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On Trees and Characters

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Abstract. A new family of trees, defined in term of Young diagrams, is introduced. Values of central characters of the symmetric group are represented as a weighted enumeration of such trees. The proof involves a new decomposition theorem for representations corresponding to general shapes.

Keywords: symmetric groups, central characters

1. Introduction

Throughout this paper the base field is of characteristic zero. Let λ and μ be partitions of n, let S^λ be the corresponding symmetric group irreducible representation, and let χ^λ_μ be the value of its character at a conjugacy class of cycle type μ. The *central character* is defined by

$$c^\lambda_\mu := |C_\mu| \cdot \frac{\chi^\lambda_\mu}{\chi^\lambda_{(1^n)}},$$

where $|C_\mu|$ is the size of the conjugacy class of cycle type μ. The central characters of the symmetric group are integer valued. The goal of this paper is to give a combinatorial interpretation to the value of the central characters of cycles $c^\lambda_{(k,1^{n-k})}$.

A new family of graphs is introduced. For any subset D of boxes in a given Young diagram $[\lambda]$ (or: of elements in the lattice $\mathbf{Z}^2$), define a *Young graph*, $\Gamma = \Gamma(D)$, as follows: $\Gamma(D)$ is the graph whose set of vertices is D, where there is an edge between two boxes in D iff they are in the same row or column, and there is no box of D between them. In other words, $((i_1, j_1), (i_2, j_2)) \in D^{\times 2}$ is an edge iff one of the following two conditions holds:

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(i) $i_1 = i_2$, and there is no j_3 such that $j_1 < j_3 < j_2$ and $(i_1, j_3) \in D$;
(ii) $j_1 = j_2$, and there is no i_3 such that $i_1 < i_3 < i_2$ and $(i_3, j_1) \in D$.

If the resulting graph is a tree we call it a *Young tree*. The *order* of a Young graph (tree) is the number of vertices in the graph.

The edges in a Young graph Γ are either horizontal or vertical (where the Young diagram is drawn in the British way; i.e. λ_1 is the length of the first upper row etc.). Denote the number of vertical edges in Γ by vert(Γ). We define the sign of Γ as

$$\text{sign}(\Gamma) := (-1)^{\text{vert}(\Gamma)}.$$

A *simple path* in Γ is a connected subgraph contained in one row or column. The length of the path p, denoted by $\ell(p)$, is the number of edges in p. A simple path p in a Young graph Γ is a *maximal simple path* if no simple path in Γ strictly contains p (as a subgraph). The weight of a Young tree T is the product

$$\text{weight}(T) := \prod_{p \text{ is a maximal simple path in } T} \ell(p)!$$

where the product is taken over all maximal simple paths in T.

Let $c_k^\lambda := c_{(k,1^{n-k})}^\lambda$ be the central character of S^λ evaluated at the conjugacy class of cycles of length k. Generalizing Suzuki's method for computing the characters of 3-cycles [12] we prove

Theorem *For any $k \le n$ and any partition $\lambda \vdash n$,*

$$c_k^\lambda = \sum_{\{T \mid T \text{ is a Young tree of order } k \text{ in } [\lambda]\}} \text{sign}(T) \cdot \text{weight}(T).$$

For example, all trees of order 2 consist of one edge, which is either horizontal or vertical. The weight of all these trees is 1. Hence, c_2^λ is equal to the number of horizontal edges in the Young diagram $[\lambda]$ minus the number of vertical edges in $[\lambda]$, which is equal to $\sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2}$, where λ'_j is the length of the j-th column. This is a well known special case of the Frobenius Trace formula [2, 4].

The proof of the main theorem combines techniques from graph theory and from the (ordinary) representation theory of the symmetric group. In particular, it involves *generalized Specht modules*, which were introduced in [6] and further studied in [8].

The rest of the paper is organized as follows. Basic properties of central characters are described in Section 2. In Section 3 two lemmas are stated. The main theorem is an immediate consequence of these lemmas. The first lemma connects the computation of central characters to weighted enumeration of Young graphs. This lemma is proved in Section 4. the second lemma shows that it suffices to consider Young trees. This lemma is proved in Section 6. The proof applies a new decomposition theorem for generalized Specht modules which is presented in Section 5.

