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Extremal Properties of Bases for Representations of Semisimple Lie Algebras

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Abstract. Let $\mathcal{L}$ be a complex semisimple Lie algebra with specified Chevalley generators. Let V be a finite dimensional representation of $\mathcal{L}$ with weight basis $\mathcal{B}$. The *supporting graph* P of $\mathcal{B}$ is defined to be the directed graph whose vertices are the elements of $\mathcal{B}$ and whose colored edges describe the supports of the actions of the Chevalley generators on V. Four properties of weight bases are introduced in this setting, and several families of representations are shown to have weight bases which have or are conjectured to have each of the four properties. The basis $\mathcal{B}$ can be determined to be *edge-minimizing* (respectively, *edge-minimal*) by comparing P to the supporting graphs of other weight bases of V. The basis $\mathcal{B}$ is *solitary* if it is the only basis (up to scalar changes) which has P as its supporting graph. The basis $\mathcal{B}$ is a *modular lattice* basis if P is the Hasse diagram of a modular lattice. The Gelfand-Tsetlin bases for the irreducible representations of $sl(n, \mathbb{C})$ serve as the prototypes for the weight bases sought in this paper. These bases, as well as weight bases for the fundamental representations of $sp(2n, \mathbb{C})$ and the irreducible "one-dimensional weight space" representations of any semisimple Lie algebra, are shown to be solitary and edge-minimal and to have modular lattice supports. Tools developed here are used to construct uniformly the irreducible one-dimensional weight space representations. Similar results for certain irreducible representations of the odd orthogonal Lie algebra $o(2n+1, \mathbb{C})$, the exceptional Lie algebra G_2, and for the adjoint and short adjoint representations of the simple Lie algebras are announced.

Keywords: semisimple Lie algebras, irreducible representations, supporting graphs

1. Introduction

In this paper we visualize a representation V with a directed graph which is defined in terms of the "supports" of the actions of the Chevalley generators relative to a chosen weight basis for V. For us, the resulting "supporting graph" (along with its associated "representation diagram") is the principal structure associated to any given weight basis for V. Supporting graphs are studied here with three purposes in mind. First, supporting graphs have been helpful in formulating certain problems from combinatorics with Lie theory (e.g. [2]). In this paper we show that any supporting graph P is the Hasse diagram for the poset defined to be the transitive closure of the directed graph P. We apply Proctor's $sl(2, \mathbb{C})$ version [17] of a technique of Stanley and Griggs to see that any connected poset arising in this way is rank symmetric, rank unimodal, and strongly Sperner. This method is used in [6] to confirm the conjecture of Reiner and Stanton that certain lattices shown to be rank symmetric and unimodal in [20] are also strongly Sperner. Second, this paper provides tools which give some direction for producing a weight basis for a given representation and for identifying the coefficients for the actions of generators on the elements of the basis. These or related techniques are used in [3–6], and Section 6 below to explicitly construct new families of weight bases. Prior to our earliest results (from 1995), there was only one construction (from 1950) of a family of representations for which the actions of the Chevalley generators on the elements of a weight basis were explicitly given [7]. Third, this paper begins to explore the combinatorial and representation theoretic consequences of producing a weight basis whose supporting graph "looks as nice as possible." We introduce four combinatorial properties which may be possessed by weight bases for representations of semisimple Lie algebras. These "extremal" properties are defined in terms of supporting graphs and appear to be possessed only by rare weight bases. In this paper and the sequels [4–6], we study particular families of weight bases in terms of these properties.

Let $\mathcal{L}$ be a semisimple Lie algebra of rank n with Chevalley generators $\{x_i, y_i, h_i\}_{i=1}^{n}$ satisfying the Serre relations. Let V be an $\mathcal{L}$-module with weight basis $\mathcal{B} = \{v_{\mathbf{x}}\}_{\mathbf{x}\in P}$, where P is an indexing set with $|P| = \dim V$. The *supporting graph* for the weight basis $\mathcal{B}$ of V is the directed graph on the vertex set P which indicates the supports of the actions of the generators as follows: a directed edge of color i is placed from index $\mathbf{s}$ to index $\mathbf{t}$ if $c_{\mathbf{t},\mathbf{s}} v_{\mathbf{t}}$ (with $c_{\mathbf{t},\mathbf{s}} \neq 0$) appears as a term in the expansion of $x_i.v_{\mathbf{s}}$ as a linear combination in the basis $\{v_{\mathbf{x}}\}$, or if $d_{\mathbf{s},\mathbf{t}} v_{\mathbf{s}}$ (with $d_{\mathbf{s},\mathbf{t}} \neq 0$) appears when we expand $y_i.v_{\mathbf{t}}$ in the basis $\{v_{\mathbf{x}}\}$. The resulting edge-colored directed graph, which is also denoted by P, is the *supporting graph* for the basis $\mathcal{B}$ of V. (One could consider the pair of graphs which describe (respectively) the supports for the actions of the x_i's and the supports for the actions of the y_i's relative to the given weight basis. However, the bases which give rise to the most combinatorially elegant supporting graphs seem to have the property that the pattern of non-zero matrix entries in the transpose of a representing matrix for y_i is the same as the pattern of non-zero matrix entries in a representing matrix for x_i. In this case the "X-supporting graph" and the "Y-supporting graph" coincide. This motivates our decision to associate to each weight basis the simpler combinatorial structure of the supporting graph.) If we attach the coefficients $c_{\mathbf{t},\mathbf{s}}$ and $d_{\mathbf{s},\mathbf{t}}$ to each edge $\mathbf{s}\xrightarrow{i}\mathbf{t}$ of the supporting graph P, then we call P the *representation diagram* for the basis $\mathcal{B}$ of V. If the edge coefficients of the representation diagram P are positive and rational, we say the basis $\mathcal{B}$ is *positive rational*. A supporting graph Q for V is *positive rational* if there exists a positive rational weight basis for V with support Q. Two weight bases which differ by only one overall scalar multiple will have the same representation diagram and the same supporting graph. Two weight bases are *diagonally equivalent* if there are orderings of these bases such that the corresponding change of basis matrix is diagonal; their supporting graphs will be the same.

A weight basis $\mathcal{B}$ for a representation V is *edge-minimizing* if the supporting graph for $\mathcal{B}$ minimizes the number of edges appearing in the supporting graph when compared to the supporting graphs for all other weight bases for V. It is *edge-minimal* if no other weight basis for V has its supporting graph appearing as an "edge-colored subgraph" (see Section 2) in the supporting graph for $\mathcal{B}$. We say that $\mathcal{B}$ is a *modular* (*distributive*) *lattice* basis if its supporting graph is the Hasse diagram for a modular (distributive) lattice. The basis $\mathcal{B}$ is *solitary* if no weight basis has the same supporting graph as $\mathcal{B}$, other than those bases that are diagonally equivalent to $\mathcal{B}$. The adjectives edge-minimizing, edge-minimal, modular lattice, and solitary will apply to supporting graphs and representation diagrams as well as to weight bases. Since it can be seen that the number of distinct possible supporting graphs for a given representation V is finite, the number of solitary weight bases for V is also finite (but conceivably zero).

Consider the adjoint representation of $sl(3, \mathbb{C})$. This simple, rank two, eight-dimensional Lie algebra has generators x_1, y_1, x_2, and y_2. Figure 1 shows representation diagrams for three different bases of $sl(3, \mathbb{C})$ under the adjoint action. In these pictures, edges are assumed to be directed "up." The number superimposed upon an edge is the color of the edge. On each edge of color i we have attached two coefficients: a coefficient going "up" for the action of x_i and a coefficient going "down" for the action of y_i. If an edge coefficient is not depicted, it is unity. One can show that any weight basis for the adjoint representation of $sl(3, \mathbb{C})$ must have one of these three graphs as its supporting graph. It is shown in Section 4 that the last two of these, the "Gelfand-Tsetlin" supporting graphs, are edge-minimizing, edge-minimal, solitary, distributive lattice supporting graphs. None of these four properties are possessed by the "maximal" support of figure 1.

In this paper and its sequels, we construct or consider several families of representations having bases which possess some or all of these extremal properties, as is summarized in Table 1. Our investigation of extremal properties has usually required explicit descriptions of the actions of generators on a specific weight basis. The only bases we know of with such explicit descriptions appear in [3–7, 14–16, 25]. Most of the bases of Table 1 are distinctive in another sense. With the exception of the bases for the G_2 representations, each

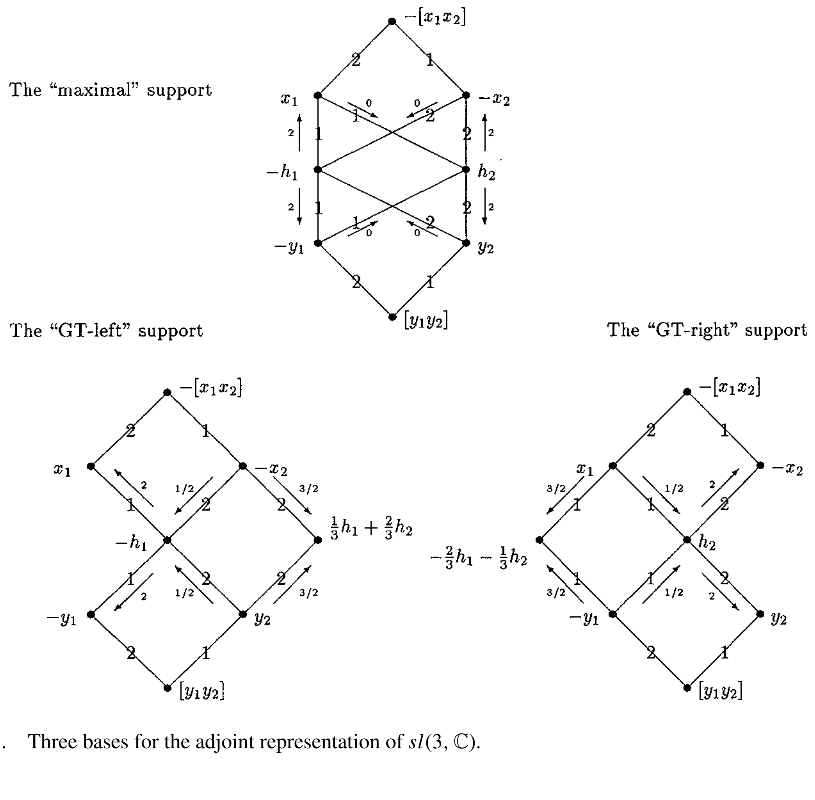

Figure 1. Three bases for the adjoint representation of $sl(3, \mathbb{C})$.

