n$ resp. In particular (ii) and (iv), (iii) and (v), (vi) and (vii) resp. are equivalent. (v) is a weakening of (iv). By induction on k we show that (iv) follows from (v). For $k = 1$ nothing is to prove. Let $k \geq 2$ and $z := z_1 \star \cdots \star z_{k-1}$. Using Sweedler's notation ([14]) we conclude:

$$\begin{aligned}
&(z_1 \star \cdots \star z_k) * x \\
&\qquad = (z \star z_k) * x \\
&\qquad = ((z \otimes z_k) *_{\otimes} x\downarrow)\,\mathsf{conv} \\
&\qquad = \sum (z * x^{(1)} \otimes z_k * x^{(2)})\,\mathsf{conv} \\
&\qquad = \sum (((z_1 \otimes \cdots \otimes z_{k-1}) *_{\otimes} x^{(1)}\downarrow^{(k-1)})\,\mathsf{conv}^{(k-1)} \otimes z_k * x^{(2)})\,\mathsf{conv} \\
&\qquad = \sum (((z_1 \otimes \cdots \otimes z_{k-1}) *_{\otimes} x^{(1)}\downarrow^{(k-1)}) \otimes z_k * x^{(2)})\,\mathsf{conv}^{(k)} \\
&\qquad = (((z_1 \otimes \cdots \otimes z_{k-1}) \otimes z_k) *_{\otimes} x\downarrow(\downarrow^{(k-1)} \otimes \mathrm{id}))\,\mathsf{conv}^{(k)} \\
&\qquad = ((z_1 \otimes \cdots \otimes z_k) *_{\otimes} x\downarrow^{(k)})\,\mathsf{conv}^{(k)}
\end{aligned}$$

[5] Cf. [9], where (vii) is shown for all $x, z \in \mathfrak{A}$ (Theorem 10).

Altogether, (ii) up to (v) are equivalent. To prove equivalence of (v) and (vii), we argue as follows: on the one hand we have for all $z_1, z_2 \in \mathcal{P}$

$$\begin{aligned}\big((x * z)\downarrow,\, z_1 \otimes z_2\big)_{\mathcal{P}\otimes\mathcal{P}} &= \big(x * z,\, z_1 \star z_2\big)_{\mathcal{P}} \\ &= \big(z,\, (z_1 \star z_2) * x\big)_{\mathcal{P}}\end{aligned}$$

and on the other hand

$$\begin{aligned}\big(x\downarrow *_{\otimes} z\downarrow,\, z_1 \otimes z_2\big)_{\mathcal{P}\otimes\mathcal{P}} &= \big(z\downarrow, (z_1 \otimes z_2) *_{\otimes} x\downarrow\big)_{\mathcal{P}\otimes\mathcal{P}} \\ &= \big(z,\, ((z_1 \otimes z_2) *_{\otimes} x\downarrow)\,\mathsf{conv}\big)_{\mathcal{P}}.\end{aligned}$$

Therefore (vii) is true, if and only if for all $z_1, z_2 \in \mathcal{P}$,

$$(z_1 \star z_2) * x = \big((z_1 \otimes z_2) *_{\otimes} x\downarrow\big)\,\mathsf{conv}$$

i.e if and only if (v) is true. Now the equivalence of (ii) up to (vii) is shown.

Assume (ii), in particular for all $k \geq 0$:

$$(\overbrace{R \star \cdots \star R}^{k}) * x = \Big((\overbrace{R \otimes \cdots \otimes R}^{k}) *_{\otimes} x\downarrow^{(k)}\Big)\widehat{\mathsf{conv}}^{(k)}.$$

Since $R * \mathcal{P} = \mathcal{O}$, we have $R^{\star k} * x \in \mathcal{O}^{\star k}$. We conclude:

$$\begin{aligned}x &= 1_{(\widehat{\mathcal{P}}, *)} * x \\ &= \exp R * x \\ &= \sum_{k\geq 0} \frac{1}{k!} R^{\star k} * x \\ &= \sum_{k=0}^{n} \frac{1}{k!} R^{\star k} * x \;\in\; \sum_{k=0}^{n} \mathcal{O}^{\star k} \;\subseteq\; \mathfrak{A} \subseteq \widehat{\mathfrak{A}}.\end{aligned}$$