2. Central characters

This section contains basic lemmas on central characters of ordinary representations, most of them are well-known.

2.1. Central characters of IC groups

In this subsection the base field is $\boldsymbol{C}$. Let G be a finite group, let C be a conjugacy class in G. Let φ be a representation of G, $\chi^\varphi(C)$—the character of φ at C, and $d_\varphi = \chi^\varphi(id)$—the degree of φ. The *central character* of φ at C is

$$c^\varphi(C) := |C| \cdot \frac{\chi^\varphi(C)}{d_\varphi}.$$

The following lemma is an immediate consequence of Schur's lemma. See e.g. ([9] Lemma 4.1, [7] Theorem 8.2.1) and references therein.

Lemma 2.1 *Let G be a finite group and let ρ be the regular representation of G. Then for any conjugacy class C in G the eigenvalues of the matrix*

$$\rho\left(\sum_{g\in C} g\right)$$

are all central characters of the irreducible representations evaluated at C.

Proof: $\sum_{g\in C} g$ is a central element in the group algebra $\boldsymbol{C}[G]$. By Schur's lemma this element acts as a scalar matrix on each irreducible representation φ. The scalar of this matrix is equal to

$$\frac{\mathrm{Trace}\left(\sum_{g\in C}\varphi(g)\right)}{d_\varphi} = \frac{1}{d_\varphi}\sum_{g\in C}\chi^\varphi(g) = |C| \cdot \frac{\chi^\varphi(C)}{d_\varphi}.$$

□

An *IC* group is a group whose characters are integer valued (see e.g. [11], p. 473).

Corollary 2.2 *The central characters of any finite IC group are integers.*

Proof: By Lemma 2.1, the central characters are eigenvalues of an integer matrix. Hence, the central characters (of any finite group) are algebraic integers, and thus they are integers whenever they belong to $\boldsymbol{Q}$. On the other hand, by definition the central characters of an *IC* group are rational. □

In particular, the symmetric group is an *IC* group. Hence, its central characters are integers. An independent proof of this assertion is given in the next subsection.

2.2. Central characters of the symmetric group

From now on the base field $\boldsymbol{C}$ may be replaced by any other field of characteristic zero. Recall a few basic notions from the representation theory of the symmetric group (cf. [5]). The *Young symmetrizer* of a Young tableau Q is the element in the group algebra $\boldsymbol{C}[S_n]$ defined by

$$e_Q := \sum_{p\in Q_r, q\in Q_c} \mathrm{sign}(q)pq,$$

where Q_c is the subgroup in S_n consisting of column preserving permutations, and Q_r is the subgroup of row preserving permutations.

The left S_n-module $C[S_n]e_Q$ is irreducible. For any given partition λ, different tableaux of shape λ define isomorphic left modules. The notation S^λ is referring to any module isomorphic to $\boldsymbol{C}[S_k]e_Q$, where Q is of shape λ. S^λ is called the *Specht module* of shape λ. The set consisting of one Specht module for each partition of n is a full set of irreducible, inequivalent representations over $\boldsymbol{C}$. Denote the central character of S^λ by c^λ.

The following lemma is used by Suzuki to compute the irreducible characters of 3-cycles [12]. Suzuki attributes it to the anoymous referee of [1].

Lemma 2.3 *Let Q be a given Young tableau of shape λ. The central character c^λ_μ is equal to the coefficient of the identity in*

$$\left(\sum_{g\in C_\mu} g\right)\cdot e_Q,$$

where C_μ is the conjugacy class of permutations of cycle type μ.

