J Algebr Comb (2007) 25:43–56
DOI 10.1007/s10801-006-0022-7

The correlation functions of vertex operators and Macdonald polynomials

Shun-Jen Cheng · Weiqiang Wang

Received: 25 October 2005 / Accepted: 26 April 2006 /
Published online: 11 July 2006

Abstract The n-point correlation functions introduced by Bloch and Okounkov have already found several geometric connections and algebraic generalizations. In this note we formulate a q,t-deformation of this n-point function. The key operator used in our formulation arises from the theory of Macdonald polynomials and affords a vertex operator interpretation. We obtain closed formulas for the n-point functions when $n = 1, 2$ in terms of the basic hypergeometric functions. We further generalize the q,t-deformed n-point function to more general vertex operators.

Keywords Correlation functions · Macdonald polynomials · Vertex operators · Hypergeometric series

1. Introduction

In [1] Bloch and Okounkov formulated an n-point correlation function on a Fock space and established a remarkable closed formula in terms of theta functions (also cf. [10]). Recently, this n-point function has found geometric connections in terms of Gromov-Witten theory [11] and Hilbert schemes of points [7], and it also affords several other algebraic generalizations (cf. [2, 9, 12]). The formulation in [1, 10] boils down to a remarkable operator $T(t)$ on the ring of symmetric functions which diagonalizes the Schur functions with explicit eigenvalues.

In this note we formulate a deformed version of the n-point functions of Bloch-Okounkov, denoted by $\widehat{F}(q_1, t_1; \ldots; q_n, t_n)$, which also depends on an additional

S.-J. Cheng
Institute of Mathematics, Academia Sinica, Taipei, Taiwan 115
e-mail: chengsj@math.ntu.edu.tw

W. Wang (✉)
Department of Mathematics, University of Virginia, Charlottesville, VA 22904
e-mail: ww9c@virginia.edu

indeterminate v associated to the energy operator. The role of $T(t)$ is replaced by an operator $\widehat{\mathfrak{B}}_{q,t}$ (cf. Garsia-Haiman [4]; see Section 2) which diagonalizes the modified Macdonald polynomials $\tilde{H}_\lambda(q,t)$ and affords a vertex operator interpretation. In Section 3 we compute the 1-point function as

$$\widehat{F}(q,t) = \frac{(vqt)_\infty}{(t)_\infty (q)_\infty}$$

where $(a)_\infty := \prod_{i=0}^{\infty}(1-av^i)$. We further found closed formulas for the 2-point functions in terms of basic hypergeometric series (Theorem 9).

From the viewpoint of vertex operators, it is also possible to further generalize the notion of the n-point function above, and we compute explicitly some cases in Section 4 (Theorems 13 and 16). We end this note in Section 5 with a discussion of open problems and possible connections.

2. Formulation of the *n*-point functions

2.1. The operators $\mathfrak{B}_{q,t}$ and $\widehat{\mathfrak{B}}_{q,t}$

Let t, q be two indeterminates. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n, we denote by $a'(\square)$ and $l'(\square)$ the *coarm* and *coleg* of a given cell $\square$ [4, 8], and denote

$$B_\lambda(q,t) := \sum_{\square \in \lambda} q^{a'(\square)} t^{l'(\square)}.$$

We set

$$\widehat{B}_\lambda(q,t) := \frac{1}{1-q} \sum_{i \geq 1} t^{i-1} q^{\lambda_i}.$$

Lemma 1. *We have*

$$B_\lambda(q,t) = \widehat{B}_\emptyset(q,t) - \widehat{B}_\lambda(q,t)$$

$$\widehat{B}_{\lambda(q,t)} \qquad \textit{where } \widehat{B}_\emptyset(q,t) = \frac{1}{(1-q)(1-t)}.$$

Proof: We calculate that

$$\begin{aligned} B_\lambda(q,t) &= \sum_{i \geq 1} \frac{t^{i-1}(1-q^{\lambda_i})}{1-q} \\ &= \frac{1}{(1-q)(1-t)} - \sum_{i \geq 1} \frac{t^{i-1} q^{\lambda_i}}{1-q} \\ &= \widehat{B}_\emptyset(q,t) - \widehat{B}_\lambda(q,t). \end{aligned}$$

□

Note that

$$B_\lambda(q,t) = B_{\lambda'}(t,q), \qquad \widehat{B}_\lambda(q,t) = \widehat{B}_{\lambda'}(t,q). \tag{1}$$

Denote by $\Lambda_{q,t}$ the ring of symmetric functions with coefficients in $\mathbb{Q}(q,t)$. Recall the Macdonald symmetric functions $P_\lambda(x;q,t)$, $Q_\lambda(x;q,t)$ from [8] and its normalized form $J_\lambda(x;q,t)$, $H_\lambda(x;q,t)$, and $\widetilde{H}_\lambda(x;q,t)$ as in [4, (8)–(11)]. We define the linear operators $\mathfrak{B}_{q,t}$ and $\widehat{\mathfrak{B}}_{q,t}$ on $\Lambda_{q,t}$ (cf. [4, (73), 74)]) by letting

$$\begin{aligned}
\mathfrak{B}_{q,t}\widetilde{H}_\lambda(x;\,q,t) &= B_\lambda(q,t)\widetilde{H}_\lambda(x;\,q,t),\\
\widehat{\mathfrak{B}}_{q,t}\widetilde{H}_\lambda(x;\,q,t) &= \widehat{B}_\lambda(q,t)\widetilde{H}_\lambda(x;\,q,t), \quad \text{for all } \lambda.
\end{aligned}$$

