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Leaves in Representation Diagrams of Bipartite Distance-Regular Graphs

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Abstract. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ and let q_{ij}^h $(0 \leq h, i, j \leq D)$ denote the Krein parameters of Γ. Pick an integer h $(1 \leq h \leq D-1)$. The *representation diagram* $\Delta = \Delta_h$ is an undirected graph with vertices $0, 1, \ldots, D$. For $0 \leq i, j \leq D$, vertices i, j are adjacent in Δ whenever $i \neq j$ and $q_{ij}^h \neq 0$. It turns out that in Δ, the vertex 0 is adjacent to h and no other vertices. Similarly, the vertex D is adjacent to $D-h$ and no other vertices. We call $0, D$ the *trivial* vertices of Δ. Let l denote a vertex of Δ. It turns out that l is adjacent to at least one vertex of Δ. We say l is a *leaf* whenever l is adjacent to exactly one vertex of Δ. We show Δ has a nontrivial leaf if and only if Δ is the disjoint union of two paths.

Keywords: primitive idempotent, eigenvalue, association scheme, Q-polynomial, antipodal

1. Introduction

In recent research on distance-regular graphs, the following theme emerges. Let Γ denote a distance-regular graph and let E and F denote primitive idempotents of Γ. When is the entrywise product $E \circ F$ a linear combination of a "small" number of primitive idempotents of Γ?

We refer the reader to the articles of MacLean [5–7], Pascasio [9–11], and the present author [4] for work on this theme. In this paper we consider the case where $E \circ F$ is a linear combination of F and one other primitive idempotent. To keep things simple, we assume Γ is bipartite. Before we state our main result, we recall a bit of notation.

Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. (Definitions are contained in the next section.) Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ. Recall that $\theta_0 = k$ and $\theta_D = -k$; we call θ_0 and θ_D the *trivial* eigenvalues of Γ. For $0 \leq i \leq D$, let E_i denote the primitive idempotent of Γ associated with θ_i. Let q_{ij}^h $(0 \leq h, i, j \leq D)$ denote the Krein parameters of Γ. Recall that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leq i, j \leq D),$$

where $\circ$ denotes entrywise multiplication.

Pick an integer h ($1 \le h \le D-1$). We recall the *representation diagram* $\Delta = \Delta_h$ [12–14]. Δ is an undirected graph with vertices $0, 1, \ldots, D$. For $0 \le i, j \le D$, vertices i and j are adjacent in Δ whenever $i \neq j$ and $q^h_{ij} \neq 0$.

It turns out that in Δ, the vertex 0 is adjacent to h and no other vertices. Similarly, the vertex D is adjacent to $D-h$ and no other vertices. We call 0 and D the *trivial* vertices of Δ.

Let l denote a vertex of Δ. As we see in the next section, l is adjacent to at least one vertex of Δ. We say l is a *leaf* whenever l is adjacent to exactly one vertex of Δ. Our main result is the following.

Theorem 1.1 *Let Γ denote a bipartite distance-regular graph with diameter $D \ge 3$ and valency $k \ge 3$. Pick an integer h ($1 \le h \le D-1$). The representation diagram Δ_h has a nontrivial leaf if and only if Δ_h is the disjoint union of two paths.*

Hypercubes and doubled Odd graphs have representation diagrams satisfying the conditions of Theorem 1.1. At diameters greater than 5, these are the only such graphs known.

2. Preliminaries

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path-length distance function ∂, and diameter $D := \max\{\partial(x, y) : x, y \in X\}$. Let k denote a nonnegative integer. We say Γ is *regular* with *valency* k whenever for all $x \in X$, $|\{z \in X : \partial(x, z) = 1\}| = k$. We say Γ is *distance-regular* whenever for all integers h, i, j ($0 \le h, i, j \le D$) and all $x, y \in X$ with $\partial(x, y) = h$, the scalar $p^h_{ij} = |\{z \in X : \partial(x, z) = i, \partial(y, z) = j\}|$ is independent of x and y. For notational convenience, set $c_i := p^i_{1i-1}$ ($1 \le i \le D$), $a_i := p^i_{1i}$ ($0 \le i \le D$), $b_i := p^i_{1i+1}$ ($0 \le i \le D-1$), and $c_0 := 0$, $b_D := 0$. For the rest of this section, suppose Γ is distance-regular. To avoid trivialities, assume $D \ge 3$ and $k \ge 3$. We observe Γ is regular with valency $k = b_0$. Further, we observe $c_i + a_i + b_i = k$ for $0 \le i \le D$.