Proof: By Lemma 2.1,

$$\left(\sum_{g\in C_\mu} g\right)\cdot e_Q = c^\lambda_\mu \cdot e_Q.$$

The coefficient of the identity in e_Q is 1. The desired result follows. □

For any $2 \le k \le n$ let C_k be the conjugacy class of the k-cycles, and let c^λ_k be the central character of S^λ at C_k. It follows from Lemma 2.3 that

Corollary 2.4 *For any Young tableau Q of shape λ, and any $2 \le k \le n$*

$$c^\lambda_k = \sum_{\{(p,q)\,|\, p\in Q_r, q\in Q_c, pq\in C_k\}} \mathrm{sign}(q).$$

3. Proof of the main theorem

The *support* of a permutation $\pi \in S_n$, $supp(\pi)$, is the set of non-fixed letters under the action of π. For any permutation $\pi \in S_n$ and any Young tableau Q of order n define

$\Gamma_Q(\pi) :=$ the Young graph whose set of vertices is the subset of boxes in Q, labeled by the letters of $supp(\pi)$.

Denote by T_k^λ the set of all Young trees of order k in the Young diagram $[\lambda]$. By Corollary 2.4, in order to prove the main theorem it suffices to prove the following two complementary lemmas:

Lemma 3.1 *For any Young tableau Q of shape λ*

$$\sum_{\{(p,q)\mid p\in Q_r,q\in Q_c,pq\in C_k,\Gamma_Q(pq)\in T_k^\lambda\}} \mathrm{sign}(q) = \sum_{\{T\mid T\in T_k^\lambda\}} \mathrm{sign}(T)\cdot \mathrm{weight}(T).$$

Lemma 3.2 *For any Young tableau Q of shape λ*

$$\sum_{\{(p,q)\mid p\in Q_r,q\in Q_c,pq\in C_k,\Gamma_Q(pq)\notin T_k^\lambda\}} \mathrm{sign}(q) = 0.$$

The following easy fact is helpful.

Fact 3.3 *Let Q be a Young tableau, and let $p \in Q_r$ and $q \in Q_c$. If $\Gamma_Q(pq)$ is not connected then pq is not a cycle.*

It follows that in order to prove Lemma 3.2 it suffices to prove the following.

Lemma 3.4 *In the above notations*:

$$\sum_{\{(p,q)\mid p\in Q_r,q\in Q_c,pq\in C_k,\Gamma_Q(pq)\text{ is connected and not a tree}\}} \mathrm{sign}(q) = 0.$$

4. Proof of Lemma 3.1

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of n. For any subset D of k boxes in the Young diagram $[\lambda]$ let $r_i(D)$ be the number of boxes of D in the i-th row of $[\lambda]$, and let $\mu^r(D)$ be the partition of k obtained by reordering the numbers $r_1(D), \ldots, r_t(D)$. Define, similarly, $c_i(D)$ to be the number of boxes of D in the i-th column of $[\lambda]$, and $\mu^c(D)$ to be the partition of k obtained by reordering these numbers. For a Young graph $\Gamma = \Gamma(D)$ denote $\mu^r(\Gamma) := \mu^r(D)$ and $\mu^c(\Gamma) := \mu^c(D)$. For a given λ-tableau Q and a subset $S \subseteq \{1, \ldots, n\}$ denote $\mu^r_Q(S) := \mu^r(D)$ and $\mu^c_Q(S) := \mu^c(D)$, where D is the subset of boxes in $[\lambda]$, which are labeled by S.

Lemma 4.1 *Let T be a Young tree of order k in a Young diagram $[\lambda]$, and let Q be a Young tableau of shape λ. For any pair (p, q), $p \in Q_r, q \in Q_c$ with $\Gamma_Q(pq) = T$, the following holds*:

$$pq \text{ is a } k\text{-cycle iff } p \text{ is of cycle type } \mu^r(T) \text{ and } q \text{ is of cycle type } \mu^c(T).$$

Proof: By induction on the number of edges in T. Clearly, if T consists of one edge the lemma holds. Assume that the lemma holds for all trees of k edges. Let T be a tree of $k+1$ edges, and let $D = T \setminus \{x\}$ be a subtree of T of k edges, where x is a vertex of degree one ("a leaf") in T. We may assume that x is connected to D by a unique horizontal edge (y, x), where $y \in D$. The proof for a vertical edge is similar.

