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The isometries of the cut, metric and hypermetric cones

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Abstract We show that the symmetry groups of the cut cone Cut_n and the metric cone Met_n both consist of the isometries induced by the permutations on $\{1,\ldots,n\}$; that is, $Is(\mathrm{Cut}_n) = Is(\mathrm{Met}_n) \simeq Sym(n)$ for $n \geq 5$. For $n = 4$ we have $Is(\mathrm{Cut}_4) = Is(\mathrm{Met}_4) \simeq Sym(3) \times Sym(4)$. This result can be extended to cones containing the cuts as extreme rays and for which the triangle inequalities are facet-inducing. For instance, $Is(\mathrm{Hyp}_n) \simeq Sym(n)$ for $n \geq 5$, where Hyp_n denotes the hypermetric cone.

Keywords Polyhedral combinatorics · Metric cone · Hypermetric cone · Symmetry group

1. Introduction and notation

The $\binom{n}{2}$-dimensional *cut cone* Cut_n is usually introduced as the conic hull of the incidence vectors of all the cuts of the complete graph on n nodes. More precisely, given a subset S of $V_n = \{1,\ldots,n\}$, the *cut* determined by S consists of the pairs (i, j) of elements of V_n such that exactly one of i, j is in S. By $\delta(S)$ we denote both the cut and its incidence vector in $\mathbb{R}^{\binom{n}{2}}$; that is, $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. By abuse of notation, we use the term cut for both the cut itself and its incidence vector, so $\delta(S)_{ij}$ are considered as coordinates of a point in $\mathbb{R}^{\binom{n}{2}}$. The cut cone Cut_n is the conic hull of

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all $2^{n-1}-1$ nonzero cuts, and the *cut polytope* cut_n is the convex hull of all 2^{n-1} cuts. The cut cone and one of its relaxation — the *metric cone* Met_n - can also be defined in terms of finite metric spaces in the following way. For all 3-sets $\{i, j, k\} \subset \{1, \ldots, n\}$, we consider the following inequalities.

$$x_{ij} - x_{ik} - x_{jk} \leq 0, \tag{1}$$

$$x_{ij} + x_{ik} + x_{jk} \leq 2. \tag{2}$$

(1) induce the $3\binom{n}{3}$ facets which define the metric cone Met_n. Then, bounding the latter by the $\binom{n}{3}$ facets induced by (2) we obtain the *metric polytope* met_n. The facets defined by (1) (resp. by (2)) can be seen as *triangle* (resp. *perimeter*) *inequalities* for distance x_{ij} on $\{1, \ldots, n\}$. While the cut cone is the conic hull of all, up to a constant multiple, {0, 1}-valued extreme rays of the metric cone, the cut polytope cut_n is the convex hull of all {0, 1}-valued vertices of the metric polytope. The link with finite metric spaces is the following: there is a natural $1-1$ correspondence between the elements of the metric cone and all the semi-metrics on n points, and the elements of the cut cone correspond precisely to the semi-metrics on n points that are isometrically embeddable into some l_1^m, see [1]. It is easy to check that such minimal m is smaller than or equal to $\binom{n}{2}$.

One of the motivations for the study of these polyhedra comes from their applications in combinatorial optimization, the most important being the MAXCUT and multicommodity flow problems. For instance, the *linear programming approach* to MAXCUT involves considering cutting planes that are needed to be added to Met_n to obtain Cut_n. These cutting planes define cones C_n such that $\text{Cut}_n \subseteq C_n \subseteq \text{Met}_n$. Perhaps the most well-known example of such a C_n is the *hypermetric cone* Hyp_n which is defined by facets induced by inequalities generalizing the triangle inequalities. For a detailed study of those polyhedra and their applications in combinatorial optimization we refer to DEZA and LAURENT [9].