2.2. The definition of n-point correlation functions

Let v be an indeterminate. For our purposes we can also think of v as a complex number with $|v| < 1$. For $r \geq 1$ we set

$$(a)_0 := 1; \quad (a)_r := \prod_{i=0}^{r-1}(1-av^i); \quad (a)_\infty := \prod_{i=0}^{\infty}(1-av^i).$$

The *energy operator* L_0 on $\Lambda_{q,t}$ is the linear operator such that $L_0 g = ng$ for every n and every symmetric function g of degree n. Given $\mathfrak{f} \in \mathrm{End}(\Lambda_{q,t})$, we consider the trace function

$$\mathrm{Tr}_v \mathfrak{f} := \mathrm{Tr}(v^{L_0}\mathfrak{f}).$$

In particular for the identity map I we have

$$\mathrm{Tr}_v I = (v)_\infty^{-1}.$$

The *n-point (correlation) functions* are defined to be

$$\begin{aligned}
F(q_1,t_1;\ldots;q_n,t_n) &:= \mathrm{Tr}_v(\mathfrak{B}_{q_1,t_1}\cdots\mathfrak{B}_{q_n,t_n}),\\
\widehat{F}(q_1,t_1;\ldots;q_n,t_n) &:= \mathrm{Tr}_v(\widehat{\mathfrak{B}}_{q_1,t_1}\cdots\widehat{\mathfrak{B}}_{q_n,t_n}).
\end{aligned}$$

We can easily convert between F and $\widehat{F}$ by Lemma 1.

There is yet another viewpoint. Let $\mathcal{P}$ be the set of all partitions, and let $f(\lambda)$ be a function on $\mathcal{P}$. We define the *v-expectation value* of f to be

$$\langle f\rangle_v := (v)_\infty \sum_{\lambda\in\mathcal{P}} f(\lambda)v^{|\lambda|},$$

assuming its convergence.

Lemma 2. *We have*

$$F(q_1, t_1; \dots, q_n, t_n) = (v)_\infty^{-1} \left\langle \prod_{k=1}^{n} B_\lambda(q_k, t_k) \right\rangle_v .$$

The same relation holds with F and B replaced by $\widehat{F}$ and $\widehat{B}$.

Proof: Note that the operators $\mathfrak{B}_{q_k,t_k}$ for different k do not commute. Let $\{s_\lambda\}$ be the Schur functions, cf. [8], and write $s_\mu = \sum_\lambda a^{(i)}_{\lambda,\mu} \widetilde{H}_\lambda(x; q_i, t_i)$ with $[a^{(i)}_{\lambda,\mu}]$ being a triangular matrix with respect to the dominance order. Then

$$\mathfrak{B}_{q_i,t_i} s_\lambda = B_\lambda(q_i, t_i) s_\lambda + \text{ lower terms},$$

and thus

$$\mathfrak{B}_{q_1,t_1} \cdots \mathfrak{B}_{q_n,t_n} s_\lambda = B_\lambda(q_1, t_1) \cdots B_\lambda(q_n,\ t_n) s_\lambda + \text{ lower terms}.$$

Therefore,

$$\mathrm{Tr}_v(\mathfrak{B}_{q_1,t_1} \cdots \mathfrak{B}_{q_n,t_n}) = \sum_\lambda B_\lambda(q_1, t_1) \cdots B_\lambda(q_n, t_n) v^{|\lambda|}.$$

□

Thanks to (1) and Lemma 2, we see that F and $\widehat{F}$ are symmetric with respect to the hyperoctahedral group $\mathbb{Z}_2^n \rtimes S_n$, where the symmetric group S_n permutes the indices i in the pairs (q_i, t_i) and the i-th copy of $\mathbb{Z}_2$ permutes q_i and t_i.

Remark 3. When $t = q^{-1}$, $\widehat{F}$ reduces to (up to a normalization) the n-point functions introduced by Bloch and Okounkov [1], where the interpretation as a v-expectation value was also made.

3. The formulas for *n*-point functions

3.1. The 1-point function

Lemma 4 ([1, Lemma 6.6]). *For a given $i \geq 1$, we have*

$$\langle q^{\lambda_i} \rangle_v = \frac{(v^i)_\infty}{(v^i q)_\infty}.$$

Proof: By conjugation symmetry of partitions and $\lambda'_i = \#\{k | \lambda_k \geq i\}$, we have

$$\langle q^{\lambda_i} \rangle_v = \langle q^{\lambda'_i} \rangle_v = \frac{(v)_\infty}{(v)_{i-1} (v^i q)_\infty} = \frac{(v^i)_\infty}{(v^i q)_\infty}.$$

□

We will use for several times the so-called q-binomial theorem (cf. [5, Appendix II.3]):

$$\sum_{r=0}^{\infty} t^r \frac{(a)_r}{(v)_r} = \frac{(at)_\infty}{(t)_\infty}, \qquad |t| < 1.$$

Theorem 5. *The* 1*-point function is given by:*

$$\widehat{F}(q,t) = \frac{(vqt)_\infty}{(q)_\infty (t)_\infty}.$$

Proof: We calculate by Lemma 4 and the q-binomial theorem that

$$\begin{aligned}\left\langle \widehat{B}_\lambda(q,t)\right\rangle_v &= (1-q)^{-1} \sum_{i=1}^{\infty} t^{i-1} \frac{(v^i)_\infty}{(v^i q)_\infty} \\ &= \frac{(v)_\infty}{(q)_\infty} \sum_{r=0}^{\infty} t^r \frac{(vq)_r}{(v)_r} = \frac{(v)_\infty (vqt)_\infty}{(q)_\infty (t)_\infty}.\end{aligned}$$

By Lemma 2, this gives rise to the 1-point function $\widehat{F}(q,t)$. □

Remark 6. When $t = q^{-1}$, Theorem 5 specializes to [1, Theorem 6.5].