Assume that $p \in Q_r, q \in Q_c, \Gamma_Q(pq) = T$ and that p is of cycle type $\mu^r(T)$. We will show that pq is a k-cycle. By definition,

$$\Gamma_Q(pq) = T \Leftrightarrow supp(pq) = Q(T),$$

where $Q(T) := \{Q(v) \mid v \in T\}$ is the set of labels of vertices of T in the given tableau Q. Also, for any $p \in Q_r$ and $q \in Q_c$, $supp(pq) = Q(T)$ implies that $supp(p) \subseteq Q(T)$. Combining this together with the assumption that $p \in Q_r$ is of cycle type $\mu^r(T)$ implies that there exists a permutation p_1 of cycle type $\mu^r(D)$ with $supp(p_1) \subseteq Q(D)$, and a transposition $\gamma = (Q(y), Q(x))$ such that

$$p = \gamma p_1.$$

In this case, for any $q \in Q_c$,

$$supp(pq) = Q(T) \Rightarrow supp(p_1 q) = Q(D).$$

By the induction hypothesis, $p_1 q$ is a $(k-1)$-cycle in S_n. On the other hand, $|supp(\gamma) \cap Q(D)| = |supp(\gamma) \cap supp(p_1 q)| = 1$. Hence, $\gamma p_1 q = pq$ is a k-cycle.

For the opposite direction, for any $p \in Q_r$ and $q \in Q_c$

$$supp(p) \subseteq supp(pq).$$

Hence, if p is not of cycle type $\mu^r_Q(supp(pq))$ then there is a row in the subset of boxes labeled by $supp(pq)$, such that the action of p on it consists of at least two cycles. Then, since there are no cycles in $\Gamma_Q(pq)$, the action of pq on $supp(pq)$ consists of at least two cycles. By definition, there is no fixed point of pq in $supp(pq)$. Hence, pq consists of at least two cycles of length >1. In particular, pq is not a k-cycle. Similarly, for q whose cycle type is not $\mu^c_Q(supp(pq))$. □

Corollary 4.2 *For any Young tree T of order k*

(i) $\text{weight}(T) = \#\{(p, q) \mid p \in Q_r, q \in Q_c, pq \in C_k, \Gamma_Q(pq) = T\}$

(ii) *For any pair (p, q) which appears in the set of the RHS of* (i)

$$\text{sign}(T) = \text{sign}(q).$$

Proof: Any pair (p, q) in the set

$$\{(p, q) \mid p \in Q_r, q \in Q_c, pq \in C_k, \Gamma_Q(pq) = T\}$$

satisfies the conditions of Lemma 4.1. Hence, pq is a k-cycle iff p is of cycle type $\mu^r(T)$ and q is of cycle type $\mu^c(T)$. The number of pairs of permutations with these cycle types (and a given support) is weight(T). This proves (i). The sign of and q of cycle type $\mu^c(T)$ is sign(T). This proves (ii). □

Lemma 3.1 is an immediate consequence of Corollary 4.2.

5. Decomposition theorem

A *generalized diagram* of order k is a subset $D \subseteq \mathbf{Z}^2$ of k lattice points (i.e., $|D| = k$). We say that Q is a *D-tableau* if the points in D are replaced by the letters $1, \ldots, k$ with no repeats.

For any D-tableau Q of order k let Q_c be the subgroup in S_k of column preserving permutations, and let Q_r is the subgroup of row preserving permutations. Define

$$r_Q := \sum_{p \in Q_r} p.$$

$$c_Q := \sum_{p \in Q_c} \operatorname{sign}(q)q.$$

$$e_Q := r_Q \cdot c_Q = \sum_{p \in Q_r, q \in Q_c} \operatorname{sign}(q)pq.$$

Then e_Q is the *Young symmetrizer* of Q, and $\mathbf{C}[S_k]e_Q$ is a left module for S_k. For any given $D \subseteq \mathbf{Z}^2$ different D-tableaux define isomorphic left ideals. Hence, we can adopt the notation S^D, referring to any module isomorphic to $\mathbf{C}[S_k]e_Q$, where Q is a D-tableau, and call it the *generalized Specht module associated with D*. Denote its character by χ^D.