2. Main result

One important feature of the metric and cut polyhedra is their very large symmetry group. We recall that the symmetry group $Is(P)$ of a polyhedron P is the group of isometries preserving P and that an isometry is a linear transformation preserving the Euclidean distance. While the symmetry groups of the cut and metric polytopes are known, the question whether the cut and metric cones admit no other isometry than the ones induced by $Sym(n)$ was open, see [7, 8, 14]. More precisely, for $n \geq 5$, $Is(\text{met}_n) = Is(\text{cut}_n)$ and both are induced by permutations on $V_n = \{1, \ldots, n\}$ and *switching reflections by a cut* and, for $n \geq 5$, we have $|Is(\text{met}_n)| = 2^{n-1}n!$, see [8]. Given a cut $\delta(S)$, the switching reflection $r_{\delta(S)}$ is defined by $y = r_{\delta(S)}(x)$ where $y_{ij} = 1 - x_{ij}$ if $(i, j) \in \delta(S)$ and $y_{ij} = x_{ij}$ otherwise.

The aim of this article is to show that $Is(\text{Cut}_n) = Is(\text{Met}_n) \simeq Sym(n)$ for $n \geq 5$ and $Is(\text{Cut}_4) = Is(\text{Met}_4) \simeq Sym(3) \times Sym(4)$. A part of Theorem (1) was conjectured in [7] and was substantiated by computer calculations of the automorphism group of the ridge graph of Met_n for $n \leq 20$. We recall that *ridge graph* of a polyhedra C_n is the graph which vertices are the facets of C_n, two facets being adjacent if and only if their intersection is a face of codimension 2 of C_n. In other words, the ridge graph of C_n is the skeleton of the dual of C_n.

Theorem 1. *The symmetry groups of the cones* Met_n *and* Cut_n *are isomorphic to* $Sym(n)$ *for* $n \geq 5$ *and to* $Sym(3) \times Sym(4)$ *for* $n = 4$.

The proof of Theorem 1 is given in Section 3. In Section 3.1 we characterize $Is(\mathrm{Met}_n)$, in Section 3.2 we show that $Is(\mathrm{Cut}_n) = Is(\mathrm{Met}_n)$ and in Section 3.3 we generalize Theorem 1 in the following way.

Theorem 2. *Let* C_n *be a cone satisfying*

(i) *the cuts are extreme rays of* C_n,
(ii) *the triangle inequalities are facet-inducing for* C_n.

Then any isometry of C_n *is induced by a permutation on* $\{1, \ldots, n\}$.

A cone C_n satisfying the condition (i) and (ii) of Theorem 2 is cone satisfying $\mathrm{Cut}_n \subseteq C_n \subseteq \mathrm{Met}_n$. Apart from Met_n and Cut_n themselves, a well-known example of such a cone C_n is the *hypermetric cone* Hyp_n defined by the following *hypermetric inequalities* (3) which generalize the triangle inequalities:

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq 0 \quad \text{with } \sum_{i=1}^{n} b_i = 1 \tag{3}$$

We recall that $Is(\mathrm{Hyp}_n)$ contains the isometries induced by the permutations on $\{1, \ldots, n\}$.

Corollary 1. *The symmetry group of* Hyp_n *is isomorphic to* $Sym(n)$ *for* $n \geq 5$ *and to* $Sym(3) \times Sym(4)$ *for* $n = 4$.

3. Proofs

We first prove Theorem 1 for Met_n by showing that its ridge graph G_n for $n > 4$ has the automorphism group $Sym(n)$; the symmetry group of Met_4 is constructed directly. We complete the proof of Theorem 1 by showing that G_n is an induced subgraph of the ridge graph of Cut_n that is invariant under the isometries of Cut_n. Finally, we prove Theorem 2 by noticing that Cut_n can be replaced by any cone C_n satisfying $\mathrm{Cut}_n \subseteq C_n \subseteq \mathrm{Met}_n$. The group-theoretic notation used in the paper can be found e.g. in [3].

3.1. The group $Is(\mathrm{Met}_n)$ for $n \geq 5$

3.1.1. $Is(\mathrm{Met}_n)$ *for* $n \geq 4$

Note that the isometries act faithfully on the facets; that is, the only isometry that stabilizes each facet of Met_n is the trivial one. As each permutation on V_n is an isometry, in order to prove the statement for $n \geq 5$, it suffices to show that the automorphism group $A = \mathrm{Aut}(G_n)$ of the ridge graph G_n is isomorphic to $Sym(n)$.

The facets of Met_n naturally correspond to $\{0, 1, -1\}$ vectors of length $\binom{n}{2}$ with one positive and two negative entries. Two triangle facets u and v are adjacent in G_n, i.e. intersect on a face of codimension 2, if and only if they are *non-conflicting*; that is, there is no position ij such that the corresponding entries u_{ij} and v_{ij} are nonzero and of opposite sign, see [5, 9].