We say Γ is *bipartite* whenever there exists a partition $X = X^+ \cup X^-$ such that no edge joins two vertices in the same cell of the partition. Observe Γ is bipartite if and only if $a_i = 0$ ($0 \le i \le D$), and in this case,

$$c_i + b_i = k \quad (0 \le i \le D). \tag{1}$$

For the rest of this section, suppose Γ is bipartite.

Let $\sim$ denote the binary relation on X such that for any $x, y \in X$, we have $x \sim y$ whenever $x = y$ or $\partial(x, y) = D$. We say Γ is *antipodal* whenever $\sim$ is an equivalence relation.

Let $\mathbb{R}$ denote the field of real numbers. By $Mat_X(\mathbb{R})$ we mean the $\mathbb{R}$-algebra consisting of all matrices whose entries are in $\mathbb{R}$ and whose rows and columns are indexed by X.

For each integer i $(0 \leq i \leq D)$, let A_i denote the matrix in $Mat_X(\mathbb{R})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{otherwise} \end{cases} \qquad (x, y \in X).$$

Abbreviate $A := A_1$. We call A the *adjacency matrix* of Γ. Let M denote the sub-algebra of $Mat_X(\mathbb{R})$ generated by A. We call M the *Bose-Mesner algebra* of Γ. By [1, Theorem 20.7], $A_0, A_1, \ldots, A_D$ is a basis for M.

By [2, Theorem 2.6.1], M has a second basis $E_0, E_1, \ldots, E_D$ such that $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq D)$. We call $E_0, E_1, \ldots, E_D$ the *(primitive) idempotents* of Γ.

Observe there exists a sequence of scalars $\theta_0, \theta_1, \ldots, \theta_D$ taken from $\mathbb{R}$ such that

$$A = \sum_{i=0}^{D} \theta_i E_i.$$

We call θ_i the *eigenvalue* of Γ associated with E_i. Note $\theta_0, \theta_1, \ldots, \theta_D$ are distinct since A generates M. Throughout this paper, we assume the eigenvalues are labeled so that $\theta_0 > \theta_1 > \cdots > \theta_D$. By [2, p. 82], $\theta_0 = k$ and $\theta_{D-i} = -\theta_i$ for $0 \leq i \leq D$. We call θ_0 and θ_D the *trivial* eigenvalues of Γ.

Let θ_h denote an eigenvalue of Γ and let E_h denote the associated idempotent. Since $A_0, A_1, \ldots, A_D$ is a basis for M, there exist real scalars $\sigma_0, \sigma_1, ..., \sigma_D$ such that

$$E_h = m_h |X|^{-1} \sum_{i=0}^{D} \sigma_i A_i, \tag{2}$$

where $m_h = \text{rank } E_h$. We call $\sigma_0, \sigma_1, ..., \sigma_D$ the *cosine sequence* associated with θ_h. By [2, p. 128],

$$c_i \sigma_{i-1} + b_i \sigma_{i+1} = \theta_h \sigma_i \quad (0 \leq i \leq D), \tag{3}$$

where σ_{-1} and σ_{D+1} denote indeterminates.

Let $\circ$ denote entrywise multiplication in $Mat_X(\mathbb{R})$ and observe

$$A_i \circ A_j = \delta_{ij} A_i \quad (0 \leq i, j \leq D). \tag{4}$$

This implies M is closed under $\circ$. Since $E_0, E_1, \ldots, E_D$ is a basis for M, there exist scalars $q_{ij}^h \in \mathbb{R}$ $(0 \leq h, i, j \leq D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h. \tag{5}$$

We call the q_{ij}^h the *Krein parameters* of Γ.

