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On Negative Orbits of Finite Coxeter Groups

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Abstract. For a Coxeter group W, X a subset of W and α a positive root, we define the negative orbit of α under X to be $\{w \cdot \alpha \mid w \in X\} \cap \Phi^-$, where Φ^- is the set of negative roots. Here we investigate the sizes of such sets as α varies in the case when W is a finite Coxeter group and X is a conjugacy class of W.

Keywords: Coxeter group, root system

1. Introduction

Suppose W is a Coxeter group, accompanied by its usual entourage of Φ, its system of roots, together with Π, Φ^+ and Φ^-, respectively the fundamental roots, the positive and the negative roots. Let X be a subset of W and let $\alpha \in \Phi^+$. We define the *negative orbit*, $X^-(\alpha)$, of α under X to be

$$X^-(\alpha) = \{w \cdot \alpha \mid w \in X\} \cap \Phi^-.$$

Set $n_X^-(\alpha) = |X^-(\alpha)|$. In this paper we consider the case when W is a finite Coxeter group and X is a conjugacy class of W, and will describe how the size of $X^-(\alpha)$ varies as α runs through Φ^+. This, we remark, is somewhat unusual as we are considering orbits under sets which are (usually) not groups. This work is, in fact, a serendipitous spin-off of [5] where a more general notion of 'Coxeter length' was introduced. Our main results are as follows.

Theorem 1.1 *Suppose W is a finite crystallographic Coxeter group. Let $\alpha = \sum_{\alpha_r \in \Pi} \lambda_r \alpha_r$ and $\beta = \sum_{\alpha_r \in \Pi} \mu_r \alpha_r$ be positive roots in the same orbit $\Phi(\alpha) := W \cdot \alpha$ of Φ. Then for a conjugacy class X of W, there exists a constant $f(X) \in \{0, \pm 1\}$ dependent only on X and $\Phi(\alpha)$ such that*

$$n_X^-(\alpha) - n_X^-(\beta) = f(X)\left(\sum_{\alpha_r \in \Pi \cap \Phi(\alpha)} \lambda_r - \sum_{\alpha_r \in \Pi \cap \Phi(\alpha)} \mu_r\right).$$

Table 1. Values of n_X^- in A_4.

Representative	$n_X^-(\alpha), \alpha \in \Phi^+$
1	0, 0, 0, 0, 0, 0, 0, 0, 0, 0
(1 2)	4, 4, 4, 4, 3, 3, 3, 2, 2, 1
(1 2) (3 4)	7, 7, 7, 7, 6, 6, 6, 5, 5, 4
(1 2 3)	6, 6, 6, 6, 6, 6, 6, 6, 6, 6
(1 2 3 4)	9, 9, 9, 9, 9, 9, 9, 9, 9, 9
(1 2 3 4 5)	6, 6, 6, 6, 7, 7, 7, 8, 8, 9
(1 2 3) (4 5)	7, 7, 7, 7, 8, 8, 8, 9, 9, 10

To better illustrate Theorem 1.1, we give $n_X^-(\alpha)$ for all positive roots α and all conjugacy classes X in the Coxeter groups A_4 (Table 1) and D_4 (Table 2). For each class a representative of the class is given, then a list of $n_X^-(\alpha)$, for $\alpha \in \Phi^+$, in increasing order of height. The 'signed cycle' notation for the class representatives is explained in Section 2.

Put slightly differently, for W a finite, simply laced crystallographic Coxeter group, if we fix a conjugacy class X, and order the roots according to height then the sequence of integers $\{n_X^-(\alpha)\}_{\alpha\in\Phi^+}$ is either constant or monotonic increasing or monotonic decreasing. This is remarkably uniform behaviour. In stark contrast we have the non-crystallographic finite Coxeter groups H_3 and H_4 which appear on the one hand chaotic yet there may still

Table 2. Values of n_X^- in D_4.

Representative	$n_X^-(\alpha), \alpha \in \Phi^+$
1	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
$(\overset{-}{1})(\overset{-}{2})(\overset{-}{3})(\overset{-}{4})$	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
$(\overset{-}{1})(\overset{-}{2})(\overset{+}{3})(\overset{+}{4})$	2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2
$(\overset{+}{1}\ \overset{+}{2})(\overset{+}{3}\ \overset{+}{4})$	2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2
$(\overset{-}{1}\ \overset{-}{2})(\overset{+}{3}\ \overset{+}{4})$	2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2
$(\overset{+}{1}\ \overset{+}{2})(\overset{+}{3})(\overset{+}{4})$	5, 5, 5, 5, 4, 4, 4, 3, 3, 3, 2, 1
$(\overset{+}{1}\ \overset{+}{2})(\overset{-}{3})(\overset{-}{4})$	5, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 9
$(\overset{+}{1}\ \overset{+}{2}\ \overset{+}{3})(\overset{+}{4})$	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8
$(\overset{-}{1}\ \overset{+}{2})(\overset{-}{3}\ \overset{+}{4})$	3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3
$(\overset{-}{1}\ \overset{+}{2})(\overset{-}{3})(\overset{+}{4})$	9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9
$(\overset{+}{1}\ \overset{+}{2}\ \overset{+}{3}\ \overset{+}{4})$	9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9
$(\overset{-}{1}\ \overset{-}{2}\ \overset{+}{3}\ \overset{+}{4})$	9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9, 9
$(\overset{-}{1}\ \overset{+}{2}\ \overset{+}{3})(\overset{-}{4})$	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8

be some underlying patterns. In Section 4 we tabulate the values of n_X^- for H_3 as well as those for B_3 and F_4. The latter two groups, on account of Φ having two W-orbits, exhibit a wider range of behaviour.

In the remainder of this section we summarize some well-known properties of finite Coxeter groups and their root systems. Let W be a finite Coxeter group and R its distinguished set of fundamental reflections. The length $l(w)$ of a non-trivial element w in W is defined to be

$$l(w) = \min\{l \in \mathbb{N} \mid w = r_1 r_2 \cdots r_l \text{ some } r_i \in R\}$$

and $l(1) = 0$. For $r, s \in R$, m_{rs} $(= m_{sr})$ denotes the order of rs (so $m_{rr} = 1$ for all $r \in R$). Let V be an $\mathbb{R}$-vector space with basis Π, where $\Pi = \{\alpha_r \mid r \in R\}$ is in one-to-one correspondence with R. For $\alpha_r, \alpha_s \in \Pi$ we define

$$\langle \alpha_r, \alpha_s \rangle = -\cos(\pi/m_{rs}),$$

and this extends to an inner product on V in the standard way. Defining, for $r \in R$, $v \in V$,

$$r \cdot v = v - 2\langle v, \alpha_r \rangle \alpha_r$$

yields a faithful action of W on V which also preserves the inner product $\langle , \rangle$ (see [4], Section 5.4). The *root system* Φ of W in V is defined to be the set $\{w \cdot \alpha_r \mid w \in W, r \in R\}$. Put $V^+ = \{\sum_{r \in R} \lambda_r \alpha_r \in V \mid \lambda_r \geq 0 \text{ for all } r \in R\}$, $\Phi^+ = \Phi \cap V^+$ and $\Phi^- = -\Phi^+$. The sets Φ^+ and Φ^- are called, respectively, the positive and negative roots of Φ and it is well known that $\Phi = \Phi^+ \dot\cup \Phi^-$ (again, see [4] Section 5.4). The elements in $\{wrw^{-1} \mid r \in R, w \in W\}$ are referred to as the reflections of W.