The concept of generalized Specht modules was introduced in [6]. In this section consider its decomposition into irreducible modules.

For every generalized diagram $D \subseteq \mathbf{Z}^2$ let width (D) be the number of nonempty columns in D, and let height (D) be the number of nonempty rows. For a D-tableau Q let width (Q) := width(D) and height(Q) := height(D).

Lemma 5.1 *For every diagram $D \subseteq \mathbf{Z}^2$ if*

$$\operatorname{width}(D) + \operatorname{height}(D) - 1 < |D|,$$

then no hook character $\chi^{(m,1^{|D|-m})}$ appears in the decomposition of χ^D into irreducible characters.

Proof: Let Q be a D-tableau, and let $k := |D|$. Assume that there is a composition factor of shape $(m, 1^{k-m})$ in $\boldsymbol{C}[S_k]e_Q$. Then there exists

$$J \subseteq \boldsymbol{C}[S_k]e_Q, \tag{5.1}$$

where J is a left ideal of $\boldsymbol{C}[S_k]$ which gives an irreducible module for the partition $(m, 1^{k-m})$.

Left P be a tableau of shape $(m, 1^{k-m})$. Then there exists a $\boldsymbol{C}[S_k]$-isomorphism ϑ from $\boldsymbol{C}[S_k]e_P$ to J. By Maschke's Theorem, there is a left ideal K of $\boldsymbol{C}[S_k]$ with $\boldsymbol{C}[S_k] = \boldsymbol{C}[S_k]e_p \oplus K$, so ϑ may be extended to be a $\boldsymbol{C}[S_k]$-homomorphism from $\boldsymbol{C}[S_k]$ to J, by letting ϑ be zero on K. Let $r := \vartheta(1)$. Then

$$\vartheta(e_P) = e_P\vartheta(1) = e_P r.$$

In particular,

$$J = \boldsymbol{C}[S_k]e_P r$$

and $e_P r \neq 0$.

It is well known that e_P^2 is a non-zero multiple of e_P (wherever P is a Young tableau whose shape is a partition). Therefore,

$$e_P J = e_P\boldsymbol{C}[S_k]e_P r \neq 0. \tag{5.2}$$

Combining (5.1) and (5.2) we obtain

$$e_P\boldsymbol{C}[S_k]e_Q \neq 0.$$

Equivalently,

$$r_P c_P\boldsymbol{C}[S_k]r_Q c_Q \neq 0. \tag{5.3}$$

Therefore,

$$r_P\boldsymbol{C}[S_k]c_Q \neq 0.$$

Hence, for some tableau Q' of same shape as Q

$$r_P c_{Q'} \neq 0. \tag{5.4}$$

This implies that the letters in the first row of P appear in different columns of Q'. In particular,

$$\text{width}(P) \leq \text{width}(Q') = \text{width}(Q).$$

Similarly, (5.3) gives

$$c_P\boldsymbol{C}[S_k]r_Q \neq 0.$$

Hence, for some tableau Q'' of same shape as Q

$$c_P r_{Q''} \neq 0. \tag{5.5}$$

This implies that the letters in the first column of P appear in different rows of Q''. In particular,

$$\text{height}(P) \leq \text{height}(Q'') = \text{height}(Q).$$

We conclude that

$$\begin{aligned} k + 1 = m + (k - m + 1) &= \text{width}(P) + \text{height}(P) \\ &\leq \text{width}(Q) + \text{height}(Q) < k + 1. \end{aligned}$$

Contradiction. □

For any set $D \subseteq \mathbf{Z}^2$ define the *height partition* $h(D) = (h_1(D), h_2(D), \ldots)$ by

$$h_i(D) := \text{number of rows in } D \text{ with at least } i \text{ boxes}.$$

Similarly, the *width partition* $w(D) = (w_1(D), w_2(D), \ldots)$ is defined by

$$w_i(D) := \text{number of columns in } D \text{ with at least } i \text{ boxes}.$$

