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# Bijective proofs of shifted tableau and alternating sign matrix identities

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**Abstract** We give a bijective proof of an identity relating primed shifted $gl(n)$-standard tableaux to the product of a $gl(n)$ character in the form of a Schur function and $\prod_{1\le i<j\le n}(x_i + y_j)$. This result generalises a number of well-known results due to Robbins and Rumsey, Chapman, Tokuyama, Okada and Macdonald. An analogous result is then obtained in the case of primed shifted $sp(2n)$-standard tableaux which are bijectively related to the product of a $t$-deformed $sp(2n)$ character and $\prod_{1\le i<j\le n}(x_i + t^2x_i^{-1} + y_j + t^2y_j^{-1})$. All results are also interpreted in terms of alternating sign matrix (ASM) identities, including a result regarding subsets of ASMs specified by conditions on certain restricted column sums.

**Keywords** Alternating sign matrices · Shifted tableaux · Schur P-functions

## 1 Introduction

The expression

$$\prod_{1\le i<j\le n}(x_i + y_j) \tag{1.1}$$

appears in a number of contexts in symmetric function theory. Given $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, when $\mathbf{y} = -\mathbf{x}$, the expression (1.1) is just

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the Vandermonde determinant that appears in Weyl's denominator formula

$$\det\left(x_i^{n-j}\right) = \prod_{1\le i<j\le n} (x_i - x_j). \tag{1.2}$$

For $\mathbf{y} = \lambda\mathbf{x}$, the expression (1.1) becomes the subject of the $\lambda$-determinant formula of Robbins and Rumsey [12]:

$$\prod_{1\le i<j\le n} (x_i + \lambda x_j) = \sum_{A\in\mathcal{A}_n} \lambda^{SE(A)}(1+\lambda)^{NS(A)} \prod_{i=1}^{n} x_i^{NE_i(A)+SE_i(A)+NS_i(A)}, \tag{1.3}$$

where the exponents are various parameters associated with alternating sign matrices and defined in Section 3. Robbins and Rumsey use different notation but do include the square ice concepts, although they use different terminology. Bressoud [2] asked for a combinatorial proof of (1.3). This was provided by Chapman [3] who generalised it to:

$$\prod_{1\le i<j\le n} (x_i + y_j) = \sum_{A\in\mathcal{A}_n} \prod_{i=1}^{n} x_i^{NE_i(A)} y_i^{SE_i(A)} (x_i + y_i)^{NS_i(A)}. \tag{1.4}$$

For $\mathbf{y} = t\mathbf{x}$, there is also the $t$-deformation of a Weyl denominator formula for $gl(n)$ due to Tokuyama [17]:

$$\prod_{i=1}^{n} x_i \prod_{1\le i<j\le n} (x_i + t x_j)\, s_\lambda(\mathbf{x}) = \sum_{ST\in\mathcal{ST}^\mu(n)} t^{\mathrm{hgt}(ST)}(1+t)^{\mathrm{str}(ST)-n}\, \mathbf{x}^{\mathrm{wgt}(ST)}, \tag{1.5}$$

where the sum is over semistandard shifted tableaux $ST$ of shape $\mu = \lambda + \delta$ with $\delta = (n, n-1, \ldots, 1)$, and where hgt, str, and wgt are parameters associated with semistandard shifted tableaux. They are defined in Section 2. Suffice to say, at this stage, that $\mathrm{wgt}(ST)$ is a vector $\mathbf{w} = (w_1, w_2 \ldots, w_n)$ and that, quite generally, $\mathbf{x}^{\mathbf{w}} = x_1^{w_1} x_2^{w_2} \cdots x_n^{w_n}$. Note also that $s_\lambda(\mathbf{x})$, the Schur function specified by the partition $\lambda$, with a suitable interpretation of the indeterminates $x_i$ for $i = 1, 2, \ldots, n$, is the character of an irreducible representation of $gl(n)$ whose highest weight is specified by the partition $\lambda$.

Here we present a general identity that unifies the results (1.2)–(1.5). This identity is our first main result and is expressed in terms of a certain generalisation of Schur $P$-functions and also in terms of the corresponding generalisation of Schur $Q$-functions. These $P$ and $Q$ functions are defined combinatorially in Section 2.

**Proposition 1.1.** *Let $\mu = \lambda + \delta$ be a strict partition of length $\ell(\mu) = n$, with $\lambda$ a partition of length $\ell(\lambda) \le n$ and $\delta = (n, n-1, \ldots, 1)$. In addition, let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$*

*and* $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. *Then*

$$P_\mu(\mathbf{x}/\mathbf{y}) = s_\lambda(\mathbf{x}) \prod_{i=1}^{n} x_i \prod_{1\le i<j\le n} (x_i + y_j),$$
$$Q_\mu(\mathbf{x}/\mathbf{y}) = s_\lambda(\mathbf{x}) \prod_{1\le i\le j\le n} (x_i + y_j), \tag{1.6}$$

*where* $P_\mu(\mathbf{x}/\mathbf{y})$ *and* $Q_\mu(\mathbf{x}/\mathbf{y})$ *are as defined in Section 2.*

A bijective proof of this Proposition, along with a number of corollaries, is provided in Section 3. The case $\mathbf{x} = \mathbf{y}$ is an example of Macdonald [8] (Ex2, p259, 2nd Edition). The case $\mathbf{y} = t\mathbf{x} = (tx_1, tx_2, \ldots, tx_n)$ is equivalent to a Weyl denominator deformation theorem due to Tokuyama [17] for the Lie algebra $gl(n)$ expressible in the form (1.5), and given a combinatorial proof by Okada [10]. The case $\lambda = 0$ is equivalent to an alternating sign matrix (ASM) identity attributed to Robbins and Rumsey [12] and proved combinatorially by Chapman [3]. The connection with ASMs is provided in Section 5, in which both (1.3) and (1.4) are shown to be simple corollaries of Proposition 1.1.

It should be pointed out that the above Proposition is restricted to the case of a strict partition $\mu$ of length $\ell(\mu) = n$. Although a similar result applying to the case $\ell(\mu) = n - 1$ may be obtained from the above by dividing both sides by $s_{1^n}(\mathbf{x}) = x_1x_2\cdots x_n$, there is no similar product formula for either $P_\mu(\mathbf{x}/\mathbf{y})$ or $Q_\mu(\mathbf{x}/\mathbf{y})$ in the case $\ell(\mu) < n - 1$.

On the other hand, the above results may all be generalised to the case of certain symplectic tableaux. The analogue of (1.1) in this setting turns out to be

$$\prod_{1\le i<j\le n} \left(x_i + t^2x_i^{-1} + y_j + t^2y_j^{-1}\right). \tag{1.7}$$

When $\mathbf{y} = -\mathbf{x}$ and $t = -1$ the expression (1.7) is a factor of the determinant that appears in Weyl's denominator formula for $sp(2n)$,

$$\det\left(x_i^{n-j+1} - x_i^{-n+j-1}\right) = \prod_{i=1}^{n}\left(x_i - x_i^{-1}\right) \prod_{1\le i<j\le n} \left(x_i + x_i^{-1} - x_j - x_j^{-1}\right). \tag{1.8}$$

More generally, for $\mathbf{y} = t\mathbf{x}$ we have [4]

$$\prod_{i=1}^{n}\left(x_i + tx_i^{-1}\right) \prod_{1\le i<j\le n} \left(x_i + t^2x_i^{-1} + tx_j + tx_j^{-1}\right) sp_\lambda(\mathbf{x};t)$$
$$= \sum_{ST\in\mathcal{ST}^\mu(n,\overline{n})} t^{\mathrm{var}(ST)+\mathrm{bar}(ST)}\,(1+t)^{\mathrm{str}(ST)-n}\,\mathbf{x}^{\mathrm{wgt}(ST)}, \tag{1.9}$$

where the sum is over semistandard shifted symplectic tableaux of shape $\mu = \lambda + \delta$ with $\delta = (n, n-1, \ldots, 1)$, and where var, bar, str and wgt are defined in Section 2. Here $sp_\lambda(\mathbf{x};t)$, once again with a suitable interpretation of the indeterminates $x_i$ for $i =$

$1, 2, \ldots, n$, is a $t$-deformation of the character $sp_\lambda(\mathbf{x})$ of the irreducible representation of the Lie algebra $sp(2n)$ whose highest weight is specified by the partition $\lambda$.

Our second main result then takes the form

**Proposition 1.2.** *Let $\mu = \lambda + \delta$ be a strict partition of length $\ell(\mu) = n$, with $\lambda$ a partition of length $\ell(\lambda) \le n$ and $\delta = (n, n-1, \ldots, 1)$. In addition, let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ and $\overline{\mathbf{y}} = (\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n)$, with $\overline{x}_i = x_i^{-1}$ and $\overline{y}_i = y_i^{-1}$ for $i = 1, 2, \ldots, n$. Then*

$$Q_\mu(\mathbf{x}/\mathbf{y}; t) = sp_\lambda(\mathbf{x}; t) \prod_{1 \le i \le j \le n} \left(x_i + t^2\overline{x}_i + y_j + t^2\overline{y}_j\right), \tag{1.10}$$

*where $Q_\mu(\mathbf{x}/\mathbf{y}; t)$ is defined in Section 2.*

Here $Q(\mathbf{x}/\mathbf{y}; t)$ is a generalisation of $Q(\mathbf{x}/\mathbf{y})$ that associates factors of $t^2$ with the barred components of $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$. Although a similar generalisation $P(\mathbf{x}/\mathbf{y}; t)$ of $P(\mathbf{x}/\mathbf{y})$ exists, as we shall see, there does not exist a corresponding identity for $P(\mathbf{x}/\mathbf{y}; t)$ that is analogous to the identity (1.10) for $Q(\mathbf{x}/\mathbf{y}; t)$.

Our paper is arranged as follows. In Section 2 the necessary background is introduced regarding both the relevant semistandard, shifted and primed tableaux, and the various $P$ and $Q$ functions and characters of $gl(n)$ and $sp(2n)$. For the $gl(n)$ case, Section 3 opens in Section 3.1 with a formal statement of the combinatorial identity upon which the first main result, Proposition 1.1, is based. A bijective proof of this identity is then provided. A detailed example appears in Section 3.2. In Section 3.3 a number of corollaries are gathered together.

Turning to the $sp(2n)$ case, the combinatorial identity necessary to establish the second main result, Proposition 1.2, is stated, bijectively proved and exemplified in Section 4. Once again two corollaries are supplied in Section 4.3, including a proof of Proposition 1.2.

Finally, in Section 5 the connection is made with alternating sign matrices and $U$-turn alternating sign matrices in the $gl(n)$ and $sp(2n)$ cases, respectively.

## 2 Background

### 2.1 $gl(n)$ tableaux

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_p > 0$ be a partition of weight $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_p$ and length $\ell(\lambda) = p$, where each $\lambda_i$ is a positive integer for all $i = 1, 2, \ldots, p$. Then $\lambda$ defines a Young diagram $F^\lambda$ consisting of $p$ rows of boxes of lengths $\lambda_1, \lambda_2 \ldots, \lambda_p$ left-adjusted to a vertical line.

A partition $\mu = (\mu_1, \mu_2, \ldots, \mu_q)$ of length $\ell(\mu) = q$ is said to be a strict partition if all the parts of $\mu$ are distinct; that is, $\mu_1 > \mu_2 > \cdots > \mu_q > 0$. A strict partition $\mu$ defines a shifted Young diagram $SF^\mu$ consisting of $q$ rows of boxes of lengths $\mu_1, \mu_2, \ldots, \mu_q$ left-adjusted this time to a diagonal line.

For any partition $\lambda$ of length $\ell(\lambda) \le n$ let $\mathcal{T}^\lambda(n)$ be the set of all semistandard tableaux $T$ obtained by numbering all the boxes of $F^\lambda$ with entries taken from the set $\{1, 2, \ldots, n\}$, subject to the usual total ordering $1 < 2 < \cdots < n$. The numbering must be such that the entries are:

T1 weakly increasing across each row from left to right;
T2 strictly increasing down each column from top to bottom;
T3 each entry $k$ may appear no lower than the $k$th row.

It will be noted that the condition T3 is redundant here, since it is implied by T2, but it will be required later. The weight of the tableau $T$ is given by $\mathrm{wgt}(T) = \mathbf{w} = (w_1, w_2, \ldots, w_n)$, where $w_k$ is the number of times $k$ appears in $T$ for $k = 1, 2, \ldots, n$. For example in the case $n = 6$, $\lambda = (3, 3, 2, 1, 1)$ we have

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & 5 & 5 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \in \mathcal{T}^{33211}(6) \quad \text{with} \quad \mathrm{wgt}(T) = (1, 1, 2, 1, 3, 2). \tag{2.11}$$

By the same token, for any strict partition $\mu$ of length $\ell(\mu) \le n$, let $\mathcal{ST}^{\mu}(n)$ be the set of all semistandard shifted tableaux $ST$ obtained by numbering all the boxes of $SF^{\mu}$ with entries taken from the set $\{1, 2, \ldots, n\}$, subject to the total ordering $1 < 2 < \cdots < n$. The numbering must be such that the entries are:

ST1 weakly increasing across each row from left to right;
ST2 weakly increasing down each column from top to bottom;
ST3 strictly increasing down each diagonal from top-left to bottom-right.

The weight of the tableau $ST$ is again given by $\mathrm{wgt}(ST) = \mathbf{w} = (w_1, w_2, \ldots, w_n)$, where $w_k$ is the number of times $k$ appears in $ST$ for $k = 1, 2, \ldots, n$.

The rules ST1-ST3 serve to exclude any $2 \times 2$ blocks of boxes all containing the same entry, and as a result, each $ST \in \mathcal{ST}^{\mu}(n)$ consists of a sequence of ribbon strips of boxes containing identical entries. Any given ribbon strip may consist of a number of disjoint connected components. Let $\mathrm{str}(ST)$ denote the total number of disjoint connected components of all the ribbon strips. Let $\mathrm{hgt}(ST)$ be the height of the tableaux, defined $\mathrm{hgt}(ST) = \sum_{k=1}^{n}(\mathrm{row}_k(ST) - \mathrm{str}_k(ST))$, where $\mathrm{row}_k(ST)$ is the number of rows of $S$ containing an entry $k$, and $\mathrm{str}_k(ST)$ is the number of connected components of the ribbon strip of $ST$ consisting of all the entries $k$.

By way of illustration, consider the case $n = 6$, $\mu = (9, 8, 6, 4, 3, 1)$ and the semi-standard shifted tableau:

$$ST = \begin{array}{ccccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 5 \\ & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 \\ & & 3 & 3 & 4 & 4 & 5 & 6 & \\ & & & 4 & 5 & 5 & 5 & & \\ & & & & 5 & 6 & 6 & & \\ & & & & & 6 & & & \end{array} \in \mathcal{ST}^{986431}(6) \ \text{with} \ \begin{array}{l} \mathrm{wgt}(ST) = (3, 5, 6, 4, 8, 5) \\ \mathrm{str}(ST) = 12, \quad \mathrm{hgt}(ST) = 6. \end{array} \tag{2.12}$$

Refining this construct, for any strict partition $\mu$ with $\ell(\mu) \le n$, let $\mathcal{PST}^{\mu}(n)$ be the set of all primed, or marked, semistandard shifted tableaux $PST$ obtained by numbering all the boxes of $SF^\mu$ with entries taken from the set $\{1', 1, 2', 2, \dots, n', n\}$, subject to the total ordering $1' < 1 < 2' < 2 < \cdots < n' < n$. The numbering must be such that the entries are:

PST1 weakly increasing across each row from left to right;
PST2 weakly increasing down each column from top to bottom;
PST3 with no two identical unprimed entries in any column;
PST4 with no two identical primed entries in any row;
PST5 with no primed entries on the main diagonal.

The weight of the tableau $PST$ is then defined to be $\mathrm{wgt}(PST) = (\mathbf{u}/\mathbf{v})$ with $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$, where $u_k$ and $v_k$ are the number of times $k$ and $k'$, respectively, appear in $PST$ for $k = 1, 2, \dots, n$.

The passage from $\mathcal{ST}^\mu(n)$ to $\mathcal{PST}^\mu(n)$ is effected merely by adding primes to the entries of each $ST \in \mathcal{ST}^\mu(n)$ in all possible ways that are consistent with PST1-5 to give some $PST \in \mathcal{PST}^\mu(n)$. The only entries for which any choice is possible are those in the lower left hand box at the beginning of each connected component of a ribbon strip. Thereafter, in that connected component of the ribbon strip, entries in the boxes of its horizontal portions are unprimed and those in the boxes of its vertical portions are primed. It should be noted that all the boxes on the main diagonal are necessarily at the lower left hand end of a connected component of a ribbon strip, but their entries remain unprimed by virtue of PST5.

To illustrate this let us assign primes to those entries of $ST$ in (2.12) for which it is essential (that is, for every entry lying immediately above the same entry) and some of those for which it is optional (those entries off the main diagonal that are at the start of any continuous strip of equal entries). This gives, for example,

$$PST = \begin{array}{ccccccccc} 1&1&1&2'&2&2&3&3&5\\ &2&2&3'&3&4'&5'&5&6'\\ &&3&3&4'&4&5'&6&\\ &&&4&5'&5&5&&\\ &&&&5&6'&6&&\\ &&&&&6&&& \end{array} \in \mathcal{PST}^{986431}(6) \quad \text{with}$$

$$\mathrm{wgt}(PST) = (3, 4, 5, 2, 5, 3/0, 1, 1, 2, 3, 2). \tag{2.13}$$

We may replace PST1-4 by identical conditions QST1-4, but discard PST5. This serves to define corresponding primed shifted tableaux $QST \in \mathcal{QST}^\mu(n)$ that now involve both primed and unprimed entries on the main diagonal.

Finally, in this $gl(n)$ context, for fixed positive integer $n$, let $\delta = (n, n-1, \dots, 1)$ and let $\mathcal{PD}^\delta(n)$ be the set of all primed shifted tableaux, $PD$, of shape $\delta$, obtained by numbering the boxes of $SF^\delta$ with entries taken from the set $\{1', 1, 2', 2, \dots, n', n\}$ in such a way that:

PD1 each unprimed entry $k$ appears only in the $k$th row;
PD2 each primed entry $k'$ appears only in the $k$th column;
PD3 there are no primed entries on the main diagonal.

The weight of the tableau $PD$ is defined by $\mathrm{wgt}(PD) = (\mathbf{u}/\mathbf{v})$ with $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, where $u_k$ and $v_k$ are the numbers of times $k$ and $k'$, respectively, appear in $PD$ for $k = 1, 2, \ldots, n$. Typically for $n = 6$ we have

$$PD = \begin{array}{cccccc} 1 & 2' & 1 & 4' & 5' & 6' \\ & 2 & 3' & 2 & 5' & 2 \\ & & 3 & 4' & 3 & 3 \\ & & & 4 & 5' & 6' \\ & & & & 5 & 5 \\ & & & & & 6 \end{array} \in \mathcal{PD}^{654321}(6) \quad \text{with}$$

$$\mathrm{wgt}(PD) = (2, 3, 3, 1, 2, 1/0, 1, 1, 2, 3, 2). \tag{2.14}$$

Since the $i$th entry on the main diagonal is always $i$ and for $i < j$ the entry in the $(i, j)$th position is either $i$ or $j'$, it is clear that

$$\sum_{PD \in \mathcal{PD}^\delta(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(PD)} = \prod_{i=1}^{n} x_i \prod_{1 \le i < j \le n} (x_i + y_j), \tag{2.15}$$

where $(\mathbf{x}/\mathbf{y})^{(\mathbf{u}/\mathbf{v})} = \mathbf{x}^\mathbf{u}\mathbf{y}^\mathbf{v} = x_1^{u_1} \cdots x_n^{u_n} y_1^{v_1} \cdots y_n^{v_n}$.