In the next two lemmas, we recall a few basic facts about the product $\circ$ and the Krein parameters.

Lemma 2.1 [9, *Lemma* 3.3, *Theorem* 3.6] *Let* $\Gamma = (X, R)$ *denote a bipartite distance-regular graph with diameter* $D \geq 3$.
(i) $E_0 \circ E_i = |X|^{-1} E_i$ *for* $0 \leq i \leq D$.
(ii) $E_D \circ E_i = |X|^{-1} E_{D-i}$ *for* $0 \leq i \leq D$.
(iii) *For* $1 \leq i, j \leq D-1$, $E_i \circ E_j$ *is not a scalar multiple of a single idempotent of* Γ.

Lemma 2.2 *Let* Γ *denote a bipartite distance-regular graph with diameter* $D \geq 3$.
(i) $q^h_{ij} = q^h_{ji}$ $(0 \leq h, i, j \leq D)$.
(ii) $m_h q^h_{ij} = m_i q^i_{jh} = m_j q^j_{hi}$ $(0 \leq h, i, j \leq D)$.
(iii) $q^h_{0j} = \delta_{hj}$ $(0 \leq h, j \leq D)$.
(iv) $q^h_{Dj} = \delta_{h,D-j}$ $(0 \leq h, j \leq D)$.
(iii) $q^{D-h}_{D-i,j} = q^h_{ij}$ $(0 \leq h, i, j \leq D)$.

Proof: (i) Immediate from (5). (ii) [2, Lemma 2.3.1(iv)] (iii) Immediate from Lemma 2.1(i). (iv) Immediate from Lemma 2.1(ii). (v) Taking the entrywise product of both sides of (5) with E_D and applying Lemma 2.1(ii), we get the result. ☐

Definition 2.3 [12] Let Γ denote a distance-regular graph with diameter D. Pick an integer h $(0 \leq h \leq D)$. We define the *representation diagram* $\Delta = \Delta_h$. Δ is an undirected graph with vertices $0, 1, \ldots, D$. For $0 \leq i, j \leq D$, vertices i and j are adjacent in Δ whenever $i \neq j$ and $q^h_{ij} \neq 0$. We sometimes say Δ is the representation diagram associated with the eigenvalue θ_h.

Let C denote a connected component of Δ. We say C is a *path* whenever there exists an ordering $v_0, v_1, \ldots, v_l$ of the vertices of C such that for $0 \leq i, j \leq l$, vertices v_i, v_j are adjacent in Δ if and only if $|i - j| = 1$.

Lemma 2.4 *Let* Γ *denote a bipartite distance-regular graph with diameter* $D \geq 3$. *With reference to Definition* 2.3, *the following hold.*
(i) *For* $0 \leq h, i, j \leq D$, *vertices* i *and* j *are adjacent in* Δ_h *if and only if* $D - i$ *and* $D - j$ *are adjacent in* Δ_h.
(ii) Δ_0 *has no edges.*
(iii) *In* Δ_D, *vertex* i $(0 \leq i \leq D)$ *is adjacent to* $D - i$ *and no other vertices.* (*If* D *is even then vertex* $D/2$ *is not adjacent to any vertices.*)
(iv) *Suppose* $h \neq 0$. *In* Δ_h, *vertex 0 is adjacent to* h *and no other vertices. Moreover, vertex* D *is adjacent to* $D - h$ *and no other vertices.*
(v) *Suppose* $1 \leq h \leq D - 1$. *Each vertex of* Δ_h *is adjacent to at least one other vertex.*

Proof: (i)–(iv) Immediate from Lemma 2.2. (v) Let i denote a vertex of Δ_h and suppose i is not adjacent to any vertices of Δ_h. By (iv) above, we find $1 \leq i \leq D - 1$. By Definition 2.3,

$q^h_{ij} = 0$ for $j \neq i$. Applying Lemma 2.2(ii), we find $q^j_{hi} = 0$ for $j \neq i$, which implies $E_h \circ E_i$ is a scalar multiple of E_i. This contradicts Lemma 2.1(iii). □

We call 0 and D the *trivial* vertices of a representation diagram.