Remains to show that (v) is a consequence of (i). We prove this under the additional assumption that x is contained in the convolution subalgebra of $\mathfrak{A}$, generated by the primitive elements of $\mathcal{O}$ [6]. By linearity we may assume that $x = y_1 \star \cdots \star y_l$ is a convolution product of primitive elements of $\mathcal{O}$. Let X be an infinite set and $\mathcal{A} := \mathcal{A}\langle X\rangle$. Call $Y_j := \mathsf{pol}\, y_j$, $Z_i := \mathsf{pol}\, z_i$ and $\mathfrak{Y} := (\mathsf{pol} \otimes \mathsf{pol})(x\downarrow)$, the endomorphisms, induced by Polya action from $z_i, y_j, x\downarrow$ on $\mathcal{A}, \mathcal{A}\otimes\mathcal{A}$ resp. Now (v) is equivalent to:

$$(Y_1 \star \cdots \star Y_l)(Z_1 \star Z_2) = \big(\mathfrak{Y}(Z_1 \otimes Z_2)\big)\,\mathsf{conv}. \tag{13}$$

On the other hand,

$$(Y_1 \star \cdots \star Y_l)\delta = \delta\mathfrak{Y},$$

[6] By [9] $\mathcal{O} = \mathsf{Prim}\,\mathfrak{A}$.

by 1.1. As an easy consequence we get

$$\begin{aligned}(Y_1 \star \cdots \star Y_l)(Z_1 \star Z_2) \\ &= (Y_1 \star \cdots \star Y_l)\delta(Z_1 \otimes Z_2)\,\mathsf{conc} \\ &= \delta\mathfrak{Y}(Z_1 \otimes Z_2)\,\mathsf{conc} \\ &= (\mathfrak{Y}(Z_1 \otimes Z_2))\,\mathsf{conv}.\end{aligned}$$

It remains to show that all elements of $\mathcal{O}$ are primitive. Let x be an element of $\mathcal{O}$. Since $R \in \widehat{\mathcal{O}}$ and $x = R * x$ on account of (12), we can apply (vi), and conclude due to (11):

$$x\downarrow = (R * x)\downarrow = R\,\widehat{\downarrow} *_{\otimes} x\downarrow = (R * x) \otimes \emptyset + \emptyset \otimes (R * x) = x \otimes \emptyset + \emptyset \otimes x. \qquad \square$$

4 Multiplication rules

Corollary 4.1 *If y is a primitive element in $(\mathcal{P}, \downarrow)$ and $x \in \widehat{\mathfrak{A}}$, then $x * y$ is also primitive. In particular,* $\mathsf{Prim}\,\mathcal{P}$ *is an $\widehat{\mathfrak{A}}$-left module.*

Proof In Sweedler's notation we have $z\downarrow = z \otimes \emptyset + \emptyset \otimes z + \sum z^{(1)} \otimes z^{(2)}$ for all $z \in \mathcal{P}$, where $z^{(1)}, z^{(2)} \neq \emptyset$. The statement now follows from 3.2, (vii). $\square$

Corollary 4.2 *If z_1, z_2 are elements of $\widehat{\mathcal{P}}$ with the property $p_0(z_1) = 0 = p_0(z_2)$, and if $x \in \widehat{\mathfrak{A}}$ is primitive, then $(z_1 \star z_2) * x = 0_{\widehat{\mathcal{P}}}$. Further, $\widehat{\mathcal{O}}$ is the set of all primitive elements of $\widehat{\mathfrak{A}}$. In particular, $\mathcal{O}$ is the set of primitive elements of $\mathfrak{A}$* (cf. [9]).

Proof From 3.2, (iii) we conclude

$$(z_1 \star z_2) * x = (z_1 * x) \star (z_2 * \emptyset) + (z_1 * \emptyset) \star (z_2 * x) = 0_{\widehat{\mathcal{P}}}.$$

The second statement follows from [9]. For convenience, we give the short argument. The end of the proof of 3.2 shows in particular:

$$\widehat{\mathcal{O}} \subseteq \mathsf{Prim}\,(\widehat{\mathfrak{A}}, \widehat{\downarrow}). \tag{14}$$

An element $x \in \widehat{\mathcal{P}}$ is primitive if and only if $p_n(x)$ is primitive for all n. In particular, $p_0(x) = 0$. Now let $x \in \mathsf{Prim}\,\widehat{\mathfrak{A}}$. Then

$$x = E * x = \exp R * x = \sum_{n \geq 0} \frac{1}{n!} R^{\star n} * x = R * x,$$

finally $\widehat{\mathcal{O}} = R * \widehat{\mathfrak{A}} \supseteq R * \mathsf{Prim}\,\widehat{\mathfrak{A}} = \mathsf{Prim}\,\widehat{\mathfrak{A}}$. $\square$

