Journal of Algebraic Combinatorics, 20, 5–16, 2004

On Certain Coxeter Lattices Without Perfect Sections

ANNE-MARIE BERGÉ berge@math.u-bordeaux.fr
Laboratoire A2X, Mathématiques, Université Bordeaux 1, 351, cours de la Libération, 33405 TALENCE Cedex, France

Received December 6, 2002; Revised May 12, 2003; Accepted June 5, 2003

Abstract. In this paper, we compute the kissing numbers of the sections of the Coxeter lattices $\mathbb{A}_n^{\frac{n+1}{2}}$, n odd, and in particular we prove that for $n \geq 7$ they cannot be perfect. The proof is merely combinatorial and relies on the structure of graphs canonically attached to the sections.

Keywords: perfect lattice, kissing number, bipartite graph

1. Introduction

A problem of recent interest is to construct integral perfect lattices with odd norm. By *lattice* we mean an additive subgroup L of a Euclidean space $(E, \cdot)$ which is additively generated by some $\mathbb{R}$-basis for E. Such a lattice is *integral* if the inner product $x \cdot y$ takes integral values on it. The *norm* of a lattice L is the minimal value M of $x \cdot x$ for $x \in L, x \neq 0$, and the vectors $\pm x \in L$ for which $x \cdot x = M$ are the *minimal vectors* of L. Their number $2s$ is the *kissing number* of L, terminology which refers to the sphere packing classically associated to the lattice L.

Perfect lattices arise in determining the densest lattice packing of spheres. A lattice L is *perfect* if it is uniquely determined up to similarity by the coordinates of its minimal vectors in one of its $\mathbb{Z}$-bases. In 1877 Korkine and Zolotareff proved that all lattices whose packing density is a local maximum (*extreme lattices*) are perfect. They also proved that a perfect lattice can be rescaled so as to be integral, and that its kissing number $2s$ satisfies

$$s \geq \frac{n(n+1)}{2},$$

where $n = \dim E$. All similarity classes of perfect lattices are now known up to dimension 7. From dimension 8 onwards, the complete classification seems out of reach. Voronoi's algorithm for perfect forms produced at this date 10916 inequivalent forms of dimension eight (for a catalogue, see http://www.math.u-bordeaux.fr/~martinet/).

An intriguing property of this list is that it contains no integral lattice of odd norm. It has recently been proved by Martinet and Venkov that *the lattice P_7^2 (in the notation of*

[4]*) is the unique integral perfect lattice of dimension* $2 \leq n \leq 9$ *having norm* 3 ([7]). Their method consists in finding for the kissing number of integral lattices of norm 3 an upper bound strictly inferior to $n(n+1)$. Note that a first 10-dimensional example of a perfect lattice having odd norm (namely 11) was recently constructed by Martinet (see [3], Section 4).

A natural method to construct integral perfect lattices having odd norm would consist in taking sections of a known one that contain a great number of its minimal vectors. About this method by sections, note that the algorithms of Batut and Martinet to "X-ray" integral lattices ([1]) showed that out of the known perfect lattices of dimension $3 \leq n \leq 8$, P_7^2 is also the unique one without perfect sections of dimension > 1.

This remarkable lattice P_7^2 belongs to an infinite sequence of perfect lattices with odd norm (when rescaled to be integral). This sequence is part of a family that Coxeter derived from the root lattices $\mathbb{A}_n$ (see [6], Section 5.2): for any dimension $n \geq 1$ and any divisor q of $n+1$ the lattice $\mathbb{A}_n^q$ is the unique sublattice of the dual lattice $\mathbb{A}_n^*$ that contains $\mathbb{A}_n$ to index q. For $n > 5$ and $q < \frac{n+1}{2}$, all these lattices have the same norm as $\mathbb{A}_n$, and are therefore perfect (and even extreme) with even norm when rescaled so as to be integral. For $q = \frac{n+1}{2}$ (n odd, $n \geq 5$), the Coxeter lattices are extreme too but with norm $\frac{2n-2}{n+1} < 2$, and their primitive integral copy has odd norm if and only if $n \equiv 3 \bmod 4$. The aim of this paper is to X-ray these lattices. In particular, as a direct consequence of the combinatorial Theorem 2 (stated and proved in Section 4), we find that for $n \geq 7$, any section of $L = \mathbb{A}_n^{\frac{n+1}{2}}$ of dimension r, $1 < r < n$, contains at most $r(r-1)+2 < r(r+1)$ minimal vectors of L. This enables us to extend to every odd dimension the property of "emptiness" noticed for the lattice $P_7^2 \sim \mathbb{A}_7^4$:

Theorem 1 *In every odd dimension n* ≥ 3, *the Coxeter lattice* $\mathbb{A}_n^{\frac{n+1}{2}}$ *has no perfect section of the same norm in dimension* >1, *except the lattice* $\mathbb{A}_5^3$ *which possesses* 15 *planar hexagonal sections.*

In Section 2 we give a description of the lattice $\mathbb{A}_n^{\frac{n+1}{2}}$ which leads to a combinatorial approach of the determination of its sections with best kissing number; this combinatorial problem is interpreted in Section 3 in terms of graphs, and solved in Section 4.

I want to thank J. Martinet for the motivation of this work, and the Reviewers for helpful suggestions and corrections.

2. A conjecture of Martinet

Let E be a Euclidean space of dimension n, and let $(e_1, \ldots, e_n)$ be a basis for the dual lattice $\mathbb{A}_n^*$ with Gram Matrix

$$\frac{1}{n+1}\begin{pmatrix} n & -1 & -1 & \cdots & -1 \\ -1 & n & -1 & \cdots & -1 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -1 & -1 & -1 & \cdots & n \end{pmatrix};$$

the minimal vectors of $\mathbb{A}_n^*$ are the $\pm e_i$, $0 \le i \le n$, where $e_0 = -(e_1 + e_2 + \cdots + e_n)$. One possible definition of the Coxeter lattice is

$$\mathbb{A}_n^{\frac{n+1}{2}} = \left\{ x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \mid (x_i) \in \mathbb{Z}^n \text{ and } \sum_i x_i \equiv 0 \bmod 2 \right\},$$

as the right-hand side defines a sublattice of index 2 in $\mathbb{A}_n^*$ containing the root lattice $\mathbb{A}_n = \langle e_i - e_0,\ 1 \le i \le n \rangle$. It then has norm $\frac{2n-2}{n+1}$ and its minimal vectors are $\pm(e_i + e_j)$, $0 \le i < j \le n$. So, to establish Theorem 1 we shall bound the number of these vectors contained in a given strict subspace of E, discarding its Euclidean structure.

In the following, E_n is a real vector space of dimension $n \ge 2$ equipped with a basis $(e_1, e_2, \ldots, e_n)$. Put

$$e_0 = -(e_1 + e_2 + \cdots + e_n).$$

For a subspace F of E_n we consider its subset

$$S_F = F \cap \{e_i + e_j, 0 \le i < j \le n\},$$

with cardinality

$$s_F = |S_F|.$$

Example A subspace F of E_n is said *canonical* if it is spanned by some vectors e_i, $0 \le i \le n$.

For a canonical subspace $F \subset E_n$ of rank r ($1 \le r \le n-1$) we have $s_F = \frac{r(r-1)}{2}$ if $r \ne n-1$ and $s_F = \frac{r(r-1)}{2} + 1$ if $r = n-1$. Indeed, up to permutations by the symmetric group S_{n+1} we may assume $F = \langle e_0, e_1, \ldots, e_{r-1} \rangle$. It then contains the $\binom{r}{2}$ vectors $e_i + e_j$, $0 \le i < j \le r-1$, and no more except if $r = n-1$, when we must add the vector $e_{n-1} + e_n = -e_0 - e_1 + \cdots - e_{n-2}$.

For any dimension $n \ge 3$ and any integer r, $1 \le r \le n-1$, we define

$$s_n(r) = \max_{F \subset E_n,\, \dim F = r} s_F.$$

Martinet ([5]) stated the following:

Conjecture

1. *For $r \ge 5$, $s_n(r)$ is equal to either $\frac{r(r-1)}{2}$ or $\frac{r(r-1)}{2} + 1$ according as $r \ne n-1$ or $r = n-1$.*
2. *For $n \ge 5$ and $r \ge 2$, we have $s_n(r) < \frac{r(r+1)}{2}$ except for $(n, r) = (5, 2)$, where $s_5(2) = 3$.*

The second part of this conjecture, applied to our lattice problem, implies Theorem 1, the value $s_5(2)$ corresponding to the hexagonal sections of the lattice $\mathbb{A}_5^3$, which are perfect indeed.