Table 1. Results for various simple Lie algebras.

Family of representations	Bases considered	Solitary?	Edge-minimal?	Modular lattice?	Edge-minimizing?
$A_n(\lambda)$ The irreducible representations of $sl(n+1, \mathbb{C})$	Both GT "left" and "right" bases	Yes: Section 4	Yes: Section 4	Yes: Section 4	Open
$C_n(\omega_k)$ The fundamental representations of $sp(2n, \mathbb{C})$	Both the "KN" and "DeC" constructions of [3]	Yes: Section 5	Yes: Section 5	Yes: [2]	Open
Irreducible one-dimensional weight space representations	The (essentially) unique weight basis	Yes: Section 6	Yes: Section 6	Yes: Section 6	Yes: Section 6
Adjoint representations of the simple Lie algebras	The n extremal bases of [4]	Yes: [4]	Yes: [4]	Yes: [4]	Yes: [4]
"Short adjoint" representations of the simple Lie algebras	The m extremal bases corresponding to the m short simple roots	Yes: [4]	Yes: [4]	Yes: [4]	Yes: [4]
$B_n(\omega_k)$ The fundamental representations of $o(2n+1, \mathbb{C})$	Both the "KN" and "DeC" constructions of [5]	Yes: [5]	Yes: [5]	Yes: [5]	Open
$B_n(k\omega_1)$ The "one-rowed" representations of $o(2n+1, \mathbb{C})$ (Largest irreducible component of the kth symmetric powers of the defining representation)	The RS and Molev bases of [6]	Yes: [6]	Yes: [6]	Yes: [6]	Open
$G_2(k\omega_1)$ The "one-rowed" representations of G_2	The RS and Molev bases of [6]	Yes	Yes	Yes: [6]	Open
$C_n(\lambda)$, $D_n(\lambda)$, $B_n(\lambda)$ The irreducible representations of $sp(2n, \mathbb{C})$, $o(2n, \mathbb{C})$, and $o(2n+1, \mathbb{C})$	Molev's bases in [14–16]	Open	Open	Open	Open

basis "restricts irreducibly" (see Section 3) under the action of a Lie subalgebra obtained by removing the generators corresponding to a certain node of the Dynkin diagram. The distributive lattice bases obtained from [6] for the irreducible representations $G_2(k\omega_1)$ do not restrict irreducibly under the action of any Lie subalgebra obtained in this way; in recent collaboration with the co-authors of that paper we have been able to show that these bases are solitary and edge-minimal.

In Section 3 of this paper we develop tools which allow us to confirm in Sections 4, 5, and 6 the entries in the first three rows of Table 1. In [4–6], we use these same techniques to confirm the results of rows four through seven. The familiar Gelfand-Tsetlin bases of [7] for the irreducible representations of $sl(n+1, \mathbb{C})$ are known to possess the distributive lattice property (e.g. [19]); in Section 4 we show they are solitary and edge-minimal. We apply this result to determine when the "left" and "right" Gelfand-Tsetlin bases for an irreducible representation of $sl(n+1, \mathbb{C})$ coincide. In Section 5 we show that the distributive lattice bases constructed in [3] for the fundamental symplectic representations are solitary and edge-minimal. In Section 5 and in [6] we use the solitary property to conclude that certain of our bases coincide with Molev's bases for certain symplectic and odd orthogonal representations. In Section 6 we uniformly construct the irreducible one-dimensional weight space representations by specifying explicit actions of the Chevalley generators in terms of weight diagram data. In Section 6 we also use the combinatorial perspective developed here to give another proof of the classification of irreducible one-dimensional weight space representations. This result obtained by Howe (Theorem 4.6.3 of Howe [8]) was recently re-derived by Stembridge [24] as a consequence of a broader classification result.

2. Definitions and preliminaries

We will only be using finite posets and directed graphs, and we will allow directed graphs to have at most one edge between any two vertices. We identify a poset with its *Hasse diagram*, the directed graph whose nodes are the elements of the poset and whose directed edges are given by the covering relations. When we depict the Hasse diagram for a poset, arrows on the edges will not be drawn; the direction of these edges is taken to be "up." A *path* $\mathcal{P}$ from $\mathbf{s}$ to $\mathbf{t}$ in a directed graph P is a sequence $\mathcal{P} = (\mathbf{s} = \mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_p = \mathbf{t})$ such that either $\mathbf{s}_{j-1} \to \mathbf{s}_j$ or $\mathbf{s}_j \to \mathbf{s}_{j-1}$ for $1 \leq j \leq p$. A *loop* in P is an edge $\mathbf{s} \to \mathbf{s}$. Let $a(\mathcal{P}) := |\{j : 1 \leq j \leq p,\ \textit{with}\ \mathbf{s}_{j-1} \to \mathbf{s}_j\}|$ be the number of *ascents* of the path $\mathcal{P}$, and let $d(\mathcal{P}) := |\{j : 1 \leq j \leq p,\ \textit{with}\ \mathbf{s}_j \to \mathbf{s}_{j-1}\}|$ be the number of *descents* of $\mathcal{P}$. See [22] for definitions of other combinatorial terms.

Let I be any set. An *edge-colored directed graph with edges colored by the set I* is a directed graph P together with a function assigning to each edge of P an element from the set I. The *dual* P^* is the set $\{\mathbf{t}^*\}_{\mathbf{t} \in P}$ together with colored edges $\mathbf{t}^* \xrightarrow{i} \mathbf{s}^*$ ($i \in I$) if and only if $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in P. If J is a subset of I, remove all edges from P whose colors are not in J; connected components of the resulting edge-colored directed graph are called *J-components* of P. Let Q be another edge-colored directed graph with edge colors from I. If the vertices of Q are a subset of the vertices of P and the edges of Q of color i are a subset of the edges of P of color i for each $i \in I$, then Q is an *edge-colored subgraph* of P. Let $P \oplus Q$ denote their disjoint union. Let $P \times Q$ be the Cartesian product $\{(\mathbf{s}, \mathbf{t}) \mid \mathbf{s} \in P, \mathbf{t} \in Q\}$ with colored edges $(\mathbf{s}_1, \mathbf{t}_1) \xrightarrow{i} (\mathbf{s}_2, \mathbf{t}_2)$ if and only if $\mathbf{s}_1 = \mathbf{s}_2$ in P with $\mathbf{t}_1 \xrightarrow{i} \mathbf{t}_2$ in Q or $\mathbf{s}_1 \xrightarrow{i} \mathbf{s}_2$ in P with $\mathbf{t}_1 = \mathbf{t}_2$ in Q. Two edge-colored directed graphs are isomorphic if there is a bijection between their vertices that preserves edges and edge colors.

For a directed graph P, a *rank function* is a surjective function $\rho : P \longrightarrow \{0, \ldots, l\}$ (where $l \geq 0$) with the property that if $\mathbf{s} \rightarrow \mathbf{t}$ in P, then $\rho(\mathbf{s}) + 1 = \rho(\mathbf{t})$. We call l the *length* of P with respect to ρ, and the set $\rho^{-1}(i)$ is the ith *rank* of P. Possessing a rank function is sufficient (but not necessary) for a directed graph to be the Hasse diagram for some poset; then we call P a *ranked poset*. A ranked poset that is connected has a unique rank function. A ranked poset P is *rank symmetric* if $|\rho^{-1}(i)| = |\rho^{-1}(l-i)|$ for $0 \leq i \leq l$. It is *rank unimodal* if there is an m such that $|\rho^{-1}(0)| \leq |\rho^{-1}(1)| \leq \cdots \leq |\rho^{-1}(m)| \geq |\rho^{-1}(m+1)| \geq \cdots \geq |\rho^{-1}(l)|$. It is *strongly Sperner* if for every $k \geq 1$, the largest union of k antichains is no larger than the largest union of k ranks. In an edge-colored ranked poset P we let $l_i(\mathbf{t})$ denote the length of the i-component of P that contains $\mathbf{t}$, and $\rho_i(\mathbf{t})$ is the rank of $\mathbf{t}$ within this component. We define the *depth* of $\mathbf{t}$ in its i-component by $\delta_i(\mathbf{t}) := l_i(\mathbf{t}) - \rho_i(\mathbf{t})$.

For semisimple Lie algebras and their representations our notation mostly follows [9]. For a *root system* Φ of rank n with *simple roots* $\{\alpha_1, \ldots, \alpha_n\}$, we let $\langle \cdot, \cdot \rangle$ denote the inner product on the Euclidean space spanned by the roots in Φ. For any root α, $\alpha^\vee$ denotes the *coroot* $\frac{2\alpha}{\langle \alpha, \alpha \rangle}$. We let $\{\omega_1, \ldots, \omega_n\}$ denote the associated *fundamental weights*. Let Λ denote the collection of *weights*, that is, the $\mathbb{Z}$-linear combinations of the fundamental weights. Let $\omega_0 := 0$ be the zero weight. Let $\mathcal{L}$ be the complex semisimple Lie algebra with *Chevalley generators* $\{x_i, y_i, h_i\}_{i=1}^n$ associated to the simple roots and satisfying the Serre relations as in Proposition 18.1 of [9]. In this paper, representations $\phi : \mathcal{L} \rightarrow gl(V)$ will be complex and finite-dimensional. Lower case x_i, y_i, and h_i denote elements of $\mathcal{L}$, and upper case X_i, Y_i, and H_i denote the corresponding images in $gl(V)$. A representation V of $\mathcal{L}$ is *non-zero* if there is a v in V and a z in $\mathcal{L}$ for which $z.v \neq 0$.