Remark We have chosen here to define root systems in terms of unit vectors. If W is crystallographic (that is, it stabilizes a lattice in $\mathbb{R}^n$), it is usual to work with a slightly different definition of the root system within which, for $r \in R$, $r \cdot \alpha$ differs from α by an integer multiple of α_r. Such root systems may require roots of different lengths. However, our results do not depend upon root length and so, since we discuss the groups H_3 and H_4, we use a definition of root system that does not require W to be crystallographic.

For $w \in W$ we define the following subset of Φ^+: $N(w) = \{\alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^-\}$.

Proposition 1.2 $l(w) = |N(w)|$ *for all* $w \in W$.

We recall the notion of *depth* and *height* of a positive root.

Definition 1.3 For each $\alpha = \sum_{r \in R} \lambda_r \alpha_r \in \Phi^+$ the *depth* of α (relative to R) is $\mathrm{dp}(\alpha) = \min\{l \in \mathbb{N} \mid w \cdot \alpha \in \Phi^- \text{ for some } w \in W \text{ with } l(w) = l\}$ and the *height* of α (relative to R) is $\mathrm{ht}(\alpha) = \sum_{r \in R} \lambda_r$.

There is a connection between depth and inner products as given in the next proposition.

Proposition 1.4 *Let $r \in R$ and $\alpha \in \Phi^+ - \{\alpha_r\}$. Then*

$$\mathrm{dp}(r \cdot \alpha) = \begin{cases} \mathrm{dp}(\alpha) - 1 & \text{if } \langle \alpha, \alpha_r \rangle > 0, \\ \mathrm{dp}(\alpha) & \text{if } \langle \alpha, \alpha_r \rangle = 0, \\ \mathrm{dp}(\alpha) + 1 & \text{if } \langle \alpha, \alpha_r \rangle < 0. \end{cases}$$

Proof: See [1], Lemma 1.7. $\square$

It can be shown that depth is a partial order on the positive roots. We end this section with the following elementary, but key, observation.

Lemma 1.5 *Suppose X is a conjugacy class of W, $r \in R$ and let α, β be positive roots for which $\beta = r \cdot \alpha$. Then*

$$n_X^-(\beta) = \begin{cases} n_X^-(\alpha) + 1 & \text{if } \alpha_r \in X \cdot \alpha, -\alpha_r \notin X \cdot \alpha \\ n_X^-(\alpha) - 1 & \text{if } \alpha_r \notin X \cdot \alpha, -\alpha_r \in X \cdot \alpha \\ n_X^-(\alpha) & \text{if either } \alpha_r \in X \cdot \alpha \text{ and } -\alpha_r \in X \cdot \alpha \\ & \text{or } \alpha_r \notin X \cdot \alpha \text{ and } -\alpha_r \notin X \cdot \alpha \end{cases}$$

Proof: Since X is a conjugacy class, we have $rXr = X$. Thus $X \cdot \beta = rXr \cdot \beta = rX \cdot \alpha$. Suppose $\alpha_r \in X \cdot \alpha$ and $-\alpha_r \notin X \cdot \alpha$. Then $r \cdot \alpha_r = -\alpha_r \in X \cdot \beta$. Also $-\alpha_r \notin X \cdot \alpha$. Hence $X^-(\beta) = rX \cdot (\alpha) \cap \Phi^- \supseteq rX^-(\alpha)$. That is $X^-(\beta) = rX^-(\alpha) \cup \{-\alpha_r\}$ and so $n_X^-(\beta) = n_X^-(\alpha) + 1$. The other parts of the lemma follow in a similar fashion. $\square$

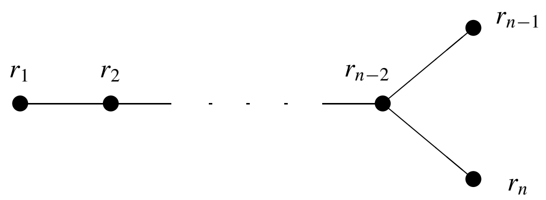
2. The classical groups

For a Coxeter group W, the Coxeter graph Γ of W is the (labelled) graph whose vertex set is R and an edge labelled m_{rs} joins $r, s \in R$ whenever $m_{rs} \geq 3$. If Γ is a connected graph, then we say that W is *irreducible*. Let $W = W_n$ be a Coxeter group of type either A_n, B_n or D_n. The corresponding Coxeter graphs are given below.

$A_n (n \geq 1)$

$B_n (n \geq 2)$

$D_n (n \geq 4)$



We may regard A_{n-1} and D_n as subgroups of B_n, and their root systems as subsystems of the root system of B_n, by considering their action on the vector space $\mathbb{R}^m$. Let $e_1, \ldots, e_n$ be an orthogonal basis of $\mathbb{R}^m$ and define $\varepsilon_i = \frac{\sqrt{2}}{2} e_i$ for $1 \leq i \leq n$. Then the permutation group S_m acts on the basis $\varepsilon_1, \ldots, \varepsilon_m$ by permuting the indices of the vectors ε_i. It is clear that, for $1 \leq i < j \leq m$, $(i \;\; j) \cdot (\varepsilon_i - \varepsilon_j) = -(\varepsilon_i - \varepsilon_j)$. Let $m = n + 1$. The group A_n is isomorphic to S_{n+1}. We may set $r_i = (i \;\; i + 1)$ for each $1 \leq i \leq n$. It is easy to check that these elements satisfy all the relations given in the Coxeter graph and do indeed generate S_{n+1}. We may now set the fundamental root corresponding to the fundamental reflection r_i to be $\varepsilon_i - \varepsilon_{i+1}$. The set of positive roots is then given by

$$\Phi^+_{A_n} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n + 1\}.$$

Now let $m = n$ and consider the 'sign change' reflection sending ε_i to $-\varepsilon_i$ and fixing all other ε_j. The set of such reflections generates a group of order 2^n, isomorphic to $(\mathbb{Z}_2)^n$. It is well known that B_n may be thought of as the semidirect product of this group and S_n. We write elements of $(\mathbb{Z}_2)^n$ as n-vectors of 0's or 1's. A general element of B_n is of the form (σ, g) with $\sigma \in S_n$ and $g = (g_1, \ldots, g_n) \in (\mathbb{Z}_2)^n$. Its action on $\mathbb{R}^n$ is given by

$$(\sigma, g) \cdot \sum_{i=1}^{n} \lambda_i \varepsilon_i = \sum_{i=1}^{n} (-1)^{g_i} \lambda_i \varepsilon_{\sigma(i)}.$$