3.2. The 2-point function

We begin with some preparation.

Lemma 7. *For fixed* $1 \le i < j$, *we have*

$$\left\langle q_1^{\lambda_i} q_2^{\lambda_j}\right\rangle_v = \frac{(v)_\infty}{(v)_{i-1}(v^i q_1)_{j-i}(v^j q_1 q_2)_\infty} = \frac{(v)_\infty}{(vq_1q_2)_\infty} \frac{(vq_1)_{i-1}(vq_1q_2)_{j-1}}{(v)_{i-1}(vq_1)_{j-1}}.$$

Proof: This is a variant of a special case of [1, (7.1)]. Similar to Lemma 4, it follows directly from $\langle q_1^{\lambda_i} q_2^{\lambda_j}\rangle_v = \langle q_1^{\lambda'_i} q_2^{\lambda'_j}\rangle_v$. □

Set

$$\begin{aligned}
T_1 &:= \frac{(v)_\infty}{(vq_1q_2)_\infty} \sum_{i=0}^{\infty} t_1^i \frac{(vq_1)_i}{(v)_i} \sum_{j=i+1}^{\infty} t_2^j \frac{(vq_1q_2)_j}{(vq_1)_j} \\
T_2 &:= \frac{(v)_\infty}{(vq_1q_2)_\infty} \sum_{i=0}^{\infty} t_2^i \frac{(vq_2)_i}{(v)_i} \sum_{j=i+1}^{\infty} t_1^j \frac{(vq_1q_2)_j}{(vq_2)_j} \\
T_3 &:= \frac{(v)_\infty}{(vq_1q_2)_\infty} \sum_{i=0}^{\infty} (t_1t_2)^i \frac{(vq_1q_2)_i}{(v)_i}.
\end{aligned}$$

Lemma 8. *We have*

$$\langle \widehat{B}_\lambda(q_1, t_1)\widehat{B}_\lambda(q_2, t_2)\rangle_v = (1-q_1)^{-1}(1-q_2)^{-1}(T_1 + T_2 + T_3).$$

Proof: By definition, we have

$$\begin{aligned}
&\langle \widehat{B}_\lambda(q_1, t_1)\widehat{B}_\lambda(q_2, t_2)\rangle_v \\
&\quad = (1-q_1)^{-1}(1-q_2)^{-1}\left\langle \sum_{i,j=1}^{\infty} t_1^{i-1} q_1^{\lambda_i} t_2^{j-1} q_2^{\lambda_j}\right\rangle_v \\
&\quad = (1-q_1)^{-1}(1-q_2)^{-1} \sum_{i,j=1}^{\infty} t_1^{i-1} t_2^{j-1} \langle q_1^{\lambda_i} q_2^{\lambda_j}\rangle_v \\
&\quad = (1-q_1)^{-1}(1-q_2)^{-1}\left(\sum_{i<j} + \sum_{i>j} + \sum_{i=j}\right) t_1^{i-1} t_2^{j-1} \langle q_1^{\lambda_i} q_2^{\lambda_j}\rangle_v
\end{aligned}$$

where the last three summands can be further identified with T_1, T_2 and T_3, respectively, using Lemmas 4 and 7. □

For $r \geq 0, a_1, \ldots, a_{r+1} \in \mathbb{C}$ and $b_1, \ldots, b_r \in \mathbb{C}$ the $(r+1, r)$-basic hypergeometric series is the series:

$${}_{r+1}\Phi_r\left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; v; z\right) := \sum_{m \geq 0} \frac{(a_1)_m (a_2)_m \cdots (a_{r+1})_m}{(v)_m (b_1)_m \cdots (b_r)_m} z^m.$$

It is assumed that the denominator is never zero, in which case it is known to converge absolutely for $|z| < 1$ (cf. [5]).