Let $\phi : \mathcal{L} \rightarrow gl(V)$ be a representation of $\mathcal{L}$, and let $\mu \in \Lambda$. A vector v in the *weight space* V_μ has weight $wt(v) := \mu$. The *weight diagram* for V is the set $\Pi(V) := \{\mu \in \Lambda \mid V_\mu \neq 0\}$, together with the partial order $\mu \leq \nu$ in $\Pi(V)$ if and only if $\nu - \mu = \sum k_i \alpha_i$, where each k_i is a non-negative integer. It can be seen that $\mu \rightarrow \nu$ in $\Pi(V)$ if and only if there is a simple root α_i for which $\mu + \alpha_i = \nu$. In this case we write $\mu \xrightarrow{i} \nu$. Following [10], let M be the finite-dimensional integrable module for $\mathcal{U}_q(\mathcal{L})$ corresponding to the representation V of $\mathcal{L}$, where $\mathcal{U}_q(\mathcal{L})$ denotes the quantized enveloping algebra associated to $\mathcal{L}$. Let A be the local ring of rational functions in $\mathbb{Q}(q)$ well-defined at $q = 0$. Let $(\mathcal{M}, \mathcal{B})$ be a *crystal base* for M, where $\mathcal{M}$ is a certain finitely generated A-module which generates M as a $\mathbb{Q}(q)$-vector space, and $\mathcal{B}$ is a certain basis for the $\mathbb{Q}$-vector space $\mathcal{M}/q\mathcal{M}$. Let $\tilde{E}_i$ and $\tilde{F}_i$ denote Kashiwara's "raising" and "lowering" operators respectively. The *crystal graph* $\mathcal{G}$ is the edge-colored directed graph whose vertices correspond to the elements of $\mathcal{B}$ and whose edges are defined by $\mathbf{s} \xrightarrow{i} \mathbf{t}$ if and only if $\tilde{E}_i \mathbf{s} = \mathbf{t}$ if and only if $\tilde{F}_i \mathbf{t} = \mathbf{s}$. (We direct crystal graph edges so they go "up.") The weight $wt(\mathbf{t})$ of an element of $\mathcal{G}$ is the same as the weight of $\mathbf{t}$ when thought of as an element of $\mathcal{B}$.

When $\mathcal{L}$ is simple of rank n, it will be convenient to identify $\mathcal{L}$ by its root system $\mathcal{X}_n$, where $\mathcal{X} \in \{A, B, C, D, E, F, G\}$. We will let $\mathcal{L}(\lambda)$ denote the equivalence class of irreducible representations of $\mathcal{L}$ with highest weight λ. So, for example, we say an irreducible representation of the Lie algebra C_n with highest weight ω_k is of type $C_n(\omega_k)$. We will also use the notation $\mathcal{L}(\lambda)$ to refer to an arbitrary irreducible representation of $\mathcal{L}$ with highest weight λ. Our numbering of the nodes of the Dynkin diagrams for the simple Lie algebras follows [9], p. 58. However, for a root system of type C_n we allow $n = 2$, and for B_n we require $n \geq 3$. The following simple linear algebra lemma will be useful later on.

Lemma 2.1 *Let V be a representation of $\mathcal{L}$, and suppose $\mu + \alpha_i = \nu$ for weights μ and ν in $\Pi(V)$. Let q (respectively, r) be the largest integer for which $\mu + q\alpha_i$ (respectively, $\mu - r\alpha_i$) is in $\Pi(V)$. If $r - q < 0$, then X_i injects V_μ into V_ν. If $r - q \geq 0$, then Y_i injects V_ν into V_μ.*

Proof: Let S_i be the subalgebra of $\mathcal{L}$ with generators $\{x_i, y_i, h_i\}$. Consider the S_i-submodule $W := \bigoplus_{p\in\mathbb{Z}} V_{\mu+p\alpha_i}$ in V. Set $j := r - q$. The weight space W_j coincides with V_μ, and $W_{j+2} = V_\nu$. Decompose W to get $W = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(s)}$, with each $W^{(k)}$ irreducible. Now $W_j = \bigoplus_{k=1}^{s} W_j^{(k)}$, and $W_{j+2} = \bigoplus_{k=1}^{s} W_{j+2}^{(k)}$. Suppose $j < 0$. If $W_j^{(k)}$ is non-empty, then so is $W_{j+2}^{(k)}$, and then $X_i(W_j^{(k)}) = W_{j+2}^{(k)}$ by standard facts about irreducible representations of $sl(2, \mathbb{C})$. Then X_i injects W_j into W_{j+2}. Similarly, if $j \geq 0$, then Y_i injects W_{j+2} into W_j. $\square$

Lattices for Sections 4 and 5. Let N be a positive integer and let λ be a shape with no more than N rows. (A "shape" is a collection of boxes arranged into left-justified rows, with each row having at least as many boxes as the row below it.) A *semistandard Young tableau T* of shape λ and with entries from $\{1, \ldots, N+1\}$ is a filling of the boxes of the shape λ with numbers from the set $\{1, \ldots, N+1\}$ so that the rows of T weakly increase (left to right) and the columns of T strictly increase (top to bottom). Let $L(N, \lambda)$ be the collection of semistandard Young tableaux of shape λ and with entries from $\{1, 2, \ldots, N+1\}$, ordered by *reverse* componentwise comparison. That is, $S \leq T$ if and only if no entry in T is larger than the corresponding entry in S. One can show that this partial order makes $L(N, \lambda)$ a distributive lattice. A tableau S is covered by a tableau T in $L(N, \lambda)$ if T is obtained from S by changing an $\boxed{i+1}$ entry in S to an $\boxed{i}$, for some i $(1 \leq i \leq N)$. In this case, attach the "color" i to the edge $S \xrightarrow{i} T$ in the Hasse diagram for $L(N, \lambda)$.

Let $1 \leq k \leq N$, and let λ be a column with k boxes. Set $L(k, N+1-k) := L(N, \lambda)$. A tableau T in $L(k, N+1-k)$ can be thought of as a k-tuple $\{T_1, \ldots, T_k\}$, where $1 \leq T_1 < \cdots < T_k \leq N+1$. So the column $T = \begin{array}{|c|}\hline 2\\ \hline 4\\ \hline 5\\ \hline\end{array}$ in $L(3, 5)$ corresponds to the 3-tuple $\{2, 4, 5\}$. Now let $1 \leq k \leq n$, and let $N = 2n - 1$. Following [2], a column T in $L(k, 2n-k)$ is *KN-admissible* if whenever $T_a = p$ and $T_b = 2n+1-p$ (where $1 \leq p \leq n$), then $a + k + 1 - b \leq p$. It is *DeC-admissible* if whenever $T_a = p$ and $T_b = 2n+1-p$ (where $1 \leq p \leq n$), we have $b+1-a \leq n+1-p$. As an example, the column $T = \{2, 4, 5\}$ is KN-admissible in $L(3, 5)$, but is DeC-*in*admissible. More elegant (but lengthier) descriptions of KN- and DeC-admissible columns appear in [2]. The KN-admissible columns were developed by Kashiwara and Nakashima in [12] to describe crystal graphs associated to the fundamental representations of $sp(2n, \mathbb{C})$. The DeC-admissible columns were used as labels to index weight bases for the fundamental representations of $sp(2n, \mathbb{C})$ ([1]; see also [21]).

We define the *symplectic lattice* $L_C^{KN}(n, \omega_k)$ (respectively, $L_C^{DeC}(n, \omega_k)$) to be the set of all KN-admissible (respectively, DeC-admissible) columns in $L(k, 2n-k)$, with the induced partial order. These posets are actually distributive sublattices of $L(k, 2n-k)$ [2]; thus they "inherit" its edge colors. We recolor the edges of the symplectic lattices by changing an edge of color i to an edge of color $2n-i$ whenever $n+1 \leq i \leq 2n-1$.

3. Supporting graphs and representation diagrams

This section presents results which expand on the definitions of "supporting graph" and "representation diagram" and which will be used to study the weight bases of this and future papers. Let P be the representation diagram for a weight basis $\{v_{\mathbf{t}}\}_{\mathbf{t}\in P}$ of a representation V of $\mathcal{L}$. We sometimes omit any reference to the associated weight basis and simply say that P is a *representation diagram for* V and that the underlying edge-colored directed graph is a *supporting graph*, or *support*, *for* V. We say that the representation diagram (or support) P *realizes the representation* V. Two supporting graphs for V are *isomorphic* if they are isomorphic as edge-colored directed graphs. The coefficients $c_{\mathbf{t},\mathbf{s}}$ (the "x-coefficient") and $d_{\mathbf{s},\mathbf{t}}$ (the "y-coefficient") are the *edge-coefficients associated to the edge* $\mathbf{s}\xrightarrow{i}\mathbf{t}$ in P. For $\mathbf{t} \in P$, we set $wt(\mathbf{t}) := wt(v_{\mathbf{t}})$, and we let $P_\mu := \{\mathbf{t} \in P \mid wt(\mathbf{t}) = \mu\}$ denote the μ-*weight space of* P.

3.1. Basic facts

Lemma 3.1 *Let V be a representation of $\mathcal{L}$.*

A. *Let P be a support for V. If $\mathbf{s}\xrightarrow{i}\mathbf{t}$ in P, then $wt(\mathbf{s}) + \alpha_i = wt(\mathbf{t})$. It follows that two vertices in P can have at most one edge between them, and in addition P has no loops.*

B. *If two weight bases for V are diagonally equivalent, and have representation diagrams P and Q respectively, then their supports are isomorphic. Moreover, the product of the "x" and "y" coefficients for an edge in P equals the product of the coefficients associated to the corresponding edge in Q.*

C. *Two weight bases for an irreducible representation V which have the same representation diagram must be scalar equivalent.*

D. *Let P be the support for a basis $\{v_{\mathbf{t}}\}$ of V, and let Q be a connected component of P. Then the linear span of $\{v_{\mathbf{s}} \mid \mathbf{s} \in Q\}$ is a submodule of V with supporting graph Q.*

E. *Let J be any subset of $\{1, \ldots, n\}$. Let P be a support for V, and let Q be any J-component of P. Then Q is the Hasse diagram for a ranked poset.*