Writing $\underline{0}$ for the identity in $(\mathbb{Z}_2)^n$, we set $r_i = ((i \;\; i + 1), \underline{0})$ for $1 \leq i \leq n - 1$. These are precisely the elements chosen to generate A_{n-1}. Let r_n be the reflection sending ε_n to $-\varepsilon_n$; in our notation r_n is $(1, (0, \ldots, 0, 1))$. All the Coxeter relations for B_n are satisfied with this choice of fundamental reflections and we may take the unit vectors $\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n$ and e_n as the set of fundamental roots. The set of positive roots is then

$$\Phi^+_{B_n} = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}.$$

For $n \geq 4$, D_n is the subgroup (of index 2) of B_n generated by S_n and the elements of $(\mathbb{Z}_2)^n$ involving an even number of sign changes. In terms of the semidirect product, it is the subgroup whose elements (σ, g) all have an even number of 1's in the expression for g. The following elements can be shown to generate D_n and obey all the relations in the D_n Coxeter graph:- $r_i = ((i\ \ i+1), \underline{0})$ for $1 \leq i \leq n-1$, and $r_n = ((n-1\ n), (0, \dots, 0, 1, 1))$. For $1 \leq i \leq n-1$ the fundamental root corresponding to r_i is $\varepsilon_i - \varepsilon_{i+1}$ and the fundamental root corresponding to r_n is $\varepsilon_{n-1} + \varepsilon_n$. The set of positive roots for D_n is

$$\Phi^+_{D_n} = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}.$$

Since both A_{n-1} and D_n may be thought of as subgroups of B_n, we will set $W = W_n$ to be A_{n-1}, B_n or D_n, and write an element of W as an element (σ, g) of B_n, though of course if $W = A_{n-1}$ then we have $g = \underline{0}$. For ease of notation, we will use a more concise way of expressing elements of W, as follows. Given $(\sigma, (g_1, \dots, g_n)) \in W$ we suppress mention of $(g_1, \dots, g_n)$ and write a plus sign above i (in its occurrence in σ) if $g_i = 0$ and a minus sign above i if $g_i = 1$. We say i is positive or negative accordingly. In this scheme, for example, the element $((1\ \ 3\ \ 2)(4), (1, 0, 0, 0))$ of B_4 will be written $(\overset{-}{1}\ \overset{+}{3}\ \overset{+}{2})(\overset{+}{4})$. Expressing σ as a product of disjoint cycles, we say that a cycle $(i_1 \cdots i_r)$ of σ is *positive* if there is an even number of minus signs above its elements, and *negative* if the number of minus signs is odd. In our example, $(\overset{-}{1}\ \overset{+}{3}\ \overset{+}{2})$ is a negative cycle, whereas $(\overset{+}{4})$ is positive. We now define the *signed cycle type* of an element of W to be the cycle type with a $(+)$ or a $(-)$ over each cycle, according as it is positive or negative (cycles of length 1 must be included). We will omit positive 1-cycles where possible, so for example $(\overset{+}{1}\ \overset{+}{2})$ will be taken to mean $(\overset{+}{1}\ \overset{+}{2})(\overset{+}{3})(\overset{+}{4}) \cdots (\overset{+}{n})$. A $*$ above a number will indicate 'plus or minus'.

We may now state the following well known result (see [3]).

Proposition 2.1

(i) *Let $W = A_n$. Then two elements of W are conjugate if and only if they have the same cycle type.*

(ii) *Elements of B_n are conjugate if and only if they have the same* signed *cycle type.*

(iii) *Conjugacy classes in D_n are parameterised by signed cycle type, with one class for each signed cycle type except in the case where the signed cycle type contains only even length, positive cycles, in which case there are two classes for each signed cycle type.*

For the remainder of this section, let W be one of A_n, B_n and D_n and X a conjugacy class of W, with w an arbitrary element of X. Let $\alpha, \beta \in \Phi^+$ with $\mathrm{dp}(\alpha) > \mathrm{dp}(\beta)$ and $r \cdot \alpha = \beta$ for some $r \in R$. We will consider the possible forms for α, that is $\varepsilon_i - \varepsilon_j$, or $\varepsilon_i + \varepsilon_j$, for some $i < j$, or e_i, and in each case, with Lemma 1.5 in mind, establishing whether or not it is possible, for some $x \in X$, to have $x \cdot \alpha = \pm\alpha_r$.

We begin with the possibility that $\alpha = \varepsilon_i - \varepsilon_j$.

Lemma 2.2 *Suppose $\alpha = \varepsilon_i - \varepsilon_j$, for some $i < j$. Then $\alpha_r \in X \cdot \alpha$ whenever w has a positive 1-cycle and a cycle of length at least 2, or, if W is either B_n or D_n, whenever w has a cycle of length at least 3. Also, $-\alpha_r \in X \cdot \alpha$ whenever w has a cycle of length at least*

3, *or, if* W *is either* B_n *or* D_n, *whenever* w *has a negative* 1*-cycle and a cycle of length at least two.*

Proof: Since $\mathrm{dp}(r \cdot \alpha) < \mathrm{dp}(\alpha)$, the only possible r are $(\overset{+}{i}\ \overset{+}{i+1})$ or $(\overset{+}{j-1}\ \overset{+}{j})$, giving $\beta = \varepsilon_{i+1} - \varepsilon_j$ or $\varepsilon_i - \varepsilon_{j-1}$ respectively. We will consider the former case, identical arguments being employed to deal with the latter. So let $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_{i+1} - \varepsilon_j$ and $r = (\overset{+}{i}\ \overset{+}{i+1})$. We wish to find some $x \in X$ for which $x \cdot \alpha = \alpha_r$, that is, $x \cdot (\varepsilon_i - \varepsilon_j) = \varepsilon_i - \varepsilon_{i+1}$. Clearly then, either $x \cdot \varepsilon_i = \varepsilon_i$ and $x \cdot \varepsilon_j = \varepsilon_{i+1}$ or $x \cdot \varepsilon_i = -\varepsilon_{i+1}$ and $x \cdot \varepsilon_j = -\varepsilon_i$.