The conjecture will result of the actual determination of all values of $s_n(r)$ and of the subspaces F which realize them. To state and prove these results, an interpretation in terms of graphs is needed.

3. Graphs associated with a subspace F of E_n

The bounds of s_F are attained at subspaces F of E generated by their subsets S_F; from now on we only consider such subspaces.

Definition 1 With a subspace F of E we associate *the graph $G = G_F$ of the relation $e_i + e_j \in F$*: its vertex set is $\{0, 1, \cdots, n\}$, and two vertices i and j are joined if $e_i + e_j$ lies in F.

To any basis $\mathcal{B} \subset S_F$ of F we attach the *subgraph $G_{\mathcal{B}} \subset G_F$ obtained by keeping only the edges ij of G_F such that $e_i + e_j \in \mathcal{B}$.*

Our aim is to compare the number of edges s_F of G_F with the number of edges $r = \dim F$ of $G_{\mathcal{B}}$.

Example For a canonical subspace F of dimension r, $3 \le r \le n-1$, *there is a basis $\mathcal{B}$ whose graph is a triangle linked to a path*: for instance the vectors $e_0 + e_1, e_1 + e_2, e_2 + e_0, e_2 + e_3, \ldots, e_{r-2} + e_{r-1}$ constitute a basis for $F = \langle e_0, \ldots, e_{r-1} \rangle$.

We now discuss the existence of cycles in the graphs G_F and $G_{\mathcal{B}}$.

Lemma 1
1. *If the vertices i and j are connected in G_F by a path of odd length, ij is an edge of G_F.*
2. *The graph $G_{\mathcal{B}}$ does not contain an even cycle of length ≥ 4.*
3. *If a connected component C of G_F contains an odd cycle, then all the vectors e_i, $i \in C$ belong to F, and C is a complete graph.*

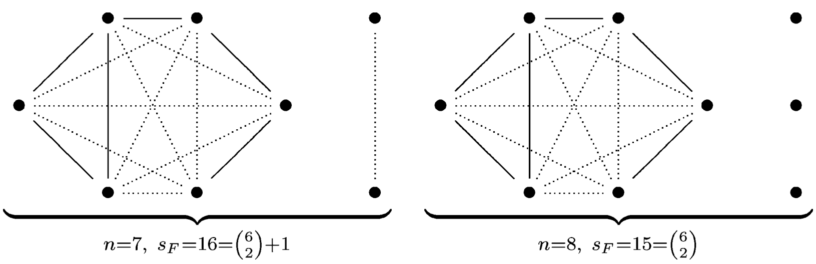

Figure 1. Graphs $G_{\mathcal{B}}$ and $G_{\mathcal{F}}$ for canonical subspaces of dimension 6 (the $s_F - r$ edges of $G_F \backslash G_{\mathcal{B}}$ appear in dotted lines).

Proof: By induction from the following relations, where $i, j, k, l \in \{0, 1, \ldots, n\}$:

$$(e_i + e_j) = (e_i + e_l) + (e_j + e_k) - (e_k + e_l),$$
$$e_i = \frac{1}{2}((e_i + e_j) - (e_j + e_k) + (e_i + e_k)),$$
$$e_i = (e_i + e_j) - e_j.$$
□

We can now characterize the canonical subspaces by their graphs.

Lemma 2 *Let F be an r-dimensional subspace of E_n ($3 \leq r \leq n-1$). Then F is canonical if and only if its graph G_F contains a complete r-graph, i.e. a graph with r vertices and $\binom{r}{2}$ edges.*

Proof: We have already seen that if F is canonical, its whole graph consists of a complete r-graph and a path of length 1 (resp. $n+1-r$ isolated vertices) if $r = n-1$ (resp. $r < n-1$).

Conversely, suppose that there is in G_F a connected component C with $|C| = r$ vertices and $\binom{r}{2}$ edges. Since C is complete of order $r \geq 3$, it contains at least one triangle; it follows from the third part of Lemma 1 that all e_i, $i \in C$ belong to F. Since $|C| = \dim F$, we conclude that $F = \langle e_i,\ i \in C \rangle$. □

4. Calculation of $s_n(r)$.

Linear type. Let F be a strict subspace of E_n and let $G_F = \bigcup_{C \in \mathcal{C}} C$ the partition of its graph into connected components. We say that the component $C \in \mathcal{C}$ is of *linear type* if the subspace

$$F_C = \langle e_i + e_j \ \text{ with } \ ij \text{ edge of } C \rangle$$

of F admits a basis $\mathcal{B}_C$ whose graph is a path.