Theorem 9. *The* 2*-point function* $\widehat{F}(q_1, t_1; q_2, t_2)$ *is equal to*

$$\begin{aligned}
&\frac{1}{(1-q_1)(1-q_2)(1-t_1t_2)} \cdot \frac{(vq_1q_2t_1t_2)_\infty}{(vt_1t_2)_\infty (vq_1q_2)_\infty} \\
&\quad \cdot\left[\frac{q_1q_2t_1t_2 - 1}{(1-q_1t_1)(1-q_2t_2)} + \frac{1}{1-q_1t_1}\, {}_3\Phi_2\left(\begin{matrix} v, q_1t_1, vq_1q_2 \\ vq_1, vq_1q_2t_1t_2 \end{matrix} ; v; t_2\right)\right. \\
&\quad \left. + \frac{1}{1-q_2t_2}\, {}_3\Phi_2\left(\begin{matrix} v, q_2t_2, vq_1q_2 \\ vq_2, vq_1q_2t_1t_2 \end{matrix} ; v; t_1\right)\right].
\end{aligned}$$

Theorem 10. *If $q_1q_2t_1t_2 = 1$, then the 2-point function $\widehat{F}(q_1, t_1; q_2, t_2)$ is equal to*

$$\frac{1}{(1-q_1)(1-q_2)(1-t_1t_2)} \cdot \frac{(v)_\infty}{(vt_1t_2)_\infty (vq_1q_2)_\infty} \cdot \\ \cdot \left[\frac{1}{1-q_1t_1} \frac{\left(vt_1^{-1}\right)_\infty \left(q_2^{-1}\right)_\infty}{(vq_1)_\infty (t_2)_\infty} + \frac{1}{1-q_2t_2} \frac{\left(vt_2^{-1}\right)_\infty \left(q_1^{-1}\right)_\infty}{(vq_2)_\infty (t_1)_\infty} \right]$$

Proof of Theorems 9 and 10: To compute the 2-point function it suffices to compute the T_i by Lemma 8. First of all,

$$\begin{aligned} T_1 &= \frac{(v)_\infty}{(vq_1)_\infty} \sum_{i=0}^{\infty} t_1^i \frac{(vq_1)_i}{(v)_i} \sum_{j=i+1}^{\infty} t_2^j \frac{(v^{j+1}q_1)_\infty}{(v^{j+1}q_1q_2)_\infty} \\ &= \frac{(v)_\infty}{(vq_1)_\infty} \sum_{i=0}^{\infty} t_1^i \frac{(vq_1)_i}{(v)_i} \sum_{j=i+1}^{\infty} t_2^j \sum_{m=0}^{\infty} \frac{(q_2^{-1})_m}{(v)_m} (v^{j+1}q_1q_2)^m \\ &= \frac{(v)_\infty}{(vq_1)_\infty} \sum_{i=0}^{\infty} t_1^i \frac{(vq_1)_i}{(v)_i} \sum_{m=0}^{\infty} \frac{(q_2^{-1})_m}{(v)_m} (vq_1q_2)^m \sum_{j=i+1}^{\infty} t_2^j v^{jm} \\ &= \frac{(v)_\infty}{(vq_1)_\infty} \sum_{i=0}^{\infty} t_1^i \frac{(vq_1)_i}{(v)_i} \sum_{m=0}^{\infty} \frac{(q_2^{-1})_m}{(v)_m} (vq_1q_2)^m \frac{(t_2v^m)^{i+1}}{1-t_2v^m} \\ &= \frac{(v)_\infty}{(vq_1)_\infty} \sum_{m=0}^{\infty} \frac{\left(q_2^{-1}\right)_m}{(v)_m} (vq_1q_2)^m \frac{t_2v^m}{1-t_2v^m} \sum_{i=0}^{\infty} (t_1t_2v^m)^i \frac{(vq_1)_i}{(v)_i} \\ &= \frac{(v)_\infty}{(vq_1)_\infty} \sum_{m=0}^{\infty} \frac{\left(q_2^{-1}\right)_m}{(v)_m} (vq_1q_2)^m \frac{t_2v^m}{1-t_2v^m} \frac{(v^{m+1}q_1t_1t_2)_\infty}{(v^m t_1t_2)_\infty} \\ &= \frac{(v)_\infty (vq_1t_1t_2)_\infty}{(vq_1)_\infty (t_1t_2)_\infty} \sum_{m=0}^{\infty} \frac{\left(q_2^{-1}\right)_m (t_1t_2)_m}{(v)_m (vq_1t_1t_2)_m} (v^2q_1q_2)^m \frac{t_2}{1-t_2v^m} \\ &= \frac{t_2}{1-t_2} \frac{(v)_\infty (vq_1t_1t_2)_\infty}{(vq_1)_\infty (t_1t_2)_\infty} \sum_{m=0}^{\infty} \frac{\left(q_2^{-1}\right)_m (t_1t_2)_m (t_2)_m}{(v)_m (vq_1t_1t_2)_m (vt_2)_m} (v^2q_1q_2)^m. \end{aligned}$$

Thus we obtain that

$$T_1 = \frac{t_2}{1-t_2} \frac{(v)_\infty (vq_1t_1t_2)_\infty}{(vq_1)_\infty (t_1t_2)_\infty} \cdot {}_3\Phi_2 \begin{pmatrix} t_1t_2,\, t_2,\, q_2^{-1} & ; v; v^2q_1q_2 \\ vt_2,\, vq_1t_1t_2 & \end{pmatrix}. \tag{2}$$

Here ${}_3\Phi_2(\begin{smallmatrix} t_1t_2,\, t_2,\, q_2^{-1} \\ vt_2,\, vq_1t_1t_2 \end{smallmatrix}; v; v^2q_1q_2)$ is a (3, 2)-hypergeometric series of type II, since

$$\frac{(vt_2)(vq_1t_1t_2)}{(t_1t_2)(t_2)\left(q_2^{-1}\right)} = v^2q_1q_2.$$

Recall Hall's transformation formula ([5], Appendix III.10):

$$ {}_3\Phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix} ; v; \frac{de}{abc}\right) = \frac{(b)_\infty (de/ab)_\infty (de/bc)_\infty}{(d)_\infty (e)_\infty (de/abc)_\infty} {}_3\Phi_2\left(\begin{matrix} d/b, e/b, de/abc \\ de/ab, de/bc \end{matrix} ; v; b\right). $$