In the first case x is forced to have a positive 1-cycle $(\overset{+}{i})$ and a cycle $(\overset{+}{j}\ \overset{*}{i+i}\ \dots)$. The fact that we require a 1-cycle means that the class X contains all elements of the same signed cycle type as w (that is, the class does not split in D_n). Hence if w has a positive 1-cycle and a cycle of length at least 2, we may manufacture an element x of the required form (choosing $*$ to be a plus or a minus as required). In the second case, where $x \cdot \varepsilon_i = -\varepsilon_{i+1}$ and $x \cdot \varepsilon_j = -\varepsilon_i$, a cycle in x of the form $(\overset{-}{j}\ \overset{-}{i}\ \overset{*}{i+1}\ \dots)$ is required. This immediately eliminates A_n from our enquiries. Suppose that W is of type B_n or D_n and that w has a cycle of length at least 3. In B_n, since all elements of the same signed cycle type are conjugate, we are done. In D_n, if the cycle is a 3-cycle then the class does not split so again we are done. If the cycle is at least a 4-cycle then w is conjugate to an element containing at least one of the cycles $(\overset{-}{j}\ \overset{-}{i}\ \overset{+}{i+1}\ \overset{+}{k}\ \dots)$, $(\overset{-}{j}\ \overset{-}{i}\ \overset{+}{i+1}\ \overset{-}{k}\ \dots)$, $(\overset{-}{j}\ \overset{-}{i}\ \overset{-}{i+1}\ \overset{+}{k}\ \dots)$ or $(\overset{-}{j}\ \overset{-}{i}\ \overset{-}{i+1}\ \overset{-}{k}\ \dots)$(for some k), any of which will suffice for the required x. Thus the classes for which we may find x such that $x \cdot \alpha = \alpha_r$ are as described in the lemma.

Suppose that $x \cdot \alpha = -\alpha_r = \varepsilon_{i+1} - \varepsilon_i$. Then either $x \cdot \varepsilon_i = \varepsilon_{i+1}$ and $x \cdot \varepsilon_j = \varepsilon_i$ or $x \cdot \varepsilon_i = -\varepsilon_i$ and $x \cdot \varepsilon_j = -\varepsilon_{i+1}$. In the first case all we require is a cycle $(\overset{+}{j}\ \overset{+}{i}\ \overset{*}{i+1}\ \dots)$. This can be arranged whenever w has a cycle of length at least 3. For the second case we must have a cycle $(\overset{-}{i})$ and a cycle $(\overset{-}{j}\ \overset{*}{i+1}\ \dots)$. This cannot occur in A_n. Since we have a 1-cycle, the class cannot split in D_n, so an appropriate x will exist whenever w has a negative 1-cycle as well as a cycle of length at least 2. □

Proposition 2.3 *Let* $W = A_n$, X *a conjugacy class of* W *and* w *an arbitrary element of* X. *Then*

$$n_X^-(\beta) = \begin{cases} n_X^-(\alpha) + 1 & \textit{if } w \textit{ is an involution with a } 1\textit{-cycle} \\ n_X^-(\alpha) - 1 & \textit{if } w \textit{ has a cycle of length at least } 3, \textit{ and no } 1\textit{-cycles} \\ n_X^-(\alpha) & \textit{otherwise.} \end{cases}$$

Proof: The root system of A_n only contains roots of the form $\varepsilon_i - \varepsilon_j$, so we may apply Lemma 2.2 to any root α. Now by Lemma 1.5, $n_X^-(\beta) = n_X^-(\alpha) + 1$ whenever $\alpha_r \in X \cdot \alpha$ and $-\alpha_r \notin X \cdot \alpha$. By Lemma 2.2 this occurs whenever w has a 1-cycle and a cycle of length at least 2, but no cycles of length 3 or above. That is, w must be an involution with a 1-cycle. The other statements follow in a similar manner. □

We now concentrate on the groups B_n and D_n:

Lemma 2.4 *Let W be of type B_n or D_n, X a conjugacy class of W and w an arbitrary element of X. Suppose additionally that $\alpha = \varepsilon_i - \varepsilon_j$. Then*

$$n_X^-(\beta) = \begin{cases} n_X^-(\alpha)+1 & \text{if } w \text{ has a positive } 1\text{-cycle but no negative } 1\text{-cycles,} \\ & \text{and all the cycles of } w \text{ are at most } 2\text{-cycles;} \\ n_X^-(\alpha) - 1 & \text{if } w \text{ has a negative } 1\text{-cycle but no positive } 1\text{-cycles,} \\ & \text{and all the cycles of } w \text{ are at most } 2\text{-cycles;} \\ n_X^-(\alpha) & \text{otherwise.} \end{cases}$$

Proof: The result follows from Lemmas 1.5 and 2.2 □

We next consider the case $\alpha = \varepsilon_i + \varepsilon_j$, $i < j$. The possibilities for r here are $(\overset{+}{i}\; \overset{+}{i+1})$ (if $i + 1 \neq j$), $(\overset{+}{j}\; \overset{+}{j+1})$, or $(\overset{-}{n})$ (in B_n), $(\overset{-}{n-1}\; \overset{-}{n})$ (in D_n).

Lemma 2.5 *Suppose $\alpha = \varepsilon_i + \varepsilon_j$, for some $i < j$ and $r = (\overset{+}{k}\; \overset{+}{k+1})$, some k. Then $\alpha_r \in X \cdot \alpha$ whenever w has a positive 1-cycle and a cycle of length at least 2, or whenever w has a cycle of length at least 3, except in the class of $(\overset{+}{1}\; \overset{+}{2}\; \overset{+}{3}\; \overset{+}{4})$ in D_4. Also, $-\alpha_r \in X \cdot \alpha$ whenever w has a cycle of length at least 3, except in the class of $(\overset{+}{1}\; \overset{+}{2}\; \overset{+}{3}\; \overset{+}{4})$ in D_4, or whenever w has a negative 1-cycle and a cycle of length at least two.*

Proof: We will assume that $\alpha_r = \varepsilon_i - \varepsilon_{i+1}$ (the other case is similar). If $x \cdot \alpha = \alpha_r$, that is $x \cdot (\varepsilon_i + \varepsilon_j) = \varepsilon_i - \varepsilon_{i+1}$, then either $x \cdot \varepsilon_i = \varepsilon_i$ and $x \cdot \varepsilon_j = -\varepsilon_{i+1}$ or $x \cdot \varepsilon_i = -\varepsilon_{i+1}$ and $x \cdot \varepsilon_j = \varepsilon_i$. We see that either x has cycles $(\overset{+}{i})$ and $(\overset{-}{j}\; \overset{*}{i+1} \ldots)$ or a cycle $(\overset{+}{j}\; \overset{-}{i}\; \overset{*}{i+1} \ldots)$. In the first case, because the conjugacy class X cannot split in D_n (we have a 1-cycle), such an x will occur in X whenever w has a positive 1-cycle and a cycle of length at least 2. In the second case, suppose w has a cycle of length at least 3. Since the sign above $*$ is arbitrary, we are done in all cases except those where there are two classes with the same signed cycle type as w, that is, elements whose cycles are all positive and of even length. If we have a cycle of length $l \geq 6$ then we may choose x to contain either $(\overset{+}{j}\; \overset{-}{i}\; \overset{-}{i+1}\; \overset{-}{k_1}\; \overset{-}{k_2}\; \overset{-}{k_3}\; \overset{+}{k_4} \ldots \overset{+}{k_l})$ or $(\overset{+}{j}\; \overset{-}{i}\; \overset{-}{i+1}\; \overset{+}{k_1} \ldots \overset{+}{k_l})$ as appropriate. If w has another cycle C then we may fix the number of minus signs in the corresponding cycle of x so that w and x are conjugate. Therefore the only exception is the conjugacy class of $(\overset{+}{1}\; \overset{+}{2}\; \overset{+}{3}\; \overset{+}{4})$ in D_4, within which it is clear that the required x cannot lie.