By way of a small variation of the above, if we replace PD1-2 by identical conditions QD1-2 and discard the condition PD3, the corresponding set $\mathcal{QD}^\delta(n)$ of primed shifted tableaux $QD$ differs from $\mathcal{PD}^\mu(n)$ only in allowing primed entries on the main diagonal. It follows that

$$\sum_{QD \in \mathcal{QD}^\delta(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QD)} = \prod_{1 \le i \le j \le n} (x_i + y_j). \tag{2.16}$$

These formulae (2.15) and (2.16) offer a combinatorial interpretation of factors appearing in the expansions (1.6) of Proposition 1.1. This will be exploited later in Section 3.

### 2.2 $sp(2n)$ tableaux

In order to establish a similar approach to Proposition 1.2 it is necessary to extend our already copious list of tableaux to encompass certain tableaux associated with the symplectic algebra $sp(2n)$. As before it is helpful to start with definitions of the various types of tableaux, both shifted and unshifted.

For any partition $\lambda$ of length $\ell(\lambda) \le n$, let $\mathcal{T}^\lambda(n, \overline{n})$ be the set of all semistandard symplectic tableaux $T$ obtained by numbering all the boxes of $F^\lambda$ with entries from the set $\{\overline{1}, 1, \overline{2}, 2, \ldots, \overline{n}, n\}$, subject to the usual total ordering $\overline{1} < 1 < \overline{2} < 2 < \cdots \overline{n} < n$. The entries are:

T1 weakly increasing across each row from left to right;
T2 strictly increasing down each column from top to bottom;
T3 $k$ or $\overline{k}$ may appear no lower than the $k$th row.

The weight of the symplectic tableau $T$ is given by $\mathrm{wgt}(T) = (\mathbf{w}) = (w_1, w_2, \dots, w_n)$, with $w_k = n_k - n_{\overline{k}}$ where $n_k$ and $n_{\overline{k}}$ are the number of times $k$ and $\overline{k}$, respectively, appear in $T$ for $k = 1, 2, \dots, n$. The parameter $\mathrm{bar}(T)$ is equal to the number of barred entries in the tableau. For example in the case $n = 5$, $\lambda = (4, 3, 3)$ we have

$$T = \begin{array}{|c|c|c|c|} \hline \overline{1} & 1 & \overline{2} & 4 \\ \hline 3 & \overline{4} & \overline{4} \\ \cline{1-3} \overline{4} & 4 & 4 \\ \cline{1-3} \end{array} \in \mathcal{T}^{433}(5, 5) \quad \text{with} \quad \begin{array}{l} \mathrm{wgt}(T) = (0, -1, 1, 0, 0) \\ \mathrm{bar}(T) = 5. \end{array} \tag{2.17}$$

For any strict partition $\mu$ of length $\ell(\mu) \le n$, let $\mathcal{ST}^{\mu}(n, \overline{n})$ be the set of all semistandard shifted symplectic tableaux $ST$ obtained by numbering all the boxes of $SF^{\mu}$ with entries taken from the set $\{\overline{1}, 1, \overline{2}, 2, \dots, \overline{n}, n\}$, subject to the total ordering $\overline{1} < 1 < \overline{2} < 2 < \cdots < \overline{n} < n$. The numbering must be such that the entries are:

ST1 weakly increasing across each row from left to right;
ST2 weakly increasing down each column from top to bottom;
ST3 strictly increasing down each diagonal from top-left to bottom-right;
ST$\overline{4}$ with $d_k \in \{k, \overline{k}\}$, where $d_k$ is the $k$th entry on the main diagonal.

The weight of the shifted symplectic tableau $ST$ is given by $\mathrm{wgt}(ST) = (w_1, w_2, \dots, w_n)$, with $w_k = n_k - n_{\overline{k}}$ where $n_k$ and $n_{\overline{k}}$ are the number of times $k$ and $\overline{k}$, respectively, appear in $ST$ for $k = 1, 2, \dots, n$. Once again it is convenient, following [5], to introduce $\mathrm{str}(ST)$ as the total number of disjoint connected components of all ribbon strips of $ST$, and $\mathrm{var}(ST) = \sum_{k=1}^{n}(\mathrm{row}_k(ST) - \mathrm{str}_k(ST) + \mathrm{col}_{\overline{k}}(ST) - \mathrm{str}_{\overline{k}}(ST))$, where $\mathrm{row}_k(ST)$ is the number of rows of $ST$ containing an entry $k$, $\mathrm{col}_{\overline{k}}(ST)$ is the number of columns containing an entry $\overline{k}$, while $\mathrm{str}_k(ST)$ and $\mathrm{str}_{\overline{k}}(ST)$ are the number of connected components of the ribbon strips of $ST$ consisting of all the entries $k$ and $\overline{k}$, respectively, and $\mathrm{bar}(ST)$ is equal to the total number of barred entries.

Typically, for $n = 5$ and $\mu = (9, 7, 6, 2, 1)$ we have

$$ST = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \overline{1} & 1 & \overline{2} & 2 & \overline{3} & \overline{3} & \overline{4} & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & \overline{2} & \overline{2} & 2 & 3 & \overline{4} & \overline{4} & 4 \\ \cline{2-8} \multicolumn{2}{c|}{} & 3 & \overline{4} & 4 & 4 & 4 & 4 \\ \cline{3-8} \multicolumn{3}{c|}{} & 4 & 4 \\ \cline{4-5} \multicolumn{4}{c|}{} & \overline{5} \\ \cline{5-5} \end{array} \in \mathcal{ST}^{97621}(5, \overline{5}) \quad \text{with}$$

$$\begin{array}{l} \mathrm{wgt}(ST) = (0, -1, 0, 4, 0) \\ \mathrm{bar}(ST) = 11, \quad \mathrm{str}(ST) = 12, \quad \mathrm{var}(ST) = 7. \end{array} \tag{2.18}$$

Refining this construct, for any strict partition $\mu$ with $\ell(\mu) \le n$, let $\mathcal{PST}^{\mu}(n, \overline{n})$ be the set of all primed semistandard shifted symplectic tableaux $PST$ obtained by numbering all the boxes of $SF^{\mu}$ with entries taken from the set $\{\overline{1}', \overline{1}, 1', 1, \overline{2}', \overline{2}, 2', 2, \dots, \overline{n}', \overline{n}, n', n\}$, subject to the total ordering

$$\overline{1}' < \overline{1} < 1' < 1 < \overline{2}' < \overline{2} < 2' < 2 < \cdots < \overline{n}' < \overline{n} < n' < n. \tag{2.19}$$

The numbering must be such that the entries are:

PST1 weakly increasing across each row from left to right;
PST2 weakly increasing down each column from top to bottom;
PST3 with no two identical unprimed entries in any column;
PST4 with no two identical primed entries in any row;
PST$\overline{5}$ with $d_k \in \{\overline{k}, k\}$, where $d_k$ is the $k$th entry on the main diagonal.

The weight of the tableau $PST$ is then defined to be $\mathrm{wgt}(PST) = (\mathbf{u}/\mathbf{v})$ with $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, where $u_k = n_k - n_{\overline{k}}$ and $v_k = n_{k'} - n_{\overline{k}'}$, with $n_k$, $n_{\overline{k}}$, $n_{k'}$ and $n_{\overline{k}'}$ are the number of times $k$, $\overline{k}$, $k'$ and $\overline{k}'$, respectively, appear in $PST$ for $k = 1, 2, \ldots, n$. In addition, let $\mathrm{bar}(PST)$ be the total number of barred entries in $PST$.

If we now replace PST1-4 by identical conditions QST1-4 and replace PST$\overline{5}$ by:

QST$\overline{5}$ with $d_k \in \{\overline{k}', \overline{k}, k', k\}$, where $d_k$ is the $k$th entry on the main diagonal.

Then once again the corresponding primed shifted tableaux $QST \in \mathcal{QST}^{\mu}(n, \overline{n})$ now have primes allowed on the main diagonal.

Typically, for $n = 5$ and $\mu = (9, 7, 6, 2, 1)$ we have

$$QST = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \overline{1} & 1 & \overline{2}' & 2' & \overline{3}' & \overline{3} & \overline{4}' & 4' & 5 \\ \hline \multicolumn{1}{c|}{} & \overline{2}' & \overline{2} & 2 & 3 & \overline{4}' & \overline{4} & 4' & \multicolumn{1}{c}{} \\ \cline{2-8} \multicolumn{2}{c|}{} & 3' & \overline{4}' & 4' & 4 & 4 & 4 & \multicolumn{1}{c}{} \\ \cline{3-8} \multicolumn{3}{c|}{} & 4' & 4 & \multicolumn{4}{c}{} \\ \cline{4-5} \multicolumn{4}{c|}{} & \overline{5}' & \multicolumn{4}{c}{} \\ \cline{5-5} \end{array} \in \mathcal{QST}^{97621}(5, \overline{5}) \quad \text{with}$$

$$\begin{aligned} &\mathrm{wgt}(QST) = (0, 0, 0, 3, 1/0, -1, 0, 1, -1), \\ &\mathrm{bar}(QST) = 11. \end{aligned} \tag{2.20}$$

To complete our set of $sp(2n)$ tableaux, for fixed positive integer $n$, let $\delta = (n, n-1, \ldots, 1)$ and let $\mathcal{PD}^{\delta}(n, \overline{n})$ be the set of all primed shifted tableaux, $PD$, of shape $\delta$, obtained by numbering the boxes of $SF^{\delta}$ with entries taken from the set $\{\overline{1}', \overline{1}, 1', 1, \overline{2}', \overline{2}, 2', 2, \ldots, \overline{n}', \overline{n}, n', n\}$ in such a way that

PD$\overline{1}$ each unprimed entry $k$ or $\overline{k}$ appears only in the $k$th row;
PD$\overline{2}$ each primed entry $k'$ or $\overline{k}'$ appears only in the $k$th column;
PD3 there are no primed entries on the main diagonal.

The weight of the tableau $PD$ is defined by $\mathrm{wgt}(PD) = (\mathbf{u}/\mathbf{v})$ with $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, where $u_k = n_k - n_{\overline{k}}$ and $v_k = n_{k'} - n_{\overline{k}'}$, with $n_k$, $n_{\overline{k}}$, $n_{k'}$ and $n_{\overline{k}'}$ are the number of times $k$, $\overline{k}$, $k'$ and $\overline{k}'$, respectively, appear in $PD$ for $k = 1, 2, \ldots, n$. In addition let $\mathrm{bar}(PD)$ be the total number of barred entries in $PD$.

With this notation, since the entry in the $i$th position on the main diagonal is either $i$ or $\overline{i}$ while for $i < j$ the entry in the $(i, j)$th position is either $i$, $\overline{i}$, $j'$ or $\overline{j}'$, it is clear that

$$\sum_{PD\in\mathcal{PD}^{\delta}(n,\overline{n})} t^{2\mathrm{bar}(PD)}(\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(PD)} = \prod_{i=1}^{n}\left(x_i + t^2\overline{x}_i\right)\prod_{1\le i<j\le n}\left(x_i + t^2\overline{x}_i + y_j + t^2\overline{y}_j\right). \tag{2.21}$$

Here the use of $t^2$ as the key parameter is due to the fact that we will later need to set $y_j = t\,x_j$ in order to recover the factors appearing in (1.9).

By way of a small variation of the above, if we replace PD$\overline{1}$-$\overline{2}$ by identical conditions QD$\overline{1}$-$\overline{2}$ and discard the condition PD3, the corresponding set $\mathcal{QD}^{\delta}(n)$ of primed shifted tableaux $QD$ differs from $\mathcal{PD}^{\mu}(n)$ only in allowing primed entries on the main diagonal.

Typically for $n=5$ we have

$$QD = \begin{array}{|c|c|c|c|c|}\hline \overline{1} & 1 & \overline{3}' & 4' & \overline{1} \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & 2 & \overline{2} \\ \cline{2-5} \multicolumn{2}{c|}{} & \overline{3}' & 3 & \overline{3} \\ \cline{3-5} \multicolumn{3}{c|}{} & 4' & 4 \\ \cline{4-5} \multicolumn{4}{c|}{} & \overline{5}' \\ \cline{5-5} \end{array} \in \mathcal{QD}^{54321}(5,\overline{5}) \quad \text{with}$$

$$\begin{aligned} &\mathrm{wgt}(QD) = (-1,2,0,1,0/0,0,-2,2,-1), \\ &\mathrm{bar}(QD) = 7. \end{aligned} \tag{2.22}$$

It follows from our definition of $\mathcal{QD}(n,\overline{n})$ that

$$\sum_{QD\in\mathcal{QD}^{\delta}(n,\overline{n})} t^{2\mathrm{bar}(QD)}(\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QD)} = \prod_{1\le i\le j\le n}(x_i + t^2\overline{x}_i + y_j + t^2\overline{y}_j). \tag{2.23}$$

These formulae (2.21) and (2.23) have been introduced so as to offer a combinatorial interpretation of factors appearing in the expansions (2.30) of Proposition 1.2. This will be exploited later in Section 3.

## 2.3 Schur's $P$ and $Q$ functions and their generalisations

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector of $n$ indeterminates and let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a vector of $n$ non-negative integers. Then $\mathbf{x}^{\mathbf{w}} = x_1^{w_1}x_2^{w_2}\cdots x_n^{w_n}$. With this notation it is well known that each partition $\lambda$ of length $\ell(\lambda)\le n$ specifies a Schur function $s_\lambda(\mathbf{x})$ with combinatorial definition:

$$s_\lambda(\mathbf{x}) = \sum_{T\in\mathcal{T}^{\lambda}(n)} \mathbf{x}^{\mathrm{wgt}(T)} \tag{2.24}$$

Similarly, each strict partition $\mu$ of length $\ell(\mu) \le n$ specifies a Schur $P$-function and a Schur $Q$-function whose combinatorial definitions take the form:

$$\begin{aligned} P_\mu(\mathbf{x}) &= \sum_{ST \in \mathcal{ST}^\mu(n)} 2^{\mathrm{str}(ST)-\ell(\mu)} \mathbf{x}^{\mathrm{wgt}(ST)}; \\ Q_\mu(\mathbf{x}) &= \sum_{ST \in \mathcal{ST}^\mu(n)} 2^{\mathrm{str}(ST)} \mathbf{x}^{\mathrm{wgt}(ST)}. \end{aligned} \tag{2.25}$$

Now let $\mathbf{z} = (\mathbf{x}/\mathbf{y}) = (x_1, x_2, \ldots, x_n/y_1, y_2, \ldots, y_n)$, where $\mathbf{x}$ and $\mathbf{y}$ are two vectors of $n$ indeterminates, and let $\mathbf{w} = (\mathbf{u}/\mathbf{v}) = (u_1, u_2, \ldots, u_n/v_1, v_2, \ldots, v_n)$ where $\mathbf{u}$ and $\mathbf{v}$ are two vectors of $n$ non-negative integers. Then let $\mathbf{z}^{\mathbf{w}} = (\mathbf{x}/\mathbf{y})^{(\mathbf{u}/\mathbf{v})} = \mathbf{x}^{\mathbf{u}}\, \mathbf{y}^{\mathbf{v}} = x_1^{u_1} \cdots x_n^{u_n}\, y_1^{v_1} \cdots y_n^{v_n}$. With this notation each strict partition $\mu$ of length $\ell(\mu) \le n$ serves to specify generalised Schur $P$ and $Q$-functions defined by:

$$\begin{aligned} P_\mu(\mathbf{x}/\mathbf{y}) &= \sum_{PST \in \mathcal{PST}^\mu(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(PST)}; \\ Q_\mu(\mathbf{x}/\mathbf{y}) &= \sum_{QST \in \mathcal{QST}^\mu(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QST)}. \end{aligned} \tag{2.26}$$

Since the maps back from $PST \in \mathcal{PST}^\mu(n)$ and from $QST \in \mathcal{QST}^\mu(n)$ to some $ST \in \mathcal{ST}^\mu(n)$ are effected merely by deleting primes, and there are no primes on the main diagonal in the case of $PST$, it follows that

$$Q_\mu(\mathbf{x}) = 2^{\ell(\mu)} P_\mu(\mathbf{x}) \quad \text{with} \quad P_\mu(\mathbf{x}) = P_\mu(\mathbf{x}/\mathbf{x}) \quad \text{and} \quad Q_\mu(\mathbf{x}) = Q_\mu(\mathbf{x}/\mathbf{x}) \tag{2.27}$$

It might be noted that $s_\lambda(\mathbf{x})$, $P_\lambda(\mathbf{x})$ and $Q_\lambda(\mathbf{x})$ are nothing other than the specialisations $P_\lambda(\mathbf{x};0)$, $P_\mu(\mathbf{x};-1)$ and $Q_\mu(\mathbf{x};-1)$, respectively, of the Hall-Littlewood functions $P_\mu(\mathbf{x};t)$ and $Q_\mu(\mathbf{x};t)$.

Turning to the symplectic case, it is well known that each partition $\lambda$ of length $\ell(\lambda) \le n$ specifies an irreducible representation of $sp(2n)$ whose character $sp_\lambda(\mathbf{x})$ may be given a combinatorial definition:

$$sp_\lambda(\mathbf{x}) = \sum_{T \in \mathcal{T}^\lambda(n,\overline{n})} \mathbf{x}^{\mathrm{wgt}(T)}. \tag{2.28}$$

This may be $t$-deformed to give

$$sp_\lambda(\mathbf{x};t) = \sum_{T \in \mathcal{T}^\lambda(n,\overline{n})} t^{2\mathrm{bar}(T)}\, \mathbf{x}^{\mathrm{wgt}(T)}. \tag{2.29}$$

In the case of a strict partition $\mu$ of length $\ell(\mu) = n$ the required generalisations of Schur $P$ and $Q$ functions take the form:

$$
\begin{aligned}
P_\mu(\mathbf{x}/\mathbf{y};t) &= \sum_{PST\in\mathcal{PST}^\mu(n,\overline{n})} t^{2\mathrm{bar}(PST)}\,(\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(PST)};\\
Q_\mu(\mathbf{x}/\mathbf{y};t) &= \sum_{QST\in\mathcal{QST}^\mu(n,\overline{n})} t^{2\mathrm{bar}(QST)}\,(\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QST)}.
\end{aligned}
\tag{2.30}
$$

With a slight abuse of notation we augment $\mathrm{wgt}(T) = \mathbf{w} = (w_1, \ldots, w_n)$ with $n$ $0'$s to give $\mathrm{wgt}(T) = (\mathbf{w}/\mathbf{0}) = (w_1, \ldots, w_n/0, \ldots, 0)$ wherever required. This means, for example, that $(\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(T)} = \mathbf{x}^{\mathrm{wgt}(T)} = x_1^{w_1}\cdots x_n^{w_n}$. More important, in what follows, both $\mathrm{wgt}(PD) + \mathrm{wgt}(T)$ and $\mathrm{wgt}(QD) + \mathrm{wgt}(T)$ are well defined.