F. *If V is irreducible, then each supporting graph for V is connected and has unique maximal and minimal elements.*

Proof: Parts A, B, and D follow from the definitions. For part C, let $\{v_{\mathbf{t}}\}_{\mathbf{t}\in P}$ and $\{w_{\mathbf{t}}\}_{\mathbf{t}\in P}$ be two weight bases with representation diagram P. Let $T : V \longrightarrow V$ be the linear map induced by $T : v_{\mathbf{t}} \mapsto w_{\mathbf{t}}$ for all $\mathbf{t} \in P$. Notice that for $1 \leq i \leq n$, $X_i(Tv_{\mathbf{s}}) = \sum_{\mathbf{t}:\mathbf{s}\xrightarrow{i}\mathbf{t}} c_{\mathbf{t},\mathbf{s}} w_{\mathbf{t}} = T(X_i v_{\mathbf{s}})$. Similarly T commutes with each Y_i. Since T therefore commutes with the action of each element of $\mathcal{L}$, by Schur's Lemma T must be a scalar multiple of the identity transformation. For part E, use part A to see that P (and therefore the J-component Q in P)

is acyclic. For a path $\mathcal{P}$ in P let $a_i(\mathcal{P})$ (respectively, $d_i(\mathcal{P})$) denote the number of ascents (respectively, descents) on edges of color i. For distinct elements $\mathbf{s}$ and $\mathbf{t}$ in Q, we write $\mathbf{s} < \mathbf{t}$ in Q if there is a path $\mathcal{P}$ in Q from $\mathbf{s}$ to $\mathbf{t}$ consisting only of ascents. This is a partial ordering on the elements of Q since Q is acyclic. It is not hard to see that $\mathbf{s}$ is covered by $\mathbf{t}$ in this partial order if and only if there is an edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in Q for some i. One can see that there is a minimal element $\mathbf{m}$ such that for any $\mathbf{t}$ in Q and any path $\mathcal{P}$ in Q from $\mathbf{m}$ to $\mathbf{t}$, the number of descents $d(\mathcal{P})$ of $\mathcal{P}$ does not exceed the number of ascents $a(\mathcal{P})$. For a path $\mathcal{P}$ from $\mathbf{m}$ to $\mathbf{t}$ in Q, we get $wt(\mathbf{t}) - wt(\mathbf{m}) = \sum_{i=1}^{n}(a_i(\mathcal{P}) - d_i(\mathcal{P}))\alpha_i$ by part A. Define $\rho(\mathbf{t}) := \sum_{i=1}^{n}(a_i(\mathcal{P}) - d_i(\mathcal{P}))$. One can now see that the definition of $\rho(\mathbf{t})$ does not depend on the path chosen from $\mathbf{m}$ to $\mathbf{t}$ and that ρ is the unique rank function for Q. For part F, if V is irreducible, part D implies that any supporting graph for V must be connected. For the remaining claim of part F, observe that a maximal (respectively, minimal) element of any supporting graph corresponds to a maximal (respectively, minimal) weight basis vector. □

The quantity $2\rho_i(\mathbf{t}) - l_i(\mathbf{t})$ introduced in the following lemma appears throughout this paper and can also be written $\rho_i(\mathbf{t}) - \delta_i(\mathbf{t})$. In [12], $\rho_i(\mathbf{t}) - \delta_i(\mathbf{t})$ is notated $\phi_i(\mathbf{t}) - \epsilon_i(\mathbf{t})$.

Lemma 3.2 *Let V be a representation of $\mathcal{L}$.*

A. *Let P be the supporting graph for a weight basis $\{v_{\mathbf{t}}\}_{\mathbf{t}\in P}$ for V. Let $1 \leq i \leq n$ and let $\mathbf{t} \in P$. Then $H_i v_{\mathbf{t}} = (2\rho_i(\mathbf{t}) - l_i(\mathbf{t}))v_{\mathbf{t}}$.* Thus, $wt(\mathbf{t}) = wt(v_{\mathbf{t}}) = \sum_{i=1}^{n}(2\rho_i(\mathbf{t}) - l_i(\mathbf{t}))\omega_i$.
B. *Elements in a connected support with the same weight have the same rank. Connected supports for the same representation have the same rank generating function.*
C. *Let P and Q be supports for an irreducible representation V. Suppose Q is an edge-colored subgraph of P. Let $\mathbf{t}$ and $\mathbf{t}'$ be corresponding elements of P and Q respectively. Then $wt(\mathbf{t}) = wt(\mathbf{t}')$.*
D. *Let P be any supporting graph for V. Let $\mu \xrightarrow{i} \nu$ in $\Pi(V)$. Then there are at least r edges between the vertex subsets P_μ and P_ν, where $r = \min(|P_\mu|, |P_\nu|)$, whose ends are mutually disjoint. In particular, there exists at least one edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in P with $wt(\mathbf{s}) = \mu$ and $wt(\mathbf{t}) = \nu$.*
E. *If V is non-zero, then there exists a connected supporting graph for V if and only if the weight diagram for V is connected.*
F. *If V is non-zero, has a weight space of dimension greater than one, and has a connected weight diagram, then it has at least two distinct supporting graphs.*

Proof: For part A, it suffices to show the following: if Q is a connected supporting graph for a representation of $sl(2, \mathbb{C})$, then $Hv_{\mathbf{t}} = (2\rho(\mathbf{t}) - l)v_{\mathbf{t}}$, where ρ is the rank function of Lemma 3.1.E for Q, and l is the length of Q. For each $\mathbf{t}$ in Q, define $m_{\mathbf{t}}$ by $Hv_{\mathbf{t}} = m_{\mathbf{t}}v_{\mathbf{t}}$. By Lemma 3.1.A, if $\mathbf{s} \to \mathbf{t}$, then $m_{\mathbf{s}}+2 = m_{\mathbf{t}}$. Let $\mathbf{x} \in Q$ with $\rho(\mathbf{x}) = 0$. Then the connectedness of Q implies that $\{m_{\mathbf{x}}, m_{\mathbf{x}}+2, \ldots, m_{\mathbf{x}}+2l\}$ is the complete list of eigenvalues for H. Since these all have the same parity, it follows from Theorem 7.2 of [9] that $m_{\mathbf{x}} = -(m_{\mathbf{x}} + 2l)$, and hence $m_{\mathbf{x}} = -l$. For any $\mathbf{t}$ in Q we have $m_{\mathbf{t}} - m_{\mathbf{x}} = 2\rho(\mathbf{t})$ by the proof of Lemma 3.1.E, whence $m_{\mathbf{t}} = 2\rho(\mathbf{t}) - l$. The second assertion of part B follows from the first. The proof of the first assertion of B is similar to the proof of Lemma 3.1.E. Similar reasoning also works in part C to show that corresponding elements $\mathbf{t}$ and $\mathbf{t}'$ in P and Q have the same weight. For part D, we apply Lemma 2.1. First, suppose X_i injects V_μ into V_ν. Set $r = |P_\mu|$. Then it is possible to find r edges $\mathbf{s}_j \xrightarrow{i} \mathbf{t}_j$ $(1 \leq j \leq r)$ for which $\mathbf{s}_j \neq \mathbf{s}_k$ in P_μ and $\mathbf{t}_j \neq \mathbf{t}_k$ in P_ν for $j \neq k$. Use a similar argument if Y_i injects V_ν into V_μ. We suppress the details of the lengthy proofs of parts E and F. In each proof, the key idea is to begin with a representation diagram and then use a local change of basis to produce a new representation diagram with the desired properties. We only use part F for the "**2** $\Rightarrow$ **1**" part of Proposition 6.3, and part E is only needed for the proof of part F. ☐

Given some representation V of $\mathcal{L}$, a Zariski topology argument can be used to show that almost all weight bases for V have the unique *maximal support* possible: If μ_1 and μ_2 are two weights for V of multiplicities m_2 and m_1 such that $\mu_2 = \mu_1 + \alpha_i$ for some simple root α_i, then there will be a total of $m_1 m_2$ edges in this maximal support between vertices of weight μ_1 and vertices of weight μ_2. The edges in the supporting graphs of Table 1 are much more sparse than the edges in the corresponding maximal supporting graph.

Lemma 3.3 *Let V be a representation of $\mathcal{L}$.*

A. *Let P be a support for V, and let Q be a support for another representation W of $\mathcal{L}$. Then the edge-colored directed graphs $P \oplus Q$, $P \times Q$, and P^* are supports for $V \oplus W$, $V \otimes W$, and V^* respectively. If P and Q are isomorphic as supports, then V and W are isomorphic representations.*

B. *Let P be a support for a representation U of $\mathcal{K}$, and let Q be a support for V. Let $\mathcal{L}$ act trivially on U, and let $\mathcal{K}$ act trivially on V. Then U and V become $\mathcal{K} \oplus \mathcal{L}$-modules, and $P \times Q$ is a supporting graph for the $\mathcal{K} \oplus \mathcal{L}$-module $U \otimes V$.*

Proof: Part B of this lemma follows from part A. For part A, the fact that $P \oplus Q$ is a supporting graph for $V \oplus W$ follows from the definitions. Now let $\{v_\mathbf{s}\}_{\mathbf{s}\in P}$ and $\{w_\mathbf{t}\}_{\mathbf{t}\in Q}$ be (respectively) bases for the representations V and W with supporting graphs P and Q. Consider the basis $\{v_\mathbf{s} \otimes v_\mathbf{t} \mid (\mathbf{s}, \mathbf{t}) \in P \times Q\}$ for $V \otimes W$. Using the fact that elements of $\mathcal{L}$ act on simple tensors according to the "Leibniz" rule, one can see that the edges of the edge-colored poset $P \times Q$ exactly describe the supports for the actions of the generators of $\mathcal{L}$ on $V \otimes W$ in this basis. Next, let $\{f_\mathbf{t}\}$ be the basis for V^* dual to the basis $\{v_\mathbf{t}\}$ for V, so $f_\mathbf{t}(v_\mathbf{x}) = \delta_{\mathbf{t},\mathbf{x}} v_\mathbf{x}$. Act on these basis vectors with elements of $\mathcal{L}$ in the usual way. By identifying the basis vector $f_\mathbf{t}$ with the element $\mathbf{t}^*$, one can see that the edges for the edge-colored poset P^* exactly describe the supports for the actions of the generators for $\mathcal{L}$ with respect to this basis. For the second claim of part A, note that Lemma 3.2.A implies that V and W will have the same formal character: $\sum_{\mu\in\Lambda}(\dim V_\mu)e(\mu) = \sum_{\mu\in\Lambda}(\dim W_\mu)e(\mu)$ in the notation of [9], Section 22.5. ☐

3.2. Producing representation diagrams and supporting graphs

With the exception of the Gelfand-Tsetlin bases and Molev's bases, all of the bases of Table 1 were obtained by first finding directed graphs which seemed likely to be candidates for supporting graphs and then "working backwards" to produce the bases. That is, in each case a representation diagram was produced without *a priori* knowledge of the associated weight basis. This process begins with an edge-colored ranked poset P with colors from

$\{1, \ldots, n\}$. Then to each edge $\mathbf{s}\xrightarrow{i}\mathbf{t}$, an "$x$" coefficient $c_{\mathbf{t},\mathbf{s}}$ and a "y" coefficient $d_{\mathbf{s},\mathbf{t}}$ are attached. An edge-colored ranked poset with coefficients so attached is called an *edge-labelled poset*. The following proposition says how to check that an edge-labelled poset P is a representation diagram for a representation of a semisimple Lie algebra. It improves on the techniques of [3]. By [11], it is not necessary to check the poset analogs of the Serre relations S^{+}_{ij} and S^{-}_{ij} in P since the representing space $V[P]$ is finite-dimensional.