A useful consequence of 3.2, (ii) is the following multiplication rule for all $\alpha_1, \cdots, \alpha_l \in \widehat{\mathcal{P}}$ and $\beta_1, \cdots, \beta_k \in \widehat{\mathcal{O}}$:

$$(\alpha_1 \star \cdots \star \alpha_l) * (\beta_1 \star \cdots \star \beta_k) = \sum_{J_1, \cdots, J_l} (\alpha_1 * \beta_{J_1}) \star \cdots \star (\alpha_l * \beta_{J_l}). \tag{15}$$

The summation is extended over all pairwise disjoint subsets $J_1, \cdots, J_l$ of $[k]$ such that $J_1 \cup \cdots \cup J_l = [k]$; if $J = \{j_1, \cdots, j_m\} \subseteq [k]$ with $j_1 < \cdots < j_m$, then $\beta_J = \beta_{j_1} \star \cdots \star \beta_{j_m}$, and if $J = \emptyset$, then $\beta_J = 1_{(\mathcal{P},\star)} = \emptyset \in \mathcal{S}_0$. Patras and Reutenauer ([9], Theorem 10) have shown that $\mathfrak{A}$ is a subalgebra of $(\mathcal{P}, *)$. This is also a consequence of (15): if $\alpha_1, \cdots, \beta_k \in \mathcal{O}$ the product in (15) is an element of $\mathfrak{A}$, since $\mathcal{O} = R * \mathcal{P}$ is a right ideal of $(\mathcal{P}, *)$. Two special cases are of interest:

$$(\alpha_1 \star \cdots \star \alpha_k) * (\beta_1 \star \cdots \star \beta_k) = \sum_{\sigma \in \mathcal{S}_k} (\alpha_1 * \beta_{1\sigma}) \star \cdots \star (\alpha_l * \beta_{l\sigma}). \tag{16}$$

If $p_0(\alpha_1) = \cdots = p_0(\alpha_l) = 0_{\widehat{\mathcal{P}}}$ and $l > k$:

$$(\alpha_1 \star \cdots \star \alpha_l) * (\beta_1 \star \cdots \star \beta_k) = 0_{\widehat{\mathcal{P}}}. \tag{17}$$

Denote some simple consequences of (15), (16) and (17):

- $R^{\star n} * R^{\star k} = 0_{\widehat{\mathcal{P}}} = I^{\star n} * R^{\star k} \quad \text{if } n > k,$
- $R^{\star n} * R^{\star n} = n! R^{\star n},$
- $I^{\star l} * R^{\star k} = l! \mathfrak{S}_k^{(l)} R^{\star k} \quad \text{if } l \le k,$
- $R * R^{\star k} = \big(\sum_{l=1}^{k} (-1)^{l-1} (l-1)! \mathfrak{S}_k^{(l)}\big) R^{\star k} = 0_{\widehat{\mathcal{P}}} \quad \text{if } 1 < k,$
- $R^{\star n} * R^{\star k} = 0_{\widehat{\mathcal{P}}} \quad \text{if } n \ne k.$

Here $\mathfrak{S}_k^{(l)}$ is a Stirling number of the second kind, i.e. the number of ways of partitioning of $[k]$ into l non-empty subsets. The fourth equation follows easily from $\mathfrak{S}_k^{(l)} = l \mathfrak{S}_{k-1}^{(l)} + \mathfrak{S}_{k-1}^{(l-1)}$ for $k > l$. As a consequence, we get:

Proposition 4.3 *The elements* $^1/_{k!} R^{\star k}$ *of* $\widehat{\mathcal{O}}$ *constitute a system of pairwise orthogonal idempotents of* $(\widehat{\mathcal{P}}, *)$, *summing up to the neutral element* $E = \exp R$ *of* $(\widehat{\mathcal{P}}, *)$ ([10]). *In particular, putting* $\mathcal{O}^{(k)} := R^{\star k} * \mathfrak{A}$,

$$\mathfrak{A} = \bigoplus_{k \ge 0} \mathcal{O}^{(k)}$$

is a direct decomposition of $(\mathfrak{A}, *)$ *into right ideals. Moreover,* $\sum_{k=1}^{n} \mathcal{O}^{(k)}$ *is an ideal of* $(\mathfrak{A}, *)$ *for all* $n \in \mathbb{N}$ *and*

$$\mathcal{O}^{(k)} = \left\langle \sum_{\sigma \in \mathcal{S}_k} \alpha_{1\sigma} \star \cdots \star \alpha_{k\sigma} \,\middle|\, \alpha_1, \cdots, \alpha_k \in \mathcal{O} \right\rangle_K . \tag{18}$$