## 3 The *gl*(*n*) bijection

### 3.1 Main result

The generalisations of the combinatorial definitions of $P_\mu(\mathbf{x})$ and $Q_\mu(\mathbf{x})$ to $P_\mu(\mathbf{x}/\mathbf{y})$ and $Q_\mu(\mathbf{x}/\mathbf{y})$, respectively, together with those of $s_\lambda(\mathbf{x})$ and the product factors appearing in (1.6), allow us to establish the validity of Proposition 1.1 by first proving the following:

**Theorem 3.1.** *Let $\mu = \lambda + \delta$ be a strict partition of length $\ell(\mu) = n$, with $\lambda$ a partition of length $\ell(\lambda) \le n$ and $\delta = (n, n-1, \ldots, 1)$. There exists a weight preserving, bijective map $\Theta$ from $\mathcal{PST}^\mu(n)$ to $(\mathcal{PD}^\delta(n), \mathcal{T}^\lambda(n))$ and from $\mathcal{QST}^\mu(n)$ to $(\mathcal{QD}^\delta(n), \mathcal{T}^\lambda(n))$ such that for all $PST \in \mathcal{PST}^\mu(n)$ and for all $QST \in \mathcal{QST}^\mu(n)$*

$$
\begin{aligned}
\Theta:\ PST &\mapsto (PD, T) \quad \textit{with} \quad \mathrm{wgt}(PST) = \mathrm{wgt}(PD) + \mathrm{wgt}(T).\\
\Theta:\ QST &\mapsto (QD, T) \quad \textit{with} \quad \mathrm{wgt}(QST) = \mathrm{wgt}(QD) + \mathrm{wgt}(T).
\end{aligned}
\tag{3.31}
$$

*with $PD \in \mathcal{PD}^\delta(n)$, $QD \in \mathcal{QD}^\delta(n)$ and $T \in \mathcal{T}^\lambda(n)$.*

**Proof:** We choose to tackle the $PST$ case first with the aim of describing a candidate map $\Theta$ and showing that it is both weight preserving and bijective.

The technique is to apply the jeu de taquin [6, 8, 13, 14, 18] to the primed entries $k'$ of $PST$ taken in turn starting with any $1'$s (actually there are none), then any $2'$s (at most one), then any $3'$s (at most two) and so on. If for fixed $k$ there is more than one $k'$ in $PST$ then these are dealt with in turn from top to bottom. The map $\Theta$ is thus expressible in the form $\Theta = \theta_{n'} \circ \cdots \circ \theta_{2'} \circ \theta_{1'}$.

We start by describing the map $\theta_{k'}$. This involves sliding each $k'$ in the north-west direction by a sequence of interchanges with either its unprimed northern or western neighbour until it reaches a position in the $k$th column either in the topmost row, or immediately below another $k'$, or immediately below some unprimed entry $i$ in the $i$th row.

This amounts to playing jeu de taquin, treating $k'$ to be strictly less than all the unprimed entries. At every stage the conditions PD1-3 apply to all entries to the left of the $k$th column, while PST1-4 apply to all entries, other than the moving $k'$, that are either in or to the right of the $k$th column. This implies that, ignoring all primed entries, the unprimed entries must satisfy the semistandardness conditions T1 and T2. In addition it is required that:

T3 each unprimed entry $i$ may appear no lower than the $i$th row;
P3 each primed entry $j'$ may appear no further to the left than the $j$th column.

Collectively, it is these conditions that ensure that each move made by $k'$ is uniquely determined, with T3 and P3 ensuring that the procedure terminates in the required manner. It should be noted that they all, including both T3 and P3, apply to each initial $PST \in \mathcal{PST}^{\mu}(n)$ with $\mu$ a strict partition of length $\ell(\mu) = n$. This is because the conditions PST1-4 imply that $i'$ and $i$ cannot lie on the same diagonal, so that on the main diagonal of length $n$ the $i$th entry $d_i$ is either $i$ or $i'$ for $i = 1, 2, \ldots, n$, with $i'$ being excluded by PST5. As a further consequence of the conditions $d_i = i$ and PST1, if $k'$ appears in the $i$th row of $PST$, then $i < k$. This condition is maintained under all interchanges of $k'$ with unprimed entries, since each $k'$ moves only north or west.

Returning now to the jeu de taquin, consider first the situation illustrated by the tableau $T_0$ in (3.32) with $k'$ not yet in the $k$th column. This is to be thought of as the subtableau surrounding a particular $k'$ at position $(i, j)$ with $j > k$ and as explained above $i < k$, awaiting its next move. For the time being we assume that $a, b, d$ are unprimed, while $c, e, f, g, h$ may be primed, or unprimed, or even absent if $k'$ is at or near either the main diagonal or the southern or eastern edge of the complete diagram. However, all the unprimed entries amongst $a, b, \ldots, h$ must, by hypothesis, satisfy the semistandardness conditions T1 and T2, as well as T3. In particular $d \geq i$.

Now for the jeu de taquin rules that define the map $\theta_{k'}$. If $d \leq b$ then $k'$ is to be interchanged with $b$ and if $d > b$ then $k'$ is to be interchanged with $d$, as shown below. In the first case $k'$ moves north and the resulting tableau $T_N$ satisfies T1-3 since $i \leq d \leq b \leq c < e$, while in the second, $k'$ moves west. This is consistent with P3 since $j > k$, and the resulting tableau $T_W$ again satisfies T1-3 since $b < d < f \leq g$, with $d \geq i$.

$$\theta_{k'} : \quad \begin{array}{c} T_0 = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & k' & e \\ \hline f & g & h \\ \hline \end{array} \\ \\ k' \text{ at } (i, j) \text{ with } i < k < j \end{array} \quad \longrightarrow \quad \begin{cases} T_N = \begin{array}{|c|c|c|} \hline a & k' & c \\ \hline d & b & e \\ \hline f & g & h \\ \hline \end{array} & \text{if} \quad b \geq d; \\ \\ T_W = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline k' & d & e \\ \hline f & g & h \\ \hline \end{array} & \text{if} \quad b < d. \end{cases} \tag{3.32}$$

Here $c, e, f, g, h$ may be primed or absent without affecting our conclusions. In the case $i = 1$ the top row $a\,b\,c$ of $T_0$ must be absent, and $T_0$ maps just to $T_W$, again without its top row $a\,b\,c$. Of course any absences from $T_0$ must leave a regular diagram, so that for example if $c$ is absent then so are $e$ and $h$, while if $g$ is present then so is $f$, unless $d$ and $g$ are on the main diagonal. Such regularity conditions apply to all subsequent diagrams. Here however, $d$ and $g$ cannot be on the main diagonal with $f$ absent, since $i < j - 1$.

On the other hand, if $k'$ is already in the $k$th column, so that $k'$ cannot be moved westward by virtue of P3, the map $\theta_{k'}$ leaves $T_0$ unaltered, that is $k'$ has reached its final resting place, unless $k'$ lies in the $i$th row of the $k$th column with an unprimed entry $b \geq i$ immediately above it. In such a case $\theta_{k'}$ acts on $T_0$ as shown below:

$$\theta_{k'}: \quad \underset{k' \text{ at } (i,k) \text{ with } i<k}{T_0 = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & k' & e \\ \hline f & g & h \\ \hline \end{array}} \quad \longrightarrow \quad T_N = \begin{array}{|c|c|c|}\hline a & k' & c \\ \hline d & b & e \\ \hline f & g & h \\ \hline \end{array} \quad \text{if} \quad b \geq i. \tag{3.33}$$

Yet again, the unprimed entries of $T_N$ satisfy T1-3, since we still have $d = i \leq b \leq c < e$. The fact that $d = i$ is a consequence of the condition PD1 that applies to the left of the $k$th column. This time $f$ would be absent if $i = k - 1$.

Now we return to the possibilities that we had previously set aside, those cases for which $a, b$ or $d$ are primed. Since the allowed moves of $k'$ are only north or west, and all primed entries $l'$ with $l > k$ are originally either south or east of each $k'$, and no interchanges of primed entries occur, then we must have $a, b, d \leq k'$. The case of a primed entry $b \leq k'$ cannot occur in the tableau $T_0$ of (3.32), since it would have already been moved leftwards to its own column before any attempt is made to move the central $k'$. The same is true of any primed entry $b < k'$ in the tableau $T_0$ of (3.33). This leaves as the only possibility $b = k'$, but if $b = k'$ then the central $k'$ has already arrived in the $k$th column immediately below another $k'$. Then, as we have already pointed out in our original description of $\theta_{k'}$, no further move is required. In addition, any primed entries $a, d < k'$ in the tableau $T_0$ of (3.32) would have been moved leftwards to their own columns, leaving just the cases $a = k'$ and $d = k'$ to consider, while in (3.33) whether or not $a$ and $d$ are primed is irrelevant, since by PD2 this primed value must be $(k-1)'$, and $k'$ has already reached its own column and any move north is unhindered by such a value of $a$ or $d$.

It follows that the only possible impediment to the movement of $k'$ in a north-westerly direction until it actually reaches the $k$th column, is the existence of another $k'$ to its immediate left, that is in $T_0$ of (3.32) we have $d = k'$, or in $T_N$ of (3.32) we have $a = k'$. That this cannot occur is a corollary of the fact that the path followed by $k'$ always remains column by column below (that is strictly south of) the path followed by any preceding $k'$. This latter path always starts south and at least as far east as the initial position of the moving $k'$ and extends westward as far as the $k$th column thereby covering all columns through which the moving $k'$ passes. To see that no horizontal pairs $k'\,k'$ may arise consider $k'$ arriving, as shown below in the diagram on the left of (3.34), at a position due south of an entry $b$ which itself lies on the path of the preceding $k'$.

$$\theta_{k'}:\ \begin{array}{|c|c|c|c|c|c|}\hline \mathbf{p} & \cdots & \mathbf{a} & \mathbf{b} & \cdots & \mathbf{q} \\ \hline r & \cdots & c & k' \\ \cline{1-4}\end{array} \quad \xrightarrow{b<c} \quad \begin{array}{|c|c|c|c|c|c|}\hline \mathbf{p} & \cdots & \mathbf{a} & \mathbf{b} & \cdots & \mathbf{q} \\ \hline r & \cdots & k' & c \\ \cline{1-4}\end{array} \tag{3.34}$$

If this path of the preceding $k'$ moves north from $b$, then there is no problem since the $k'$ can follow the same path north or move west without violating the strictly south condition. On the other hand the path of the preceding $k'$ may move west along the indicated boldface track from $q$ to $p$. In doing so, it must at one stage have displaced $b$ from its original position at the site of $a$, immediately above $c$ and satisfying the T2 condition $b < c$. This condition then ensures that the $k'$ must itself move west as shown in (3.34). It therefore stays south of the path of the preceding $k'$ that passes through the position of $a$. This implies that the path of $k'$ must always stay strictly south of the path of the preceding $k'$, thereby excluding the possibility $d = k'$ in (3.32) and and also $a = k'$ in (3.32). This ensures that each $k'$ will eventually reach and ascend the $k$th column by means of a sequence of moves of type (3.32)–(3.33). Furthermore, since in the initial $PST$ its path starts strictly north of the $k$th row and it only moves north or west it never reaches the main diagonal.

Following the action of $\theta_{k'}$ each unprimed entry $k$ on the main diagonal of $PST$ therefore remains fixed, and all $k'$s are in the $k$th column along with distinct unprimed entries $j$ with $1 \le j \le k$. If $k'$ appears in the $i$th row, then $i$ cannot appear above it, since $k'$ would then move north as in (3.33), and cannot appear below it by virtue of T3. It follows that the unprimed entries $j$ in the $k$th column do not include the row numbers of $k'$. Since they are distinct and $1 \le j \le k$, they must include all the other row numbers, and be arranged in strictly increasing order in accordance with T2. This means that each unprimed entry in the $k$th column lies in its own row. Since the primed entries in this column are all $k'$s all the entries in the $k$th column satisfy PD1-3.

Iterating this procedure for all $k = 1, 2, \ldots, n$ results in all primed entries being moved to the first $k$ columns of $SF^\mu$ along with some unprimed entries, collectively satisfying PD1-3 in this region of shape $SF^\delta$, and leaving only unprimed entries, all satisfying T1-3, in the right hand region of shape $F^\lambda$. In fact, with the absence of primed entries in this region, T3 is redundant since it is implied by T2. Thus the result of applying $\Theta$ to $PST \in \mathcal{PST}^\mu$ is a semistandard tableau $T \in \mathcal{T}^\lambda(n)$ of shape $\lambda$ juxtaposed to a primed tableau $PD \in \mathcal{PD}^\delta(n)$. This map is necessarily weight preserving since every individual step is a simple interchange which does not alter the number of $k$s or $k'$s for any $k$.

To show that this map $\Theta$ is bijective it should be noted that each step may be reversed. One starts by juxtaposing an arbitrary pair of tableaux $PD \in \mathcal{PD}^\delta(n)$ and $T \in \mathcal{T}^\lambda(n)$ to create a tableaux of shape $SF^\mu$ with $\mu = \lambda + \delta$. Then for each $k$ taken in turn from $n$ to $n-1$ down to 1 one applies $\theta_{k'}^{-1}$ to all the primed entries $k'$; that is to say one reverses the action of $\theta_{k'}$ by playing jeu de taquin in the reverse direction with primed entries $k'$ treated in turn from bottom to top, moving each one in a south-easterly direction with $k'$ now assumed to be larger than $i$ for $i = 1, \ldots, k-1$ but less than $i$ for $i = k, k+1, \ldots, n$ with the conditions T1-3 applying to all unprimed entries at all times. For example in (3.35) if both $e$ and $g$ are unprimed and less than $k$, this leads unambiguously from $T_0$ to $T_E$ if $e < g$ and from $T_0$ to $T_S$ if $e \ge g$, with all unprimed entries satisfying the semistandardness conditions T1 and T2 since in addition $b \le c < e < g$ in $T_E$ and $d < f \le g \le e$ in $T_S$.

$$\theta_{k'}^{-1}:\ \ T_0 = \begin{array}{|c|c|c|}\hline a&b&c\\\hline d&k'&e\\\hline f&g&h\\\hline\end{array} \qquad \longrightarrow \qquad \begin{cases} T_E = \begin{array}{|c|c|c|}\hline a&b&c\\\hline d&e&k'\\\hline f&g&h\\\hline\end{array} & \text{if} \quad e < g; \\ \\ T_S = \begin{array}{|c|c|c|}\hline a&b&c\\\hline d&g&e\\\hline f&k'&h\\\hline\end{array} & \text{if} \quad e \ge g. \end{cases} \tag{3.35}$$

In addition, given that the entries of $T_0$ satisfy T3 and P3, then so do those of $T_E$ and $T_S$ since unprimed entries move north or west, while $k'$ moves south or east.

More importantly, returning to the original jeu de taquin moves illustrated in (3.32), this reversed jeu de taquin is such that if the conditions T1–T3 are satisfied by the entries in $T_0$, then the reverse process leads directly from $T_W$ to $T_0$ since $d < f$ and from $T_N$ to $T_0$ since $b \le c$. Thus the original steps along each of the $k'$ paths are retraced precisely. The same is true of the map (3.33).

The only task remaining is to show that the endpoints of these retraced paths results in an element $PST$ of $\mathcal{PST}^{\mu}$. If $T_0$ is such that $k'$ has reached its endpoint then neither $e$ nor $g$ is unprimed and $< k$. If either $e$ or $g$ is absent, or primed or unprimed but $\ge k$, then this poses no problem. The $k'$ in $T_0$ is simply stopped from moving east or south, respectively. If $g = k'$ there is again no problem since the rules PST1-5 allow two (or more) $k'$s in the same column. It is only the case $e = k'$ that produces a violation, in this case of PST4. Fortunately this case is excluded by the following argument analogous to that which led to the strictly south property of the original jeu de taquin. Now we require a strictly north property. The argument goes as follows. The fact that the strictly north property applies to the reverse jeu de taquin follows from a consideration of the following diagram in which a $k'$ south westerly path meets a preceding west–east path, indicated by means of boldface entries, passing from $p$ to $q$ through the positions of $a$ and $b$. The existence of the latter requires that $a$ must initially have been immediately south of $c$, so that $c < a$. This in turn implies that $k'$ moves eastwards staying strictly north of the preceding path as shown below:

$$\theta_k^{-1}:\ \ \begin{array}{|c|c|c|c|c|c|}\cline{3-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & k' & c & \cdots & s\\\hline \mathbf{p} & \cdots & \mathbf{a} & \mathbf{b} & \cdots & \mathbf{q}\\\hline\end{array} \quad \overset{a > c}{\longrightarrow} \quad \begin{array}{|c|c|c|c|c|c|}\cline{3-6} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & c & k' & \cdots & s\\\hline \mathbf{p} & \cdots & \mathbf{a} & \mathbf{b} & \cdots & \mathbf{q}\\\hline\end{array} \tag{3.36}$$

Proceeding in this way, the process terminates when each $k'$ has moved as far east and south as the jeu de taquin allows. If in the above diagram $q$ represents the final position of the preceding $k'$, then all elements to the right of $q$ must necessarily be greater than $k'$. This means that although the higher $k'$ may move to a column further to the right than that of $q$, it always remains in a higher row than the preceding $k'$. This ensures that no two $k'$s can appear in the same row. This is sufficient, when taken together with T1-2 and the fact that each primed entry has reached its end point, to show that the resulting primed shifted tableau satisfies PST1-4. Since we had already noted that the diagonal entries are always unprimed, PST5 is also satisfied.

It follows that $\Theta^{-1}$ is well defined and maps the juxtaposition of any pair of tableaux $PD \in \mathcal{PD}^\delta(n)$ and $T \in \mathcal{T}^\lambda(n)$ to a unique $PST \in \mathcal{PST}^\mu(n)$. Thus the original map $\Theta$ from $\mathcal{PST}^\mu(n)$ to $(\mathcal{PD}^\delta(n), \mathcal{T}^\lambda(n))$ is indeed bijective. Since it is also weight preserving, as argued earlier, this completes the proof of the $PST$ case in Theorem 3.1.