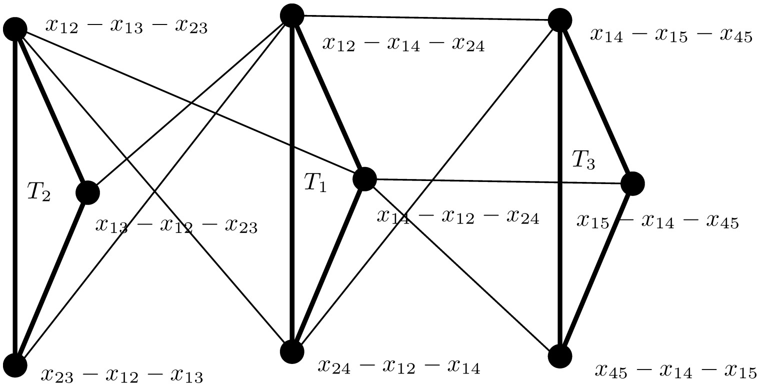

Fig. 1 Adjacencies between Triangles $T_1 = \{1, 2, 4\}$, $T_2 = \{1, 2, 3\}$, and $T_3 = \{1, 4, 5\}$

As already observed in [5], instead of working with G_n, it appears to be easier to work with its complement $\bar{G}_n$ which has the same automorphism group. Observe that if two vertices u and v are conflicting then they have either exactly one nonzero entry in common, or all the three nonzero entries in common. Indeed, if $u_{ik}u_{jk}u_{ij}v_{ij} \neq 0$ and $v_{ik'}v_{jk'} \neq 0$, then either $k = k'$, and the latter holds, or $k \neq k'$, and the former holds.

The subgraph induced on the neighbours of a vertex u of $\bar{G}_n$ is isomorphic to the disjoint union of $n - 3$ hexagons on a common edge (u', u''), see [5]. It is easy to see that u, u', and u'' all have the same zero entries. From now on let us refer to such a type of 3-clique in $\bar{G}_n$ as a *Triangle*. The Triangles $\{u, u', u''\}$ form an orbit of A, as the number of common neighbours of an edge of Triangle is $n - 2$, bigger than for the edges of another types, where it is just 2.

Let us look at the edges between any two given Triangles T_1 and T_2. If they do not share a common nonzero entry then there are no edges in between. Otherwise there are exactly 4 edges (see Fig. 1), and, ignoring the edges of T_1 and T_2, the subgraph induced on their 6 vertices is the disjoint union of two 2-paths. This implies that every $g \in A$ stabilizing T_1 and T_2 either fixes T_1 pointwise, or induces an element of order 2 on the vertices of T_1. Let T_3 be a third Triangle having a common entry with T_1, the different one than the common entry of T_1 and T_3. Then the 2-path between T_1 and T_3 with the middle point in T_1 does not intersect the 2-path between T_1 and T_2 with the middle point in T_1. Hence every $g \in A$ stabilizing T_1, T_2 and T_3 fixes T_1 pointwise. Therefore if $g \in A$ stabilizes all the Triangles, then g is the identity; that is, A acts faithfully on the set of Triangles.

Let us define the graph Γ_n on the Triangles, two Triangles are adjacent if there is an edge of $\bar{G}_n$ joining a vertex of the first Triangle with a vertex of the second Triangle (so there are 4 edges forming the disjoint union of 2-paths that join them). To complete the proof, it suffices to show that $\mathrm{Aut}(\Gamma_n) \cong A$. The latter is in fact not true for $n = 6$, and we shall treat this case separately.

Note that Γ_n is naturally isomorphic to the first subconstituent of the Johnson scheme $J(n, 3)$, in other words, the graph with the vertex set $\binom{V_n}{3}$, two vertices adjacent if the corresponding 3-subsets intersect in a 2-subset. Automorphism groups of these graphs are described e.g. in [10, 12], and were known at least since [13]. We give here a self-contained treatment, as the particular case we are dealing with is a simple one. Note that Γ_5 is the

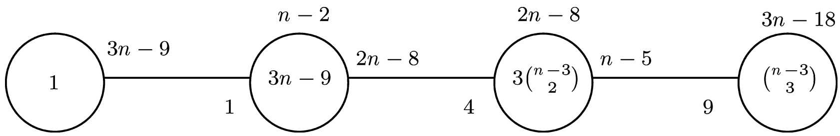

Fig. 2 The distribution diagram of Γ_n

complement of the Petersen graph, and it is well-known that its automorphism group is $Sym(5)$. This completes the proof for $n = 5$.