Lemma 2.5 *Let* Γ *denote a bipartite distance-regular graph with diameter* $D \geq 3$ *and valency* $k \geq 3$. *The following are equivalent for* $1 \leq h \leq D - 1$.
(i) Δ_h *is not connected.*
(ii) Γ *is antipodal and h is even.*
Suppose (i)–(ii) *hold. Then* Δ_h *has two connected components, one consisting of the even vertices and one consisting of the odd vertices.*

Proof: Let $\sigma_0, \sigma_1, ..., \sigma_D$ denote the cosine sequence associated with θ_h.

(i) → (ii) Since Δ_h is not connected and by [2, Proposition 2.11.1], $\sigma_i = 1$ for some i ($1 \leq i \leq D$). By [2, Proposition 4.4.7], Γ is antipodal and $\sigma_D = 1$. Now h is even by [2, p. 142].

(ii) → (i) By [2, p. 142], $\sigma_D = 1$. Now Δ_h is not connected by [2, Proposition 2.11.1].

Suppose (i)–(ii) hold. We already mentioned $\sigma_D = 1$. By [2, Proposition 4.4.7], $\sigma_i \neq 1$ for $1 \leq i \leq D - 1$. Now by [2, Proposition 2.11.1], Δ_h has two components. By [2, p. 413], $q^h_{ij} = 0$ if one of i and j is even and the other is odd. The result follows. □

Example 2.6 Let Γ denote a bipartite distance-regular graph with diameter 3 and valency $k \geq 3$. With reference to Definition 2.3, the following hold.

(i) Δ_1 is the path 0, 1, 2, 3.
(ii) Suppose Γ is not antipodal. Then Δ_2 is the path 0, 2, 1, 3.
(iii) Suppose Γ is antipodal. Then Δ_2 is the disjoint union of the paths 0, 2 and 1, 3.

Proof: (i) By Lemma 2.4(iv), vertex 0 is adjacent to 1 and no other vertices. Also, vertex 3 is adjacent to 2 and no other vertices. By Lemma 2.5, Δ_1 is connected, so 1 is adjacent to 2 and we are done.

(ii), (iii) Similar to the proof of (i). □

3. Leaves

Definition 3.1 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Fix h ($1 \leq h \leq D - 1$) and let $\Delta = \Delta_h$ denote a representation diagram of Γ. Let l denote a vertex of Δ. By Lemma 2.4(v), l is adjacent to at least one vertex of Δ. We say l is a *leaf* whenever l is adjacent to exactly one vertex of Δ. Observe l is a leaf if and only if there exists an idempotent F of Γ with $F \neq E_l$ such that

$$E_h \circ E_l \in \text{Span}\,\{E_l, F\}. \tag{6}$$

By Lemma 2.4, l is a leaf if and only if $D - l$ is a leaf. Also, the trivial vertices 0 and D of Δ are leaves.

Theorem 3.2 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Let Δ_h $(1 \leq h \leq D-1)$ denote a representation diagram of Γ. Suppose Δ_h has at least one nontrivial leaf. Then the following hold.*

(i) *Δ_h is the disjoint union of two paths, one consisting of the even vertices and one consisting of the odd vertices.*

(ii) *Γ is antipodal and h is even.*

Proof: We abbreviate $\Delta := \Delta_h$.

(i) If $D = 3$ the result follows from Example 2.6, so suppose $D \geq 4$. Let $\sigma_0, \sigma_1, ..., \sigma_D$ denote the cosine sequence associated with θ_h. We show there exists $\beta \in \mathbb{R}$ such that $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1}$ is independent of i for $1 \leq i \leq D-1$.