The only difference between the $PST$ and $QST$ cases is the fact that in the latter case primed entries are allowed on the main diagonal. This is reflected in the same distinction between $PD$ and $QD$ on the right of the above formulae. In fact it is not difficult to see that the map $\Theta$ preserves the entries on the main diagonal in both cases; that is, just as the main diagonal of $PST$ coincides with that of $PD$, where $\Theta: \ PST \mapsto (PD, T)$, so the main diagonal of $QST$, complete with any primes, coincides with that of $QD$, where $\Theta: \ QST \mapsto (QD, T)$. This observation is sufficient to complete the proof of Theorem 3.1. $\square$

## 3.2 Example

This bijection is illustrated by the map from $PST$ of (2.13) to $PD$ of (2.14) and $T$ of (2.11); that is,

$$PST = \begin{array}{ccccccccc} 1 & 1 & 1 & 2' & 2 & 2 & 3 & 3 & 5 \\ & 2 & 2 & 3' & 3 & 4' & 5' & 5 & 6' \\ & & 3 & 3 & 4' & 4 & 5' & 6 & \\ & & & 4 & 5' & 5 & 5 & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \longleftrightarrow PD = \begin{array}{cccccc} 1 & 2' & 1 & 4' & 5' & 6' \\ & 2 & 3' & 2 & 5' & 2 \\ & & 3 & 4' & 3 & 3 \\ & & & 4 & 5' & 6' \\ & & & & 5 & 5 \\ & & & & & 6 \end{array}, \quad T = \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 5 & 5 \\ 4 & 6 & \\ 5 & & \\ 6 & & \end{array} \tag{3.37}$$

The paths traced out by the primed entries $k'$ of $PST$ as they move northwest as far as but no further than the $k$th column are illustrated by means of boldface entries in the tableaux shown below:

First moving the single $2'$ under the map $\theta_{2'}$ gives:

$$\begin{array}{ccccccccc} 1 & \mathbf{1} & \mathbf{1} & \mathbf{2'} & 2 & 2 & 3 & 3 & 5 \\ & 2 & 2 & 3' & 3 & 4' & 5' & 5 & 6' \\ & & 3 & 3 & 4' & 4 & 5' & 6 & \\ & & & 4 & 5' & 5 & 5 & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \rightarrow \begin{array}{ccccccccc} 1 & \mathbf{2'} & \mathbf{1} & \mathbf{1} & 2 & 2 & 3 & 3 & 5 \\ & 2 & 2 & 3' & 3 & 4' & 5' & 5 & 6' \\ & & 3 & 3 & 4' & 4 & 5' & 6 & \\ & & & 4 & 5' & 5 & 5 & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \tag{3.38}$$

Under $\theta_{3'}$ the only $3'$ moves just one step west where it has, as required, reached the 3rd column. It does not move north because the entry 1 immediately above already lies in its own row:

$$
\begin{array}{ccccccccc}
1 & 2' & 1 & 1 & 2 & 2 & 3 & 3 & 5 \\
 & 2 & \mathbf{2} & \mathbf{3'} & 3 & 4' & 5' & 5 & 6' \\
 & & 3 & 3 & 4' & 4 & 5' & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
\quad \to \quad
\begin{array}{ccccccccc}
1 & 2' & 1 & 1 & 2 & 2 & 3 & 3 & 5 \\
 & 2 & \mathbf{3'} & \mathbf{2} & 3 & 4' & 5' & 5 & 6' \\
 & & 3 & 3 & 4' & 4 & 5' & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
\tag{3.39}
$$

There are two $4'$s. Under $\theta_{4'}$ the upper one must be moved first and then the lower one:

$$
\begin{array}{ccccccccc}
1 & 2' & 1 & \mathbf{1} & \mathbf{2} & 2 & 3 & 3 & 5 \\
 & 2 & 3' & 2 & \mathbf{3} & \mathbf{4'} & 5' & 5 & 6' \\
 & & 3 & 3 & 4' & 4 & 5' & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
\quad \to \quad
\begin{array}{ccccccccc}
1 & 2' & 1 & \mathbf{4'} & \mathbf{1} & 2 & 3 & 3 & 5 \\
 & 2 & 3' & 2 & \mathbf{2} & \mathbf{3} & 5' & 5 & 6' \\
 & & 3 & 3 & 4' & 4 & 5' & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
$$

$$
= \quad
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & 1 & 2 & 3 & 3 & 5 \\
 & 2 & 3' & 2 & 2 & 3 & 5' & 5 & 6' \\
 & & 3 & \mathbf{3} & \mathbf{4'} & 4 & 5' & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
\quad \to \quad
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & 1 & 2 & 3 & 3 & 5 \\
 & 2 & 3' & 2 & 2 & 3 & 5' & 5 & 6' \\
 & & 3 & \mathbf{4'} & \mathbf{3} & 4 & 5' & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
\tag{3.40}
$$

There are three $5'$s to deal with in turn from top to bottom using $\theta_{5'}$, but the last of these is already in the 3rd column and directly below a 3 in the 3rd row, and so does not move:

$$
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & \mathbf{1} & \mathbf{2} & \mathbf{3} & 3 & 5 \\
 & 2 & 3' & 2 & 2 & 3 & \mathbf{5'} & 5 & 6' \\
 & & 3 & 4' & 3 & 4 & 5' & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
\quad \to \quad
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & \mathbf{5'} & \mathbf{1} & \mathbf{2} & 3 & 5 \\
 & 2 & 3' & 2 & 2 & 3 & \mathbf{3} & 5 & 6' \\
 & & 3 & 4' & 3 & 4 & 5' & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
$$

$$
= \quad
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & 5' & 1 & 2 & 3 & 5 \\
 & 2 & 3' & 2 & \mathbf{2} & \mathbf{3} & 3 & 5 & 6' \\
 & & 3 & 4' & 3 & \mathbf{4} & \mathbf{5'} & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
\quad \to \quad
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & 5' & 1 & 2 & 3 & 5 \\
 & 2 & 3' & 2 & \mathbf{5'} & \mathbf{2} & 3 & 5 & 6' \\
 & & 3 & 4' & 3 & \mathbf{3} & \mathbf{4} & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
\tag{3.41}
$$

Then we deal with the two $6'$s by applying $\theta_{6'}$ to give

$$
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & 5' & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{5} \\
 & 2 & 3' & 2 & 5' & 2 & 3 & 5 & \mathbf{6'} \\
 & & 3 & 4' & 3 & 3 & 4 & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
\;\to\;
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & 5' & \mathbf{6'} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\
 & 2 & 3' & 2 & 5' & 2 & 3 & 5 & \mathbf{5} \\
 & & 3 & 4' & 3 & 3 & 4 & 6 & \\
 & & & 4 & 5' & 5 & 5 & & \\
 & & & & 5 & 6' & 6 & & \\
 & & & & & 6 & & &
\end{array}
$$

$$
=\;
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & 5' & 6' & 1 & 2 & 3 \\
 & 2 & 3' & 2 & 5' & 2 & 3 & 5 & 5 \\
 & & 3 & 4' & 3 & 3 & 4 & 6 & \\
 & & & 4 & 5' & \mathbf{5} & 5 & & \\
 & & & & 5 & \mathbf{6'} & 6 & & \\
 & & & & & 6 & & &
\end{array}
\;\to\;
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & 5' & 6' & 1 & 2 & 3 \\
 & 2 & 3' & 2 & 5' & 2 & 3 & 5 & 5 \\
 & & 3 & 4' & 3 & 3 & 4 & 6 & \\
 & & & 4 & 5' & \mathbf{6'} & 5 & & \\
 & & & & 5 & \mathbf{5} & 6 & & \\
 & & & & & 6 & & &
\end{array}
\tag{3.42}
$$

This results in the juxtaposition of $PD$ from (2.14) and $T$ from (2.11) as claimed:

$$
\begin{array}{ccccccccc}
1 & 2' & 1 & 4' & 5' & 6' & 1 & 2 & 3 \\
 & 2 & 3' & 2 & 5' & 2 & 3 & 5 & 5 \\
 & & 3 & 4' & 3 & 3 & 4 & 6 & \\
 & & & 4 & 5' & 6' & 5 & & \\
 & & & & 5 & 5 & 6 & & \\
 & & & & & 6 & & &
\end{array}
\;\equiv\;
\begin{array}{cccccc}
1 & 2' & 1 & 4' & 5' & 6' \\
 & 2 & 3' & 2 & 5' & 2 \\
 & & 3 & 4' & 3 & 3 \\
 & & & 4 & 5' & 6' \\
 & & & & 5 & 5 \\
 & & & & & 6
\end{array}
\;\cdot\;
\begin{array}{ccc}
1 & 2 & 3 \\
3 & 5 & 5 \\
4 & 6 & \\
5 & & \\
6 & &
\end{array}
\tag{3.43}
$$

### 3.3 Corollaries

By associating $x_k$ and $y_k$ to each entry $k$ and $k'$, respectively, in the various tableaux $PST$, $QST$, $PD$, $QD$ and $T$ appearing in Theorem 3.1 we immediately have the following corollary.

**Corollary 3.2.** *Let $\mu = \lambda + \delta$ be a strict partition of length $\ell(\mu) = n$, with $\lambda$ a partition of length $\ell(\lambda) \le n$ and $\delta = (n, n-1, \dots, 1)$.*

$$
\begin{aligned}
\sum_{PST\in\mathcal{PST}^\mu(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(PST)} &= \sum_{PD\in\mathcal{PD}^\delta(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(PD)} \sum_{T\in\mathcal{T}^\lambda(n)} \mathbf{x}^{\mathrm{wgt}(T)}; \\
\sum_{QST\in\mathcal{QST}^\mu(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QST)} &= \sum_{QD\in\mathcal{QD}^\delta(n)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QD)} \sum_{T\in\mathcal{T}^\lambda(n)} \mathbf{x}^{\mathrm{wgt}(T)}.
\end{aligned}
\tag{3.44}
$$

Thanks to the definitions of $P(\mathbf{x}/\mathbf{y})$ and $Q(\mathbf{x}/\mathbf{y})$ given in (2.26), the identities (2.15) and (2.16) and the combinatorial definition of $s_\lambda(\mathbf{x})$ given in (2.24), the above result is nothing other than our first main result, Proposition 1.1.

Other corollaries follow as special cases of these results. Setting $\lambda = 0$ we obtain

$$P_\delta(\mathbf{x}/\mathbf{y}) = s_{1^n}(\mathbf{x}) \prod_{1\le i<j\le n} (x_i + y_j) \quad \text{and} \quad Q_\delta(\mathbf{x}/\mathbf{y}) = \prod_{1\le i\le j\le n} (x_i + y_j). \tag{3.45}$$

Further specialisation to the case $\mathbf{y} = \mathbf{x}$ leads to a result given by Macdonald [8](Sec. III.8, Ex. 3 p 259):

$$P_\delta(\mathbf{x}) = s_\delta(\mathbf{x}) \quad \text{and} \quad Q_\delta(\mathbf{x}) = 2^n\, s_\delta(\mathbf{x}), \tag{3.46}$$

where use has been made of the fact that

$$\prod_{i=1}^n x_i \prod_{1\le i<j\le n} (x_i + x_j) = s_{1^n}(\mathbf{x}) \prod_{1\le i<j\le n} \frac{\left(x_i^2 - x_j^2\right)}{(x_i - x_j)} = s_{1^n}(\mathbf{x}) s_{\delta/1^n}(\mathbf{x}) = s_\delta(\mathbf{x}), \tag{3.47}$$

where the last step is true when, as here, $\mathbf{x}$ has $n$ components $x_1, x_2, \dots, x_n$.

More generally, if $\mu = \lambda + \delta$ for any partition $\lambda$ of length $\ell(\lambda) \le n$, but $\mathbf{y} = \mathbf{x}$ we have another result due to Macdonald [8](Sec. III.8, Ex. 2 p 259):

$$P_{\lambda+\delta}(\mathbf{x}) = s_\delta(\mathbf{x})\, s_\lambda(\mathbf{x}). \tag{3.48}$$

where (3.47) and (2.27) have been applied directly to the $\mathbf{y} = \mathbf{x}$ case of (1.6).

On the other hand the case $\mathbf{y} = t\mathbf{x} = (tx_1, tx_2, \dots, tx_n)$ of (1.6) is equivalent to (1.5), the $t$-deformation of Weyl's denominator formula for the Lie algebra $gl(n)$ due to Tokuyama [17]:

**Corollary 3.3.**

$$\prod_{i=1}^n x_i \prod_{1\le i<j\le n} (x_i + t x_j)\, s_\lambda(\mathbf{x}) = \sum_{ST\in\mathcal{ST}^\mu(n)} t^{\mathrm{hgt}(ST)}(1+t)^{\mathrm{str}(ST)-n}\, \mathbf{x}^{\mathrm{wgt}(ST)}, \tag{3.49}$$

**Proof:** While there is a combinatorial proof of this result due to Okada [10], it follows immediately from Theorem 3.1 by setting $y_k = tx_k$ for all $k = 1, 2, \dots, n$, noting that deleting primes from the entries $k'$ in each $PST \in \mathcal{PST}^\mu(n)$ gives a shifted tableaux $ST \in \mathcal{ST}^\mu(n)$ with a factor of $t$ arising from each primed entry of $PST$, and observing that these must occur in precisely those boxes contributing to $\mathrm{hgt}(ST)$ and are optional, thereby giving rise to a factor of $(1+t)$ in those $\mathrm{str}(ST) - n$ boxes at the lower left hand end of all continuous strips of identical entries other than those starting on the main diagonal.

The remaining corollaries mentioned in the Introduction are the formulae (1.3) of Robbins and Rumsey [12] and (1.4) of Chapman [3]. These require for their elucidation a link with alternating sign matrices. This is provided in Section 5. □

## 4 The $sp(2n)$ bijection

### 4.1 Main result

The analogue in the symplectic case of Theorem 3.1 is the following:

**Theorem 4.1.** *Let $\mu = \lambda + \delta$ be a strict partition of length $\ell(\mu) = n$, with $\lambda$ a partition of length $\ell(\lambda) \leq n$ and $\delta = (n, n-1, \dots, 1)$. There exists a weight and barred weight preserving, bijective map $\Phi$ from $\mathcal{QST}^{\mu}(n, \overline{n})$ to $(\mathcal{QD}^{\delta}(n, \overline{n}), \mathcal{T}^{\lambda}(n, \overline{n}))$ such that for all $QST \in \mathcal{QST}^{\mu}(n, \overline{n})$*

$$\Phi : \ QST \ \mapsto \ (QD, T) \quad \textit{with} \quad \begin{cases} \mathrm{wgt}(QST) = \mathrm{wgt}(QD) + \mathrm{wgt}(T) \\ \\ \mathrm{bar}(QST) = \mathrm{bar}(QD) + \mathrm{bar}(T) \end{cases} \tag{4.50}$$

*and $QD \in \mathcal{QD}^{\delta}(n, \overline{n})$ and $T \in \mathcal{T}^{\lambda}(n, \overline{n})$.*

**Proof:** The Theorem is proved by the identification of a suitable map $\Phi$ that it is both weight preserving and bijective. The underlying procedure is the same as before, in that the jeu de taquin is applied successively to all primed entries of $QST$ dealing in sequence with all entries $\overline{k}'$ and then $k'$ for $k = 1, 2 \dots, n$. In the case of the $\overline{k}'$s there is no impediment to moving all these entries to the $k$th column by means of the jeu de taquin, but something slightly more subtle is required in the case of the $k'$s. It may be necessary to invoke two new weight preserving transformations.

The structure of $\Phi$ is such that $\Phi = \phi_{n'} \circ \phi_{\overline{n}'} \circ \cdots \circ \phi_{2'} \circ \phi_{\overline{2}'} \circ \phi_{1'} \circ \phi_{\overline{1}'}$. Here $\phi_{\overline{k}'}$ differs from $\theta_{k'}$ only in that the jeu de taquin is played with $\overline{k}'$s rather than the $k'$s, while $\phi_{k'} = \chi_{k'} \circ \psi_{k'}$ where $\psi_{k'}$ differs from $\theta_{k'}$ only if the final step of the path of $k'$ into the $i$th row of the $k$th column is blocked by an entry $\overline{k}'$. In such a situation the horizontal pair of entries $\overline{k}' k'$ in the $i$th row is replaced by the horizontal pair $i\,\overline{i}$. Having moved all the $k'$s into the $k$th column or annihilated them as above, there may remain in the $k$th column vertical pairs $\overline{i}\,i$. It is then necessary to invoke $\chi_{k'}$. This replaces the lowest such pair, for which $i$ is necessarily in the $i$th row, by a vertical pair $k'\overline{k}'$, and then moves the resulting $k'$ north as far as possible whilst still satisfying T$\overline{3}$, and then acts in the same way on the next lowest vertical pair $\overline{j}\,j$, replacing them by another vertical pair $k'\overline{k}'$, and so on. Having removed all unprimed vertical pairs in this way any remaining unbarred entries $i$ or $\overline{i}$, but not both, lie in their own $i$th row, as required for consistency with QD$\overline{1}$.

To demonstrate that these various maps are well defined, we exhibit the relevant individual steps as below, first for the $\phi_{\overline{k}'}$ case. The starting point is one in which the action of $\phi_{(k-1)'}$ will have already left the conditions QD$\overline{1}$-$\overline{2}$ satisfied to the left of the $k$th column, and QST1-$\overline{5}$ satisfied in the $k$th column and to its right. It should be noted that for the unprimed entries QST1-4 subsume T1-2. In addition, we require at every stage that

T$\overline{3}$ each unprimed entry $i$ and $\overline{i}$ may appear no lower than the $i$th row;
P$\overline{3}$ each primed entry $j'$ and $\overline{j}'$ may appear no further to the left than the $j$th column.

These conditions $\mathrm{T}\overline{3}$ and $\mathrm{P}\overline{3}$ apply automatically to each initial $QST \in \mathcal{QST}(n,\overline{n})$ with $\mu$ a strict partition of length $\ell(\mu)=n$, and it is important that they remain satisfied at all subsequent stages since they ensure that the procedures terminate appropriately. Notice also that thanks to $\mathrm{QST}\overline{5}$ and QST1, if $k'$ or $\overline{k}'$ appears in the $i$th row of $QST$, then $i \le k$. This condition is maintained under all interchanges of $k'$ and $\overline{k}'$ with unprimed entres since $k'$ and $\overline{k}'$ move only north or west. In fact for $\overline{k}'$, QST1 implies that the case $i=k$ only occurs if $\overline{k}'$ is on the main diagonal, otherwise $i<k$.

Turning then to the action of $\phi_{\overline{k}'}$, consider the tableau $T_0$ in (4.51) below, with $\overline{k}'$ not yet in the $k$th column. Then $T_0$ is to be thought of as the subtableau surrounding a particular $\overline{k}'$ after the jeu de taquin has been applied to all $\overline{k}'$s appearing initially above the $\overline{k}'$ in question, moving them into the $k$th column. Some further steps of the jeu de taquin may have already been applied to the central $\overline{k}'$ and the diagram is intended to indicate under what conditions its next move is north or west, given that it currently lies at position $(i,j)$ with $j>k$ and $i<k$.