Proposition 3.4 *Let P be an edge-labelled (ranked) poset with edge colors from $\{1, \ldots, n\}$. Let $V[P]$ be the complex vector space freely generated by $\{v_{\mathbf{t}}\}_{\mathbf{t}\in P}$, and for $1 \le i \le n$ define linear maps X_i and Y_i on $V[P]$ by*

$$X_i v_{\mathbf{s}} = \sum_{\mathbf{t}:\mathbf{s}\xrightarrow{i}\mathbf{t}} c_{\mathbf{t},\mathbf{s}} v_{\mathbf{t}} \quad \text{and} \quad Y_i v_{\mathbf{t}} = \sum_{\mathbf{s}:\mathbf{s}\xrightarrow{i}\mathbf{t}} d_{\mathbf{s},\mathbf{t}} v_{\mathbf{s}}.$$

Then $V[P]$ is a representation of $\mathcal{L}$ with Lie algebra map $\mathcal{L} \to gl(V[P])$ induced by $x_i \mapsto X_i$ and $y_i \mapsto Y_i$ and P is a representation diagram for the representation $V[P]$ if and only if (1) $[X_i, Y_j] = 0$ for $i \neq j$; (2) $[X_i, Y_i]v_{\mathbf{t}} = (2\rho_i(\mathbf{t}) - l_i(\mathbf{t}))v_{\mathbf{t}}$ for $1 \le i \le n$ and for each $\mathbf{t}$ in P; and (3) for $1 \le i \le n$, we have $2\rho_i(\mathbf{s}) - l_i(\mathbf{s}) + \langle \alpha_j, \alpha_i^\vee\rangle = 2\rho_i(\mathbf{t}) - l_i(\mathbf{t})$ whenever $\mathbf{s}\xrightarrow{j}\mathbf{t}$ with $i \neq j$.

Proof: Set $H_i := [X_i Y_i]$. In the forward direction, conclusion (1) is immediate, and (2) is just Lemma 3.2.A. Suppose $\mathbf{s}\xrightarrow{j}\mathbf{t}$ and let $1 \le i \le n$. Set $m_i(\mathbf{r}) := 2\rho_i(\mathbf{r}) - l_i(\mathbf{r})$ for any $\mathbf{r}$ in P. Note that $[H_i X_j](v_{\mathbf{s}}) = \sum_{\mathbf{t}':\mathbf{s}\xrightarrow{j}\mathbf{t}'} c_{\mathbf{t}',\mathbf{s}}(m_i(\mathbf{t}') - m_i(\mathbf{s}))v_{\mathbf{t}'}$. But $[H_i, X_j] = \langle \alpha_j, \alpha_i^\vee\rangle X_j$. Thus $c_{\mathbf{t},\mathbf{s}}(m_i(\mathbf{t}) - m_i(\mathbf{s})) = c_{\mathbf{t},\mathbf{s}}\langle \alpha_j, \alpha_i^\vee\rangle$. An argument using $[H_i Y_j]$ shows that $d_{\mathbf{s},\mathbf{t}}(m_i(\mathbf{s}) - m_i(\mathbf{t})) = -d_{\mathbf{s},\mathbf{t}}\langle \alpha_j, \alpha_i^\vee\rangle$. Now one of $c_{\mathbf{t},\mathbf{s}}$ or $d_{\mathbf{s},\mathbf{t}}$ is non-zero, so $2\rho_i(\mathbf{t}) - l_i(\mathbf{t}) - 2\rho_i(\mathbf{s}) + l_i(\mathbf{s}) = \langle \alpha_j, \alpha_i^\vee\rangle$, which is conclusion (3).

For the converse we must show that the Serre relations (S1), (S2), (S3), (S^{+}_{ij}), and (S^{-}_{ij}) from [9] Proposition 18.1 hold for X_i, Y_i, and H_i. (S1) is obvious. (S2) follows from assumptions (1) and (2) of the proposition statement. (S3) follows from computations similar to the previous paragraph, together with the observation that $2\rho_i(\mathbf{s}) - l_i(\mathbf{s}) + 2 = 2\rho_i(\mathbf{t}) - l_i(\mathbf{t})$ whenever $\mathbf{s}\xrightarrow{i}\mathbf{t}$. In Proposition B.1 of [11], it is observed that the integrable finite-dimensional $\mathcal{U}_q(\mathcal{L})$-modules are the same as the integrable finite-dimensional $\hat{\mathcal{U}}_q(\mathcal{L})$-modules, where $\hat{\mathcal{U}}_q(\mathcal{L})$ has the same generators as $\mathcal{U}_q(\mathcal{L})$ but without the quantum analogs of the Serre relations (S^{+}_{ij}) and (S^{-}_{ij}). At $q = 1$ this means that finite-dimensional $\hat{\mathcal{L}}$-modules are the same as the finite-dimensional $\mathcal{L}$-modules, where $\hat{\mathcal{L}}$ is the Lie algebra with the same generators as $\mathcal{L}$ but without the Serre relations (S^{+}_{ij}) and (S^{-}_{ij}). To see this, let $\phi : \hat{\mathcal{L}} \to gl(V[P])$ be the representation induced by $x_i \mapsto X_i$ and $y_i \mapsto Y_i$. Then $\mathrm{im}\phi$ is a finite-dimensional $\hat{\mathcal{L}}$-module via $w.\phi(z) := [\phi(w), \phi(z)]$. Let $S_i := \mathrm{span}\{x_i, y_i, h_i\}$ in $\hat{\mathcal{L}}$. Observe that $\phi(y_j)$ $(i \neq j)$ is a maximal vector under the action of S_i on $\mathrm{im}\phi$. The S_i-submodule W of $\mathrm{im}\phi$ generated by $\phi(y_j)$ is finite-dimensional and standard cyclic, and therefore irreducible. But $h_i.\phi(y_j) = \phi([h_i y_j]) = -\langle \alpha_j, \alpha_i^\vee\rangle\phi(y_j)$, so W has dimension $1 - \langle \alpha_j, \alpha_i^\vee\rangle$. Thus we kill $\phi(y_j)$ if we act on it by y_i in succession $1 - \langle \alpha_j, \alpha_i^\vee\rangle$ times. Therefore $\phi(\mathrm{ad}(y_i)^{1-\langle \alpha_j,\alpha_i^\vee\rangle}(y_j)) = 0$. Similarly $\phi(\mathrm{ad}(x_i)^{1-\langle \alpha_j,\alpha_i^\vee\rangle}(x_j)) = 0$. $\square$

Tableaux or other combinatorial objects are often used to "explain" the weight multiplicities of a representation. Sometimes obvious partial orders on these objects will produce supporting graphs for the representation. We say a set of objects P with weight rule $wt : P \to \Pi(V)$ *splits the multiplicities* of a representation V if $|wt^{-1}(\mu)| = \dim(V_\mu)$ for each weight μ for V. If P is also an edge-colored directed graph with colors from $\{1, \ldots, n\}$ and such that $wt(\mathbf{s}) + \alpha_i = wt(\mathbf{t})$ whenever $\mathbf{s} \xrightarrow{i} \mathbf{t}$, then we say that *the edges in P preserve weights*. Any supporting graph for a representation V splits the multiplicities of V, and its edges preserve weights. The following result can make Proposition 3.4 easier to apply in practice. Part (1) of this proposition formulates rank symmetry and unimodality results due to Dynkin in the language of edge-colored posets; it can be used to obtain rank symmetry and unimodality results for posets that are not known to satisfy the representation diagram condition of Proposition 3.11.

Proposition 3.5 *Let V be a representation of $\mathcal{L}$. Let P be an edge-colored directed graph with weight rule $wt : P \to \Pi(V)$. For any* $\mathbf{t}$ *in P, write $wt(\mathbf{t}) = \sum_{i=1}^{n} m_i(\mathbf{t})\omega_i$.* (1) *Suppose that P is connected, splits the multiplicities of V, and that the edges of P preserve weights. Then P is the Hasse diagram for a rank symmetric and rank unimodal poset.* (2) *In addition to* (1), *suppose that for each* $\mathbf{t}$ *in P and for each i, we have $m_i(\mathbf{t}) = 2\rho_i(\mathbf{t}) - l_i(\mathbf{t})$. Then $2\rho_i(\mathbf{s}) - l_i(\mathbf{s}) + \langle \alpha_j, \alpha_i^\vee \rangle = 2\rho_i(\mathbf{t}) - l_i(\mathbf{t})$ whenever $\mathbf{s} \xrightarrow{j} \mathbf{t}$ for $1 \le i \ne j \le n$. Moreover, whenever $\mu \xrightarrow{i} \nu$ is an edge in the weight diagram $\Pi(V)$, there exists an edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in P with $wt(\mathbf{s}) = \mu$ and $wt(\mathbf{t}) = \nu$.*

Proof: Apply the argument in the proof of Lemma 3.1.E to the directed graph P to see that P is the Hasse diagram for a ranked poset. The action of a "principal three-dimensional subalgebra" can be applied to obtain the remaining conclusions of part (1) (see for example [18] and the references therein). For part (2), assume that $2\rho_i(\mathbf{r}) - l_i(\mathbf{r}) = \langle wt(\mathbf{r}), \alpha_i^\vee \rangle$ for any $\mathbf{r}$ in P and any i. If $\mathbf{s} \xrightarrow{j} \mathbf{t}$ in P, then a simple calculation shows $2\rho_i(\mathbf{t}) - l_i(\mathbf{t}) = 2\rho_i(\mathbf{s}) - l_i(\mathbf{s}) + \langle \alpha_j, \alpha_i^\vee \rangle$. Now suppose that $\mu \xrightarrow{i} \nu$ in $\Pi(V)$. We wish to show that there exist $\mathbf{s}$ and $\mathbf{t}$ in P for which $\mathbf{s} \xrightarrow{i} \mathbf{t}$ with $wt(\mathbf{s}) = \mu$ and $wt(\mathbf{t}) = \nu$. If not, then any $\mathbf{s}$ in P of weight μ is maximal in its i-component. Thus $2\rho_i(\mathbf{s}) - l_i(\mathbf{s})$ is non-negative. Similarly, any $\mathbf{t}$ in P of weight ν is minimal in its i-component, and hence $2\rho_i(\mathbf{t}) - l_i(\mathbf{t})$ is non-positive. But this contradicts the fact that $2\rho_i(\mathbf{t}) - l_i(\mathbf{t}) = \langle wt(\mathbf{t}), \alpha_i^\vee \rangle = \langle wt(\mathbf{s}), \alpha_i^\vee \rangle + 2 = 2\rho_i(\mathbf{s}) - l_i(\mathbf{s}) + 2$. □

The next result follows easily from standard facts about crystal graphs. Thus the crystal graph $\mathcal{G}$ associated to an irreducible representation V has enough vertices of correct weight and its edges are oriented in the manner needed for $\mathcal{G}$ to serve as a supporting graph for V. However, Proposition 6.3 shows that $\mathcal{G}$ can serve as a support for V only when all weight spaces of V are one-dimensional.