By assumption, Δ has a nontrivial leaf. Let us denote this leaf by l. Let t denote the vertex of Δ to which l is adjacent. Apparently, $t \neq l$ and there exist $\epsilon, \zeta \in \mathbb{R}$ with $\zeta \neq 0$ such that

$$E_h \circ E_l = \epsilon E_l + \zeta E_t. \tag{7}$$

Let $\rho_0, \rho_1, \ldots, \rho_D$ and $\tau_0, \tau_1, \ldots, \tau_D$ denote the cosine sequences associated with θ_l and θ_t, respectively. We use (2) to eliminate E_h, E_l and E_t from (7) and then apply (4). In the result, we equate coefficients of A_i and simplify to find that for $0 \leq i \leq D$,

$$\sigma_i \rho_i = x\rho_i + y\tau_i, \tag{8}$$

where $x = |X|m_h^{-1}\epsilon$ and $y = |X|m_t m_h^{-1} m_l^{-1}\zeta$. Note $y \neq 0$ because $\zeta \neq 0$.

We use (8) for $0 \leq i \leq 4$. Repeatedly applying (3) and (1), we find for $0 \leq i \leq 4$ that $\sigma_i = f_i(\theta_h)$, $\rho_i = f_i(\theta_l)$ and $\tau_i = f_i(\theta_t)$, where the functions f_i are given by

$$f_0(\lambda) = 1, \quad f_1(\lambda) = \frac{\lambda}{k}, \quad f_2(\lambda) = \frac{\lambda^2 - k}{kb_1}, \tag{9}$$

$$f_3(\lambda) = \frac{\lambda^3 - (k + c_2b_1)\lambda}{kb_1b_2}, \quad f_4(\lambda) = \frac{\lambda^4 - (k + c_2b_1 + c_3b_2)\lambda^2 + c_3kb_2}{kb_1b_2b_3}. \tag{10}$$

We set $i = 0, 1$ in (8) to obtain two linear equations in x and y. To solve this system, we first verify the coefficient matrix is nonsingular. This coefficient matrix is

$$\begin{pmatrix} \rho_0 & \tau_0 \\ \rho_1 & \tau_1 \end{pmatrix}. \tag{11}$$

Evaluating the determinant of (11) using (9), we find this determinant equals $(\theta_t - \theta_l)k^{-1}$. This is nonzero, so the coefficient matrix is nonsingular. We now solve the system of equations to find in view of (9) that

$$x = \frac{\theta_h\theta_l - \theta_t k}{k(\theta_l - \theta_t)}, \quad y = \frac{\theta_l(k - \theta_h)}{k(\theta_l - \theta_t)}. \tag{12}$$

Note θ_l is a factor of y and so cannot be zero. We set $i = 2$ in (8), apply (9) and (12) and solve for θ_t to get

$$\theta_t = \frac{\theta_l^2\theta_h + \theta_l^2 - \theta_h k - k^2}{b_1\theta_l}. \tag{13}$$

We set $i = 3$ in (8), apply (9)–(10), (12) and (13), and simplify to find

$$\frac{\left(k^2 - \theta_h^2\right)\left(k^2 - \theta_l^2\right)}{\theta_l k^2 b_1^3 b_2^2}\left((b_2 - (c_2 - 1)\theta_h)\theta_l^2 - b_2(k + \theta_h)\right) = 0. \tag{14}$$

The fraction is nonzero and $b_2(k + \theta_h) \neq 0$, so $b_2 - (c_2 - 1)\theta_h \neq 0$. We solve (14) for θ_l^2 to get

$$\theta_l^2 = \frac{(k + \theta_h)b_2}{b_2 - (c_2 - 1)\theta_h}. \tag{15}$$

We set $i = 4$ in (8) and apply (9)–(10), (12), (13) and (15) to find the left side minus the right is

$$\frac{\left(k^2 - \theta_h^2\right)(k + \theta_h)\left(k^2 - \theta_l^2\right)c_2}{\theta_l^2(b_2 - (c_2 - 1)\theta_h)^2 k^2 b_1^2 b_2 b_3^2} \tag{16}$$

times

$$(b_2 - b_3)\theta_h^3 + (b_2 - b_3c_2)\theta_h^2 + \left(2b_3b_2 - b_3c_2b_2 - b_2^2\right)\theta_h + b_2^2(b_3 - 1). \tag{17}$$

Since (16) is nonzero, (17) must be zero.