Assume first that $a$, $b$ and $d$ are all unprimed, with $d \ge \overline{i}$ by virtue of $\mathrm{T}\overline{3}$. If $d \le b$ then $\overline{k}'$ is to be interchanged with $b$ and if $d>b$ then $\overline{k}'$ is to be interchanged with $d$ as shown below. In the first case $\overline{k}'$ moves north and the resulting tableau $T_N$ satisfies T1-2 and $\mathrm{T}\overline{3}$ since $\overline{i} \le d \le b \le c < e$, while in the second $\overline{k}'$ moves west and the resulting tableau $T_W$ satisfies T1-2 and $\mathrm{T}\overline{3}$ since $b<d<f\le g$ and $\overline{i}\le d$. In addition, in both $T_N$ and $T_W$ the condition $\mathrm{P}\overline{3}$ is satisfied, since $j>k$.

$$\phi_{\overline{k}'}: \quad \begin{array}{c} T_0 = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & \overline{k}' & e \\ \hline f & g & h \\ \hline \end{array} \\[2ex] \overline{k}' \text{ at } (i,j) \text{ with } i<k<j \end{array} \quad \longrightarrow \quad \begin{cases} T_N = \begin{array}{|c|c|c|}\hline a & \overline{k}' & c \\ \hline d & b & e \\ \hline f & g & h \\ \hline \end{array} & \text{if} \quad d \le b \\[4ex] T_W = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline \overline{k}' & d & e \\ \hline f & g & h \\ \hline \end{array} & \text{if} \quad b < d \end{cases} \tag{4.51}$$

The case $i=k$ has been excluded since this only arises if $\overline{k}'$ is on the main diagonal, that is $i=j$, but $j>k$. Once again if $i=1$ then the top row $a\,b\,c$ of $T_0$ is necessarily absent and $T_0$ just maps to $T_W$ without its top row. As in the case of (3.32), the maps to $T_N$ and $T_W$ are unaffected if any one or more of $c, e, f, g, h$ are primed or absent, provided that the regularity of the diagram is maintained. The fact that $i<k<j$ precludes the absence of $f$.

If $\overline{k}'$ is already in the $k$th column, so that $\overline{k}'$ cannot move westward by virtue of $\mathrm{P}\overline{3}$, the following illustrates the only allowed move of $\overline{k}'$ northwards:

$$\phi_{\overline{k}'}: \quad \begin{array}{c} T_0 = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & \overline{k}' & e \\ \hline f & g & h \\ \hline \end{array} \\[2ex] \overline{k}' \text{ at } (i,k) \text{ with } i<k \end{array} \quad \longrightarrow \quad T_N = \begin{array}{|c|c|c|}\hline a & \overline{k}' & c \\ \hline d & b & e \\ \hline f & g & h \\ \hline \end{array} \quad \text{if} \quad \overline{i} \le b \tag{4.52}$$

Notice that in order to satisfy T$\overline{3}$ the map from $T_0$ to $T_N$ only takes place if $b \geq \overline{i}$, otherwise $T_0$ is unaltered and $\overline{k}'$ occupies the site in the $i$th row of the $k$th column, until disturbed by any incoming $k'$s, as we shall see later. This time the case $i = k$ is excluded from (4.52) because in $T_0$ we must have $b < \overline{k} = \overline{i}$, whilst the map from $T_0$ to $T_N$ requires $b \geq \overline{i}$. In both $T_0$ and $T_N$ the condition QD$\overline{1}$ applies to the left hand column. In $T_0$ the conditions T1-$\overline{3}$ apply to unprimed elements to the right of this column. The same is then true of $T_N$ since we have $\overline{i} \leq b \leq c < e$. The cases for which $a$, $b$ or $d$ are primed may be dealt with precisely as in the $\theta_{k'}$ case of Section 3 with $k'$ replaced by $\overline{k}'$. This time the only possible impediment to the movement of $\overline{k}'$ in a north-westerly direction until it reaches the $k$th column is the existence of another $\overline{k}'$ to its immediate left. However, by the same argument as that used in the $\theta_{k'}$ case, this cannot occur because the path followed by $\overline{k}'$ always remains strictly south of the path followed by all preceding $\overline{k}'$s. As in the case of $k'$s moving under the action of $\theta_{k'}$, any $\overline{k}'$ that is not on the main diagonal in the initial QST cannot reach that diagonal under the action of $\phi_{\overline{k}'}$ since its path necessarily starts north of the $k$th row and it subsequently moves only north or west. Of course in the initial QST, unlike any PST, a primed entry $\overline{k}'$ may appear on the main diagonal.

Having completed the jeu de taquin moves for all $\overline{k}'$s and moved them into and as far north as possible in the $k$th column under the action of $\phi_{\overline{k}'}$, the conditions QD$\overline{1}$-$\overline{2}$ are satisfied to the left of the $k$th column, while, with the exception of the entries $\overline{k}'$ that are already in the $k$th column, the conditions QST1-$\overline{5}$ are satisfied by the remaining entries in the $k$th column and to its right. As usual T$\overline{3}$ and P$\overline{3}$ apply to all entries.

It remains to deal with any $k'$s in $QST$ using $\phi_{k'} = \chi_{k'} \circ \psi_{k'}$. The action of $\psi_{k'}$ is carried out in the same way as before with the diagrams of (4.51) and (4.52) just altered by changing $\overline{k}'$ to $k'$, provided that $d$ is unprimed. The only difference is that the condition $i < k$ now has to be replaced by $i \leq k$. In the case of (4.51) these changes give:

$$\psi_{k'}: \quad \begin{array}{c} T_0 = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & k' & e \\ \hline f & g & h \\ \hline \end{array} \\[2em] k' \text{ at } (i,j) \text{ with } i \leq k < j \end{array} \longrightarrow \begin{cases} T_N = \begin{array}{|c|c|c|}\hline a & k' & c \\ \hline d & b & e \\ \hline f & g & h \\ \hline \end{array} & \text{if} \quad d \leq b \\[3em] T_W = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline k' & d & e \\ \hline f & g & h \\ \hline \end{array} & \text{if} \quad b < d \end{cases} \tag{4.53}$$

A new feature may arise in this case. If $i = k = j - 1$ then $f$ is absent and in $T_0$ the entries $d$ and $g$ lie on the main diagonal of the full shifted diagram. It follows from QST1 and QST$\overline{5}$ that $d \in \{\overline{i'}, \overline{i}, i'\}$ with $i = k$, and from QST$\overline{5}$ that $g \in \{\overline{j}', \overline{j}, j', j\}$ with $j = k + 1$. This means that if $b < \overline{k}$ then an unprimed entry $d = \overline{k}$ on the main diagonal will be replaced by the primed entry $k'$ under the map from $T_0$ to $T_W$. Notice also that if $i < k = j - 1$ then $f$ is present and if primed must be either $k'$ or $\overline{k'}$ by virtue of QD$\overline{2}$. If $d > b$ the map from $T_0$ to $T_W$ then yields a vertical pair $k'\,k'$ or $k'\,\overline{k}'$, respectively. The first case never causes any problem, but in the second it is important to realise that it will only occur if $d \leq i$ for otherwise $d$ would have been exchanged with $f = \overline{k}'$ in the $(i+1)$th row of $T_0$ as part of the action of $\phi_{\overline{k}'}$.

If $d$ is primed then $d \neq k'$ since $d = k'$ would give a horizontal pair $k'k'$, and this cannot occur since the path of the second $k'$ must stay strictly south of the path of the first $k'$. To deal with the case $d = \overline{k}'$, the following map is required, whereby $k'$ either moves north to give $T_N$ or moves west, annihilating the neighbouring $\overline{k}'$ and creating the horizontal pair $i\,\overline{i}$, to give $T_W$.

$$\psi_{k'}: \quad \begin{array}{c} T_0 = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline \overline{k}' & k' & e \\ \hline f & g & h \\ \hline \end{array} \\ \\ k' \text{ at } (i, k+1) \text{ with } i \leq k \end{array} \quad \longrightarrow \quad \begin{cases} T_N = \begin{array}{|c|c|c|}\hline a & k' & c \\ \hline \overline{k}' & b & e \\ \hline f & g & h \\ \hline \end{array} & \text{if} \quad \overline{i} \leq b \\ \\ T_H = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline i & \overline{i} & e \\ \hline f & g & h \\ \hline \end{array} & \text{if} \quad b < \overline{i} \end{cases} \tag{4.54}$$

The fact that the pair $i\,\overline{i}$ violates T1 in $T_H$ is not a problem because the $i$ lies in the $k$th column of what will become $QD$ and the $\overline{i}$ lies either in the $(k+1)$th column of $QD$ or the first column of $T$. In neither case does the condition T1 apply to both $i$ and $\overline{i}$. If they are both in $QD$ they automatically satisfy the condition QD$\overline{1}$, and if one is in $QD$ and the other in $T$ then $i$ satisfies QD$\overline{1}$ and $\overline{i}$ satisfies T$\overline{3}$, with all elements to its right satisfying QST1, as required. The $i$ in $QD$ subsequently remains fixed, while the $\overline{i}$ may move east along the same row under subsequent interchanges, but not south, since it is constrained by T$\overline{3}$. It might also be remarked that if $i = k$ the map from $T_0$ to $T_H$, with $f$ necessarily absent, changes the main diagonal primed entry $\overline{k}'$ to an unprimed $k$.

In (4.53), if $b$ is primed then we must have $b = j' \leq k'$ in which case it would have been moved into its own column to the left of the $k'$ in $T_0$ under the action of $\phi_{\overline{j}'}$. Thus $b$ must be unprimed. Furthermore, if $a$ is unprimed then $a < \overline{i} < i$, since if this were not the case the action of $\phi_{\overline{k}'}$ would have required that $a$ and $\overline{k}'$ be interchanged in $T_0$. The conditions $a < i$ and $b < \overline{i}$, with $i$ and $\overline{i}$ in the $i$th row, then ensure that $T_H$ satisfies both T2 and T$\overline{3}$. Finally, if $a$ is primed then $a = \overline{k}'$ or $k'$, since otherwise it would not have remained in the $k$th column. In the first case $T_0$ may be mapped to either $T_N$ or $T_H$, as in (4.53) with $a = \overline{k}'$, but in the second case $T_0$ is necessarily mapped to $T_H$ with $a = k'$, since $T_N$ is ruled out by the strictly south property of $k'$ paths.

If $k'$ is already in the $k$th column, then thanks to P$\overline{3}$ it cannot move westwards, and (4.55) illustrates the only allowed move of $k'$ northwards:

$$\psi_{k'}: \quad \begin{array}{c} T_0 = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & k' & e \\ \hline f & g & h \\ \hline \end{array} \\ \\ k' \text{ at } (i, k) \text{ with } i \leq k \end{array} \quad \longrightarrow \quad T_N = \begin{array}{|c|c|c|}\hline a & k' & c \\ \hline d & b & e \\ \hline f & g & h \\ \hline \end{array} \quad \text{if} \quad \overline{i} \leq b, \tag{4.55}$$

with $b < k$ unprimed. Notice that as a consequence of QD$\overline{1}$ we must have $d = i$ or $\overline{i}$, so that in the case $d = i$ and $b = \overline{i}$, we again obtain a horizontal pair $i\,\overline{i}$, this time in $T_N$ with both $i$ and $\overline{i}$ in what is destined to become QD. Furthermore, in the case $i = k$, so that $d$, $f$ and $g$ are all absent, if $b = \overline{k}$ then the primed entry $k'$ on the main diagonal is replaced by the unprimed entry $\overline{k}$ under the map from $T_0$ to $T_N$. If $b$ were to be primed it must already be in its own column with $b = \overline{k}'$ or $k'$. In either case the $k'$ below $b$ does not move.

Proceeding in this way all $k'$'s are moved into and up the $k$th column as far as possible by the action of $\psi_{k'}$. In the columns to the left of the $k$th, the conditions QD$\overline{1}$-$\overline{2}$ apply, and in the $k$th column and to its right the conditions QST1-$\overline{5}$ apply along with T$\overline{3}$. However in the $k$th column itself although QD$\overline{2}$ applies, QD$\overline{1}$ may not yet apply, since one may still have one or more unprimed pairs say $i$ and $\overline{i}$, in the $k$th column. Every such pair lies in consecutive rows. To see this it should be noted that no unprimed entry can lie between $\overline{i}$ and $i$ in the same column, by virtue of T2. Furthermore, no primed entry can lie between $\overline{i}$ and $i$, since any such entries $k'$ or $\overline{k}'$ will have been moved above $\overline{i}$ by means of the action of $\psi_{k'}$ or $\phi_{\overline{k}'}$, as appropriate. Such moves are always allowed since $i$ lies in some row $j$, with $j \leq i$ by virtue of T$\overline{3}$, and there is no impediment to $\overline{i}$ moving down the $k$th column to the $(j-1)$th row by means of interchanges with as many $k'$s or $\overline{k}'$s as necessary.

Having completed the action of $\psi_{k'}$ we consider the action of $\chi_{k'}$. It acts first on the lowest of the above pairs, that is the pair with the highest value of $i$. This pair must be such that $\overline{i}$ and $i$ lie in the $(i-1)$th and $i$th rows, respectively. To see this let them lie in the $(j-1)$th and $j$th rows with $j \leq i$ by virtue of T$\overline{3}$. Then following the action of $\psi_{k'}$ all unprimed entries below $i$ are unpaired and must lie in their own row, whether they are barred or unbarred, by virtue of the argument already given in the $\theta_{k'}$ case. Thus the entry immediately below $i$ is either $j+1$, $\overline{j+1}$, $k'$ or $\overline{k}'$. In all four cases the condition T2 and the rule for shifting $k'$ and $\overline{k}'$ as far north as is consistent with T$\overline{3}$ imply that $i \leq j$. Since $j \leq i$ it follows that $j = i$, as claimed. The action of $\chi_{k'}$ is then to map this vertical pair $\overline{i}\,i$ in $T_0$, with $i$ in the $i$th row, to $k'\,\overline{k}'$ in $T_V$:

$$\chi_{k'}: \quad T_0 = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & \overline{i} & e \\ \hline f & i & h \\ \hline \end{array} \quad \longrightarrow \quad T_V = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & k' & e \\ \hline f & \overline{k}' & h \\ \hline \end{array} \tag{4.56}$$

$i$ at $(i,k)$ with $i \leq k$

In these diagrams $c, e, f, g, h$ may be either unprimed, primed or absent. If present $e \geq \overline{i}$ and $h \geq i$, thanks to QST1 applied to $T_0$. Of course a violation of QST1 occurs in $T_V$ if $k' > e$ or $\overline{k}' > h$, but this is no problem because the $k$th column will form part of $QD$ and is only subject to the conditions QD$\overline{1}$-$\overline{2}$. It is important to recognise that $T_V$ with its vertical pair $k'\,\overline{k}'$ can only be arrived at as shown, because if it were thought to have arisen through an interchange of $e$ and $k'$ under the action of $\psi_{k'}$, as discussed following (4.53), this could not be the case since the $e$ in the $k$th column with $e \geq \overline{i}$ would have already been interchanged with the $\overline{k}'$ immediately beneath it through the action of $\phi_{\overline{k}'}$. Notice that if $i = k$ in (4.56) then $f$ is absent and an unprimed entry $k$ on the main diagonal is replaced by a primed entry $\overline{k}'$.

Returning to the action of $\chi_{k'}$, the $k'\overline{k}'$ pair created in the $k$th column, with $\overline{k}'$ in the $i$th row may be such that the $k'$ can move north without violating $\mathrm{T}\overline{3}$. This will be the case if the entry $b$ immediately above $k'$ is unprimed and equal to either $\overline{i-1}$ or $i-1$, then the action of $\chi_{k'}$ is extended as shown below:

$$\chi_{k'}:\quad \underset{\overline{k}' \text{ at } (i,k) \text{ with } i\le k}{T_V = \begin{array}{|c|c|c|}\hline a & b & c\\\hline d & k' & e\\\hline f & \overline{k}' & h\\\hline\end{array}} \longrightarrow T_N = \begin{array}{|c|c|c|}\hline a & k' & c\\\hline d & b & e\\\hline f & \overline{k}' & h\\\hline\end{array} \quad \text{if} \quad b = i-1 \text{ or } \overline{i-1} \tag{4.57}$$

Recall from (4.57) that if $e$ exists then $e \ge \overline{i}$. In addition $c \le e$ thanks to QST2, so that $b \le e$, thereby maintaining consistency with QST1 in $T_N$. Notice that $b \le c$ in $T_V$, so that $T_N$ cannot have arisen by interchanging $k'$ and $c$. The restriction of $b$ to just two values comes about first, because of its origin in $T_0$ of (4.56) for which T2 implies $b < \overline{i}$, and second, because $\mathrm{T}\overline{3}$ implies $b \ge \overline{i-1}$ in $T_N$. It should be noted that in both the allowed cases the entry $b$ has arrived in $T_N$ at its own, $(i-1)$th row, and cannot be moved lower. This means that the $\overline{k}'$ entry of any newly created vertical pair $k'\overline{k}'$ necessarily remains fixed, since it cannot be exchanged with the $b$ in $T_N$ by virtue of $\mathrm{T}\overline{3}$. On the other hand $k'$, still under the action of $\chi_{k'}$, may be moved further up the $k$th column by means of interchanges with unprimed elements, constrained as usual by the condition T2 and $\mathrm{T}\overline{3}$. Thus in (4.57) the symbols $a, b, c$ may may be interpreted as columns of entries as shown below:

$$\chi_{k'}:\quad \underset{\overline{k}' \text{ at } (i,k) \text{ with } i\le k}{T_V = \begin{array}{|c|c|c|}\hline a_1 & b_1 & c_1\\\hline a_2 & b_2 & c_2\\\hline \vdots & \vdots & \vdots\\\hline a_r & b_r & c_r\\\hline d & k' & e\\\hline f & \overline{k}' & h\\\hline\end{array}} \longrightarrow T_N = \begin{array}{|c|c|c|}\hline a_1 & k' & c_1\\\hline a_2 & b_1 & c_2\\\hline \vdots & \vdots & \vdots\\\hline a_r & \vdots & c_r\\\hline d & b_r & e\\\hline f & \overline{k}' & h\\\hline\end{array} \quad \text{if} \quad b_r = i-1 \text{ or } \overline{i-1} \tag{4.58}$$

with each $b_s$ unprimed, but either barred or unbarred and subject to the QST1 condition $b_s \le c_s$ if $c_s$ is present, the T2 condition $b_s < b_{s+1}$ if $s < r$, as well as the $\mathrm{T}\overline{3}$ condition $\overline{i-1-r+s} \le b_s$. As before, if $e$ is present, then $e \ge \overline{i}$ and $b_r \le c_r \le e$. The final resting place of the $k'$ is either in the topmost row, or immediately below a primed entry $k'$ or $\overline{k}'$, or in the $l$th row immediately below an unprimed entry $b < \overline{l}$ that is prevented by the condition $\mathrm{T}\overline{3}$ from being interchanged with the moving $k'$.

The process is then repeated for the next lowest vertical pair $\overline{j}\,j$ in the $k$th column with $j$ in the $j$th row, which could lie between $k'$ and $\overline{k}'$ separated as in (4.58), until all such pairs are eliminated. The final result is that every unprimed entry $i$ or $\overline{i}$ in the $k$th column lies in the $i$th row, with all other entries in this column equal to $k'$ or $\overline{k}'$. Thus following the action of $\chi_{k'}$ all entries in the $k$th column and to its left satisfy the conditions QD$\overline{1}$ and QD$\overline{2}$, while all entries to the right of the $k$th column are subject to QST1-$\overline{5}$.