Lemma 3.6 *Let V be an irreducible representation of $\mathcal{L}$. With the weight rule of Section* 2, *the crystal graph $\mathcal{G}$ associated to V is a connected edge-colored directed graph which satisfies the hypotheses of parts* (1) *and* (2) *of Proposition* 3.5.

3.3. Restricting to the action of a subalgebra

For any $J \subset \{1, 2, \ldots, n\}$ the (semisimple) subalgebra $\mathcal{K}$ with Chevalley generators $\{x_i, y_i, h_i\}_{i \in J}$ is a *Levi subalgebra* of $\mathcal{L}$. Let P be a supporting graph for a representation V of $\mathcal{L}$. Let Q be the edge-colored subgraph obtained from P by removing all edges whose colors are not in the set J. Observe that Q is a supporting graph for the $\mathcal{K}$-module V. A connected component of Q is called a *$\mathcal{K}$-component* of P. An element $\mathbf{t}$ of P is *$\mathcal{K}$-maximal* if it is maximal in some $\mathcal{K}$-component of P. Write $wt(\mathbf{t}) = \sum_{i=1}^{n} m_i^{\mathbf{t}} \omega_i$. The *$\mathcal{K}$-weight* of $\mathbf{t}$ is $wt_{\mathcal{K}}(\mathbf{t}) = \sum_{i \in J} m_i^{\mathbf{t}} \omega_i$. We say that P (or any weight basis with support P) *restricts irreducibly under the action of* $\mathcal{K}$ if the connected components of Q realize *irreducible* representations of $\mathcal{K}$. More generally, consider a "chain" of Levi subalgebras $\mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{m-1} \subset \mathcal{L}_m = \mathcal{L}$. For the supporting graph P, form diagrams $Q_{m-1}, \ldots, Q_2, Q_1$ by successively removing edges from P as described above. We say that P (or any associated weight basis) *restricts irreducibly for the chain of subalgebras* $\mathcal{L}_1 \subset \cdots \subset \mathcal{L}_{m-1} \subset \mathcal{L}_m = \mathcal{L}$ if the connected components of Q_i realize *irreducible* representations of $\mathcal{L}_i$, where $1 \leq i \leq m-1$. The following lemmas are used to show that bases considered in Sections 4 and 5 and in forthcoming papers have the solitary and edge-minimal properties.

Lemma 3.7 (Branching Lemmas) *Let V be a representation of $\mathcal{L}$.*

A. *Let $\mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m$ be a chain of Levi subalgebras of $\mathcal{L} := \mathcal{L}_m$. Let P be a supporting graph for V that restricts irreducibly for this chain of subalgebras. Suppose that distinct $\mathcal{L}_{i-1}$-maximal elements in any $\mathcal{L}_i$-component of P have distinct $\mathcal{L}_i$-weights. Suppose that each irreducible component in the decomposition of V as an $\mathcal{L}_1$-module has only one possible supporting graph. Then P is solitary and edge-minimal, and a weight basis for V restricts irreducibly for the chain of subalgebras $\mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m$ if and only if it has supporting graph P.*

B. *Let P be the supporting graph for a weight basis $\{v_{\mathbf{s}}\}_{\mathbf{s} \in P}$ of V. Let $\mathcal{K}$ be a Levi subalgebra of $\mathcal{L}$, and suppose that P restricts irreducibly under the action of $\mathcal{K}$. Suppose P has the property that if $\{w_{\mathbf{s}}\}_{\mathbf{s} \in P}$ is any weight basis for V with support P and if $\mathbf{t}$ is any $\mathcal{K}$-maximal element of P, then $w_{\mathbf{t}}$ is a scalar multiple of $v_{\mathbf{t}}$. Suppose that the $\mathcal{K}$-components of P are solitary as supports for representations of $\mathcal{K}$. Then P is solitary as a support for the $\mathcal{L}$-module V.*

C. *Suppose V is irreducible, and let P and Q be supports for V. Suppose that Q is an edge-colored subgraph of P. Let $\mathcal{K}$ be a Levi subalgebra of $\mathcal{L}$, and suppose P restricts irreducibly under the action of $\mathcal{K}$. If the $\mathcal{K}$-components of P are edge-minimal, then the $\mathcal{K}$-components of P and the $\mathcal{K}$-components of Q are the same.*

Proof: For part A, let $\{v_{\mathbf{s}}\}_{\mathbf{s} \in P}$ be any weight basis for V with support P. Let Q be the supporting graph for another weight basis $\{w_{\mathbf{t}}\}_{\mathbf{t} \in Q}$ which also restricts irreducibly for the chain of subalgebras. We show that $\{w_{\mathbf{t}}\}_{\mathbf{t} \in Q}$ is diagonally equivalent to $\{v_{\mathbf{s}}\}_{\mathbf{s} \in P}$. Regard V as an $\mathcal{L}_{m-1}$-module, and suppose $\mathcal{L}_{m-1}(\mu)$ occurs with multiplicity $k > 0$ in the decomposition of V. Let $\{w_{\mathbf{t}_1}, \ldots, w_{\mathbf{t}_k}\}$ be the $\mathcal{L}_{m-1}$-maximal vectors of $\mathcal{L}_{m-1}$-weight μ. Also, let $\mathbf{s}_1, \ldots, \mathbf{s}_k$ be $\mathcal{L}_{m-1}$-maximal elements of P of $\mathcal{L}_{m-1}$-weight μ. Now the vector subspace of V of $\mathcal{L}_{m-1}$-maximal vectors of $\mathcal{L}_{m-1}$-weight μ has dimension k and is spanned by $\{v_{\mathbf{s}_1}, \ldots, v_{\mathbf{s}_k}\}$. But the

$\mathcal{L}_m$-weights of the $v_{\mathbf{s}_i}$'s are distinct, so no non-trivial linear combination of these vectors can again be an $\mathcal{L}_m$-weight vector. Thus each $w_{\mathbf{t}_i}$ is a scalar multiple of some $v_{\mathbf{s}_j}$. Apply the same argument to the $\mathcal{L}_{i-1}$-maximal elements inside the $\mathcal{L}_i$-components of P, for $2 \leq i \leq m$. Thus, for each $\mathcal{L}_{i-1}$-maximal $\mathbf{s}$ in P, there is a corresponding $\mathcal{L}_{i-1}$-maximal element $\mathbf{t}$ in Q so that $\mathbf{s}$ and $\mathbf{t}$ have the same $\mathcal{L}_{i-1}$-weight and $v_{\mathbf{s}}$ and $w_{\mathbf{t}}$ differ only by some scalar factor. The only elements of P and Q we have not yet accounted for are the non-maximal elements of the $\mathcal{L}_1$-components. Pick corresponding $\mathcal{L}_1$-maximal elements $\mathbf{s}$ in P and $\mathbf{t}$ in Q. In particular, $\mathbf{s}$ and $\mathbf{t}$ have the same $\mathcal{L}_1$-weight, so their $\mathcal{L}_1$-components ($C(\mathbf{s})$ and $C(\mathbf{t})$ respectively) realize the same irreducible representation of $\mathcal{L}_1$. But by hypothesis, there is only one possible supporting graph for this irreducible $\mathcal{L}_1$-module. Thus if $\mathbf{s}'$ in $C(\mathbf{s})$ corresponds to $\mathbf{t}'$ in $C(\mathbf{t})$, then we see that $v_{\mathbf{s}'}$ and $w_{\mathbf{t}'}$ only differ by a scalar factor. So $\{w_{\mathbf{t}}\}_{\mathbf{t}\in Q}$ is diagonally equivalent to $\{v_{\mathbf{s}}\}_{\mathbf{s}\in P}$. In particular, P is solitary.

Now suppose Q is a support for V and is contained in P as an edge-colored subgraph. We claim that Q restricts irreducibly for the chain of subalgebras $\mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m$. Indeed, each $\mathcal{L}_i$-component of Q is contained in an $\mathcal{L}_i$-component of P, where $1 \leq i \leq m-1$. Now there are exactly as many $\mathcal{L}_i$-components of P as there are factors in the decomposition of the $\mathcal{L}_i$-module V. Thus, there is exactly one $\mathcal{L}_i$-component of Q contained in any $\mathcal{L}_i$-component of P. It follows that each $\mathcal{L}_i$-component of Q realizes an irreducible representation of $\mathcal{L}_i$. But now any basis with support Q restricts irreducibly for the chain of subalgebras $\mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m$, and the previous paragraphs imply that $Q = P$. Therefore P is edge-minimal.

For part B, let $\{w_{\mathbf{s}}\}_{\mathbf{s}\in P}$ be any other weight basis with support P. Let $\mathbf{t}$ be $\mathcal{K}$-maximal, and let $P_{\mathbf{t}}$ be the $\mathcal{K}$-component of P containing $\mathbf{t}$. By hypothesis the basis elements $v_{\mathbf{t}}$ and $w_{\mathbf{t}}$ only differ by some scalar factor. Then $span\{v_{\mathbf{x}} \mid \mathbf{x} \in P_{\mathbf{t}}\} = span\{w_{\mathbf{x}} \mid \mathbf{x} \in P_{\mathbf{t}}\}$ as subspaces and as irreducible $\mathcal{K}$-submodules of V. But since $P_{\mathbf{t}}$ is solitary as a support for $\mathcal{K}$, we see that each $v_{\mathbf{x}}$ ($\mathbf{x}$ in $P_{\mathbf{t}}$) only differs from $w_{\mathbf{x}}$ by some scalar factor. It follows that $\{v_{\mathbf{s}}\}_{\mathbf{s}\in P}$ and $\{w_{\mathbf{s}}\}_{\mathbf{s}\in P}$ are diagonally equivalent. For part C, argue as in the final paragraph of the proof of part A that each $\mathcal{K}$-component of P contains one and only one $\mathcal{K}$-component of Q. Thus the $\mathcal{K}$-maximal elements of Q are exactly the same as those of P. By Lemma 3.2.C, corresponding $\mathcal{K}$-maximal elements in P and Q will have the same $\mathcal{L}$-weight, and hence the same $\mathcal{K}$-weight. It follows that corresponding $\mathcal{K}$-components of P and Q realize the same irreducible representation of $\mathcal{K}$. But since each $\mathcal{K}$-component C of P is edge-minimal, then the corresponding $\mathcal{K}$-component of Q must be the same as C. □