By [3, Lemma 9.3] and since (17) is zero, there exists $\beta \in \mathbb{R}$ such that $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1}$ is independent of i for $1 \leq i \leq D - 1$. Now, by [4, Theorem 5.4], either Δ is a path or Δ is as in (i). Since Δ has a nontrivial leaf, Δ cannot be a path. So Δ is as in (i), as desired.

(ii) By (i), Δ is not connected. Now the result follows by Lemma 2.5. □

Example 3.3 Let Γ denote a bipartite antipodal distance-regular graph with diameter 4 and valency $k \geq 3$. With reference to Definition 2.3, the following hold.

(i) Δ_2 is the disjoint union of the paths 0, 2, 4 and 1, 3.
(ii) Suppose $h = 1$ or $h = 3$. Then Δ_h has no nontrivial leaves.

Proof: (i) Immediate from Lemma 2.4(iv) and Lemma 2.5.

(ii) Since h is odd, Δ_h has no nontrivial leaves by Theorem 3.2. □

Example 3.4 Let Γ denote a bipartite antipodal distance-regular graph with diameter 5 and valency $k \geq 3$. With reference to Definition 2.3, the following hold.

(i) Δ_2 is the disjoint union of the paths 0, 2, 4 and 5, 3, 1.
(ii) Δ_4 is the disjoint union of the paths 0, 4, 2 and 5, 1, 3.
(iii) Suppose $h = 1$ or $h = 3$. Then Δ_h has no nontrivial leaves.

Proof: (i) By Lemma 2.5, the even vertices of Δ_2 comprise a connected component. This component consist of the path 0, 2, 4 since vertex 0 is adjacent to 2 but not 4 by Lemma 2.4(iv). Now the odd vertices form the path 5, 3, 1 by Lemma 2.4(i).

(ii) Similar to the proof of (i).

(iii) Since h is odd, Δ_h has no nontrivial leaves by Theorem 3.2. □

In the next lemma and the following two examples, we recall some information about the D-cube.

Lemma 3.5 [2, *Section* 9.2] *Let* Γ *denote the* D*-cube.*
(i) Γ *is antipodal.*
(ii) *For* $0 \leq h, i, j \leq D$, $q^h_{ij} \neq 0$ *if* $|i - j| = h$ *and* $q^h_{ij} = 0$ *if* $|i - j| > h$.

Example 3.6 Let D denote an odd integer with $D \geq 3$ and let Γ denote the D-cube. With reference to Definition 2.3, the following hold.

(i) Δ_2 is the disjoint union of the paths $0, 2, 4, \ldots, D-1$ and $1, 3, 5, \ldots, D$.
(ii) Δ_{D-1} is the disjoint union of the paths $0, D-1, 2, D-3, \ldots$ and $D, 1, D-2, 3, \ldots$.
(iii) Suppose $h \neq 2$ and $h \neq D-1$. Then Δ_h has no nontrivial leaves.

Proof: (i) Γ is antipodal by Lemma 3.5(i), so by Lemma 2.5, Δ_2 has two connected components, one consisting of the even vertices and one consisting of the odd vertices. For $0 \leq i, j \leq D$, we see by Lemma 3.5(ii) that $q^2_{ij} \neq 0$ if $|i - j| = 2$ and $q^2_{ij} = 0$ if $|i - j| > 2$. The result follows.