Similar considerations show that there is an element x of X such that $x \cdot \alpha = -\alpha_r$ whenever w has a cycle of length at least 3, again except in the class of $(\overset{+}{1}\; \overset{+}{2}\; \overset{+}{3}\; \overset{+}{4})$ in D_4, or whenever w has a negative 1-cycle and a cycle of length at least two. □

In the group D_n, it only remains to consider $\alpha = \varepsilon_i - \varepsilon_j$ and $r = (\overset{-}{n-1}\; \overset{-}{n})$. The only way that r can reduce the depth of α is if j is either $n - 1$ or n. To clarify this, we give the relevant part of the root system below, with the roots arranged by depth (which coincides with height here):

$$\begin{tikzcd}[column sep=small]
 & \varepsilon_i + \varepsilon_{n-1} \arrow[dl, "(\overline{n-1}\;\overline{n})"'] \arrow[dr, "(\overset{+}{n-1}\;\overset{+}{n})"] & \\
\varepsilon_i - \varepsilon_n \arrow[dr, "(\overset{+}{n-1}\;\overset{+}{n})"'] & & \varepsilon_i + \varepsilon_n \arrow[dl, "(\overline{n-1}\;\overline{n})"] \\
 & \varepsilon_i - \varepsilon_{n-1} &
\end{tikzcd}$$

We have therefore to consider the cases $\alpha = \varepsilon_i + \varepsilon_{n-1}$ and $\alpha = \varepsilon_i + \varepsilon_n$; the results are given in the next lemma. The proof is similar to those seen above, and is therefore omitted.

Lemma 2.6 *Let $W = D_n$, $\alpha = \varepsilon_i + \varepsilon_j$ ($j = n-1$ or n), and $r = (\overline{n-1}\;\overline{n})$. Then $\alpha_r \in X \cdot \alpha$ whenever w has a positive 1-cycle and a cycle of length at least 2, or whenever w has a cycle of length at least 3. Also, $-\alpha_r \in X \cdot \alpha$ whenever w has a cycle of length at least 3, or whenever w has a negative 1-cycle and a cycle of length at least two.*

We now have

Proposition 2.7 *Let $W = D_n$, with X a conjugacy class of W and w an arbitrary element of X. If w has a cycle of length at least 3 then $n_X^-(\beta) = n_X^-(\alpha)$. If not, then we have*

$$n_X^-(\beta) = \begin{cases} n_X^-(\alpha) + 1 & \text{if } w \text{ has a 2-cycle, a positive 1-cycle but no negative 1-cycles,} \\ n_X^-(\alpha) - 1 & \text{if } w \text{ has a 2-cycle, a negative 1-cycle but no positive 1-cycles,} \\ n_X^-(\alpha) & \text{otherwise.} \end{cases}$$

Proof: The result follows from Lemmas 1.5, 2.4, 2.5 and 2.6. □

We are now left only with the group B_n.

Lemma 2.8 *Let $W = B_n$ and X a conjugacy class of W. If $\alpha = \varepsilon_i \pm \varepsilon_j$ ($i < j$) and $r = (\overline{n})$ then $\pm\alpha_r \notin X \cdot \alpha$. In addition, if $\alpha = e_i$ then, for any r with $\mathrm{dp}(r \cdot \alpha) < \mathrm{dp}(\alpha)$, $\pm\alpha_r \notin X \cdot \alpha$.*

Proof: There are two W-orbits in the root system of B_n. One contains all roots of the form $\pm\varepsilon_i \pm \varepsilon_j$ and the other, all roots of the form $\pm e_i$. So there is no $x \in W$ such that $x \cdot (\varepsilon_i \pm \varepsilon_j) = \pm e_n$. Similarly, the only fundamental reflection r which can reduce the depth of $\alpha = e_i$ is $r = (\overset{+}{i}\;\overset{+}{i+1})$, so it is impossible to have $\pm\alpha_r \in X \cdot \alpha$. □

Proposition 2.9 *Let $W = B_n$, X a conjugacy class of W and w an arbitrary element of X. If either $\alpha = \varepsilon_i$ for some i, $r = (\overline{n})$ or w has a cycle of length at least 3, then*

$n_X^-(\beta) = n_X^-(\alpha)$. *If none of these holds then*

$$n_X^-(\beta) = \begin{cases} n_X^-(\alpha)+1 & \textit{if } w \textit{ has a } 2\textit{-cycle, a positive } 1\textit{-cycle but no negative } 1\textit{-cycles,} \\ n_X^-(\alpha)-1 & \textit{if } w \textit{ has a } 2\textit{-cycle, a negative } 1\textit{-cycle but no positive } 1\textit{-cycles,} \\ n_X^-(\alpha) & \textit{otherwise.} \end{cases}$$

Proof: The proof follows from Lemmas 1.5, 2.4, 2.5 and 2.8. □

Theorem 2.10 *Theorem* 1.1 *holds for* A_n, B_n *and* D_n.

Proof: By Propositions 2.3 and 2.7, if $W = A_n$ or D_n we see that the number $f(X) := n_X^-(\alpha) - n_X^-(\beta)$ depends only on the conjugacy class X, and that $f(X) \in \{-1, 0, 1\}$. Once X is fixed then, we may calculate $n_X^-(\tilde{\alpha})$ for the (unique) highest root $\tilde{\alpha}$, and then apply a sequence of fundamental roots r to $\tilde{\alpha}$ until we reach our chosen root α. Each time r decreases the height, we subtract $f(X)$ from the current value of n_X^-. Therefore for all $\gamma = \sum \alpha_r \in \Pi \nu_r \alpha_r \in \Phi^+$, $n_X^-(\alpha) - n_X^-(\gamma) = f(X)(\sum_{\alpha_r \in \Pi} \lambda_r - \sum_{\alpha_r \in \Pi} \nu_r)$. In particular if $\mathrm{ht}(\alpha) = \mathrm{ht}(\gamma)$ then $n_X^-(\alpha) = n_X^-(\gamma)$. If $W = B_n$ then there are two W-orbits of the root system Φ. By Proposition 2.9, $n_X^-(e_i)$ is constant for all i. In the other orbit, for $\beta = r \cdot \alpha$, there are two possibilities. Unless $r = (\bar{n})$, then as before $f(X) := n_X^-(\beta) - n_X^-(\alpha)$ depends only on the conjugacy class X. If $r = (\bar{n})$ then $n_X^-(\alpha) = n_X^-(\beta)$, but in this situation α and β only differ by some multiple of α_r. Therefore for all $\gamma = \sum \alpha_r \in \Pi \nu_r \alpha_r \in \Phi^+$, $n_X^-(\alpha) - n_X^-(\gamma) = f(X)(\sum_{\alpha_r \in \Pi} \lambda_r - \sum_{\alpha_r \in \Pi} \nu_r)$, as required. □

3. The exceptional groups

This section is largely devoted to establishing some criteria which will enable easier calculation of $n_X^-(\alpha)$, $\alpha \in \Phi^+$ in the case of the exceptional groups E_6, E_7 and E_8. However all the results hold for D_n as well, so we have included it.