Repeating this procedure for all $k = 1, 2, \ldots, n$ results, as required, in the juxtaposition of a primed tableau $QD \in \mathcal{PD}^\delta(n,\overline{n})$ and an unprimed tableau $T \in \mathcal{T}^\lambda(n,\overline{n})$. It should be noted that each individual step of the maps constituting $\Phi$ is either a simple interchange or a map from a horizontal pair $\overline{k}'k'$ to a horizontal pair $i\,\overline{i}$, or a map from a vertical pair $\overline{i}\,i$ to a vertical pair $k'\overline{k}'$. In each case the weight of each pair is zero, and the number of barred entries is one. It follows that such steps are all weight and barred weight preserving, so that (4.50) is always satisfied.

To show that the map $\Phi$ from $QST \in \mathcal{QST}^\mu(n,\overline{n})$ to $(QD, T)$ with $QD \in \mathcal{PD}^\delta(n,\overline{n})$ and $T \in \mathcal{T}^\lambda(n,\overline{n})$ is bijective it is sufficient to note that each step of the map $\Phi$ may be reversed and that if $\Phi^{-1}$ is defined by such a reversal of each step, then the action of $\Phi^{-1}$ on the juxtaposition of any $QD \in \mathcal{PD}^\delta(n,\overline{n})$ and any $T \in \mathcal{T}^\lambda(n,\overline{n})$ always leads to some $QST \in \mathcal{QST}^\mu(n,\overline{n})$.

To see this note that since $\Phi = \phi_{n'} \circ \phi_{\overline{n}'} \circ \cdots \circ \phi_{2'} \circ \phi_{\overline{2}'} \circ \phi_{1'} \circ \phi_{\overline{1}'}$ with $\phi_{k'} = \chi_{k'} \circ \psi_{k'}$, then $\Phi^{-1} = \phi_{\overline{1}'}^{-1} \circ \phi_{1'}^{-1} \circ \phi_{\overline{2}'}^{-1} \circ \phi_{2'}^{-1} \circ \cdots \circ \phi_{\overline{n}'}^{-1} \circ \phi_{n'}^{-1}$ with $\phi_{k'}^{-1} = \psi_{k'}^{-1} \circ \chi_{k'}^{-1}$. The action of the $\phi_{\overline{k}'}^{-1}$ is defined by the action of $\theta_{k'}^{-1}$ illustrated in (3.35) with $k'$ replaced by $\overline{k}'$, while that of $\psi_{k'}^{-1}$ coincides precisely with that of $\theta_{k'}^{-1}$ together with additional transformations of the type

$$\psi_{k'}^{-1}: \quad \underset{i \text{ at } (i,k) \text{ with } i \le k}{T_0 = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline i & \overline{i} & e \\ \hline f & g & h \\ \hline\end{array}} \quad \longrightarrow \quad T_E = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline \overline{k}' & k' & e \\ \hline f & g & h \\ \hline\end{array} \quad \text{if } b < \overline{i} \tag{4.59}$$

where, as indicated, the pair $i\,\overline{i}$ lies in the $i$th row of the tableau $T_0$, with $i$ in the $k$th column. The fact that we necessarily have $b < \overline{i}$ ensures that this is the exact inverse of the map $\psi_{k'}$ that takes $T_0$ to $T_W$ in (4.53). It is important to note that when applying $\psi_{k'}^{-1}$, the map of any $i\,\overline{i}$ to $\overline{k}'\,k'$ in the $i$th row must be carried out before moving any $k'$s that may appear in the tableau higher than the $i$th row.

Similarly, the basic action of $\chi_{k'}^{-1}$, which must be applied before $\psi_{k'}^{-1}$ comes into play, lowers $k'$s in the $k$th column in accordance with the map

$$\chi_{k'}^{-1}: \quad \underset{\overline{k}' \text{ at } (i,k) \text{ with } i \le k}{T_0 = \begin{array}{|c|c|c|}\hline a & k' & c \\ \hline d & b & e \\ \hline f & \overline{k}' & h \\ \hline\end{array}} \quad \longrightarrow \quad T_S = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & k' & e \\ \hline f & \overline{k}' & h \\ \hline\end{array} \tag{4.60}$$

if $e \ge \overline{i}$ and $b \le c$ and $b = (i-1)$ or $\overline{i-1}$. This is the precise inverse of (4.57). Once again $a, b, c$ may be replaced, as in (4.58), by certain vertical sequences of $r$ entries with each $b_s$ unprimed.

Once the $k'$ is adjacent to $\overline{k}'$ in the $k$th column, then $\chi_{k'}^{-1}$ acts again as follows:

$$\chi_{k'}^{-1}: \quad \underset{\overline{k}' \text{ at } (i,k) \text{ with } i \leq k}{T_S = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & k' & e \\ \hline f & \overline{k}' & h \\ \hline \end{array}} \quad \longrightarrow \quad T_I = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & \overline{i} & e \\ \hline f & i & h \\ \hline \end{array} \tag{4.61}$$

It is the fact that $e \geq \overline{i}$ that ensures that the vertical pair $k'\,\overline{k}'$ is of the type created by $\chi_{k'}$, and is not to be confused with one created by $\psi_{k'}$. In fact if $e < \overline{i}$ then $\chi_{k'}^{-1}$ cannot map as above, so it simply leaves $T_S$ unchanged, but then subject to $\psi_{k'}^{-1}$ which acts as follows

$$\psi_{k'}^{-1}: \quad \underset{\overline{k}' \text{ at } (i,k) \text{ with } i \leq k}{T_S = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & k' & e \\ \hline f & \overline{k}' & h \\ \hline \end{array}} \quad \longrightarrow \quad T_E = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & k' \\ \hline f & \overline{k}' & h \\ \hline \end{array} \tag{4.62}$$

In applying $\chi_{k'}^{-1}$ as in (4.60) or (4.61) one starts the action with the most northerly vertical pair, $k'$ and $\overline{k}'$, either simply adjacent as in $T_V$ of (4.56) or separated by a sequence of unprimed entries as in $T_V$ of (4.58). This is in contrast to the subsequent action of $\psi_{k'}^{-1}$ which acts on the most southerly $k'$ first. Following the action of $\chi_{k'}^{-1}$ it should be pointed out that $\psi_{k'}^{-1}$ can produce vertical pairs $k'\,\overline{k}'$ through a map of the type (4.59) either with $a = k'$ as it stands or with $a$ subsequently replaced by $k'$ as a result of the further action of $\psi_{k'}$ on a higher $k'$ in the $k$th column. In either case (4.59) requires that $b < \overline{i}$. This constraint precludes any confusion with a vertical pair of the type created by $\chi_{k'}$. That could only be removed using $\chi_{k'}^{-1}$ as in (4.61), which requires $e \geq i$, where $e$ in (4.61) is the counterpart of $b$ in (4.59).

Finally, it is necessary to show that all $k'$s and $\overline{k}'$s are mapped by $\phi_{k'}^{-1}$ and $\phi_{\overline{k}'}^{-1}$ to endpoints consistent with the conditions QST1-4 and QST$\overline{5}$ on all primed tableaux $QST \in \mathcal{QST}^{\mu}(n,\overline{n})$. First it should be noted that QST3 and QST$\overline{5}$ are satisfied throughout the application of $\Phi^{-1}$. Furthermore, a violation of QST1 or QST2 by any $k'$ or $\overline{k}'$ simply means that application of $\phi_{k'}^{-1}$ or $\phi_{\overline{k}'}^{-1}$, respectively, has not been completed. The argument used in Section 3 regarding reverse paths staying strictly north of one another in any given column, is sufficient to ensure that no two identical primed entries may appear in the same row, thereby ensuring that the final condition QST3 is always satisfied. Thus the image of $\Phi^{-1}$ of any pair $(QD, T)$ with $QD \in \mathcal{QD}^{\delta}(n,\overline{n})$ and $T \in \mathcal{T}^{\lambda}(n,\overline{n})$ is some $QST \in \mathcal{QST}^{\mu}(n,\overline{n})$.

This implies that the original map $\Phi$ is bijective. Since it is also both weight and barred weight preserving, this completes the proof of Theorem 4.1. $\square$

It should be pointed out that, unlike the $gl(n)$ case, a corresponding result does not apply to $PST \in \mathcal{PST}^{\mu}(n,\overline{n})$ in the $sp(2n)$ case because of the necessity of using (4.53) and (4.56). In (4.53) if $i = k = j-1$, $d = \overline{k}$ and $b < \overline{k}$ then $\overline{k}$ is replaced by $k'$ on the main diagonal, while in (4.56) if $i=k$ then $k$ is replaced by $\overline{k}'$ on the main diagonal. These transformations are in direct violation of PD3. Similarly, primed entries on the main diagonal may be replaced by unprimed entries through the use of (4.54) and (4.55). In (4.54) if $i=k$ and $b<\overline{k}$ then $\overline{k}'$ is replaced by $k$ on the main diagonal, while in (4.55) if $i=k$ and $b=\overline{k}$ then $k'$ is replaced by $\overline{k}$ on the main diagonal.

## 4.2 Example

The bijection $\Phi$ is illustrated by the following map for $n=5$ and $\mu=(9,7,6,2,1)$:

$$QST=\begin{array}{ccccccccc} \overline{1} & 1' & \overline{2}' & 2' & \overline{3}' & 3 & \overline{4} & \overline{4} & 5\\ & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\ & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\ & & & 4 & 4 & & & & \\ & & & & 5' & & & & \end{array} \longrightarrow QD=\begin{array}{ccccc} 1' & 1 & \overline{3}' & 4' & \overline{1}\\ & \overline{2}' & \overline{2} & 4' & 5'\\ & & \overline{3} & \overline{4}' & \overline{3}\\ & & & 4 & 4\\ & & & & 5' \end{array},\quad T=\begin{array}{cccc} \overline{1} & \overline{4} & \overline{4} & 5\\ 3 & 4 & 4 & \\ \overline{4} & 5 & 5 & \end{array} \tag{4.63}$$

Once again we indicate by means of boldface entries both the paths traced out by elements $\overline{k}'$ and $k'$ as they move to the $k$th column under the action of $\phi_{\overline{k}'}$ and $\psi'_k$, respectively, as well as annihilations and creation of $\overline{k}'k'$ pairs under $\psi_{k'}$ and $\chi_{k'}$, respectively.

There are no $\overline{1}'$s, so we first move the single $1'$ under the action of $\psi_{1'}$ as shown:

$$\begin{array}{ccccccccc} \mathbf{\overline{1}} & \mathbf{1'} & \overline{2}' & 2' & \overline{3}' & 3 & \overline{4} & \overline{4} & 5\\ & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\ & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\ & & & 4 & 4 & & & & \\ & & & & 5' & & & & \end{array} \longrightarrow \begin{array}{ccccccccc} \mathbf{1'} & \mathbf{\overline{1}} & \overline{2}' & 2' & \overline{3}' & 3 & \overline{4} & \overline{4} & 5\\ & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\ & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\ & & & 4 & 4 & & & & \\ & & & & 5' & & & & \end{array} \tag{4.64}$$

The application of $\phi'_{\overline{2}}$ then gives

$$\begin{array}{ccccccccc} 1' & \mathbf{\overline{1}} & \mathbf{\overline{2}'} & 2' & \overline{3}' & 3 & \overline{4} & \overline{4} & 5\\ & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\ & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\ & & & 4 & 4 & & & & \\ & & & & 5' & & & & \end{array} \longrightarrow \begin{array}{ccccccccc} 1' & \mathbf{\overline{2}'} & \mathbf{\overline{1}} & 2' & \overline{3}' & 3 & \overline{4} & \overline{4} & 5\\ & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\ & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\ & & & 4 & 4 & & & & \\ & & & & 5' & & & & \end{array} \tag{4.65}$$

where there is no possibility of moving the lower $\overline{2}'$. Then the application of $\psi_{2'}$ on the only $2'$ involves first a transposition and then the replacement of the resulting horizontal pair $\overline{2}'\,2'$ in the first row by $1\,\overline{1}$:

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
1' & \overline{2}' & \bar{\mathbf{1}} & \mathbf{2'} & \overline{3}' & 3 & \overline{4} & \overline{4} & 5 \\
 & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\
 & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\
 & & & 4 & 4 & & & & \\
 & & & & 5' & & & &
\end{array}
\longrightarrow
\begin{array}{|c|c|c|c|c|c|c|c|c|}
1' & \overline{2}' & \mathbf{2'} & \bar{\mathbf{1}} & \overline{3}' & 3 & \overline{4} & \overline{4} & 5 \\
 & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\
 & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\
 & & & 4 & 4 & & & & \\
 & & & & 5' & & & &
\end{array}
$$

$$
=
\begin{array}{|c|c|c|c|c|c|c|c|c|}
1' & \bar{\mathbf{2}}' & \mathbf{2'} & \overline{1} & \overline{3}' & 3 & \overline{4} & \overline{4} & 5 \\
 & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\
 & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\
 & & & 4 & 4 & & & & \\
 & & & & 5' & & & &
\end{array}
\longrightarrow
\begin{array}{|c|c|c|c|c|c|c|c|c|}
1' & \mathbf{1} & \bar{\mathbf{1}} & \overline{1} & \overline{3}' & 3 & \overline{4} & \overline{4} & 5 \\
 & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\
 & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\
 & & & 4 & 4 & & & & \\
 & & & & 5' & & & &
\end{array}
\tag{4.66}
$$

The single $\overline{3}'$ is moved as shown to the 3rd column under the action of $\psi_{3'}$:

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
1' & 1 & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \bar{\mathbf{3}}' & 3 & \overline{4} & \overline{4} & 5 \\
 & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\
 & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\
 & & & 4 & 4 & & & & \\
 & & & & 5' & & & &
\end{array}
\longrightarrow
\begin{array}{|c|c|c|c|c|c|c|c|c|}
1' & 1 & \bar{\mathbf{3}}' & \bar{\mathbf{1}} & \bar{\mathbf{1}} & 3 & \overline{4} & \overline{4} & 5 \\
 & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & 4' & 4 & 4 & \\
 & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\
 & & & 4 & 4 & & & & \\
 & & & & 5' & & & &
\end{array}
\tag{4.67}
$$

Next, the single $4'$ is moved under $\phi_{4'}$ as follows:

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
1' & 1 & \overline{3}' & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \mathbf{3} & \overline{4} & \overline{4} & 5 \\
 & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & \mathbf{4'} & 4 & 4 & \\
 & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\
 & & & 4 & 4 & & & & \\
 & & & & 5' & & & &
\end{array}
\longrightarrow
\begin{array}{|c|c|c|c|c|c|c|c|c|}
1' & 1 & \overline{3}' & \mathbf{4'} & \bar{\mathbf{1}} & \bar{\mathbf{1}} & \overline{4} & \overline{4} & 5 \\
 & \overline{2}' & \overline{2} & \overline{3} & \overline{3} & \mathbf{3} & 4 & 4 & \\
 & & \overline{3} & 3 & \overline{4} & 5' & 5 & 5 & \\
 & & & 4 & 4 & & & & \\
 & & & & 5' & & & &
\end{array}
\tag{4.68}
$$

There are no $\overline{4}'$s. However, the 4th column contains the pair $\overline{3}\,3$ which must be replaced by $4'\,\overline{4}'$ under the action of $\chi_{4'}$:

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
1' & 1 & \overline{3}' & 4' & \overline{1} & \overline{1} & \overline{4} & \overline{4} & 5 \\
 & \overline{2}' & \overline{2} & \bar{\mathbf{3}} & \overline{3} & 3 & 4 & 4 & \\
 & & \overline{3} & \mathbf{3} & \overline{4} & 5' & 5 & 5 & \\
 & & & 4 & 4 & & & & \\
 & & & & 5' & & & &
\end{array}
\longrightarrow
\begin{array}{|c|c|c|c|c|c|c|c|c|}
1' & 1 & \overline{3}' & 4' & \overline{1} & \overline{1} & \overline{4} & \overline{4} & 5 \\
 & \overline{2}' & \overline{2} & \mathbf{4'} & \overline{3} & 3 & 4 & 4 & \\
 & & \overline{3} & \bar{\mathbf{4}}' & \overline{4} & 5' & 5 & 5 & \\
 & & & 4 & 4 & & & & \\
 & & & & 5' & & & &
\end{array}
\tag{4.69}
$$

There are no $\overline{5}'$s and it is then important to notice that one does not replace the vertical pair $\overline{4}\,4$ in the 5th column by $5'\,\overline{5}'$ under the action of $\chi_{5'}$ because one must first apply $\psi_{5'}$ to the two $5'$s. In any case the premature action of $\chi_{5'}$ would lead to two $5'$s in the same row, which is forbidden. Instead, the algorithm dictates that one first acts on the two $5'$s with $\psi_{5'}$. The uppermost $5'$ is moved into the 5th column and up that column until it is just below the entry $\overline{1}$ which cannot be moved into the second row. This leaves both the $\overline{3}$ and 4 in their own rows in the 5th column, and no $\overline{4}\,4$ pair.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline 1' & 1 & \overline{3}' & 4' & \overline{1} & \overline{1} & \overline{4} & \overline{4} & 5 \\
\hline \multicolumn{1}{c|}{} & \overline{2}' & \overline{2} & 4' & \mathbf{\overline{3}} & 3 & 4 & 4 & \multicolumn{1}{c}{} \\
\cline{2-8} \multicolumn{2}{c|}{} & \overline{3} & \overline{4}' & \mathbf{\overline{4}} & \mathbf{5'} & 5 & 5 & \multicolumn{1}{c}{} \\
\cline{3-8} \multicolumn{3}{c|}{} & 4 & 4 & \multicolumn{4}{c}{} \\
\cline{4-5} \multicolumn{4}{c|}{} & 5' & \multicolumn{4}{c}{} \\
\cline{5-5}
\end{array}
\longrightarrow
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline 1' & 1 & \overline{3}' & 4' & \overline{1} & \overline{1} & \overline{4} & \overline{4} & 5 \\
\hline \multicolumn{1}{c|}{} & \overline{2}' & \overline{2} & 4' & \mathbf{5'} & 3 & 4 & 4 & \multicolumn{1}{c}{} \\
\cline{2-8} \multicolumn{2}{c|}{} & \overline{3} & \overline{4}' & \mathbf{\overline{3}} & \mathbf{\overline{4}} & 5 & 5 & \multicolumn{1}{c}{} \\
\cline{3-8} \multicolumn{3}{c|}{} & 4 & 4 & \multicolumn{4}{c}{} \\
\cline{4-5} \multicolumn{4}{c|}{} & 5' & \multicolumn{4}{c}{} \\
\cline{5-5}
\end{array} \tag{4.70}$$

The second $5'$ does not move under the action of $\psi_{5'}$ since the 4 immediately above it cannot move into the 5th row. The final result can then be seen to be, as claimed, the juxtaposition of a primed tableaux $QD \in \mathcal{QD}^{54321}(5,\overline{5})$ and an unprimed tableaux $T \in \mathcal{T}^{433}(5,\overline{5})$:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline 1' & 1 & \overline{3}' & 4' & \overline{1} & \overline{1} & \overline{4} & \overline{4} & 5 \\
\hline \multicolumn{1}{c|}{} & \overline{2}' & \overline{2} & 4' & 5' & 3 & 4 & 4 & \multicolumn{1}{c}{} \\
\cline{2-8} \multicolumn{2}{c|}{} & \overline{3} & \overline{4}' & \overline{3} & \overline{4} & 5 & 5 & \multicolumn{1}{c}{} \\
\cline{3-8} \multicolumn{3}{c|}{} & 4 & 4 & \multicolumn{4}{c}{} \\
\cline{4-5} \multicolumn{4}{c|}{} & 5' & \multicolumn{4}{c}{} \\
\cline{5-5}
\end{array}
\equiv
\begin{array}{|c|c|c|c|c|}
\hline 1' & 1 & \overline{3}' & 4' & \overline{1} \\
\hline \multicolumn{1}{c|}{} & \overline{2}' & \overline{2} & 4' & 5' \\
\cline{2-5} \multicolumn{2}{c|}{} & \overline{3} & \overline{4}' & \overline{3} \\
\cline{3-5} \multicolumn{3}{c|}{} & 4 & 4 \\
\cline{4-5} \multicolumn{4}{c|}{} & 5' \\
\cline{5-5}
\end{array}
\cdot
\begin{array}{|c|c|c|c|}
\hline \overline{1} & \overline{4} & \overline{4} & 5 \\
\hline 3 & 4 & 4 & \multicolumn{1}{c}{} \\
\cline{1-3} \overline{4} & 5 & 5 & \multicolumn{1}{c}{} \\
\cline{1-3}
\end{array} \tag{4.71}$$

## 4.3 Corollaries

By associating $x_k$, $t^2\,\overline{x}_k$, $y_k$ and $t^2\,\overline{y}_k$ to each entry $k$, $\overline{k}$, $k'$ and $\overline{k}'$, respectively, in the various tableaux $QST$, $QD$ and $T$ appearing in Theorem 4.1 we immediately have the following corollary.