The above two-term transformation formula holds for $|b| < 1$ and $|de/abc| < 1$. Applying this transformation formula to (2) and cancelling terms with $(t_2)_\infty = (1 - t_2)(vt_2)_\infty$, we rewrite (2) as

$$ \begin{aligned} T_1 &= t_2 \frac{(v)_\infty (v^2 q_1)_\infty (v^2 q_1 q_2 t_1 t_2)_\infty}{(vq_1)_\infty (t_1 t_2)_\infty (v^2 q_1 q_2)_\infty} {}_3\Phi_2\left(\begin{matrix} v, vq_1 t_1, v^2 q_1 q_2 \\ v^2 q_1, v^2 q_1 q_2 t_1 t_2 \end{matrix} ; v; t_2\right) \\ &= \frac{t_2}{1 - vq_1} \frac{(v)_\infty (v^2 q_1 q_2 t_1 t_2)_\infty}{(t_1 t_2)_\infty (v^2 q_1 q_2)_\infty} \cdot {}_3\Phi_2\left(\begin{matrix} v, vq_1 t_1, v^2 q_1 q_2 \\ v^2 q_1, v^2 q_1 q_2 t_1 t_2 \end{matrix} ; v; t_2\right) \\ &= \frac{t_2}{1 - vq_1} \frac{1 - vq_1 q_2}{1 - vq_1 q_2 t_1 t_2} \frac{(v)_\infty (vq_1 q_2 t_1 t_2)_\infty}{(t_1 t_2)_\infty (vq_1 q_2)_\infty} \cdot {}_3\Phi_2\left(\begin{matrix} v, vq_1 t_1, v^2 q_1 q_2 \\ v^2 q_1, v^2 q_1 q_2 t_1 t_2 \end{matrix} ; v; t_2\right) \\ &= \frac{1}{1 - q_1 t_1} \frac{(v)_\infty (vq_1 q_2 t_1 t_2)_\infty}{(t_1 t_2)_\infty (vq_1 q_2)_\infty} \cdot \sum_{m=0}^{\infty} \frac{(v)_{m+1} (q_1 t_1)_{m+1} (vq_1 q_2)_{m+1}}{(v)_{m+1} (vq_1)_{m+1} (vq_1 q_2 t_1 t_2)_{m+1}} t_2^{m+1} \\ &= \frac{1}{1 - q_1 t_1} \frac{(v)_\infty (vq_1 q_2 t_1 t_2)_\infty}{(t_1 t_2)_\infty (vq_1 q_2)_\infty} \cdot \left[{}_3\Phi_2\left(\begin{matrix} v, q_1 t_1, vq_1 q_2 \\ vq_1, vq_1 q_2 t_1 t_2 \end{matrix} ; v; t_2\right) - 1 \right] \\ &= \frac{1}{1 - t_1 t_2} \frac{(v)_\infty (vq_1 q_2 t_1 t_2)_\infty}{(vt_1 t_2)_\infty (vq_1 q_2)_\infty} \cdot \left[\frac{1}{1 - q_1 t_1} {}_3\Phi_2\left(\begin{matrix} v, q_1 t_1, vq_1 q_2 \\ vq_1, vq_1 q_2 t_1 t_2 \end{matrix} ; v; t_2\right) - \frac{1}{1 - q_1 t_1} \right]. \end{aligned} $$

Since T_2 is the same as T_1 after switching of variables $t_1 \leftrightarrow t_2$, $q_1 \leftrightarrow q_2$, we have

$$ \begin{aligned} T_2 = \frac{1}{1 - t_1 t_2} &\cdot \frac{(v)_\infty (vq_1 q_2 t_1 t_2)_\infty}{(vt_1 t_2)_\infty (vq_1 q_2)_\infty} \\ &\cdot \left[\frac{1}{1 - q_2 t_2} {}_3\Phi_2\left(\begin{matrix} v, q_2 t_2, vq_1 q_2 \\ vq_2, vq_1 q_2 t_1 t_2 \end{matrix} ; v; t_1\right) - \frac{1}{1 - q_2 t_2} \right]. \end{aligned} $$

Note in addition by the q-binomial theorem that

$$ T_3 = \frac{1}{1 - t_1 t_2} \cdot \frac{(v)_\infty (vq_1 q_2 t_1 t_2)_\infty}{(vt_1 t_2)_\infty (vq_1 q_2)_\infty}. $$

Therefore,

$$T_1 + T_2 + T_3 = \frac{1}{1-t_1t_2} \cdot \frac{(v)_\infty (vq_1q_2t_1t_2)_\infty}{(vt_1t_2)_\infty (vq_1q_2)_\infty} \cdot \left[\frac{q_1q_2t_1t_2 - 1}{(1-q_1t_1)(1-q_2t_2)} + \frac{1}{1-q_1t_1} {}_3\Phi_2 \left(\begin{matrix} v, q_1t_1, vq_1q_2 \\ vq_1, vq_1q_2t_1t_2 \end{matrix} ; v; t_2 \right) + \frac{1}{1-q_2t_2} {}_3\Phi_2 \left(\begin{matrix} v, q_2t_2, vq_1q_2 \\ vq_2, vq_1q_2t_1t_2 \end{matrix} ; v; t_1 \right) \right].$$

Recall by Lemma 2 that $\widehat{F}(q_1, t_1; q_2, t_2) = (v)_\infty^{-1} \cdot \langle \widehat{B}_\lambda(q_1, t_1) \widehat{B}_\lambda(q_2, t_2) \rangle_v$. This together with Lemma 8 proves Theorem 9.