We say that F itself is of *linear type* if, apart from isolated vertices, every component of G_F is of linear type. We label the type by the sequence of the lengths of the paths, the zeros representing the isolated vertices.

For example, figure 2 shows the four possible graph structures for $r = 2$ (the graph of a basis $\mathcal{B} \subset \cup \mathcal{B}_C$ appears in continuous lines).

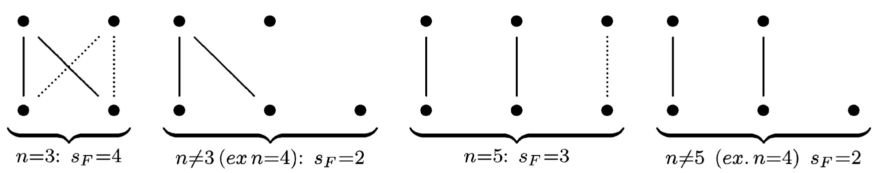

Figure 2. Linear types [2], [2, 0, 0], [1, 1, 1] and [1, 1, 0].

Theorem 2 below shows in particular that the invariant s_F assumes its greatest value for subspaces which are either canonical or (in low dimension) of linear type.

Theorem 2 *Let F be an r-dimensional* $(1 \leq r \leq n-1)$ *subspace of E_n. Then*

1. *For $r \geq 4$, we have*

$$s_F \leq \begin{cases} \dfrac{r(r-1)}{2} & \text{if } r \neq n-1, \\ \dfrac{r(r-1)}{2}+1 & \text{if } r = n-1, \end{cases} \tag{1}$$

except for $r=4$, $n=5$, F of linear type [4] *where $s_F = 9$. Equality in* (1) *holds only when F is either canonical or of one of the following linear types*:
$r=4$: $n \geq 6$, *type* $[4,0,0,\ldots]$ *or* $n=7$, *type* $[3,1,1]$;
$r=5$: $n=7$, *type* $[5,1]$;
$r=6$: $n=7$, *type* $[6]$.

2. *For $r=1$,*
$n \neq 3$: $s_n(1)=1$ *attained at type* $[1,0,\ldots]$,
$n=3$: $s_3(1)=2$ *attained at type* $[1,1]$.

3. *For $r=2$,*
$n \neq 3,5$: $s_n(2)=2$, *at types* $[1,1,0,\ldots]$ *and* $[2,0,\ldots]$,
$s_3(2)=4$ *attained at type* $[2]$,
$s_5(2)=3$ *attained at type* $[1,1,1]$.

4. *For $r=3$,*
$n \neq 5$: $s_n(3)=4$, *attained at linear types* $[3,0,0,\ldots]$ *and* $[1,1,1,1]$ (*if* $n=7$), *and at canonical hyperplanes* (*if* $n=4$);
$n=5$: $s_5(3)=5$ *attained at type* $[3,1]$.

Going back to Euclidean lattices we can interpret some maximal values of s in low dimensions. We first note that the value $s_3(2)$ corresponds to square sections of the cubic lattice $\mathbb{A}_3^2$, the set S_F consisting of two pairs of orthogonal vectors. The sections of $\mathbb{A}_5^3$ which realize the maximum $s_5(2)=3$ (resp. $s_5(4)=9$, resp. $s_5(3)=5$) are similar to the perfect lattice $\mathbb{A}_2$ (resp. to $\mathbb{A}_2 \otimes \mathbb{A}_2$, resp. to the "fragile" lattice of crystallography, see [6], Section 9.5). In dimension 7, there are coincidences, due to the multiple embeddings of the lattice $\mathbb{A}_7^4 \sim \mathbb{E}_7^*$ into $\mathbb{A}_7^*$; for instance, canonical as well as linear type [6] hyperplanes correspond to sections of $\mathbb{E}_7^*$ similar to the isodual lattice $\mathbb{D}_6^+$. This phenomenon does not occur for $n=9$.