For $n > 5$, the graph Γ_n is distance-regular of diameter 3, see e.g. [2] for this notion.

The subgraph Ω induced on the neighbourhood $\Gamma_n(v)$ of a vertex v is isomorphic to the line graph of $K_{3,n-3}$.

If $n > 6$ then the automorphism group of Ω is isomorphic to $Sym(3) \times Sym(n-3)$. It suffices to show that the stabilizer of v in $\mathrm{Aut}(\Gamma_n)$ acts faithfully on $\Gamma_n(v)$, as then it will coincide with the one in $\mathrm{Aut}(\Gamma_n) \cap Sym(n)$. Let H be the kernel of the latter action. As any vertex at distance 2 from v is adjacent to exactly 4 vertices in $\Gamma_n(v)$, and any two different vertices at distance 2 from v are adjacent to different 4-sets of vertices in $\Gamma_n(v)$, the vertices at distance 2 from v are all fixed by H. In particular $\Gamma_n(u)$ is fixed by H for any $u \in \Gamma_n(v)$. Hence all the vertices at distance 2 from u are fixed, too, and $H = 1$, as claimed. This completes the proof for $n > 6$.

The graph Γ_6 is a double antipodal cover of K_{10}, and $\mathrm{Aut}(\Gamma_6) \cong 2 \times Sym(6)$. We show that nevertheless only $Sym(6)$ arises as $\mathrm{Aut}(\bar{G}_6)$. Indeed, the normal subgroup $H = Sym(2)$ interchanges simultaneously all the vertices of Γ_6 at the maximal distance; they correspond to Triangles of $\bar{G}_6$ with no common nonzero coordinate. On the other hand, H must act on $\bar{G}_6$. Observe that the pointwise stabilizer in $Sym(6)$ of the facet $t_1 = (x_{12}, -x_{13}, -x_{23})$, that belongs to a Triangle T_1, acts transitively on the three facets with coordinates x_{45}, x_{46} and x_{56} forming a Triangle T_2. On the other hand H must map t_1 to one of these three latter facets, as it must interchange T_1 and T_2. This means that H does not commute with the action of $Sym(6)$ on the vertices of $\bar{G}_6$, contradicting the assumption that $H \times Sym(6)$ acts there. This completes the proof in the remaining case $n = 6$.

3.1.2. Is(Met$_4$)

Here we shall construct the group of isometries of Met$_4$ as the reflection group (see e.g. [11]) $A_2 \times A_3 \cong Sym(3) \times Sym(4)$. Let us recall the definition of a reflection $s(\alpha)$ with respect to the hyperplane orthogonal to the vector $0 \neq \alpha \in V$, with V the Euclidean with the scalar product $\langle , \rangle$.

$$s(\alpha)v = v - 2\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha, \quad v \in V. \tag{4}$$

Since Met$_4$ = Cut$_4$, the 7 extreme rays of Met$_4$ are just the 7 nonzero cuts of the graph K_4. In other words, a cut $V_1 \cup V_2$ of K_4, i.e., a partition of the vertices of K_4 into two parts V_1 and V_2, corresponds to the semimetric d satisfying $d(x, y) = 1$ for x and y from different parts of the cut, and $d(x, y) = 0$ otherwise. Writing the above-the-main-diagonal contents of 4×4 symmetric matrices as vectors, the cuts $r_1, \ldots, r_7$ are as follows:

$$\begin{array}{rcccccccl} r_1 = & (0 & 0 & 1 & 0 & 1 & 1) & & \\ r_2 = & (0 & 1 & 1 & 1 & 1 & 0) & & \\ r_3 = & (0 & 1 & 0 & 1 & 0 & 1) & & \\ r_4 = & (1 & 0 & 1 & 1 & 0 & 1), & & (r_4, r_5) = s(0, -1, 1, 1, -1, 0) \\ r_5 = & (1 & 1 & 0 & 0 & 1 & 1) & & \\ r_6 = & (1 & 0 & 0 & 1 & 1 & 0) & & \\ r_7 = & (1 & 1 & 1 & 0 & 0 & 0) & & \end{array}$$

Under the action of $Sym(4)$, there are 2 orbits, one of 4 vectors r_1, r_3, r_6, r_7 with 3 nonzero entries, and the other of the remaining 3 vectors.