Note that $h(D) = \mu^r(D)'$ and $w(D) = \mu^c(D)'$, where $\mu^r(D)$ and $\mu^c(D)$ are defined as in the beginning of Section 4, and μ' is the conjugate of the partition μ. Let $\preceq$ be the dominance order on partitions. Namely, $\lambda \preceq \mu$ iff for all i $\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j$. Generalizing the proof of Lemma 5.1 gives

Theorem 5.2 *For any generalized diagram $D \subseteq \mathbf{Z}^2$ the following holds*:
If the irreducible character χ^ν appears as a composition factor of χ^D then

$$h(D)' \preceq \nu \preceq w(D),$$

where $h(D)'$ is the conjugate of the height partition $h(D)$, and $w(D)$ is the width partition.

Proof: Let Q be a D-tableau and let P be a Young tableau of shape ν. By (5.4) there is a D-tableau, Q', for which, for any j, the letters in the j-th row of P appear in different columns of Q'. This implies that for any i

$$\sum_{j=1}^{i} \nu_j \leq \sum_{j=1}^{i} w_j(D).$$

Similarly, by (5.5), for some D-tableau Q'' the letters in the j-th column of P appear in different rows of Q''. Hence, for any i

$$\sum_{j=1}^{i} \nu_j' \leq \sum_{j=1}^{i} h_j(D).$$

Implying $\nu' \preceq h(D)$, or equivalently, $h(D)' \preceq \nu$. □

Note: The classical Littlewood-Richardson rule, which provides a combinatorial method for decomposing outer products of Specht modules, gives, as well, the composition series for Specht modules of skew shapes (cf. [6], Theorem 3.1; [11], (7.60)). A characteristic free version was given in [6], and extended by Reiner and Shimozono [8] to column-convex shapes. In particular, Theorem 5.2 for column-convex shapes is a consequence of ([8], Theorem 1).

6. Proof of Lemma 3.4

For any subset D of boxes in a Young diagram $[\lambda]$, let its *width*, width(D), be the number of nonempty columns in D, and let the *height*, height(D), be the number of nonempty rows in D. Recall the definition of the graph $\Gamma(D)$ (where either $D \subseteq [\lambda]$ or $D \subseteq \mathbf{Z}^2$) from Section 1.

Lemma 6.1 *Let D be a subset in a Young diagram, whose Young graph $\Gamma(D)$ is connected. Then $\Gamma(D)$ is not a tree iff*

$$\text{width}(D) + \text{height}(D) - 1 < |D|.$$

Proof: First, note that for any subset $D \subseteq [\lambda]$, whose Young graph is connected, width (D) $+$ height$(D) - 1 \leq |D|$. The lemma states that equality holds iff $\Gamma(D)$ is a tree.

The lemma is proved by induction on the order of D. If $|D| = 1$ the lemma is obvious.

Assume that the lemma holds for all subsets of $[\lambda]$ of cardinality less than k. Let D be a subset of order k. Consider two complementary cases:

(1) $\Gamma(D)$ is connected but not a tree. Let $x \in D$ be a box which lies in a cycle of $\Gamma(D)$. Adding x to the set $D \backslash x$ adds zero to the total number of rows and columns, in which boxes of the subset appear. Note that this number is equal to the sum of the width and the height. By the induction hypothesis we are done.

(2) $\Gamma(D)$ is a tree. Let $x \in D$ be a leaf in $\Gamma(D)$. Then $D = T \cup x$, where $\Gamma(T)$ is a tree of order $k - 1$. There are two options: There are vertices of T in the row of x, but no vertices of T in its column. Or: there are vertices of T in the column of x, but no vertices in its row. In both cases, adding x to the set T adds one to the total number of rows and columns. Hence,

$$\text{width}(D) + \text{height}(D) = \text{width}(T) + \text{height}(T) + 1 = |T| = |D| - 1.$$

The second equality follows from the induction hypothesis. □

The following lemma is well known.