The rest of the paper is devoted to the proof of Theorem 2. Let

$$G_F = \bigcup_{C \in \mathcal{C}} C$$

be the partition of the graph of F into connected components, where *at most one C is complete with* $|C| \geq 3$ (by Lemma 2 it corresponds to the canonical subspace $F_C = \langle e_i, i \in C \rangle$ of F).

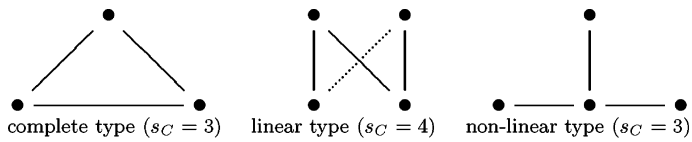

Figure 3. Connected components C such that $r_C = 3$.

For a component $C \in \mathcal{C}$ we denote by

$c = |C|$ the number of vertices of C,
s_C the number of edges of C (or *size* of C),
r_C (*rank of* C) the dimension of $F_C = \langle e_i + e_j, ij$ edge of $C\rangle$. (Of course for isolated vertices $c = 1$ and $s_C = r_C = 0$.)

For example there are three possible components of rank 3.

Contribution of a component. Lemma 2 settles this question for complete components. We now describe the other cases.

Lemma 3 *Let C be a non-complete component of G_F, with $c \geq 2$.*
1. *There exists an integer d_C, $0 \leq d_C \leq c-2$, $d_c \equiv c$ mod 2, such that*

$$s_C = \frac{c^2 - d_C^2}{4} \leq \left\lfloor \frac{c^2}{4} \right\rfloor.$$

2. *F_C admits a basis whose graph is a path linked to a star of degree $d_C + 1$, and its dimension is*

$$r_C = \begin{cases} c-1 & \text{if } c \leq n, \\ c-2 & \text{if } c = n+1\ (\text{which requires } n \text{ odd and } d_C = 0). \end{cases}$$

3. *$s_C = \lfloor \frac{c^2}{4} \rfloor$ only if C is of linear type.*
4. *The following conditions are equivalent*:
   (i) $\sum_{i \in C} e_i \in F_C$
   (ii) $d_C = 0$
   (iii) *C is of linear type with an even number of vertices.*

Proof:

1. Since C is not complete, it does not contain odd cycles. It is thus bipartite (see [2], I.2, Theorem 4), and even by Lemma 1, C is a *complete bipartite graph, i.e. there exists a partition $C = V_0 \cup V_1$ of C such that ij is an edge of C if and only if i and j are in distinct sets V_k*, as we now prove. Indeed, given $i, j \in C$, the lengths of two paths i–j are congruent modulo 2 (otherwise, they would form an odd cycle); then V_0 and

V_1 are the equivalence classes for the equivalence relation $i\mathcal{R}j$ if $i = j$ or if i and j are connected by an even path. Clearly two neighbours in C belong to distinct classes; conversely, if i and j are in distinct classes, there are connected by a path of odd length, and by Lemma 1, ij is an edge of C. We conclude that $s_C = |V_0||V_1| = \frac{c+d_C}{2}\frac{c-d_C}{2}$ where $d_C = ||V_0| - |V_1||$; thus we recover Mantel's bound $\lfloor c^2/4 \rfloor$ for graphs without triangles.

2. From Lemma 1 it follows that the subgraph $G_\mathcal{B}$ associated with any basis of F_C does not contain any cycle. Thus its connected components are trees, $G_\mathcal{B} = T_1 \cup T_2 \cup \cdots \cup T_m$ say. We then have $r_C = \sum_i(|T_i| - 1) = |G_\mathcal{B}| - m \leq c - 1$. Actually, in the case $c = n + 1$ (i.e. $G_F = C$), we must have $r < n = c - 1$, since otherwise $F_C = E_n$ would be canonical. We now define for F_C a standard basis $\mathcal{B}_C$ whose graph is a tree depending only on d_C.