In the case when $q_1q_2t_1t_2 = 1$, the above expression for $T_1 + T_2 + T_3$ can be further simplified to be

$$\begin{aligned} &\frac{1}{1-t_1t_2} \cdot \frac{(v)_\infty^2}{(vt_1t_2)_\infty (vq_1q_2)_\infty} \cdot \left[\frac{1}{1-q_1t_1} {}_3\Phi_2 \left(\begin{matrix} v, q_1t_1, vq_1q_2 \\ vq_1, v \end{matrix} ; v; t_2 \right) + \frac{1}{1-q_2t_2} {}_3\Phi_2 \left(\begin{matrix} v, q_2t_2, vq_1q_2 \\ vq_2, v \end{matrix} ; v; t_1 \right) \right] \\ &= \frac{1}{1-t_1t_2} \cdot \frac{(v)_\infty^2}{(vt_1t_2)_\infty (vq_1q_2)_\infty} \cdot \left[\frac{1}{1-q_1t_1} {}_2\Phi_1 \left(\begin{matrix} q_1t_1, vq_1q_2 \\ vq_1 \end{matrix} ; v; t_2 \right) + \frac{1}{1-q_2t_2} {}_2\Phi_1 \left(\begin{matrix} q_2t_2, vq_1q_2 \\ vq_2 \end{matrix} ; v; t_1 \right) \right]. \end{aligned} \tag{3}$$

Thanks to $q_1q_2t_1t_2 = 1$, the two (2, 1)-basic hypergeometric series are of the form ${}_2\Phi_1(\begin{smallmatrix} a, b \\ c \end{smallmatrix} ; v; c/ab)$. Now by Heine's formula (cf. [5, Appendix II.8])

$${}_2\Phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; v; c/ab \right) = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty}, \quad |b| < 1, \left| \frac{c}{ab} \right| < 1,$$

the expression (3) for $T_1 + T_2 + T_3$ becomes

$$\frac{1}{1-t_1t_2} \cdot \frac{(v)_\infty^2}{(vt_1t_2)_\infty (vq_1q_2)_\infty} \cdot \left[\frac{1}{1-q_1t_1} \frac{(vt_1^{-1})_\infty (q_2^{-1})_\infty}{(vq_1)_\infty (t_2)_\infty} + \frac{1}{1-q_2t_2} \frac{(vt_2^{-1})_\infty (q_1^{-1})_\infty}{(vq_2)_\infty (t_1)_\infty} \right].$$

This together with Lemmas 2 and 8 completes the proof of Theorem 10. □

Remark 11. It follows from the proof above that the convergence of the 2-point function is guaranteed by assuming that $|t_1| < 1$, $|t_2| < 1$, $|vq_1q_2| < 1$, $|v| < 1$, and by excluding the values for t_i, q_i which make the denominators of the (3, 2)-basic hypergeometric series and other denominators in the above theorems vanish.

4. A generalization via vertex operators

4.1. 1-point function of the zero-mode of a vertex operator

Consider the Heisenberg algebra generated by I and $\mathfrak{a}_n$, $n \in \mathbb{Z}$, with the commutation relations (where κ is a constant):

$$[\mathfrak{a}_m, \mathfrak{a}_n] = \kappa m \delta_{m,-n} \mathrm{I}.$$

The Fock space B is the irreducible representation of the Heisenberg algebra generated by a (highest weight) vector $|0\rangle$ such that $\mathrm{I}|0\rangle = |0\rangle$ and $\mathfrak{a}_n|0\rangle = 0$ for $n \geq 0$. The Fock space B has a linear basis $\mathfrak{a}_{-\lambda} := \mathfrak{a}_{-\lambda_1}\mathfrak{a}_{-\lambda_2}\cdots|0\rangle$, where $\lambda = (\lambda_1, \lambda_2, \ldots)$ runs over all partitions. Below we identify B with the ring of symmetric function Λ by identifying $\mathfrak{a}_{-\lambda}$ with the power-sum symmetric functions p_λ.

Introduce the following deformed vertex operator

$$V(z; q_1, t_1, q_2, t_2) = \exp\left(\sum_{k\geq 1} \left(q_1^k - q_2^k\right)\mathfrak{a}_{-k}\frac{z^k}{k}\right) \exp\left(\sum_{k\geq 1} \left(t_2^k - t_1^k\right)\mathfrak{a}_{k}\frac{z^{-k}}{k}\right).$$

Write

$$V(z; q_1, t_1, q_2, t_2) = \sum_{m\in\mathbb{Z}} V_m(q_1, q_2, t_1, t_2) z^m.$$

Remark 12. When $\kappa = 1$, $q_2 = t_2 = 1$, and write $q = q_1$ and $t = t_1$, the operator V_0 provides a vertex operator realization for $\widehat{\mathfrak{B}}_{q,t}$:

$$\widehat{\mathfrak{B}}_{q,t} = \frac{1}{(1-q)(1-t)} \cdot V_0(q, 1, t, 1). \tag{4}$$

This formula in a λ-ring form (in different notations) appears in the study of Macdonald polynomials by Garsia and Haiman [4, (73)]. In this sense, Theorem 13 below is a generalization of Theorem 5 (with different proofs). The formula (4) for $t = q^{-1}$ is equivalent to a formula of Lascoux and Thibon [6, Prop. 3.3].