Proposition 3.8 *Let U and V be irreducible representations for semisimple Lie algebras $\mathcal{K}$ and $\mathcal{L}$ respectively, with respective supporting graphs P and Q. If P and Q are both solitary (respectively, edge-minimal, positive rational, modular lattice) supports for U and V, then $P \times Q$ is a solitary (respectively edge-minimal, positive rational, modular lattice) support for the $\mathcal{K} \oplus \mathcal{L}$-module $U \otimes V$.*

Proof: Let $\{u_{\mathbf{s}}\}_{\mathbf{s}\in P}$ (respectively, $\{v_{\mathbf{t}}\}_{\mathbf{t}\in Q}$) be a weight basis for U (respectively, V) with support P (respectively, Q). The basis of simple tensors $\{u_{\mathbf{s}} \otimes v_{\mathbf{t}}\}_{(\mathbf{s},\mathbf{t})\in P\times Q}$ will have the edge-colored directed graph $P \times Q$ as its support. If the edge-coefficients for the basis $\{u_{\mathbf{s}}\}_{\mathbf{s}\in P}$ for U and for the basis $\{v_{\mathbf{t}}\}_{\mathbf{t}\in Q}$ for V are positive rational, then the edge-coefficients for $\{u_{\mathbf{s}} \otimes v_{\mathbf{t}}\}_{(\mathbf{s},\mathbf{t})\in P\times Q}$ will be as well. If P and Q are modular lattices, then $P \times Q$ is also a modular lattice since "meets" and "joins" can be formed componentwise in $P \times Q$. Suppose P and Q are both edge-minimal, and suppose a support S for $U \otimes V$ is contained in $P \times Q$ as an edge-colored subgraph. Apply Lemma 3.7.C to S and $P \times Q$ to see that their $\mathcal{K}$-components are the same. The $\mathcal{K}$-components of $P \times Q$ are just the copies of P in this product of graphs. Likewise, we also see that S and $P \times Q$ have the same $\mathcal{L}$-components (each of which is a copy of Q). Thus, $S = P \times Q$ as edge-colored graphs. Since this is true for any such S, it follows that $P \times Q$ is edge-minimal.

Suppose now that P and Q are solitary, and let $\{w_{(\mathbf{s},\mathbf{t})}\}_{(\mathbf{s},\mathbf{t})\in P\times Q}$ be another weight basis for $U \otimes V$ that has support $P \times Q$. Let $\mathbf{m}$ be the unique maximal element in P and $\mathbf{m}'$ be maximal in Q. The maximal vector $w_{(\mathbf{m},\mathbf{m}')}$ must be a scalar multiple of $u_{\mathbf{m}} \otimes v_{\mathbf{m}'}$. Let $Q_{\mathbf{m}} = \{(\mathbf{m}, \mathbf{t})\}_{\mathbf{t}\in Q}$ be the $\mathcal{L}$-component of $P \times Q$ that has $(\mathbf{m}, \mathbf{m}')$ as its maximal element. Now each of $span\{u_{\mathbf{m}} \otimes v_{\mathbf{t}}\}_{\mathbf{t}\in Q}$ and $span\{w_{(\mathbf{m},\mathbf{t})}\}_{\mathbf{t}\in Q}$ is isomorphic to V as an $\mathcal{L}$-module, and they have the same maximal vector (up to some scalar). Then they coincide as subspaces of $U \otimes V$. Since $Q_{\mathbf{m}} \cong Q$ is solitary, then each basis vector $w_{(\mathbf{m},\mathbf{t})}$ ($\mathbf{t} \in Q$) is just a scalar multiple of $u_{\mathbf{m}} \otimes v_{\mathbf{t}}$. Observe that if $(\mathbf{s}, \mathbf{t})$ is $\mathcal{K}$-maximal in $P \times Q$, then $\mathbf{s} = \mathbf{m}$. Also, the $\mathcal{K}$-components of $P \times Q$ are just copies of P. Then $P \times Q$ and the basis of simple tensors for $U \otimes V$ satisfy the hypotheses of Lemma 3.7.B, which implies that $P \times Q$ is a solitary support for $U \otimes V$. □

We conjecture that the edge-minimizing analog of this result is also true. This is related to the question: if P and Q are edge-minimizing supports for representations U and V of $\mathcal{L}$, is $P \oplus Q$ an edge-minimizing support for the representation $U \oplus V$ of $\mathcal{L}$? For evidence in the affirmative, see Proposition 3.10 below.

3.4. The rank one case

The weight spaces of $A_1(k\omega_1)$ each have dimension one, so there is only one weight basis up to diagonal equivalence. By Lemma 3.1.B, there is only one possible support for $A_1(k\omega_1)$. This support is automatically solitary, edge-minimal, and edge-minimizing. The explicit basis of Section 7 of [9] has positive rational (in fact integral) support. This support is easily seen to be a chain of length k, which is a distributive lattice. The following lemma characterizes all possible representation diagrams for the irreducible representations of $sl(2, \mathbb{C})$.

Lemma 3.9 *An edge-labelled poset P with all edges having the same color is a representation diagram for $A_1(k\omega_1)$ if and only if P is a chain of length k and the product of the edge coefficients on an edge $\mathbf{s} \to \mathbf{t}$ is $r(k + 1 - r)$, where r is the rank of $\mathbf{t}$.*

Proof: In the forward direction we have already observed that P must be a chain of length k. Let $\mathbf{t}_r$ denote the unique element of P of rank r ($0 \le r \le k$). Let $c_{r,r-1}$ and $d_{r-1,r}$ be the "x" and "y" coefficients on the edge $\mathbf{t}_{r-1} \to \mathbf{t}_r$ (where $1 \le r \le k$). Now $Hv_{\mathbf{t}_0} = -kv_{\mathbf{t}_0}$ (Lemma 3.2.A) while $[X, Y](v_{\mathbf{t}_0}) = XYv_{\mathbf{t}_0} - YXv_{\mathbf{t}_0} = -c_{1,0}d_{0,1}v_{\mathbf{t}_0}$, and so $c_{1,0}d_{0,1} = k$. To see that $c_{r,r-1}d_{r-1,r} = r(k + 1 - r)$ for $\mathbf{t}_{r-1} \to \mathbf{t}_r$, induct on r. For the converse, check condition (2) of Proposition 3.4 with a simple computation. □

The following proposition implies that the connected components of an edge-minimizing supporting graph for a representation V of $sl(2, \mathbb{C})$ correspond to the irreducible components in the decomposition of V.

Proposition 3.10 *Let P be a supporting graph for some representation of $sl(2, \mathbb{C})$. Then P is edge-minimizing if and only if P is a direct sum of chains.*

Proof: Let V be a representation for $sl(2, \mathbb{C})$ with supporting graph P that is a direct sum of chains. Set $P_i := \{\mathbf{t} \in P \mid wt(\mathbf{t}) = i\}$ for any integer i, so $|P_i| = \dim(V_i)$. It is easy to see that there are precisely r edges between P_i and P_{i+2}, where $r = \min(|P_i|, |P_{i+2}|)$. By Lemma 3.2.D, this is the least number of edges we can have between the i and $i+2$ weight spaces in any support for V. So P is an edge-minimizing support for V. Now suppose Q is another edge-minimizing support for V. Since P has the minimum number of edges between P_i and P_{i+2} allowed by Lemma 3.2.D, the graph Q must have the minimum number of edges between Q_i and Q_{i+2}. Let $i \geq 0$. Since Y injects V_{i+2} into V_i by Lemma 2.1, each element in Q_{i+2} covers at least one element in Q_i, and hence exactly one element. And since $\dim Y(V_{i+2}) = \dim V_{i+2}$, we see that for each $\mathbf{t}$ in Q_{i+2}, there is a *unique* element in Q_i covered by $\mathbf{t}$. Similarly, one can show that when $i < 0$, then for each $\mathbf{s}$ in Q_i, there is a unique element in Q_{i+2} that covers $\mathbf{s}$. Taken together, these say that each element of Q is covered by at most one other element, and covers at most one other element. So Q is a direct sum of chains. □

Inside any semisimple Lie algebra $\mathcal{L}$ with Chevalley generators $\{x_i, y_i, h_i\}_{i=1}^n$ are certain three-dimensional subalgebras called *principal TDS's*. One such principal TDS is spanned by $x := \sum c_i x_i$, $y := \sum y_i$, and $h := \sum c_i h_i$, where

$$c_i := 4 \sum_{j=1}^{n} \frac{\langle \omega_i, \omega_j \rangle}{\langle \alpha_j, \alpha_j \rangle}.$$

Each c_i is positive since $\langle \omega_i, \omega_j \rangle \geq 0$ for $1 \leq j \leq n$. It can be seen that $[xy] = h$, $[hx] = 2x$, and $[hy] = -2y$, so that $\{x, y, h\}$ are Chevalley generators for a copy of $sl(2, \mathbb{C})$ inside $\mathcal{L}$. Let P be a representation diagram for a representation of $\mathcal{L}$. Then P becomes a representation diagram for $sl(2, \mathbb{C})$ under the induced action of this principal TDS if we multiply the "x-coefficients" on each edge of color i by c_i and then change all the edge colors to black. If P is connected, then in the language of [17], x acts as an order-raising operator and y acts as a lowering operator on the vector space $V[P]$ spanned by $\{v_{\mathbf{t}}\}_{\mathbf{t} \in P}$. Now apply Proctor's "Peck Poset Theorem" [17] to get:

Proposition 3.11 *Let P be a connected supporting graph for a representation V of $\mathcal{L}$. Then P is the Hasse diagram for a rank symmetric, rank unimodal, and strongly Sperner poset.*

4. The Gelfand-Tsetlin bases

For an irreducible $gl(n+1,\mathbb{C})$-module, it is known that the Gelfand-Tsetlin basis [7] is "determined by" the restrictions to the "upper left" subalgebras $gl(1,\mathbb{C}) \subset \cdots \subset gl(n,\mathbb{C}) \subset gl(n+1,\mathbb{C})$. A second Gelfand-Tsetlin basis is determined by the restrictions to the "lower right" subalgebras $gl(n+1,\mathbb{C}) \supset gl(n,\mathbb{C}) \supset \cdots \supset gl(1,\mathbb{C})$. View A_k inside A_n as the Levi subalgebra generated by $\{x_i, y_i, h_i\}_{i=1}^{k}$; that is, A_k is the subalgebra whose generators correspond to the k leftmost nodes of the Dynkin diagram for A_n. Let A'_k be the subalgebra inside A_n generated by $\{x_i, y_i, h_i\}_{i=n+1-k}^{n}$. Let V be an irreducible A_n-module. Unlike the $gl(n+1,\mathbb{C})$ case, an irreducible A_{n-1}-module can appear with multiplicity in the decomposition of the A_{n-1}-module V. We use combinatorial arguments to see that that the Gelfand-Tsetlin bases for V are nonetheless uniquely determined by the restrictions $A_1 \subset \cdots \subset A_{n-1} \subset A_n$ (respectively, $A_n \supset A'_{n-1} \supset \cdots \supset A'_2 \supset A'_1$). In Theorem 4.4 we show these bases are solitary and edge-minimal. We use the combinatorics of their respective supporting graphs to determine when the two Gelfand-Tsetlin bases coincide (Corollary 4.5).