Proof Put $\mathcal{O}^{\star n} := \overbrace{\mathcal{O} \star \cdots \star \mathcal{O}}^{n}$, then by (17)

$$\mathcal{O}^{\star n} = E * \mathcal{O}^{\star n} = \sum_{k \ge 0} \frac{1}{k!} R^{\star k} * \mathcal{O}^{\star n} = \sum_{k=0}^{n} R^{\star k} * \mathcal{O}^{\star n} \subseteq \sum_{k=0}^{n} \mathcal{O}^{(k)},$$

therefore $\sum_{k=0}^{n} \mathcal{O}^{\star k} = \sum_{k=0}^{n} \mathcal{O}^{(k)}$. Applying $R^{\star n}$ on this equation by Polya action we conclude $R^{\star n} * \mathcal{O}^{\star n} = \mathcal{O}^{(n)}$. The assertion (18) now follows from (16). By (17),

$$\mathcal{O}^{\star l} * \mathcal{O}^{\star k} = \{0_{\widehat{\mathcal{P}}}\} \quad \text{if } l > k.$$

In particular, $\sum_{k=1}^{n} \mathcal{O}^{(k)}$ is an ideal in $(\mathfrak{A}, *)$. □

5 The endomorphism Θ of $\mathfrak{A}$

The next statement describes another combinatorial property of the Patras-Reutenauer algebra.

Theorem 5.1 *Let $x \in \mathfrak{A}_n$, then $(x\mathsf{w})_J\,\mathsf{st}$ is an element of $\mathfrak{A}_{|J|}$ and depends only on $|J|$, for all $J \subseteq [n]$.*

Proof As a convolution algebra, $\mathfrak{A}$ is generated by it's primitive elements ([9] or 4.2). If $\alpha_1, \cdots, \alpha_k \in \mathsf{Prim}\,\mathfrak{A}$ then

$$(\alpha_1 \star \cdots \star \alpha_k) \downarrow = \sum_{J \subseteq [k]} \alpha^J \otimes \alpha^{[k]\setminus J},$$

where $\alpha^J = \alpha_{j_1} \star \cdots \star \alpha_{j_l}$ for $J = \{j_1, \cdots, j_l\}$, $j_1 < \cdots < j_l$. Therefore $\mathfrak{A}\downarrow \subseteq \mathfrak{A} \otimes \mathfrak{A}$ and $\mathfrak{A}_n \downarrow \subseteq \bigoplus \mathfrak{A}_j \otimes \mathfrak{A}_{n-j}$. Let $x = \sum_{\sigma \in \mathcal{S}_n} k_\sigma \sigma$, then by (3)

$$x \downarrow = \sum_{j=0}^{n} \sum_{\sigma \in \mathcal{S}_n} k_\sigma \big((\sigma\mathsf{w})_{[j]}\,\mathsf{st} \otimes (\sigma\mathsf{w})_{[n]\setminus[j]}\,\mathsf{st}\big). \tag{19}$$

For all $J \subseteq [n]$ with $|J| = n-1$ now follows by 1.3:

$$\begin{aligned}
(x\mathsf{w})_J\,\mathsf{st} \otimes 1_{\mathcal{S}_1} &= \Big(\sum_{\sigma \in \mathcal{S}_n} k_\sigma (\sigma\mathsf{w})_J\,\mathsf{st}\Big) \otimes 1_{\mathcal{S}_1} \\
&= \sum_{\sigma \in \mathcal{S}_n} k_\sigma \big((\sigma\mathsf{w})_J\,\mathsf{st} \otimes (\sigma\mathsf{w})_{[n]\setminus J}\,\mathsf{st}\big) \\
&= \sum_{\sigma \in \mathcal{S}_n} k_\sigma \big((\sigma\mathsf{w})_{[n-1]}\,\mathsf{st} \otimes (\sigma\mathsf{w})_{[n]\setminus[n-1]}\,\mathsf{st}\big) \\
&= \Big(\sum_{\sigma \in \mathcal{S}_n} k_\sigma (\sigma\mathsf{w})_{[n-1]}\,\mathsf{st}\Big) \otimes 1_{\mathcal{S}_1} \\
&= (x\mathsf{w})_{[n-1]}\,\mathsf{st} \otimes 1_{\mathcal{S}_1}.
\end{aligned}$$