Proposition 3.1 *Suppose* W *is one of* D_n, E_6, E_7 *or* E_8. *Let* $\alpha \in \Phi$ *and* $\mathrm{Stab}(\alpha) = \{w \in W \mid w \cdot \alpha = \alpha\}$. *Then* $\mathrm{Stab}(\alpha)$ *acts transitively on the sets* $\{\beta \in \Phi \mid \langle \alpha, \beta \rangle = \frac{1}{2}\}$ *and* $\{\beta \in \Phi \mid \langle \alpha, \beta \rangle = -\frac{1}{2}\}$.

Proof: It suffices to prove the result for the highest root, $\tilde{\alpha}$. In each of the Coxeter groups we are considering, there is exactly one fundamental root, α_r say, which is not orthogonal to $\tilde{\alpha}$. Suppose that $\alpha \in \Phi$ is such that $\langle \alpha, \tilde{\alpha} \rangle = \frac{1}{2}$. Writing $\alpha = \sum_{s \in R} \lambda_s \alpha_s$ we have, as $\langle \tilde{\alpha}, \alpha_r \rangle = \frac{1}{2}$, $\frac{1}{2} = \sum_{s \in R} \lambda_s \langle \alpha_s, \tilde{\alpha} \rangle = \frac{1}{2} \lambda_r$. Thus $\lambda_r = 1$.

Since W acts transitively on Φ, there exists $w \in W$ of minimal length such that $w \cdot \alpha_r = \alpha$. We will show by induction on $l(w)$ that there exists $v \in \mathrm{Stab}(\tilde{\alpha})$ with $v \cdot \alpha_r = \alpha$. If $l(w) = 0$ then we are done. Assume then that $l(w) > 0$. Then $w = sw'$ for some $s \in R$, where $l(w) = 1 + l(w')$. Let $\gamma = w' \cdot \alpha_r$. Then by induction there exists $v' \in \mathrm{Stab}(\tilde{\alpha})$ such that

$\gamma = v' \cdot \alpha_r$. We may set $v = sv'$ unless $s = r$. However in this case:

$$\begin{aligned}
\frac{1}{2} = \langle \tilde{\alpha}, \alpha \rangle &= \langle \tilde{\alpha}, r \cdot \gamma \rangle \\
&= \langle \tilde{\alpha}, \gamma \rangle - 2\langle \alpha_r, \gamma \rangle \langle \tilde{\alpha}, \alpha_r \rangle \\
&= \langle (v')^{-1} \cdot \tilde{\alpha}, \alpha_r \rangle - \langle \alpha_r, \gamma \rangle \\
&= \frac{1}{2} - \langle \alpha_r, \gamma \rangle
\end{aligned}$$

So $\langle \alpha_r, \gamma \rangle = 0$. Hence $\alpha = w \cdot \alpha_r = r \cdot \gamma = \gamma$ and we may set $v = v'$. Therefore, we have shown that whenever $\langle \tilde{\alpha}, \alpha \rangle = \frac{1}{2}$ there exists $v \in \mathrm{Stab}(\tilde{\alpha})$ such that $v \cdot \alpha_r = \alpha$. By symmetry, whenever $\langle \tilde{\alpha}, \alpha \rangle = -\frac{1}{2}$, there exists $v \in \mathrm{Stab}(\tilde{\alpha})$ such that $v \cdot (-\alpha_r) = \alpha$. The result follows immediately. □

Proposition 3.2 *Let W be one of D_n, E_6, E_7 and E_8. Suppose that $\alpha, \beta \in \Phi^+$ and $r \in R$ with $\alpha = r \cdot \beta$ and $\mathrm{dp}(\alpha) > \mathrm{dp}(\beta)$. Let X be a conjugacy class of W and w an arbitrary element of X. If there is a root γ such that $\langle \gamma, w \cdot \gamma \rangle = \frac{1}{2}$, then there exists $x \in X$ such that $x \cdot \alpha = \alpha_r$. If there is a root γ such that $\langle \gamma, w \cdot \gamma \rangle = -\frac{1}{2}$, then there exists $x \in X$ such that $x \cdot \alpha = -\alpha_r$.*

Proof: Suppose α, β, and r are as stated and that there is a root γ such that $\langle \gamma, w \cdot \gamma \rangle = \frac{1}{2}$. Then because W acts transitively on Φ, we have $\alpha = z \cdot \gamma$ for some $z \in W$. Writing $x' = zwz^{-1}$ we see that $\langle \alpha, x' \cdot \alpha \rangle = \langle \gamma, w \cdot \gamma \rangle = \frac{1}{2}$. Now $\mathrm{dp}(\alpha) > \mathrm{dp}(\beta)$ and so, by Proposition 1.4, $\langle \alpha, \alpha_r \rangle = \frac{1}{2}$. By Proposition 3.1, there exists $y \in \mathrm{Stab}(\tilde{\alpha})$ such that $y(x' \cdot \alpha) = \alpha_r$. Now setting $x = yx'y^{-1}$ we find $x \cdot \alpha = yx'y^{-1} \cdot \alpha = yx' \cdot \alpha = \alpha_r$. An identical argument applies for the case $\langle \gamma, w \cdot \gamma \rangle = -\frac{1}{2}$. □

Corollary 3.3 *Let X be a conjugacy class of W where W is one of D_n, E_6, E_7 and E_8. Suppose α, β are positive roots in Φ with $\mathrm{dp}(\alpha) > \mathrm{dp}(\beta)$ and $\alpha = r \cdot \beta$ for some $r \in R$. Let $x \in X$ be arbitrary. Assume that for some $\gamma \in \Phi$ we have $\langle \gamma, x \cdot \gamma \rangle = \frac{1}{2}$, but that there is no $\delta \in \Phi$ for which $\langle \delta, x \cdot \delta \rangle = -\frac{1}{2}$. Then $n_X^-(\beta) = n_X^-(\alpha) + 1$. Conversely, if there is no $\gamma \in \Phi$ for which $\langle \gamma, x \cdot \gamma \rangle = \frac{1}{2}$, but $\langle \delta, x \cdot \delta \rangle = -\frac{1}{2}$ for some root δ, then $n_X^-(\beta) = n_X^-(\alpha) - 1$. In all other cases $n_X^-(\beta) = n_X^-(\alpha)$.*

Proof: The proof follows immediately from Lemma 1.5 and Proposition 3.2. □

A manual check shows that Theorem 1.1 holds for G_2. Employing MAGMA [2] we obtain the n_X^- values for F_4 as given in (4.3). Thus combining Theorem 2.10, Corollary 3.3 and (4.3) yields Theorem 1.1.