Theorem 13. *We have*

$$\langle V_0(q_1, q_2, t_1, t_2)\rangle_v = \left[\frac{(q_1t_1v)_\infty (q_2t_2v)_\infty}{(q_1t_2v)_\infty (q_2t_1v)_\infty}\right]^\kappa.$$

Proof: Let us denote $\Delta := V_0(q_1, q_2, t_1, t_2)$. For a partition $\lambda = (r^{m_r})_{r\geq 1}$ with m_r parts equal to r, $p_\lambda = \prod_{r\geq 1} \mathfrak{a}_{-r}^{m_r}|0\rangle$. To compute the trace $\mathrm{Tr}_B v^{L_0}\Delta$, we will compute the projection of Δp_λ to the one-dimensional subspace $\mathbb{C}p_\lambda$ (with respect to the basis

p_μ's). A similar method has been also used in [3].

$$\text{projection of } \Delta p_\lambda = \sum_{\substack{(n_r)\\ n_r\le m_r \text{ for all } r}} \left(\prod_{r\ge1} \frac{\left(q_1^r - q_2^r\right)^{n_r} \mathfrak{a}_{-r}^{n_r}}{r^{n_r} n_r!}\right)\left(\prod_{r\ge1} \frac{\left(t_2^r - t_1^r\right)^{n_r} \mathfrak{a}_{r}^{n_r}}{r^{n_r} n_r!}\right)\cdot \prod_{r\ge1} \mathfrak{a}_{-r}^{m_r}|0\rangle$$
$$= \sum_{\substack{(n_r)\\ n_r\le m_r \text{ for all } r}} \prod_{r\ge1} \frac{\binom{m_r}{n_r} r^{n_r} n_r! \kappa^{n_r}\left(q_1^r - q_2^r\right)^{n_r}\left(t_2^r - t_1^r\right)^{n_r}}{\left(r^{n_r} n_r!\right)^2}\cdot \prod_{r\ge1} \mathfrak{a}_{-r}^{m_r}|0\rangle.$$

Therefore, we have

$$\mathrm{Tr}_B(v^{L_0}\Delta) = \sum_{\substack{(m_r),(n_r)\\ n_r\le m_r \text{ for all } r}} \prod_{r\ge1} \frac{\binom{m_r}{n_r} r^{n_r} n_r! \kappa^{n_r}\left(q_1^r - q_2^r\right)^{n_r}\left(t_2^r - t_1^r\right)^{n_r} v^{rm_r}}{(r^{n_r} n_r!)^2}$$
$$= \prod_{r\ge1}\sum_{(n_r)} \frac{\left(\kappa\left(q_1^r - q_2^r\right)\left(t_2^r - t_1^r\right)\right)^{n_r}}{r^{n_r} n_r!} \sum_{\substack{(m_r)\\ m_r\ge n_r \text{ for all } r}} \binom{m_r}{n_r} v^{rm_r}.$$

Using the simple binomial identity for $n \ge 0$,

$$\sum_{m\ge n}\binom{m}{n}x^m = \frac{x^n}{(1-x)^{1+n}},$$

we have

$$\mathrm{Tr}_B(v^{L_0}\Delta) = \prod_{r\ge1}\sum_{(n_r)} \frac{\left(\kappa\left(q_1^r - q_2^r\right)\left(t_2^r - t_1^r\right)\right)^{n_r}}{r^{n_r} n_r!} \frac{(v^r)^{n_r}}{(1-v^r)^{1+n_r}}$$
$$= (v)_\infty^{-1}\cdot \exp\left(\sum_{r\ge1} \frac{\kappa v^r\left(q_1^r - q_2^r\right)\left(t_2^r - t_1^r\right)}{r(1-v^r)}\right).$$

Hence,

$$\begin{aligned}\langle\Delta\rangle_v &= (v)_\infty \mathrm{Tr}_B(v^{L_0}\Delta)\\ &= \exp\left(\sum_{r\ge1}\sum_{n\ge1}\frac{\kappa}{r}[(q_1t_2v^n)^r + (q_2t_1v^n)^r - (q_1t_1v^n)^r - (q_2t_2v^n)^r]\right)\\ &= \exp\left(\kappa\sum_{n\ge1}(\ln(1-q_1t_1v^n)(1-q_2t_2v^n) - \ln(1-q_1t_2v^n)(1-q_2t_1v^n))\right)\\ &= \left[\frac{(q_1t_1v)_\infty(q_2t_2v)_\infty}{(q_1t_2v)_\infty(q_2t_1v)_\infty}\right]^\kappa.\end{aligned}$$

□

4.2. The n-point function of a vertex operator

It turns out that it is fairly easy to compute the n-point function of the full vertex operator $V(z; s, t, u, w)$ in contrast to the n-point function of its zero-mode (for $n \geq 2$). We first recall a standard lemma.