Since the summand for $j = n-1$ in (19) is an element of $\mathfrak{A}_{n-1} \otimes \mathfrak{A}_1 = \mathfrak{A}_{n-1} \otimes 1_{\mathcal{S}_1}$, we conclude

$$(x\mathsf{w})_J\,\mathsf{st} \otimes 1_{\mathcal{S}_1} = a \otimes 1_{\mathcal{S}_1} = (x\mathsf{w})_{[n-1]}\,\mathsf{st} \otimes 1_{\mathcal{S}_1},$$

for some suitable $a \in \mathfrak{A}_{n-1}$. A simple argument, using the basis $\mathcal{S}$ of $\mathcal{P}$, shows

$$(x\mathsf{w})_J\,\mathsf{st} = a = (x\mathsf{w})_{[n-1]}\,\mathsf{st}.$$

We have proved the statement for $|J| = n-1$. Let L, L' be different subsets of $[n]$ such that $|L| = |L'| \le n-2$. Take $i \in L \setminus L'$ and $j \in L' \setminus L$, put $L'' := (L \setminus \{i\}) \cup \{j\}$, then $i \notin L' \cup L''$ and $|L \cup L''| = |L \cup \{j\}| = |L| + 1 \le n-1$. Therefore exist $J_1, J_2 \subseteq [n]$ with the property $|J_1| = n-1 = |J_2|$ and $L' \cup L'' \subseteq J_1$, $L \cup L'' \subseteq J_2$. If $M \subseteq N \subseteq [n]$, then for all $\sigma \in \mathcal{S}_n$ we have the *transitivity rule*:

$$(\sigma\mathsf{w})_M\,\mathsf{st} = ((\sigma\mathsf{w})_N)_M)\,\mathsf{st} = ((\sigma\mathsf{w})_N\varphi)_{M\varphi}\,\mathsf{st} = (((\sigma\mathsf{w})_N\,\mathsf{st})\mathsf{w})_{M\varphi}\,\mathsf{st}, \qquad (20)$$

where φ is an algebra endomorphism of $\mathcal{A}\langle\mathbb{N}\rangle$, such that φ induces the uniquely determined order isomorphism of N onto $\{1, \cdots, |N|\}$. For example,

$$(x\mathsf{w})_L\,\mathsf{st} = (((x\mathsf{w})_{J_1}\,\mathsf{st})\mathsf{w})_{L\varphi}\,\mathsf{st}.$$

Because $(x\mathsf{w})_{J_1}\,\mathsf{st} = (x\mathsf{w})_{J_2}\,\mathsf{st} \in \mathfrak{A}_{n-1}$, we conclude by induction

$$(x\mathsf{w})_L\,\mathsf{st} = (x\mathsf{w})_{L''}\,\mathsf{st} = (x\mathsf{w})_{L'}\,\mathsf{st} \in \mathfrak{A}_{|L|}. \qquad \square$$

We define a linear mapping $\Theta : \mathcal{P} \to \mathcal{P}$, $\sigma \mapsto \sigma\Theta := (\sigma\mathsf{w})_{[n-1]}\,\mathsf{st}$, for all $\sigma \in \mathcal{S}_n$. Roughly spoken, $\sigma\Theta$ emerges from $\sigma \in \mathcal{S}_n$ by striking out the letter n in the image line of σ, for example, $13247865\Theta = 1324765$. Then $\mathfrak{A}\Theta \subseteq \mathfrak{A}$ by 5.1. As a matter of fact, $(x\mathsf{w})_J\,\mathsf{st} = x\Theta^{n-|J|}$, for all $x \in \mathfrak{A}_n$ and all $J \subseteq [n]$.

Proposition 5.2 $\Theta : \mathfrak{A} \to \mathfrak{A}$ *is a homomorphism for the inner product. In particular,* $((x * y)\mathsf{w})_J\,\mathsf{st} = (x\mathsf{w})_J\,\mathsf{st} * (y\mathsf{w})_J\,\mathsf{st}$, *for all* $x, y \in \mathfrak{A}_n$ *and all* $J \subseteq [n]$.