We construct 5 reflections that generate the group $Sym(3) \times Sym(4)$ acting on set (5) of vectors. Each reflection is determined by the hyperplane, generated by 5 linearly independent elements of (5). More precisely, the reflection $s(\alpha)$ acting as the permutation (r_i, r_j) is given by an $\alpha \neq 0$ in the kernel of the 5×6 matrix $\left(\begin{smallmatrix} r_{\ell_1} \\ \vdots \\ r_{\ell_5} \end{smallmatrix}\right)$, where $\ell_j \neq i, j$. Note that it is necessary that r_i and r_j lie in the same orbit of $Sym(4)$. For instance, α for (r_4, r_5) is given in (5).

It is straightforward to check that indeed the 5 reflections just described generate the group $A_2 \times A_3$, and that they act on the rays r_i's in (5). Moreover, it suffices to check that one of these reflections acts on the rays, as together with the already present $Sym(4)$ they generate the whole group in question.

To complete the proof of Theorem 1 in this case, it suffices to refer to the fact that the ridge graph G_4 of Met_4 is isomorphic to the line graph of $K_{3,4}$ (cf. [5]), and thus the symmetry group cannot be bigger than its automorphism group $Sym(3) \times Sym(4)$.

3.2. The group $Is(\text{Cut}_n)$ for $n \geq 4$

As $\text{Cut}_4 = \text{Met}_4$, we can assume that $n > 4$. First, we remind that the maximal size facets of Cut_n are the triangle facets given in (1) and that a pair of triangle facets are adjacent in the ridge graph of Cut_n if and only if they are adjacent in the ridge graph of Met_n.

Lemma 1 ([6]). *Any facet of* Cut_n *contains at most* $3 \cdot 2^{n-3} - 1$ *extreme rays (cuts) with equality if and only if it is a triangle facet.*

Lemma 2 ([5]). *A pair of triangle facets of* Cut_n *intersect on a face of codimension 2 if and only if they are non-conflicting.*

Lemmas 1 and 2 imply that the ridge graph of Met_n is an induced subgraph in the ridge graph of Cut_n that is invariant under any isometry of Cut_n. Therefore we have $Is(\text{Cut}_n) = Is(\text{Met}_n)$.

3.3. The group $Is(C_n)$ for $\text{Cut}_n \subseteq C_n \subseteq \text{Met}_n$

Let C_n be a cone having, among others, the cuts as extreme rays and for which, among others, the triangle inequalities as facet-inducing. As $\text{Cut}_4 = \text{Met}_4$, we can assume that $n > 4$. In the same way as for Lemma 1, Lemma 3 can be directly deduced from a similar statement for the cut polytope cut_n, see [4] and also [9, Proposition 26.3.12].

Lemma 3. *Let C_n be a cone satisfying* $\mathrm{Cut}_n \subseteq C_n \subseteq \mathrm{Met}_n$*, any facet of C_n contains at most* $3 \cdot 2^{n-3} - 1$ *cuts with equality if and only if it is a triangle facet.*

Since $\mathrm{Cut}_n \subseteq C_n \subseteq \mathrm{Met}_n$ and triangle facets are adjacent in the ridge graphs of both Cut_n and Met_n if and only if they are non-conflicting, we have the following.

Lemma 4. *Let C_n be a cone satisfying* $\mathrm{Cut}_n \subseteq C_n \subseteq \mathrm{Met}_n$*, a pair of triangle facets of C_n intersect on a face of codimension 2 if and only if they are non-conflicting.*

As in Section 3.2, Lemmas 3 and 4 imply that the ridge graph of Met_n is an induced subgraph in the ridge graph of C_n that is invariant under any isometry of C_n. This completes the proof of Theorem 2.

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