We discuss further the groups E_6, E_7 and E_8. Note that Theorem 1.1 implies that in these groups, whenever $\mathrm{ht}(\alpha) = \mathrm{ht}(\beta)$ then $n_X^-(\alpha) = n_X^-(\beta)$. Corollary 3.3 allows us to decide, for a conjugacy class X, whether n_X^- is increasing, decreasing or constant by examining just one element of X. This has been done again using the computer algebra package MAGMA [2] and we summarize our conclusions below. In E_6, for eight of the 25 conjugacy classes,

Table 3. Classes X for which n_X^- is not constant in E_6.

n_X^- decreasing with height		n_X^- increasing with height	
Cycle type	Γ	Cycle type	Γ
$1^{30} \cdot 2^{21}$	A_1	3^{24}	A_2^3
$1^{12} \cdot 2^{30}$	A_1^2	$1^6 \cdot 3^{22}$	A_2^2
4^{18}	$D_4(a_1)$	$2^6 \cdot 4^{15}$	$A_3 \times A_1^2$
6^{12}	$E_6(a_2)$	$2^3 \cdot 6^{11}$	$A_5 \times A_1$

Table 4. Classes X for which n_X^- is not constant in E_7.

n_X^- decreasing with height		n_X^- increasing with height	
Cycle type	Γ	Cycle type	Γ
$1^{60} \cdot 2^{33}$	A_1	$1^2 \cdot 2^{62}$	A_1^6
$1^{26} \cdot 2^{50}$	A_1^2	$1^4 \cdot 2^{61}$	A_1^5
$1^6 \cdot 4^{30}$	$D_4(a_1)$	3^{42}	A_2^3
$1^4 \cdot 2^1 \cdot 4^{30}$	A_3^2	$2^3 \cdot 4^{30}$	$A_3^2 \times A_1$
6^{21}	$E_7(a_4)$	$1^2 \cdot 2^2 \cdot 4^{30}$	$D_4(a_1) \times A_1$
$3^2 \cdot 6^{20}$	$E_6(a_2)$	$3^8 \cdot 6^{17}$	$A_5 \times A_2$

$n_X^-(\alpha)$ is not constant for all $\alpha \in \Phi^+$. In Table 3 we give the cycle types of representatives of these classes, viewed as permutations of the roots, along with the type of the graph Γ associated with each class, using the notation of [3].

In E_7, $n_X^-(\alpha)$ is constant on 48 of its 60 conjugacy classes. The other classes are described in Table 4.

In E_8 there are 112 classes, for 96 of which $n_X^-(\alpha)$ is constant. The other classes are described in Table 5.

Table 5. Classes X for which n_X^- is not constant in E_8.

n_X^- decreasing with height		n_X^- increasing with height	
Cycle type	Γ	Cycle type	Γ
$1^{126} \cdot 2^{57}$	A_1	$1^2 \cdot 2^{119}$	A_1^7
$1^{60} \cdot 2^{90}$	A_1^2	$1^4 \cdot 2^{118}$	A_1^6
$1^{24} \cdot 4^{54}$	$D_4(a_1)$	3^{80}	A_2^4
$1^6 \cdot 2^9 \cdot 4^{54}$	$A_3^2 \times A_1$	$1^6 \cdot 3^{78}$	A_2^3
6^{40}	$E_8(a_8)$	$2^{12} \cdot 4^{54}$	$A_3^2 \times A_1^2$
$2^3 \cdot 6^{39}$	$E_7(a_4) \times A_1$	$1^2 \cdot 2^{11} \cdot 4^{54}$	$D_4(a_1) \times A_1$
$1^6 \cdot 3^6 \cdot 6^{36}$	$E_6(a_2)$	5^{48}	A_4^2
10^{24}	$E_8(a_6)$	$2^3 \cdot 3^8 \cdot 6^{35}$	$E_6(a_2) \times A_2$

4. The groups H_3, B_3 and F_4

4.1. *Values of n_X^- in H_3*

In Table 6, we give the values of n_X^- for conjugacy classes of H_3. The roots are ordered according to depth. More precisely, denote the fundamental roots by α_r, α_s and α_t with $m_{rs} = 5, m_{st} = 3$. Then we write each positive root as a triple (μ_r, μ_s, μ_t) where the μ are the coefficients of the fundamental roots. Let $\lambda := (1 + \sqrt{5})/2$. The roots are ordered as follows: $(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, \lambda, 0), (\lambda, 1, 0), (0, 1, 1), (\lambda, \lambda, 0), (1, \lambda, \lambda), (\lambda, 1, 1)$, $(\lambda, \lambda, \lambda)$, $(\lambda, \lambda^2, 1)$, $(\lambda, \lambda^2, \lambda)$, $(\lambda^2, \lambda^2, 1)$, $(\lambda^2, \lambda^2, \lambda)$, $(\lambda^2, 2\lambda + 1, \lambda)$. A representative of each conjugacy class is given (where w_0 denotes the central longest element of H_3), along

Table 6. Values of n_X^- in H_3.

Representative	Cycle type	$n_X^-(\alpha), \alpha \in \Phi^+$
1	1^{30}	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
w_0	2^{15}	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
rt	$1^2 \cdot 2^{14}$	7, 7, 7, 8, 8, 8, 7, 9, 9, 8, 10, 9, 11, 10, 9
r	$1^4 \cdot 2^{13}$	7, 7, 7, 6, 6, 6, 7, 5, 5, 6, 4, 5, 3, 4, 5
st	3^{10}	8, 8, 8, 7, 7, 9, 8, 6, 8, 7, 7, 8, 8, 9, 10
$rsrs$	5^6	6, 6, 6, 7, 7, 5, 8, 8, 6, 9, 7, 8, 6, 7, 8
rs	5^6	6, 6, 6, 5, 5, 5, 4, 4, 4, 3, 3, 2, 2, 1, 0
w_0st	6^5	8, 8, 8, 9, 9, 7, 8, 10, 8, 9, 9, 8, 8, 7, 6
w_0rs	10^3	6, 6, 6, 7, 7, 7, 8, 8, 8, 9, 9, 10, 10, 11, 12
rst	10^3	6, 6, 6, 5, 5, 7, 4, 4, 6, 3, 5, 4, 6, 5, 4

Table 7. Values of n_X^- in B_3.