Lemma 14. *We have*

$$\exp\left(\frac{\left(t_i^k - s_i^k\right) z_i^{-k} \mathfrak{a}_k}{k}\right) \exp\left(\frac{\left(u_j^k - w_j^k\right) z_j^k \mathfrak{a}_{-k}}{k}\right)$$
$$= \exp\left(\frac{\kappa\left(t_i^k - s_i^k\right)\left(u_j^k - w_j^k\right) z_j^k z_i^{-k}}{k}\right) \exp\left(\frac{\left(u_j^k - w_j^k\right) z_j^k \mathfrak{a}_{-k}}{k}\right) \exp\left(\frac{\left(t_i^k - s_i^k\right) z_i^{-k} \mathfrak{a}_k}{k}\right).$$

Lemma 15. *We have*

$$\prod_{i=1}^{n} V(z_i; s_i, t_i, u_i, w_i)$$
$$= \exp\left(\kappa \sum_{k\geq 1} \frac{\sum_{1\leq i<j\leq n} \left(t_i^k - s_i^k\right)\left(u_j^k - w_j^k\right) z_j^k z_i^{-k}}{k}\right) \exp\left(\sum_{k\geq 1} \frac{\sum_{i=1}^{n} \left(u_i^k - w_i^k\right) z_i^k \mathfrak{a}_{-k}}{k}\right)$$
$$\times \exp\left(\sum_{k\geq 1} \frac{\sum_{i=1}^{n} \left(t_i^k - s_i^k\right) z_i^{-k} \mathfrak{a}_k}{k}\right).$$

Proof: Follows from applying Lemma 14 repeatedly. □

Theorem 16. *We have*

$$\left\langle \prod_{i=1}^{n} V(z_i; s_i, t_i, u_i, w_i) \right\rangle_v = \prod_{1\leq i<j\leq n} \left[\frac{\left(1 - t_i w_j z_i^{-1} z_j\right)\left(1 - s_i u_j z_i^{-1} z_j\right)}{\left(1 - t_i u_j z_i^{-1} z_j\right)\left(1 - s_i w_j z_i^{-1} z_j\right)}\right]^{\kappa} \tag{5}$$

$$\times \prod_{i,j=1}^{n} \left[\frac{\left(t_i w_j z_i^{-1} z_j\right)_\infty \left(s_i u_j z_i^{-1} z_j\right)_\infty}{\left(t_i u_j z_i^{-1} z_j\right)_\infty \left(s_i w_j z_i^{-1} z_j\right)_\infty}\right]^{\kappa}. \tag{6}$$

Proof: Using Lemma 15 and the projection technique as used in the proof of Theorem 13, the trace $\mathrm{Tr}(v^{L_0}\prod_{i=1}^{n} V(z_i; s_i, t_i, u_i, w_i))$ can be shown to be

$$(v)_\infty^{-1} \cdot \exp\left(\kappa \sum_{k\geq 1} \frac{\sum_{1\leq i<j\leq n} \left(t_i^k - s_i^k\right)\left(u_j^k - w_j^k\right) z_j^k z_i^{-k}}{k}\right)$$
$$\times \exp\left(\kappa \sum_{k\geq 1} \frac{v^k \sum_{j=1}^{n} \left(u_j^k - w_j^k\right) z_j^k \cdot \sum_{i=1}^{n} \left(t_i^k - s_i^k\right) z_i^{-k}}{k(1-v^k)}\right). \tag{7}$$

It is a simple algebraic manipulation to rewrite the first exponential in (7) as the product (5) and the second exponential in (7) as the product (6). □

Remark 17. Theorem 16 can be regarded as a generalization of [9, Theorem 3.1]. It specializes when $n = 1$ to Theorem 13. For $n \geq 2$, the n-point correlation function for a vertex operator differs from that for the zero-mode of a vertex operator. While the correlation functions for the zero-mode of a vertex operator has more direct connections with other fields, it is much more difficult to calculate.

5. Discussions

In this Note, we have formulated the n-point correlation functions which are generalizations of [1], and found closed formulas when $n = 1, 2$. We then formulated and computed some related n-point functions of vertex operators. In a way, this Note raises more questions than we could answer. Let us list some open problems and connections below:

1. The symmetric functions which are the eigenvectors for $V_0(q_1, q_2, t_1, t_2)$ are common generalizations of the Macdonald polynomials and Jack polynomials (with Jack parameter κ). It is interesting to study them in detail and in particular to see if they have Schur-positivity etc.
2. Calculate the n-point correlation functions for general n. The simple closed formulas obtained in this Note for $n = 1, 2$ suggests a nice general solution, which will be a generalization of the remarkable formula found in [1].
3. The n-point functions of [1] afford geometric interpretations in terms of Gromov-Witten theory of an elliptic curve and Hilbert schemes of points on the affine plane. We speculate that our n-point functions have similar interpretations using equivariant K-theory formulations.
4. The function $B_\lambda(q, t)$ (after normalization) can be regarded as a probability measure on the set of partitions, which generalizes those studied actively in literature (cf. [10] and the references therein).

Acknowledgements We thank George Andrews for a helpful email correspondence. S.-J.C. is partially supported by an NSC grant of the R.O.C., and he thanks University of Virginia for hospitality, where part of this work was carried out. W.W. is partially supported by NSA and NSF.

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