**Corollary 4.2.** *Let $\mu = \lambda + \delta$ be a strict partition of length $\ell(\mu) = n$, with $\lambda$ a partition of length $\ell(\lambda) \le n$ and $\delta = (n, n-1, \dots, 1)$.*

$$\sum_{QST \in \mathcal{QST}^{\mu}(n)} t^{2\mathrm{bar}(QST)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QST)} = \sum_{QD \in \mathcal{QD}^{\delta}(n)} t^{2\mathrm{bar}(QD)} (\mathbf{x}/\mathbf{y})^{\mathrm{wgt}(QD)} \sum_{T \in \mathcal{T}^{\lambda}(n)} t^{2\mathrm{bar}(T)} \mathbf{x}^{\mathrm{wgt}(T)}. \tag{4.72}$$

Thanks to the definition of $Q(\mathbf{x}/\mathbf{y};t)$ given in (2.30), the identity (2.23) and the combinatorial definition of the $t$-deformation $sp_\lambda(\mathbf{x};t)$ of symplectic characters given in (2.29), the above result is precisely our second main result Proposition 1.2.

Other corollaries follow as special cases of these results. Setting $\lambda = 0$ we obtain

$$Q_\delta(\mathbf{x}/\mathbf{y};t) = \prod_{1 \le i \le j \le n} \left(x_i + t^2\,\overline{x}_i + y_j + t^2\,\overline{y}_j\right). \tag{4.73}$$

On the other hand the case $\mathbf{y} = t\mathbf{x} = (tx_1, tx_2, \dots, tx_n)$ of (1.10) is equivalent to the $t$-deformation of Weyl's denominator formula (1.9) for the Lie algebra $sp(2n)$ derived elsewhere [4, 5] by much more circuitous means.

**Corollary 4.3.**

$$\prod_{i=1}^{n} (x_i + t\,\overline{x}_i) \prod_{1\le i<j\le n} (x_i + t^2\,\overline{x}_i + t\,x_j + t\,\overline{x}_j)\; sp_\lambda(\mathbf{x};t)$$
$$= \sum_{ST\in\mathcal{ST}^\mu(n,\overline{n})} t^{\mathrm{var}(ST)+\mathrm{bar}(ST)}(1+t)^{\mathrm{str}(ST)-n}\;\mathbf{x}^{\mathrm{wgt}(ST)}. \tag{4.74}$$

**Proof:** First it should be noted that

$$\prod_{1\le i\le j\le n} (x_i + t^2\,\overline{x}_i + t\,x_j + t\,\overline{x}_j)$$
$$= \prod_{i=1}^{n} (1+t)\,(x_i + t\,\overline{x}_i) \prod_{1\le i<j\le n} (x_i + t^2\,\overline{x}_i + t\,x_j + t\,\overline{x}_j)\,, \tag{4.75}$$

which includes a factor $(1+t)^n$.

Then it suffices to recognise that deleting primes from the entries $k'$ and $\overline{k}'$ in $QST \in \mathcal{QST}^\mu(n,\overline{n})$ gives a symplectic shifted tableau $ST \in \mathcal{ST}^\mu(n,\overline{n})$ with a factor of $t$ arising from each primed entry since $y_k = tx_k$ and $t^2 y_{\overline{k}}^{-1} = tx_{\overline{k}}^{-1}$. Additional factors of $t^2$ arise from each barred entry $k$ since these are associated with $t^2 x_{\overline{k}}^{-1}$. Compulsorily primed entries in each primed shifted tableau $QST$ corresponding to a fixed shifted tableau $ST$ appear once and only once in each row of each connected strip of identical entries, whether barred or unbarrred, while the leftmost lowest entry of each connected strip may each be primed or unprimed. To summarise, each connected component of a $k$-strip gives rise to a factor $(1+t)\,t^{\mathrm{row}_k-1}$, where $\mathrm{row}_k$ is the number of rows of the $k$-strip component, and each component of a $\overline{k}$-strip gives rise to a factor of $(1+t^{-1})\,t^{2\mathrm{bar}_{\overline{k}}-\mathrm{row}_{\overline{k}}+1} = (1+t)\,t^{\mathrm{bar}_{\overline{k}}+\mathrm{col}_{\overline{k}}-1}$ where $\mathrm{bar}_{\overline{k}}$ is the length of the component of the $\overline{k}$-strip, while $\mathrm{row}_{\overline{k}}$ and $\mathrm{col}_{\overline{k}}$ are the numbers of rows and columns, respectively, that it occupies, with $\mathrm{bar}_{\overline{k}} = \mathrm{row}_{\overline{k}} + \mathrm{col}_{\overline{k}} - 1$. Combining all these factors for $k = 1, 2, \ldots, n$ gives $(1+t)^{\mathrm{str}(ST)}\,t^{\mathrm{var}(ST)+\mathrm{bar}(ST)}$, as required, since as defined earlier $\mathrm{var}(ST) = \sum(\mathrm{row}_k + \mathrm{col}_{\overline{k}} - 1)$ where the sum is over all connected components of all $k$ and $\overline{k}$ strips, for all $k$, and $\mathrm{bar}(ST) = \sum \mathrm{bar}_{\overline{k}}$ is the total number of barred entries in $ST$. □

Another significant corollary involves a link with $U$-turn alternating sign matrices. This is provided in Section 5.

## 5 Connection to alternating sign matrices

### 5.1 $gl(n)$ case

In this section we show how to map from shifted tableaux, $ST$, to alternating sign matrices. Using the analogous relationship for primed shifted tableaux, $PST$, a result of Chapman [3] is a straightforward consequence of Theorem 3.1.

An alternating sign matrix (ASM) is an $n \times n$ matrix filled with $0'$s, $1'$s, and $-1'$s such that the first and last nonzero entries of each row and column are $1'$s and the nonzero entries within a row or column alternate in sign. There is a famous formula, conjectured by Mills, Robbins, and Rumsey [9] and proved by Zeilberger [19], that counts the number of ASM of size $n$ as $\prod_{j=0}^{n-1} (3j+1)!\big/(n+j)!$. See also Bressoud [1].

We work with a generalisation of ASM called $\mu$-ASM introduced by Okada [11] that can be associated with semistandard shifted tableaux. The new feature here is that the alternating sign matrix is no longer square. Its row sums are all 1 but its column sums are 1 or 0 according as the column label is or is not a part of some partition $\mu$. To be more precise, given a partition $\mu$ with distinct parts and such that $\ell(\mu) = n$ and $\mu_1 \le m$ for some $m \ge n$, the set $\mathcal{A}^\mu(n)$ of $\mu$-alternating sign matrices is the set of $n \times m$ matrices $A = (a_{iq})_{1\le i\le n, 1\le q\le m}$ that satisfy the following conditions:

ASM1 $a_{iq} \in \{-1, 0, 1\}$ for $1 \le i \le n$, $1 \le q \le m$;
ASM2 $\sum_{q=p}^{m} a_{iq} \in \{0, 1\}$ for $1 \le i \le n$, $1 \le p \le m$;
ASM3 $\sum_{i=j}^{n} a_{iq} \in \{0, 1\}$ for $1 \le j \le n$, $1 \le q \le m$
ASM4 $\sum_{q=1}^{m} a_{iq} = 1$ for $1 \le i \le n$;
ASM5 $\sum_{i=1}^{n} a_{iq} = 1$ if $q = \mu_j$ for some $j$; or $\sum_{i=1}^{n} a_{iq} = 0$ otherwise; for $1 \le q \le m$.

In what follows we also require $U$-turn alternating sign matrices, UASMs, and their generalisation $\mu$–UASMs that are associated with semistandard shifted symplectic tableaux [4, 5]. In fact the bijection between semistandard shifted tableaux $ST \in \mathcal{ST}^\mu(n)$ and $\mu$–ASMs $A \in \mathcal{A}^\mu(n)$ for any fixed $m \ge \mu_1$, is a special case of a bijection between semistandard shifted symplectic tableaux $ST \in \mathcal{ST}^\mu(n, \overline{n})$ and $\mu$–UASMs $A \in \mathcal{A}^\mu(n, \overline{n})$ [5] for the same $m$. In what follows we always fix $m = \mu_1$ so as to avoid redundant columns of zeros at the extreme right of each alternating sign matrix.

Briefly, in the $\mu$-ASM case, we associate to each semistandard shifted tableaux $ST \in \mathcal{ST}^\mu(n)$ of shape $\mu$ with $\ell(\mu) = n$ and $\mu_1 = m$ an $n \times m$ matrix $M(ST)$ filled with the entries from $ST$ together with zeros such that if there is an $i$ on diagonal $j$ of $ST$ (where the main diagonal is diagonal 1 and the last box in the first row is in diagonal $\mu_1 = m$), then there is an $i$ in row $i$, column $j$ of the matrix. All other entries are zero.

For example, in the case $n = 6$ and $\mu = (9, 8, 6, 4, 3, 1)$ a given semistandard shifted tableau $ST$ of shape $\mu$ yields a $6 \times 9$ matrix, $M(ST)$, as shown:

$$ST = \begin{array}{cccccccccccccc} 1&1&1&2&3&3&4&4&4&&&&& \\ &2&2&2&3&4&5&5&5&&&&& \\ &&3&4&4&4&5&6&&&&&& \\ &&&4&5&5&6&&&&&&& \\ &&&&5&6&6&&&&&&& \\ &&&&&6&&&&&&&& \end{array} \Longrightarrow M(ST) = \begin{bmatrix} 1&1&1&0&0&0&0&0&0 \\ 2&2&2&2&0&0&0&0&0 \\ 3&0&0&3&3&3&0&0&0 \\ 4&4&4&4&4&0&4&4&4 \\ 5&5&5&0&5&5&5&5&0 \\ 6&6&6&6&0&6&0&0&0 \end{bmatrix} \quad (5.76)$$

A primed semistandard shifted tableau $PST \in \mathcal{PST}^{\mu}(N)$ yields a similar matrix $M(PST)$ in the same way:

$$PST = \begin{array}{ccccccccc} 1 & 1 & 1 & 2' & 3' & 3 & 4 & 4 & 4 \\ & 2 & 2 & 2 & 3' & 4' & 5' & 5 & 5 \\ & & 3 & 4' & 4 & 4 & 5 & 6 & \\ & & & 4 & 5' & 5 & 6' & & \\ & & & & 5 & 6' & 6 & & \\ & & & & & 6 & & & \end{array} \Longrightarrow M(PST) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2' & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 3' & 3' & 3 & 0 & 0 & 0 \\ 4 & 4' & 4 & 4 & 4' & 0 & 4 & 4 & 4 \\ 5 & 5' & 5 & 0 & 5 & 5' & 5 & 5 & 0 \\ 6 & 6' & 6 & 6' & 0 & 6 & 0 & 0 & 0 \end{bmatrix} \tag{5.77}$$

where as we shall see it is possible to distinguish various types of entry 0 as characterised by their set of nearest non-vanishing neighbours.

Each of these matrices can be converted into a $\mu$-alternating sign matrix by replacing the rightmost entry of each continuous sequence of nonzero entries by a 1 and each zero immediately to the left of a nonzero entry by $-1$, leaving all other entries 0. In the case of the above example we obtain in this way

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{A}^{986431}(6) \tag{5.78}$$

Square ice provides a further refinement of the relationship between shifted tableaux and $\mu$-ASM. Square ice is a directed graph that models the orientation of oxygen and hydrogen molecules in frozen water. The vertices are laid out in an $n \times m$ grid and each vertex has two incoming and two outgoing edges in a north, south, east, west orientation. The square ice graph corresponding to (5.78) appears in Fig. 1.

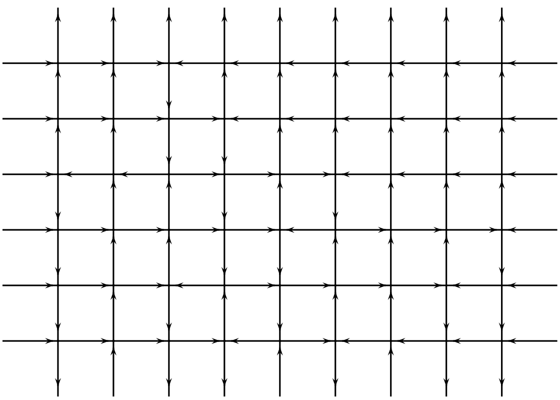
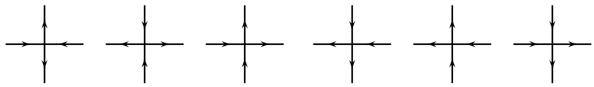


**Fig. 1** Square ice for Eq. (5.78)



$WE(+1)$ $NS(-1)$ $NE(0)$ $SW(0)$ $NW(0)$ $SE(0)$

**Fig. 2** Square ice and corresponding compass points and ASM entries. Figure adapted from Kuperberg [7]

At each vertex there are six possible orientations of the four directed edges (see Fig. 2). These orientations may be specified by the pairs of compass points giving the directions of the incoming edges. In this way the above square ice graph is specified by a corresponding "compass points" matrix:

$$CM = \begin{bmatrix} NE & NE & WE & NW & NW & NW & NW & NW & NW \\ NE & NE & SE & WE & NW & NW & NW & NW & NW \\ WE & NW & NS & SE & NE & WE & NW & NW & NW \\ SE & NE & NE & SE & WE & NS & NE & NE & WE \\ SE & NE & WE & NS & SE & NE & NE & WE & SW \\ SE & NE & SE & WE & NS & WE & NW & SW & SW \end{bmatrix} \tag{5.79}$$

The bijection between compass point matrices, square ice graphs and $\mu$-ASMs is provided by the following correspondences:

The horizontal orientation (with both horizontal edges directed in), $WE$, corresponds to each entry $+1$ in $A$, and the vertical orientation (with both vertical edges directed in), $NS$, corresponds to each entry $-1$ in $A$; the other four orientations, $NE$, $SW$, $NW$ and $SE$ correspond to the entries 0 in $A$. Accordingly there are northwest zeros (with edges pointing in the north and west directions), southwest zeros, northeast zeros, and southeast zeros. Northwest zeros are those whose nearest nonzero neighbour to the right, if it has one, is $-1$, and whose nearest nonzero neighbour below, if it has one, is 1. Southwest zeros are those whose nearest nonzero neighbour to the right, if it has one, is $-1$, and whose nearest nonzero neighbour below, if it has one, is $-1$. Northeast zeros are those whose nearest nonzero neighbour to the right is 1, and whose nearest nonzero neighbour below, if it has one, is 1. Southeast zeros are those whose nearest nonzero neighbour to the right is 1, and whose nearest nonzero neighbour below, if it has one, is $-1$.

The compass points matrices $CM$ can then be associated to the set of all primed shifted tableaux $PST$ that may be obtained by adding primes to the entries of the unprimed tableau $ST$. For example, the entries $NE$ in the $k$th row are associated with an entry $k$ in $PST$ and correspondingly to a weight factor $x_k$. The entries $SE$ in the $k$th row are associated with an entry $k'$ in $PST$ and correspondingly to a weight factor $y_k$. The entries $NS$ in the $k$th row are associated with the two possible labels $k$ and $k'$ of the first box of each connected component of $\mathrm{str}_k(PST)$ other than the one starting on the main diagonal. Correspondingly each $NS$ in row $k$ is associated with a weight factor $(x_k + y_k)$. It should be pointed out that the main diagonal is not included at all in the compass points matrix so that the first column corresponds to the second diagonal and indeed in general, column $k$ of $CM$ corresponds to diagonal $k+1$ of

$PST$. This implies that the above weighting excludes the weight $x_1x_2\cdots x_n$ arising from the entries $1,2,\ldots,n$ on the main diagonal of each $PST$.

Combining the weight factors we have a total weight associated with each $A\in\mathcal{A}^\mu(n)$ given by

$$\prod_{k=1}^{n} x_k^{NE_k(A)}\, y_k^{SE_k(A)}\, (x_k+y_k)^{NS_k(A)} \tag{5.80}$$

where $NE_k(A)$, $SE_k(A)$ and $NS_k(A)$ are the numbers of entries $NE$, $SE$ and $NS$ in the $k$th row of the compass matrix $CM(A)$ corresponding to $A$.