   Put $c = 2p + d_C$ so that the vertex classes of C have respectively p and $p + d_C$ elements; up to permutation by S_{n+1} we may assume them to be

$$\{2k - 1, 1 \leq k \leq p\} \quad \text{and} \quad \{2k - 2, 1 \leq k \leq p\} \cup \{2p + k, 0 \leq k \leq d_C - 1\}.$$

   Then the subspace F_C contains the following $c - 1$ vectors:

$$f_i = \begin{cases} e_{i-1} + e_i & \text{for } 1 \leq i \leq 2p - 1, \\ e_{2p-1} + e_i & \text{for } 2p \leq i \leq c - 1. \end{cases}$$

   For any $(\lambda_i) \in \mathbb{R}^{c-1}$ we have

$$\sum_{i=1}^{c-1} \lambda_i f_i = \lambda_1 e_0 + \sum_{1}^{2p-2} (\lambda_i + \lambda_{i+1}) e_i + \left(\sum_{2p-1}^{c-1} \lambda_i \right) e_{2p-1} + \sum_{2p}^{c-1} \lambda_i e_i.$$

   For any $\lambda \in \mathbb{R}$ we then have the equivalence

$$\sum_{i=1}^{c-1} \lambda_i f_i = \lambda \sum_{i \in C} e_i \Leftrightarrow \begin{cases} \lambda_i = 0 & \text{if } i \in \{1, \ldots, 2p - 1\} \text{ is even}, \\ \lambda_i = \lambda & \text{if } i \in \{1, \ldots, 2p - 1\} \text{ is odd} \\ & \text{or if } i \geq 2p, \\ d_C \lambda = 0. \end{cases} \tag{*}$$

   If $c \leq n$, the e_i, $i \in C$, are independent, thus from (*) with $\lambda = 0$ we obtain that the $c - 1$ vectors f_i are independent, and since $r_C \leq c - 1$, they constitute a basis for F_C, whose rank is $r_C = c - 1$.

   If $c = n + 1$, we know that $r_C \leq c - 2$, and the $c - 1$ vectors f_i must satisfy a non-trivial relation $\sum_{1 \leq i \leq c-1} \lambda_i f_i = 0$. On the other hand, there exists, up to multiplication by a scalar, a unique non-trivial relation between the e_i, $i \in C$: $e_0 + e_1 + \cdots + e_n = 0$. Therefore, using (*) with $\lambda \neq 0$, we obtain $d_C = 0$ and thus $n = 2p - 1$. Conversely, if $d_C = 0$, the n vectors $f_i = e_{i-1} + e_i$, $i = 1, \ldots, n$ satisfy the "unique" relation $f_n = -f_1 - f_3 - \cdots - f_{n-2}$, and $f_1, f_2, \ldots, f_{n-1}$ constitute a basis for $F_C = F$. Its graph is a path of $c - 1 = n$ vertices (which does not span C).

3. It is clear from the previous parts of the lemma, as s_C attains Mantel's bound if and only if $d_C = 0$ or 1.
4. It follows immediately from (*) with $\lambda = 1$.

□

We now compare Mantel's bound $\lfloor \frac{c^2}{4} \rfloor$ to $\binom{r_C}{2}$. The differences $\binom{r_C}{2} - \lfloor \frac{(r_C+1)^2}{4} \rfloor$ and $\binom{r_C}{2} - \frac{(r_C+2)^2}{4}$ are increasing functions of r_C. We can thus state the following.

Lemma 4 *Let C be a non-complete component of G_F of positive rank. Its size s_C and rank r_C satisfy*

$$s_C - \binom{r_C}{2} = \begin{cases} 3 & \textit{if } (n, r_C) = (3,2) \textit{ or } (5,4), \\ 1 & \textit{if } (n, r_C) = (7,6) \textit{ or if } r_C \leq 3 \ (\textit{linear type}), \\ 0 & \textit{if } C \textit{ is complete, or if } r_C = 3 \ (\textit{non-linear type}) \\ & \quad \textit{or if} r_C = 4 \ (\textit{linear type}, n \geq 6), \end{cases}$$

and $s_C < \binom{r_C}{2}$ otherwise.

Right now, Theorem 2 is proved for subspaces F whose graphs contain exactly one component of positive rank. In particular these F realize the bounds $s_3(2)$ (figure 2), $s_n(4)$ (figure 4) and $s_7(6)$ (figure 5).

From now on, we suppose that the graph $G_F = \bigcup_{C \in \mathcal{C}} C$ of F contains at least two components of positive rank.

Dimensions. Consider $x = e_i + e_j \in S_F$. The indices i and j belong to the same connected component C, and thus the vector x belongs to the corresponding subspace F_C. Since S_F spans F, we have $F = \sum F_C$. We first discuss whether this sum is direct.