(ii) We mentioned Γ is antipodal, so by Lemma 2.5, Δ_{D-1} has two connected components, one consisting of the even vertices and one consisting of the odd vertices. For $0 \leq i, j \leq D$, we see by Lemma 3.5(ii) that $q^1_{ij} \neq 0$ if $|i - j| = 1$ and $q^1_{ij} = 0$ if $|i - j| > 1$. Applying Lemma 2.2(v), we then get $q^{D-1}_{D-i,j} \neq 0$ if $|i - j| = 1$ and $q^{D-1}_{D-i,j} = 0$ if $|i - j| > 1$. The result follows.

(iii) Follows from Theorem 3.2, [4, Theorem 5.4]and [3, Example 17.1] □

Example 3.7 Let D denote an even integer with $D \geq 4$ and let Γ denote the D-cube. With reference to Definition 2.3, the following hold.

(i) Δ_2 is the disjoint union of the paths $0, 2, 4, \ldots, D$ and $1, 3, 5, \ldots, D-1$.
(ii) Suppose $h \neq 2$. Then Δ_h has no nontrivial leaves.

Proof: (i) By Lemma 2.5, Δ_2 has two connected components, one consisting of the even vertices and one consisting of the odd vertices. For $0 \leq i, j \leq D$, we see by Lemma 3.5(ii) that $q^2_{ij} \neq 0$ if $|i - j| = 2$ and $q^2_{ij} = 0$ if $|i - j| > 2$. The result follows.

(ii) Follows from Theorem 3.2, [4, Theorem 5.4]and [3, Example 17.1]. □

Theorem 3.8 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Let $\Delta = \Delta_h$ $(1 \leq h \leq D-1)$ denote a representation diagram of Γ. Suppose Δ has a nontrivial leaf. Suppose Γ is not one of Examples* 2.6, 3.3, 3.4, 3.6, 3.7. *Then either* (a) *D is odd and $h = D-1$ or* (b) *$D \equiv 1(\text{mod} 4)$ and $h = (D-1)/2$. In case* (a) *the nontrivial leaves are $(D-1)/2$ and $(D+1)/2$. In case* (b) *the nontrivial leaves are* 1 *and $D-1$.*

Proof: Let l denote a nontrivial leaf of Δ. Then $D-l$ is also a leaf. Replacing l by $D-l$ if necessary, we assume $l \leq D/2$. Recall $E_h \circ E_l$ is a linear combination of E_l and one other idempotent of Γ. By [7, Lemma 4.4], Γ is either 2-homogeneous in the sense of Nomura [8] or taut in the sense of MacLean [7].

First suppose Γ is 2-homogeneous. By [8, Theorem 1.2], either Γ is antipodal with $D \leq 5$ or Γ is the D-cube. But this implies Γ is as in one of the examples we excluded.

Next suppose Γ is taut and D is even. Set $d := D/2$. By [7, Theorem 4.3], l is either 1 or d. By [7, Corollary 6.6], $l = d-1$. Combining these facts, we find $l = 1$ and $d = 2$. Now $D = 2d = 4$. Recall Γ is antipodal by Theorem 3.2(ii). Now Γ is as in Example 3.3(ii), which is a contradiction.

Finally, suppose Γ is taut and D is odd. Set $d := (D-1)/2$. By Theorem 3.3(ii), h is even. By [7, Theorem 4.3], the ordered pair (h, l) is one of $(D-1, d)$, $(d, 1)$, and $(D-d, 1)$. First suppose $(h, l) = (D-1, d)$. Then we have (a). Next suppose $(h, l) = (d, 1)$. Since $D = 2d+1$ and since $d = h$ is even, we have (b). Finally, suppose $(h, l) = (D-d, 1)$. By [7, Theorem 6.2, Corollary 6.3], $E_{D-d} \circ E_1 \in \text{Span}\,\{E_R, E_S\}$ for some R, S such that $1 < S < R$. But this contradicts (6). □

Note 3.9 The doubled Odd graph $2.\text{O}_\text{k}$ is the only known graph for which (a) holds in Theorem 3.8. There is no known graph for which (b) holds.

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