Proof Let $x = \sum_{\sigma\in\mathcal{S}_n} k_\sigma \sigma$ and $y = \sum_{\sigma\in\mathcal{S}_n} l_\sigma \sigma$ be elements of $\mathfrak{A}_n$. Put $x * y = \sum_{\sigma\in\mathcal{S}_n} m_\sigma \sigma$. Recall that $\mathfrak{A}$ is a subalgebra of $(\mathcal{P}, *)$. By Schocker's theorem (or [9], Theorem 10) $(x * y)\downarrow = x\downarrow *_\otimes y\downarrow$, and we conclude by 1.3

$$\begin{aligned}
&((x * y)\mathsf{w})_{[n-1]}\,\mathsf{st} \otimes 1_{\mathcal{S}_1}\\
&\quad = \sum_{\sigma\in\mathcal{S}_n} m_\sigma \big((\sigma\mathsf{w})_{[n-1]}\,\mathsf{st} \otimes (\sigma\mathsf{w})_{[n]\setminus[n-1]}\,\mathsf{st}\big)\\
&\quad = \sum_{\rho,\tau\in\mathcal{S}_n} k_\rho l_\tau \big((\rho\mathsf{w})_{[n-1]}\,\mathsf{st} * (\tau\mathsf{w})_{[n-1]}\,\mathsf{st} \otimes 1_{\mathcal{S}_1}\big)\\
&\quad = \Big(\big(\sum_{\rho\in\mathcal{S}_n} k_\rho (\rho\mathsf{w})_{[n-1]}\,\mathsf{st}\big) * \big(\sum_{\tau\in\mathcal{S}_n} k_\tau (\tau\mathsf{w})_{[n-1]}\,\mathsf{st}\big)\Big) \otimes 1_{\mathcal{S}_1}\\
&\quad = \big((x\mathsf{w})_{[n-1]}\,\mathsf{st} * (y\mathsf{w})_{[n-1]}\,\mathsf{st}\big) \otimes 1_{\mathcal{S}_1},
\end{aligned}$$

therefore $(x * y)\Theta = x\Theta * y\Theta$. $\square$

Proposition 5.3 *The mapping* Θ *induces a derivation of the convolution algebra* $(\mathfrak{A}, \star)$.

Proof Let $n \in \mathbb{N}$, $x \in \mathfrak{A}_j$ and $y \in \mathfrak{A}_{n-j}$. Put $v := 1.2.\cdots.n \in \mathbb{N}^*$ and $u := 1.2.\cdots.(n-1) \in \mathbb{N}^*$. Then

$$(x \star y)\mathsf{w} = (x \star y)1.2.\cdots.n = \sum_{\substack{J\subseteq[n]\\|J|=j}} (xv_J)(yv_{[n]\setminus J}),$$

for example by (7). Observe that $(x \star y)\Theta = x \star y\Theta + x\Theta \star y$ is equivalent to $(x \star y)\mathsf{w}_{[n-1]} = (x \star y)\Theta\mathsf{w} = (x \star y\Theta)\mathsf{w} + (x\Theta \star y)\mathsf{w}$. On the one hand, we have

$$\begin{aligned}
&(x \star y)\mathsf{w}_{[n-1]} \\
&\quad= \sum_{\substack{J\subseteq[n]\\|J|=j}} \big((xv_J)(yv_{[n]\setminus J})\big)_{[n-1]} \\
&\quad= \sum_{\substack{n\notin J\subseteq[n]\\|J|=j}} (xv_J)(yv_{[n]\setminus J})_{[n-1]} + \sum_{\substack{n\in J\subseteq[n]\\|J|=j}} (xv_J)_{[n-1]}(yv_{[n]\setminus J}).
\end{aligned}$$

For $n \in J \subseteq [n]$ with $|J| = j$ call Φ any algebra endomorphism of $\mathcal{A}\langle\mathbb{N}\rangle$, such that Φ induces the uniquely determined order preserving bijection of J onto $[j]$. Then,

$$x\Theta u_{J\setminus\{n\}}\Phi = x\Theta\mathsf{w} = (x\mathsf{w})_{[j-1]} = (xv_J)_{[n-1]}\Phi,$$

therefore $x\Theta u_{J\setminus\{n\}} = (xv_J)_{[n-1]}$, analogously $y\Theta u_{[n]\setminus J} = (yv_{[n]\setminus J})_{[n-1]}$ if $n \notin J$. To finish the proof, we conclude on the other hand