Lemma 5 *We have $r = \sum_C r_C - \delta$ with $\delta = 0$ or 1, where $\delta = 1$ if and only if all non-complete components $C \in \mathcal{C}$ have linear type and odd rank.*

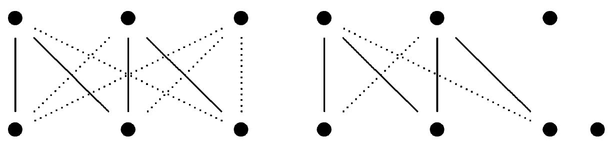

Figure 4. Linear types [4] ($s_F = 9$) and [4, 0, 0] ($s_F = 6$).

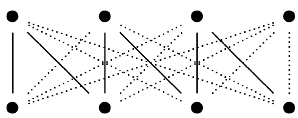

Figure 5. Linear type [6] ($s_F = 16$).

Proof: Note that for $|C| \geq 2$ the typical vector of F_C can be written $x_C = \sum_{i \in C} \lambda_i e_i$. Since any relation of dependence between the e_i has the form $\lambda(e_0 + e_1 + \cdots + e_n) = 0$ for some $\lambda \in \mathbb{R}$, and since $\mathcal{C}$ is a partition of $\{0, 1, \ldots, n\}$, we have the following equivalence:

$$\left(\sum_{C \in \mathcal{C}} x_C = 0,\ x_C \in F_C \right) \Longleftrightarrow \left(\exists \lambda \in \mathbb{R} \mid \forall C \in \mathcal{C} : x_C = \lambda \sum_{i \in C} e_i \right).$$

Using the last part of Lemma 3, we are left with two possibilities:

(1) there is an isolated vertex or a non-complete component with invariant $d_C \neq 0$: the above λ is null, and $F = \oplus F_C$.
(2) every non-complete component C has order $|C| \geq 2$ and invariant $d_C = 0$: then for all $C \in \mathcal{C}$, the nonzero vector $x_C = \sum_{i \in C} e_i$ belongs to F_C, and we have a non-trivial relation $\sum x_C = 0$. As this is up to scale the only one, we have $\dim(\sum F_C) = \sum \dim F_C - 1$.

□

Proof of Theorem 2: We have to compare the size $s_F = \sum r_C$ of G_F to the dimension $r = \sum r_C - \delta$ of F.

Type $[1, \ldots, 1, 0, \ldots, 0]$ *case*: the graph G_F consists of $1 \leq k \leq \frac{n+1}{2}$ paths of length 1 and of $n + 1 - k$ isolated vertices; it then has $s_F = k$ edges, while by Lemma 5, $r = k - \delta$, where $\delta = 1$ if and only if there are no isolated vertices i.e. $n = 2k - 1 = 2r + 1$. Thus

$$s_F = r + \delta = \begin{cases} r & \text{if } n \geq 2r,\ n \neq 2r + 1 \\ r + 1 & \text{if } n = 2r + 1 \end{cases}$$

is $< \binom{r}{2}$ if and only if $r \geq 4$. In contrast, this linear type realizes the values $s_n(1)$, $s_n(2)$ (in particular $s_5(2) = 3$) and $s_7(3)$.
General case: $\max_C r_C = r_0 \geq 2$. From Lemma 4 we deduce

$$s_F = \sum s_C \leq \sum \binom{r_C}{2} + k$$

where k denotes the number of components C of linear type and rank $r_C \leq 3$. Now, writing $\sum (r_C^2 - r_C) = (\sum r_C)^2 - \sum r_C - 2 \sum_{C \neq C'} r_C r_{C'}$ where $\sum r_C = r + \delta$ by Lemma 5, we obtain the inequality

$$\binom{r}{2} - s_F \geq \sum_{C \neq C'} r_C r_{C'} - k - \delta r, \tag{3}$$

which, by Lemma 4, is strict if there is a non-complete component of rank $r_C > 4$.

If $\delta = 0$, (3) implies $s_F \leq \binom{r}{2}$ (equality only for $r = 3$) as stated in Theorem 2. Indeed

if $k \leq 2$: $\sum_{C \neq C'} r_C r_{C'} - k \geq r_0 - k \geq r_0 - 2 \geq 0$, equality holding only for F of type [2, 1] (and $r = 3$);
if $k \geq 3$: $\sum_{C \neq C'} r_C r_{C'} - k \geq r_0(k-1) - k \geq 2(k-1) - k \geq 1$.