Thanks to the connection already made between $PST$s and weighted $ST$s, the following is then an immediate corollary of Proposition 1.1:

**Corollary 5.1.** *Let $\mu=\lambda+\delta$ be a strict partition of length $\ell(\mu)=n$, with $\lambda$ a partition of length $\ell(\lambda)\le n$ and $\delta=(n,n-1,\ldots,1)$. Then for all $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ we have*

$$\prod_{1\le i<j\le n}(x_i+y_j)\, s_\lambda(\mathbf{x}) = \sum_{A\in\mathcal{A}^\mu(n)}\prod_{k=1}^{n} x_k^{NE_k(A)} y_k^{SE_k(A)}(x_k+y_k)^{NS_k(A)}. \tag{5.81}$$

This generalises a result of Chapman [3]. In his original paper he weights by column instead of row so the parameters in his paper correspond to the transpose matrix. Now setting $\lambda=0$ so that $\mu=\delta$, and noting that $\mathcal{A}^\delta(n)=\mathcal{A}(n)$, the set of all $n\times n$ ASMs, we have

**Corollary 5.2** (Chapman [3]). *For all $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ and $\mathbf{y}=(y_1,y_2,\ldots,y_n)$ we have*

$$\prod_{1\le i<j\le n}(x_i+y_j) = \sum_{A\in\mathcal{A}(n)}\prod_{k=1}^{n} x_k^{NE_k(A)} y_k^{SE_k(A)}(x_k+y_k)^{NS_k(A)}. \tag{5.82}$$

Corollary 5.1 has a further consequence:

**Corollary 5.3.** *Let $\mu=\lambda+\delta$ be a strict partition of length $\ell(\mu)=n$, with $\lambda$ a partition of length $\ell(\lambda)\le n$ and $\delta=(n,n-1,\ldots,1)$. For any $m$ for which $m>n$ and $\mu_1\le m$, let $\mathcal{A}(n,m,\mu)\subseteq\mathcal{A}(m)$ be the subset consisting of those ASMs, $C$, whose top $n$ rows constitute an ASM, $A$, in $\mathcal{A}^\mu(n)$. Then for all $(x_1,\ldots,x_n,x_{n+1},\ldots,x_m)$ and $(y_1,\ldots,y_n,y_{n+1},\ldots,y_m)$ we have*

$$\begin{aligned}\prod_{1\le i<j\le n}(x_i+y_j)\;\; s_\lambda(x_1,\ldots,x_n)\prod_{n+1\le i<j\le m}(x_i+y_j)\;\; s_\kappa(y_{n+1},\ldots,y_m)\\ =\sum_{C\in\mathcal{A}(n,m,\mu)}\prod_{k=1}^{m} x_k^{NE_k(C)} y_k^{SE_k(C)}(x_k+y_k)^{NS_k(C)},\end{aligned} \tag{5.83}$$

*where $\kappa$ is the conjugate of the complement of $\lambda$ with respect to the rectangular partition $((m-n)^n)$, that is $\kappa=((m-n)^n/\lambda)'$.*

**Proof:** Let the top $n$ rows of $C$ and the bottom $(m-n)$ rows of $C$, reversed in order, form the matrices $A$ and $B$, respectively. Then the application of Corollary 5.1 to $A$ gives the contribution made by the top $n$ rows of each $C$ on the right hand side of (5.83) in the form of the first two factors on the left hand side. Similarly, the remaining two factors on the left hand side arise from the contribution of the bottom $(m-n)$ rows of each $C$. To see this one applies Corollary 5.1 to $B$, but this time with $\mu = \lambda + \delta$ replaced by $\nu = \kappa + \epsilon$, where $\epsilon = (m-n, m-n-1, \ldots, 1)$, and with $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ replaced by $(y_m, \ldots, y_{n+1})$ and $(x_m, \ldots, x_{n+1})$, respectively. It only remains to relate $\lambda$ and $\kappa$. Since the parts of $\mu$ and $\nu$ specify those columns of $A$ and $B$, respectively, whose column sums are 1, and $A$ and $B$ are constructed from an ASM $C$, all these parts must be distinct and together constitute $(m, m-1, \ldots, 1)$. It follows that the union of $\{\lambda_i + n - i + 1 \,|\, i = 1, \ldots, n\}$ and $\{\kappa_j + (m-n) - j + 1 \,|\, j = 1, \ldots, m-n\}$ must be $\{1, \ldots, m\}$. However, it is well known [8]p3 that the complement of $\{\lambda_i + n - i + 1 \,|\, i = 1, \ldots, n\}$ in $\{1, \ldots, m\}$ is $\{n + k - \lambda'_k \,|\, k = 1, \ldots, m-n\}$. By setting $k = m - n - j + 1$ it can then be seen that $\kappa_j = n - \lambda'_{m-n-j+1}$ for $j = 1, \ldots, m-n$, so that $\kappa = (n^{m-n}/\lambda' = ((m - n)^n/\lambda)'$, as claimed. □

Alternatively, Corollary 5.3 may be proved bijectively by taking each primed shifted tableau $PST$ specified by some $C \in \mathcal{A}(n, m, \mu)$ and using the jeu de taquin, first as described above, to move all entries $k'$ with $1 \le k \le n$ north-west to the $k$th column, and then in an analogous manner, to move all entries $k$ with $n+1 \le k \le m$ south-east to the $k$th row. The result is a pair of triangular subtableaux, both of type $PD$ but with all entries $k$ and $k'$ such that $k \le n$ in one case and $k > n$ in the other, together with a pair of semistandard tableaux, one of shape $\lambda$ with unprimed entries subject to the order relation $1 < 2 < \cdots < n$ and the other of shape $\kappa$ with primed entries subject to the order relation $m' < (m-1)' < \cdots < (n+1)'$.

By way of an example, let $m = 6$ and consider the following $6 \times 6$ ASM

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathcal{A}(6)\,. \tag{5.84}$$

Taking $n = 2$, the top two rows of $C$ constitute $A$, and the bottom four rows of $C$, reversed in order, constitute $B$, where:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{bmatrix} \in \mathcal{A}^{4,2}(2) \quad \text{and}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{A}^{6,5,3,1}(4). \tag{5.85}$$

The superscripts on $\mathcal{A}^{4,2}(2)$ and $\mathcal{A}^{6,5,3,1}(4)$ specify those columns of $A$ and $B$, respectively, having column sums 1. They indicate that $A \in \mathcal{A}(2,4,\mu)$ with $\mu = (4,2)$ so that $\lambda = (2,1)$, while $\nu = (6,5,3,1)$ so that $\kappa = (2,2,1)$. This is in accordance with the formula $\kappa = (4^2/\lambda)' = (3,2)' = (2,2,1)$.

The compass point matrix corresponding to $C$ takes the form:

$$CM = \begin{bmatrix} NE & NE & WE & NW & NW & NW \\ NE & WE & NS & WE & NW & NW \\ WE & NS & WE & SW & NW & NW \\ SE & NE & SE & SE & WE & NW \\ SE & WE & SW & SW & NS & WE \\ SE & SE & SE & SE & WE & SW \end{bmatrix} \tag{5.86}$$

so that the contribution of $C$ to the right hand side of (5.83) is

$$x_1^2\, x_2\, (x_2 + y_2)\, (x_3 + y_3)\, x_4\, y_4^3\, y_5\, (x_5 + y_5)\, y_6^4. \tag{5.87}$$

The three factors $(x_k + y_k)$ arise from three entries $NS$ in $CM$, that themselves arise from the three entries $-1$ in $C$. There must therefore be precisely $2^3$ primed shifted tableaux $PST$ corresponding to $C$. Choosing just one of these for illustrative purposes, the use of the jeu de taquin to move all $k'$s with $k \le 2$ north-west and all $k$s with $k > 2$ south-east, gives the following bijective map:

$$PST = \begin{array}{cccccc} 1 & 1 & 1 & 2' & 4' & 5 \\ & 2 & 2 & 3 & 4' & 6' \\ & & 3 & 4' & 4 & 6' \\ & & & 4 & 5' & 6' \\ & & & & 5 & 6' \\ & & & & & 6 \end{array} \longleftrightarrow \begin{array}{cccccc} 1 & 2' & 1 & 1 & 4' & 6' \\ & 2 & 2 & 4' & 5' & 6' \\ & & 3 & 4' & 3 & 6' \\ & & & 4 & 4 & 6' \\ & & & & 5 & 5 \\ & & & & & 6 \end{array} \tag{5.88}$$

The corresponding contribution to the left hand side of (5.83) is then given by

$$x_1^2\, x_2\, y_2\, x_3\, x_4\, y_4^3\, x_5\, y_5\, y_6^4 = y_2 \cdot x_1^2\, x_2 \cdot x_3\, x_4\, y_4\, x_5\, y_6^2 \cdot y_4^2\, y_5\, y_6^2, \tag{5.89}$$

where the arrangement of the terms on the right exhibits the contributions to each of the four factors constituting the left hand side of (5.83). Both tableaux in (5.88) may be displayed, as shown below, in terms of suitably re-oriented subtableaux involving entries $k$ and $k'$, with all $k \le 2$ in one case, and all $k > 2$ on the other.

$$\begin{array}{cccc} 1 & 1 & 1 & 2' \\ & 2 & 2 & \end{array} \cdot \begin{array}{cccccc} 6 & 6' & 6' & 6' & 6' & 5 \\ & 5 & 5' & 4 & 4' & 4' \\ & & 4 & 4' & 3 & \\ & & & 3 & & \end{array} \longleftrightarrow \begin{array}{cc} 1 & 2' \\ & 2 \end{array} \cdot \begin{array}{cc} 1 & 1 \\ 2 & \end{array} \cdot \begin{array}{cccc} 6 & 5 & 6' & 6' \\ & 5 & 4 & 3 \\ & & 4 & 4' \\ & & & 3 \end{array} \cdot \begin{array}{cc} 6' & 6' \\ 5' & 4' \\ 4' & \end{array} \tag{5.90}$$

This illustrates the outcome of applying the jeu de taquin to primed shifted tableaux corresponding to the submatrices $A$ and $B$ of the ASM $C$. The resulting contribution of the four final tableaux to the left hand side of (5.83) is then confirmed to be as given on the right hand side of (5.89).

### 5.2 $sp(2n)$ case

The symplectic case involves a modified alternating sign matrix called a $U$-turn $\mu$-ASM or $\mu$-UASM. Informally, the $U$-turn condition means that two consecutive rows and the $U$-turn between them must follow the $\mu$-ASM summation rules, ASM2-5; that is, the cumulative sum must be zero or one, and the total sum must be one. These $\mu$-UASM were first defined in Hamel and King [4] where they were called sp$(2n)$-generalised alternating sign matrices. They are discussed at length in Hamel and King [5]. A formal definition is as follows:

Let $\mu$ be a partition of length $\ell(\mu)=n$, all of whose parts are distinct, and for which $\mu_1 \le m$. Then the matrix $A=(a_{iq})_{1\le i\le 2n, 1\le q\le m}$ is said to belong to the set $\mathcal{UA}^{\mu}(2n)$ of $\mu$-alternating sign matrices with a U–turn boundary if its elements $a_{iq}$ satisfy the conditions:

| | | |
|---|---|---|
| UA1 | $a_{iq} \in \{-1,0,1\}$ | for $1\le i\le 2n$, $1\le q\le m$; |
| UA2 | $\sum_{q=p}^{m} a_{iq} \in \{0,1\}$ | for $1\le i\le 2n$, $1\le p\le m$; |
| UA3 | $\sum_{i=j}^{2n} a_{iq} \in \{0,1\}$ | for $1\le j\le 2n$, $1\le q\le m$. |
| UA4 | $\sum_{q=1}^{m}(a_{2i-1,q}+a_{2i,q})=1$ | for $1\le i\le n$; |
| UA5 | $\sum_{i=1}^{2n} a_{iq} = \begin{cases} 1 & \text{if } q=\mu_k \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$ | for $1\le q\le m$, $1\le k\le n$. |

Again so as to avoid redundant columns of zeros at the extreme right of $A$ we set $m=\mu_n$ in what follows.

In the case $\mu=\delta=(n,n-1,\dots,1)$ and $m=n$, for which UA5 becomes $\sum_{i=1}^{2n} a_{iq}=1$ for $1\le q\le n$, this definition is such that the set of $\mu$-UASM coincides with the set of U–turn alternating sign matrices, UASMs, defined by Kuperberg [7].

As noted above, Hamel and King [5] established a bijection between $\mu$-UASM and semistandard shifted symplectic tableaux. An example of this association is illustrated below in the case $n=5$ and $\mu=(9,7,6,2,1)$:

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\bar{1} & 1 & \bar{2} & 2 & \bar{3} & \bar{3} & \bar{4} & 4 & 5 \\
\hline
\multicolumn{1}{c|}{} & \bar{2} & \bar{2} & 2 & 3 & \bar{4} & \bar{4} & 4 & \multicolumn{1}{c}{} \\
\cline{2-8}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & \bar{4} & 4 & 4 & 4 & 4 & \multicolumn{1}{c}{} \\
\cline{3-8}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{4-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \bar{5} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{5-5}
\end{array}
\Longrightarrow
\begin{bmatrix}
\bar{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{2} & \bar{2} & \bar{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{3} & \bar{3} & 0 & 0 & 0 \\
3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{4} & 0 & 0 & \bar{4} & \bar{4} & \bar{4} & 0 & 0 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 0 \\
\bar{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{bmatrix}
$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{5.91}$$

where the columns are labeled from left to right $1, 2, \ldots, 9 = m = \mu_1$, and the rows from top to bottom $\overline{1}, 1, \overline{2}, 2, \ldots, \overline{5}, 5 = n$.

The translation to square ice is also natural and just requires a modification of the left boundary by the insertion of a $U$-turn. The square ice graph in Fig. 3 corresponds to the above $\mu$-UASM matrix. The same example appeared in Hamel and King [5].

In this symplectic case the bijection from the $U$-turn $\mu$-ASMs to U–turn square ice graphs is precisely as before, with entries $+1$ and $-1$ mapped to $WE$ and $NS$ vertex orientations, and $NE$, $SW$, $NW$ and $SE$ entries 0 distinguished by their nearest non-zero neighbouring entries. This map is encoded in the corresponding compass points

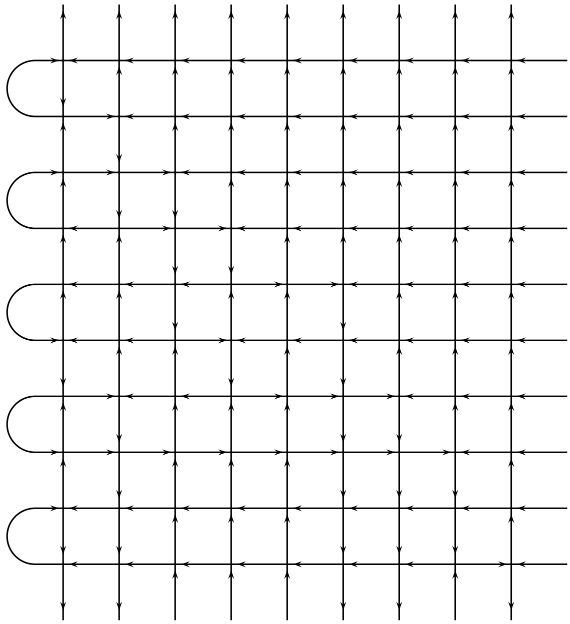


**Fig. 3** Square ice with $U$-turn boundary

matrix. For the above example, this takes the form:

$$CM = \begin{bmatrix} WE & NW & NW & NW & NW & NW & NW & NW & NW \\ NS & WE & NW & NW & NW & NW & NW & NW & NW \\ NE & SE & WE & NW & NW & NW & NW & NW & NW \\ NW & NS & SE & WE & NW & NW & NW & NW & NW \\ NW & NW & SW & NS & NE & WE & NW & NW & NW \\ WE & NW & NS & WE & NW & SW & NW & NW & NW \\ NS & WE & NW & NS & NE & SE & WE & NW & NW \\ NE & SE & NE & NE & NE & SE & SE & WE & NW \\ WE & SW & NW & NW & NW & SW & SW & SW & NW \\ SW & SW & NW & NW & NW & SW & SW & NS & WE \end{bmatrix} \tag{5.92}$$

Then we can generate a weighting in the same manner as for the $gl(n)$ case, with the $k$th column of $CM$ corresponding to the $(k+1)$th diagonal of $QST$. In this case we have unbarred entries corresponding to even rows and barred entries corresponding to odd rows. An entry $NE$ in row $k$ is associated to an entry $k$ in $QST$ and correspondingly to a weight factor $x_k$. An entry $SE$ in row $k$ is associated to an entry $k'$ in $QST$ and correspondingly to a weight factor $y_k$. An entry $NE$ in row $\overline{k}$ is associated to an entry $\overline{k}$ in $QST$ and correspondingly to a weight factor $t^2 x_k^{-1}$. An entry $SE$ in row $\overline{k}$ is associated to an entry $\overline{k}'$ in $QST$ and correspondingly to $t^2 y_k^{-1}$. An entry $NS$ in row $k$ is associated to the two possible labels $k$ or $k'$ of the first box of each connected component of $\mathrm{str}_k(QST)$ (other than one starting on the main diagonal) and correspondingly to a weight $(x_k + y_k)$, while an entry $NS$ in row $\overline{k}$ is associated to the two possible labels $\overline{k}$ or $\overline{k}'$ of the first box of each connected component of $\mathrm{str}_{\overline{k}}(QST)$ (other than one starting on the main diagonal) and correspondingly to a weight $t^2(\overline{x}_k + \overline{y}_k)$. Combining the weight factors we have

$$\prod_{k=1}^{n} x_k^{NE_k(A)} \big(t^2\overline{x}_k\big)^{NE_{\overline{k}}(A)} y_k^{SE_k(A)} \big(t^2\overline{y}_k\big)^{SE_{\overline{k}}(A)} \big(x_k+y_k\big)^{NS_k(A)} \big(t^2\overline{x}_k+t^2\overline{y}_k\big)^{NS_{\overline{k}}(A)} \tag{5.93}$$

where $SE_k(A)$, $NE_k(A)$, $NS_k(A)$ (resp. $SE_{\overline{k}}(A)$, $NE_{\overline{k}}(A)$, $NS_{\overline{k}}(A)$) are the numbers of entries $SE$, $NE$, $NS$ in row $k$ (resp. $\overline{k}$) of the compass matrix $CM(A)$.

We then have the following immediate corollary of Proposition 1.2:

**Corollary 5.4.** *Let $\mu = \lambda + \delta$ be a strict partition of length $\ell(\mu) = n$, with $\lambda$ a partition of length $\ell(\lambda) \le n$ and $\delta = (n, n-1, \ldots, 1)$. Then for all $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ we have*

$$\begin{aligned} &\prod_{1\le i<j\le n} \big(x_i + t^2\overline{x}_i + y_j + t^2\overline{y}_j\big) sp_\lambda(\mathbf{x};t) \\ &\quad = \sum_{A\in\mathcal{UA}(2n)} \prod_{k=1}^{n} x_k^{NE_k(A)} \big(t^2\overline{x}_k\big)^{NE_{\overline{k}}(A)} y_k^{SE_k(A)} \big(t^2\overline{y}_k\big)^{SE_{\overline{k}}(A)} \\ &\qquad \times (x_k + y_k)^{NS_k(A)} \big(t^2\overline{x}_k + t^2\overline{y}_k\big)^{NS_{\overline{k}}(A)}. \end{aligned} \tag{5.94}$$

This Corollary is a generalisation of Theorem 6.4 of Hamel and King [5]. This Theorem 6.4 may be recovered from Corollary 5.4 by setting $\mathbf{y} = t\mathbf{x}$, exploiting the bijection between compass point matrices $CM(A)$ and the U-turn $\mu$-ASM's $A$, and noting that the number of entries $NS$ and $WE$ in any row of $CM$ are either the same or differ by one according to the nature, barred or unbarred, of the corresponding entry on the main diagonal of the associated semistandard shifted symplectic tableau $ST$. Note also that Theorem 6.4 includes the weighting for the main diagonal on each side of the equation, whereas Corollary 5.4 does not.

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