Representative	$n_X^-(\alpha), \alpha \in \Phi^+$
1	0, 0, 0, 0, 0, 0; 0, 0, 0
$(\overset{-}{1})(\overset{-}{2})(\overset{-}{3})$	1, 1, 1, 1, 1, 1; 1, 1, 1
$(\overset{-}{1})(\overset{-}{2})(\overset{+}{3})$	2, 2, 2, 2, 2, 2; 1, 1, 1
$(\overset{-}{1})(\overset{+}{2})(\overset{+}{3})$	1, 1, 1, 1, 1, 1; 1, 1, 1
$(\overset{+}{1}\ \overset{+}{2})(\overset{+}{3})$	3, 3, 3, 2, 2, 1; 2, 2, 2
$(\overset{+}{1}\ \overset{+}{2})(\overset{-}{3})$	3, 3, 3, 4, 4, 5; 3, 3, 3
$(\overset{+}{1}\ \overset{+}{2}\ \overset{+}{3})$	4, 4, 4, 4, 4, 4; 2, 2, 2
$(\overset{-}{1}\ \overset{+}{2})(\overset{-}{3})$	3, 3, 3, 4, 4, 5; 3, 3, 3
$(\overset{-}{1}\ \overset{+}{2})(\overset{+}{3})$	3, 3, 3, 2, 2, 1; 2, 2, 2
$(\overset{-}{1}\ \overset{+}{2}\ \overset{+}{3})$	4, 4, 4, 4, 4, 4; 2, 2, 2

with its cycle type as a permutation of roots. Although n_X^- is constant across fundamental roots (which is also the case for H_4), there seems to be little other obvious structure to the values taken by $n_X^-(\alpha)$.

4.2. *Values of n_X^- in B_3*

Let $r_1 = (\overset{+}{1}\ \overset{+}{2})(\overset{+}{3})$, $r_2 = (\overset{+}{1})(\overset{+}{2}\ \overset{+}{3})$, $r_3 = (\overset{-}{3})$ and write α_i for α_{r_i}. Suppose that α, β are positive roots both of the form $\varepsilon_i \pm \varepsilon_j$, for some $i < j$. Let $\alpha = \sum_{i=1}^{3} \lambda_i \alpha_i$, and $\beta = \sum_{i=1}^{3} \mu_i \alpha_i$. Then Theorem 1.1 implies that $n_X^-(\alpha) = n_X^-(\beta)$ whenever $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$. Table 7 illustrates this fact for group B_3. For each conjugacy class X of B_3, a representative of X is given, followed by $n_X^-(\alpha)$ for each $\alpha \in \Phi^+$. The roots are ordered

Table 8. Values of n_X^- in F_4.

Cycle type	Γ	$n_X^-(\alpha), \alpha \in \Phi^+$	
1^{48}	$\emptyset$	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
2^{24}	A_1^4	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
$1^2 \cdot 2^{23}$	$A_1^2 \times \tilde{A}_1$	5, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 9	4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4
$1^2 \cdot 2^{23}$	A_1^3	4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4	5, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 9
$1^{18} \cdot 2^{15}$	$\tilde{A}_1$	5, 5, 5, 5, 4, 4, 4, 3, 3, 3, 2, 1	3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3
$1^{18} \cdot 2^{15}$	A_1	3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3	5, 5, 5, 5, 4, 4, 4, 3, 3, 3, 2, 1
$1^8 \cdot 2^{20}$	A_1^2	4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4	4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4
$1^4 \cdot 2^{22}$	$A_1 \times \tilde{A}_1$	12,12,12,12,12,12,12,12,12,12,12,12	12,12,12,12,12,12,12,12,12,12,12,12
3^{16}	$A_2 \times \tilde{A}_2$	4, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 8	4, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 8
$1^6 \cdot 3^{14}$	A_2	3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8
$1^6 \cdot 3^{14}$	$\tilde{A}_2$	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8	3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3
4^{12}	$D_4(a_1)$	3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3	3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3
$2^4 \cdot 4^{10}$	$A_3 \times \tilde{A}_1$	8, 8, 8, 8, 9, 9, 9, 10, 10, 10, 11, 12	8, 8, 8, 8, 9, 9, 9, 10, 10, 10, 11, 12
$1^8 \cdot 4^{10}$	B_2	7, 7, 7, 7, 6, 6, 6, 5, 5, 5, 4, 3	7, 7, 7, 7, 6, 6, 6, 5, 5, 5, 4, 3
$1^2 \cdot 2^3 \cdot 4^{10}$	A_3	11,11,11,11,11,11,11,11,11,11,11,11	4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4
$1^2 \cdot 2^3 \cdot 4^{10}$	$B_2 \times A_1$	4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4	11,11,11,11,11,11,11,11,11,11,11,11
6^8	$F_4(a_1)$	4, 4, 4, 4, 3, 3, 3, 2, 2, 2, 1, 0	4, 4, 4, 4, 3, 3, 3, 2, 2, 2, 1, 0
$2^3 \cdot 6^7$	$C_3 \times A_1$	4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8
$2^3 \cdot 6^7$	D_4	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8	4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4
$1^2 \cdot 2^2 \cdot 6^7$	B_3	11,11,11,11,11,11,11,11,11,11,11,11	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8
$1^2 \cdot 2^2 \cdot 6^7$	C_3	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8	11,11,11,11,11,11,11,11,11,11,11,11
$2^3 \cdot 3^6 \cdot 6^4$	$A_2 \times \tilde{A}_1$	12,12,12,12,12,12,12,12,12,12,12,12	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8
$2^3 \cdot 3^6 \cdot 6^4$	$\tilde{A}_2 \times A_1$	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8	12,12,12,12,12,12,12,12,12,12,12,12
8^6	B_4	11,11,11,11,11,11,11,11,11,11,11,11	11,11,11,11,11,11,11,11,11,11,11,11
12^4	F_4	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8	8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8

within orbits, as follows $\alpha_1, \alpha_2, \alpha_2 + \sqrt{2}\alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \sqrt{2}\alpha_3, \alpha_1 + 2\alpha_2 + \sqrt{2}\alpha_3$; $\alpha_3, \alpha_3 + \sqrt{2}\alpha_2, \alpha_3 + \sqrt{2}\alpha_2 + \sqrt{2}\alpha_1$.

4.3. Values of n_X^- in F_4

In Table 8, $n_X^-(\alpha)$ is given for the roots, which are ordered in the manner suggested in Theorem 1.1: short roots first, ordered by height, then long roots, ordered by height. For each conjugacy class, the cycle type of its elements as permutations of the roots is given, along with the associated graph Γ (again, using the notation of [3]).

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