From now on we suppose $\delta = 1$: all non-complete $C \in \mathcal{C}$ are of linear type with odd ranks. In particular, we have $r_0 \geq 3$. We write $G_F = C_0 \cup C_1 \cup \cdots \cup C_m$ $(m \geq 1)$ with $r_0 \geq r_1 \geq r_m \geq 1$ and $\sum |C_i| = n + 1$.

We shall use for $r + 1 = r_0 + r_1 + \cdots + r_m$ and $\sum r_C r_{C'} = r_0 r_1 + \cdots$ the estimations

$$r + 1 \geq r_0 + k - 1, \tag{4}$$
$$\sum r_C r'_C \geq r_0(r + 1 - r_0). \tag{5}$$

Note that the equality in (4) (resp. (5)) holds if and only if F is of linear type [3, 1, ..., 1] (resp. $m = 1$). We then obtain the estimation

$$\binom{r}{2} - s_F \geq M,$$

with

$$M = (r - r_0)(r_0 - 2) - 2,$$

where $r - r_0 = r_1 + \cdots + r_m - 1 \geq m - 1 \geq 0$ and $r_0 - 2 \geq 1$. We then have $M \geq -2$. We even obtain $M > 0$ (i.e. $s_F < \binom{r}{2}$) if $r - r_0 \geq 3$. We now concentrate on the three cases $0 \leq r - r_0 \leq 2$.

(a) $r = r_0$: $G_F = C_0 \cup C_1$, with $r_1 = 1$. If C_0 is complete, F is a canonical hyperplane and $s_F = \binom{r}{2} + 1$ as asserted in Theorem 2. If C_0 is linear of rank 3, 5, 7, ..., it follows from Lemma 3 that $s_F = 1 + s_{C_0}$ is equal to $1 + (r_0 + 1)^2/4 = 5, 10, 17, \ldots$, strictly smaller than $\binom{r}{2}$ except for the cases [3, 1] (which realizes the maximum $s_5(3)$) and [5, 1] (which realizes $s_n(5)$), see figure 6.
(b) $r = r_0 + 1$: $G_F = C_0 \cup C_1 \cup C_2$, with $r_1 = r_2 = 1$. Since (5) is no more an equality, we obtain $\binom{r}{2} - s_F \geq M + 1 = r_0 - 3 \geq 0$, where the equality requires that equality (4) holds, i.e. that F is of type [3, 1, 1], which indeed realizes the maximum $s_7(4) = 6$ (figure 7).
(c) $r = r_0 + 2$, i.e. $r_1 + \cdots + r_m = 3$. There are two occurrences of this situation: $m = 3, r_1 = r_2 = r_3 = 1$, or $m = 1, r_1 = 3$. In the first case, equality (5) does not

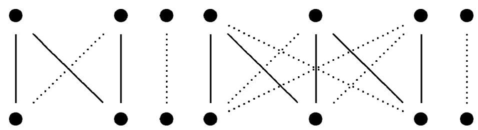

Figure 6. Linear types [3, 1] and [5, 1].

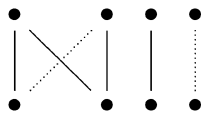

Figure 7. Type [3, 1, 1].

hold. In the second case, (4) does not hold. Anyway, we have $\binom{r}{2} - s_F \geq M + 1 = 2r_0 - 5 > 0$.

This completes the proof of Theorem 2. □

References

1 C. Batut and J. Martinet, "Radiographie des réseaux parfaits", *Experimental Math.* **3** (1994), 39–49.
2 B. Bollobás, *Modern Graph Theory*, Graduate texts in Mathematics **184**, Springer-Verlag, Heidelberg, 1998.
3 A.M. Bergé and J. Martinet, " Symmetric groups and lattices", *Monatschefte für Math.* **140**(3) (2003), 179–195.
4 J.H. Conway and N.J.A. Sloane, "Low-dimensional lattices. III. Perfect forms", *Proc. Royal Soc. London A* **418** (1988), 43–80.
5 J. Martinet, personal communication.
6 J. Martinet, *Perfect Lattices in Euclidean Spaces*, Grundlehren **327**, Springer-Verlag, Heidelberg, 2003.
7 J. Martinet and B. Venkov, "On integral lattices having an odd minimum," *Algebra and Analysis*, Saint-Petersburg **16**(3) (2004), 198–237.