Lemma 6.2 ([10], Lemma 4.10.3) *Let λ be a partition of k, which is not of hook shape. Then*

$$\chi_k^\lambda = 0,$$

where χ_k^λ is the value of the character of S^λ at cycles of length k.

Lemma 6.3 *Let $D \subseteq \mathbf{Z}^2$ be a set of order k, and let Q be a D-tableau. If $\Gamma(D)$ is connected but not a tree then*

$$\sum_{\{(p,q)\in S_k^{\times 2} \mid p\in Q_r, q\in Q_c, pq\in C_k\}} \operatorname{sign}(q) = 0.$$

Proof: Let ρ_Q be the S_k-representation $\mathbf{C}[S_k]e_Q$. By Schur's lemma, $\rho_Q(\sum_{g\in C_k} g)$ is diagonalizable, and its spectrum consists of the central characters of the irreducible representations, which appear in the decomposition of ρ_Q, evaluated at k-cycles (see Proof of Lemma 2.1).

$\Gamma(D)$ is not a tree. Hence, by Lemma 6.1 width$(D)+$height$(D)-1 < |D|$. It follows from Lemma 5.1 that no hooks appear in the decomposition of ρ_Q into irreducibles. Combining this with Lemma 6.2 shows that all eigenvalues of $\rho_Q(\sum_{g\in C_k} g)$ are zero. It follows that

$$\left(\sum_{g\in C_k} g\right)\cdot e_Q = \rho_Q\left(\sum_{g\in C_k} g\right)e_Q = 0.$$

In particular, the coefficient of the identity in $(\sum_{g\in C_k} g)e_Q$ is zero. Namely,

$$\sum_{\{(p,q)\in S_k^{\times 2} \mid p\in Q_r, q\in Q_C, pq\in C_k\}} \operatorname{sign}(q) = 0.$$

□

Corollary 6.4 *Let P be a Young tableau of shape λ, and let D be a subset of k boxes in the Young diagram $[\lambda]$. If $\Gamma(D)$ is not a tree then*

$$\sum_{\{(p,q)\in S_n^{\times 2} \mid p\in P_r, q\in P_c, pq\in C_k, \Gamma_P(pq)=\Gamma(D)\}} \operatorname{sign}(q) = 0.$$

Proof: Let $S_k(D)$ be the symmetric group on the letters $P(D) = \{P(v) \mid v \in D\}$. Let Q be the restriction of P on the subset D, i.e., Q is the D-tableau, in which for any $v \in D$, $Q(v) := P(v)$.

For any $p \in P_r$ and $q \in P_c$, $supp(p) \subseteq supp(pq)$ and $supp(q) \subseteq supp(pq)$. Hence, $\Gamma_P(pq) = \Gamma(D)$ implies that $p \in Q_r$ and $q \in Q_c$. On the other hand, for any $p \in Q_r$ and $q \in Q_c$, if pq is a k-cycle then $\Gamma_P(pq) = \Gamma(D)$. Therefore,

$$\begin{aligned}&\sum_{\{(p,q)\in S_n^{\times 2} \mid p\in P_r, q\in P_c, pq\in C_k, \Gamma_P(pq)=\Gamma_P(D)\}} \operatorname{sign}(q)\\ &= \sum_{\{(p,q)\in S_k(D)^{\times 2} \mid p\in Q_r, q\in Q_c, pq\in C_k\}} \operatorname{sign}(q) = 0.\end{aligned}$$

The last equality follows from Lemma 6.3. □

Lemma 3.4 follows:

$$\sum_{\{(p,q)\mid p\in Q_r,q\in Q_c,pq\in C_{k,n},\Gamma_Q(pq)\text{ is connected and not a tree}\}} \text{sign}(q)$$
$$= \sum_{\{D\mid D\subseteq[\lambda]\text{ and }|D|=k\}} \sum_{\{(p,q)\mid p\in Q_r,q\in Q_c,pq\in C_k,\Gamma_Q(pq)=\Gamma_Q(D)\}} \text{sign}(q) = 0.$$

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