$$\begin{aligned}
&(x \star y\Theta)\mathsf{w} + (x\Theta \star y)\mathsf{w} \\
&\quad= \sum_{\substack{L\subseteq[n-1]\\|L|=j}} (xu_L)(y\Theta u_{[n-1]\setminus L}) + \sum_{\substack{M\subseteq[n-1]\\|M|=j-1}} (x\Theta u_M)(yu_{[n-1]\setminus M}) \\
&\quad= \sum_{\substack{n\notin J\subseteq[n]\\|J|=j}} (xu_J)(y\Theta u_{[n]\setminus J}) + \sum_{\substack{n\in J\subseteq[n]\\|J|=j}} (x\Theta u_{J\setminus\{n\}})(yv_{[n]\setminus J}) \\
&\quad= \sum_{\substack{n\notin J\subseteq[n]\\|J|=j}} (xu_J)((yv_{[n]\setminus J})_{[n-1]}) + \sum_{\substack{n\in J\subseteq[n]\\|J|=j}} ((xv_J)_{[n-1]})(yv_{[n]\setminus J}),
\end{aligned}$$

and that was to be shown. □

Next we prove that Θ induces an epimorphism of $\mathfrak{A}_n$ onto $\mathfrak{A}_{n-1}$. In particular, $(\mathfrak{A}_{n-1}, *)$ is isomorphic to a factor algebra of $(\mathfrak{A}_n, *)$. We first describe a system of generators for the vector space $\mathfrak{A}$. Recall that $\mathcal{O}$ is a homogeneous subspace of $\mathfrak{A}$, i.e.

$$\mathcal{O} = \bigoplus_{k\geq 1} \mathcal{O}_n,$$

where $\mathcal{O}_n := \mathcal{O} \cap K\mathcal{S}_n$. The elements of $\mathcal{O}_n$ are called homogeneous of degree n. Obviously, the set of all products $\alpha_1 \star \cdots \star \alpha_n$, where $\alpha_1, \cdots, \alpha_n$ are homogeneous elements of $\mathcal{O}$, is a system of linear generators of $\mathfrak{A}$. Call $\mathcal{G}$ the set of all products

$$\alpha_{1,1} \star \cdots \star \alpha_{1,j_1} \star \alpha_{2,1} \star \cdots \star \alpha_{2,j_2} \star \cdots \star \alpha_{l,1} \star \cdots \star \alpha_{l,j_l}, \tag{21}$$

where $l \in \mathbb{N}$ and $\alpha_{i,k} \in \mathcal{O}_i$. Let $\mathcal{G}^n$, $\mathcal{G}_n$ resp. be the set of all such elements with the property $j_1 + j_2 + \cdots + j_l = n$, $1 \cdot j_1 + 2 \cdot j_2 + \cdots + l \cdot j_l = n$ resp., then $\mathcal{G}_n \subseteq \mathfrak{A}_n$. Finally put $\mathcal{O}^{<n>}$ the subspace generated by $\mathcal{G}^n$. By (16), (17) and 4.3 we get:

$$R^{\star n} * \mathcal{O}^{<m>} = \begin{cases} \mathcal{O}^{(n)} & \text{if } n = m, \\ \{0_{\mathcal{P}}\} & \text{if } n > m. \end{cases}$$

We conclude

$$\sum_{k=0}^{n} \mathcal{O}^{<k>} = E \sum_{k=0}^{n} \mathcal{O}^{<k>} = \sum_{j=0}^{n} (j!)^{-1} R^{\star j} \sum_{k=0}^{n} \mathcal{O}^{<k>} \supseteq \sum_{k=0}^{n} \mathcal{O}^{(k)}.$$

In particular, $\mathfrak{A} = \sum_{k\geq 0} \mathcal{O}^{<k>}$ by 4.3, i.e. $\mathcal{G}$ generates the vector space $\mathfrak{A}$, and $\mathcal{G}_n$ is a generating system for $\mathfrak{A}_n$, since $\mathfrak{A} = \bigoplus \mathfrak{A}_n$. Recall the Specht-Wever element

$$\omega_n = \sum_{\pi \in \mathcal{V}_n} (-1)^{1\pi^{-1}-1} \pi,$$

where $\mathcal{V}_n$ is the set of all valley permutations in $\mathcal{S}_n$ (cf. [3]). A permutation π is in $\mathcal{V}_n$ if

$$1\pi > 2\pi > \cdots > k\pi > (k+1)\pi = 1 < (k+2)\pi < \cdots < n\pi.$$

Polya action of ω_n creates left normed Lie monomials:

$$\omega_n x_1 \cdots x_n = [\cdots [[x_1, x_2], x_3] \cdots, x_n],$$

for all words $x_1 \cdots x_n$ of length n over an alphabet X, e.g. by [6]. By the Dynkin-Specht-Wever theorem we have $\omega_n * \omega_n = n \cdot \omega_n$. Since the left normed Lie monomials generate the Lie algebra $\mathcal{L}\langle X \rangle$, we conclude from $\omega_n * \sigma = (\omega_n(\sigma \mathsf{w}))\,\mathsf{st}$:

$$\mathcal{O}_n = \omega_n * K\mathcal{S}_n = \omega_n * \mathfrak{A}_n.$$

In particular, $\mathcal{O}_1 = \omega_1 * K\mathcal{S}_1 = \langle \omega_1 \rangle_K$. Therefore we may assume

$$\alpha_{1,1} = \cdots = \alpha_{1,j_1} = \omega_1,$$

for all elements (21) of $\mathcal{G}_n$. From the definition of ω_n easily follows $\omega_1 \Theta = \emptyset \in \mathcal{S}_0$ and

$$\omega_n \Theta = 0_{\mathcal{P}} \quad \text{for } n \geq 2,$$

in particular $\mathcal{O}_n = \omega_n * \mathfrak{A}_n \subseteq \ker\Theta$ for $n \geq 0$ by 5.2, where $\ker\Theta$ denotes the kernel of Θ. Together with 5.3 this leads to

$$\omega_1^{\star j_1} \star \alpha_{2,1} \star \cdots \star \alpha_{2,j_2} \star \cdots \star \alpha_{l,1} \star \cdots \star \alpha_{l,j_l}\Theta$$
$$= j_1\, \omega_1^{\star(j_1-1)} \star \alpha_{2,1} \star \cdots \star \alpha_{2,j_2} \star \cdots \star \alpha_{l,1} \star \cdots \star \alpha_{l,j_l}.$$

Therefore $\mathcal{G}\Theta$ generates $\mathfrak{A}$ and $\mathcal{G}_n\Theta$ generates $\mathfrak{A}_{n-1}$. For all $a \in \mathfrak{A}_{n-1}$ and all $b = \sum_{\sigma\in\mathcal{S}_n} c_\sigma \sigma \in \mathfrak{A}_n$,

$$(\omega_1 \star a) * b = \big((\omega_1 \otimes a) *_{\otimes} b{\downarrow}\big)\,\mathsf{conv}$$
$$= (\omega_1 * \big(\sum_{\sigma\in\mathcal{S}_n} c_\sigma\big)1_{\mathcal{S}_1}) \star (a * b\Theta) \in \omega_1 \star \mathfrak{A}_{n-1},$$

by 3.2. Observing that $\Theta : \omega_1 \star \mathfrak{A}_{n-1} \to \mathfrak{A}_{n-1}$ is a bijection, we have shown:

Proposition 5.4 $\Theta : \mathfrak{A} \to \mathfrak{A}$ *is surjective. In particular,* $\mathfrak{A}_n\Theta = \mathfrak{A}_{n-1}$. *Furthermore,* $\omega_1 \star \mathfrak{A}_{n-1}$ *is a right ideal in* $(\mathfrak{A}_n, *)$ *and*

$$\mathfrak{A}_n = (\omega_1 \star \mathfrak{A}_{n-1}) \oplus (\mathfrak{A}_n \cap \ker\Theta).$$

6 Concluding remarks

By work of Loïc Foissy $\mathsf{Prim}\,\mathcal{P}$ is a free Lie algebra with respect to convolution [4]. $\mathcal{O}$ is a Lie subalgebra of $\mathsf{Prim}\,\mathcal{P}$, since $(\mathfrak{A}, \star, \downarrow)$ is a Bialgebra and $\mathcal{O} = \mathsf{Prim}\,\mathfrak{A}$ ([9]). Therefore $\mathcal{O}$ is a free Lie algebra by a theorem of Shirshov/Witt (cf. [11]). As a consequence, $\mathfrak{A}$ is a free associative algebra, since $\mathfrak{A}$ is a universal enveloping algebra of $\mathcal{O}$ ([9]). The direct decomposition of $\mathfrak{A}$ in 4.3 then follows from the Poincaré-Birkhoff-Witt theorem. Schocker's result leads to the *ideal* properties of the subspaces $\mathcal{O}^{(l)}$. The homomorphism Θ, described in 5.2, induces a homomorphism between Solomon's algebras $\mathcal{D}_n$ and $\mathcal{D}_{n-1}$, which was first studied in [1].

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