Throughout this section, $\lambda = a_1\omega_1 + a_2\omega_2 + \cdots + a_n\omega_n$ denotes a dominant weight, and $\lambda^{sym} := a_n\omega_1 + a_{n-1}\omega_2 + \cdots + a_1\omega_n$. Let *shape*($\lambda$) be the shape with a_n columns of length n, a_{n-1} columns of length $n-1$, etc. The *Gelfand-Tsetlin lattice* $L_A^{\text{GT-left}}(n,\lambda)$ is the edge-colored distributive lattice $L(n, \mathit{shape}(\lambda))$ of Section 2. We define the *GT-left basis* for $A_n(\lambda)$ to be the version of the GT basis obtained in [16]. As Proctor observed in [19], $L_A^{\text{GT-left}}(n,\lambda)$ is the supporting graph for the GT-left basis. Attach the positive rational coefficients of [16] to the edges of $L_A^{\text{GT-left}}(n,\lambda)$. Let L denote the edge-colored poset dual to $L_A^{\text{GT-left}}(n,\lambda)$. For an edge $\mathbf{t}^* \xrightarrow{i} \mathbf{s}^*$ in L attach coefficients $c_{\mathbf{s}^*,\mathbf{t}^*} := d_{\mathbf{s},\mathbf{t}}$ and $d_{\mathbf{t}^*,\mathbf{s}^*} := c_{\mathbf{t},\mathbf{s}}$. The *Gelfand-Tsetlin lattice* $L_A^{\text{GT-right}}(n,\lambda)$ is the edge-labelled distributive lattice L, after the edges have been re-colored by the rule $i \mapsto n+1-i$, where $1 \le i \le n$. It will be convenient to identify the vertex in $L_A^{\text{GT-right}}(n,\lambda)$ associated to the semistandard tableau T as an element T' in $L(n, \mathit{shape}(\lambda^{sym}))$, where the $(a_1 + \cdots + a_n + 1 - i)$ column of T' is just the setwise complement of the ith column of T in $\{1,\ldots,n+1\}$. One can see that $L_A^{\text{GT-right}}(n,\lambda)$ is a representation diagram for some basis of $A_n(\lambda)$. This basis is unique up to an overall scalar by Lemma 3.1.C. Call this the *GT-right basis*. The adjectives "left" and "right" are motivated by Theorem 4.4 below. (The GT-right basis is also easily obtained from the GT-left basis by acting on V with the image of A_n under the outer automorphism of A_n induced by the Dynkin diagram.) We record these observations in the following proposition.

Proposition 4.1 *The GT-left and GT-right bases are positive rational bases for the representation $A_n(\lambda)$ with distributive lattice supporting graphs $L_A^{\text{GT-left}}(n,\lambda)$* and *$L_A^{\text{GT-right}}(n,\lambda)$ respectively.*

Corollary 4.2 *For T in $L_A^{\text{GT-left}}(n,\lambda)$* or *$L_A^{\text{GT-right}}(n,\lambda)$, let m_j^T denote the number of j entries in T. If T is in $L_A^{\text{GT-left}}(n,\lambda)$, then the weight of T is $\sum_{i=1}^{n}(m_i^T - m_{i+1}^T)\omega_i$. If T is in $L_A^{\text{GT-right}}(n,\lambda)$, then the weight of T is $\sum_{i=1}^{n}(m_{n-i}^T - m_{n+1-i}^T)\omega_i$.*

Proof: From the edge-coloring rule for $L_A^{\text{GT-left}}(n,\lambda)$ it follows that $\rho_i(T)$ is the number of columns of T with an i but without an $i+1$. Also, $l_i(T)$ is $\rho_i(T)$ plus the number of columns of T with an $i+1$ but without an i. Then $2\rho_i(T) - l_i(T) = m_i^T - m_{i+1}^T$. Use a similar argument for $L_A^{\text{GT-right}}(n, \lambda)$. Now apply Lemma 3.2.A. □

Lemma 4.3 *Let S be a semistandard Young tableau which is A_{n-1}-maximal in $L_A^{\text{GT-left}}(n, \lambda)$, and set $\mu = wt_{A_{n-1}}(S)$. Then the A_{n-1}-component containing S is isomorphic to $L_A^{\text{GT-left}}(n-1, \mu)$. Moreover, there is no other A_{n-1}-maximal tableau in $L_A^{\text{GT-left}}(n, \lambda)$ with the same A_n-weight as S.*

Proof: One can see that a tableau S in $L_A^{\text{GT-left}}(n, \lambda)$ will be A_{n-1}-maximal if and only if each column of S of length i has entries $\{1, 2, \ldots, i\}$ or $\{1, 2, \ldots, i-1, n+1\}$. Let b_i be the number of columns of S of length i which do not have an $n+1$ entry; then S has $a_i - b_i$ columns of length i with an $n+1$ entry. So $m_i^S = b_i + \sum_{j=i+1}^{n} a_j$ when $1 \leq i \leq n$, and $m_{n+1}^S = \sum_{j=1}^{n}(a_j - b_j)$. By Corollary 4.2 the A_{n-1}-weight of S is $\mu = wt_{A_{n-1}}(S) = \sum_{i=1}^{n-1}(b_i + a_{i+1} - b_{i+1})\omega_i$. The shape corresponding to the A_{n-1}-weight μ can be obtained from S by removing all boxes with an $n+1$ entry and all columns of length n which do not have an $n+1$ entry. Now observe that the A_{n-1}-component containing S is just $L_A^{\text{GT-left}}(n-1, \mu)$. Suppose T is another A_{n-1}-maximal element, and suppose S and T have the same A_n-weight. Let c_i be the number of columns of T of length i which do not have an $n+1$ entry. Then $b_i + a_{i+1} - b_{i+1} = c_i + a_{i+1} - c_{i+1}$ for $1 \leq i \leq n$, so $b_1 - c_1 = b_2 - c_2 = \cdots = b_n - c_n$. But $m_n^S - m_{n+1}^S = m_n^T - m_{n+1}^T$ gives us $(b_1 - c_1) + \cdots + (b_{n-1} - c_{n-1}) + 2(b_n - c_n) = 0$. In light of the previous statement, we see that $b_i = c_i$ for $1 \leq i \leq n$, so $S = T$. □

Theorem 4.4 *The GT-left and GT-right bases for $A_n(\lambda)$ are solitary and edge-minimal. The GT-left (respectively, GT-right) basis is the unique weight basis for $A_n(\lambda)$ up to diagonal equivalence which restricts irreducibly for the chain of subalgebras $A_1 \subset \cdots \subset A_{n-1} \subset A_n$ (respectively, $A_n \supset A'_{n-1} \supset \cdots \supset A'_2 \supset A'_1$).*

Proof: In light of Lemma 4.3, apply Lemma 3.7.A to $L_A^{\text{GT-left}}(n, \lambda)$. The result for the GT-right basis follows from the result for the GT-left basis. □

As an application, we use the combinatorics of the supports $L_A^{\text{GT-left}}(n, \lambda)$ and $L_A^{\text{GT-right}}(n, \lambda)$ to determine when the GT-left and GT-right bases coincide.

Corollary 4.5 *The GT-left and GT-right bases for $A_n(\lambda)$ are diagonally equivalent if and only if λ is a multiple of a fundamental weight.*

Proof: Here we identify a dominant weight μ with its corresponding shape $shape(\mu)$. In the forward direction, we first decide when $L_A^{\text{GT-left}}(n, \lambda)$ will restrict irreducibly under the action of A'_{n-1}. Begin by removing all edges of color 1 from $L_A^{\text{GT-left}}(n, \lambda)$. Two tableaux are in the same A'_{n-1}-connected component if and only if they have the same number of 1 entries. Let P^r be the connected component consisting of all tableaux with exactly r boxes containing the entry 1. Removing these boxes, the tableaux in P^r can be thought of as semistandard Young tableaux of *skew shape* λ/r with entries from $\{2, \ldots, n+1\}$

(see [23]). By the "skew version" of Pieri's Rule ([23], Chapter 7, Corollary 15.9), each P^r will correspond to an irreducible A'_{n-1}-module if and only if λ has rectangular shape. (Otherwise, when $r = 1$ there will be more than one possible ν for which $\nu \subset \lambda$ and such that λ/ν is a *horizontal strip* of size $r = 1$.) This proves that λ must be a multiple of a fundamental weight.

For the converse, it suffices to produce a bijection ϕ from $L_A^{GT\text{-}left}(n, m\omega_k)$ to $L(n, (m\omega_k)^{sym})$ that takes edges of color i to edges of color $n+1-i$, and vice-versa. For S in $L_A^{GT\text{-}left}(n, m\omega_k)$ form $\phi(S)$ in $L(n, m\omega_{n+1-k})$ as follows: the ith column of $\phi(S)$ is obtained from the ith column of S by taking its complement in the set $\{1, \ldots, n+1\}$, and then changing an entry j to $n+2-j$. $\square$

5. Bases for the fundamental representations of $sp(2n, \mathbb{C})$

Let $1 \leq k \leq n$. The main result of [3] was:

Theorem 5.1 *The symplectic distributive lattices L_C^{KN} and L_C^{DeC} are positive rational supporting graphs for the* kth *fundamental representation of* $sp(2n, \mathbb{C})$.

We call the corresponding weight bases specified in [3] the *KN basis* and the *De Concini basis* for $C_n(\omega_k)$. Theorem 5.4 below states that these bases are solitary and edge-minimal. As with the Gelfand-Tsetlin bases, the key is to observe that these bases for $C_n(\omega_k)$ are well-behaved with respect to the action of certain subalgebras of C_n. The following is Lemma 5.2 of [2]:

Lemma 5.2 *Let T be a column tableau in* $L_C^{KN}(n, \omega_k)$ or $L_C^{DeC}(n, \omega_k)$, *and let m_i^T be the number of i entries in T. Then the weight of T is* $\sum_{i=1}^{n-1} \left(m_i^T - m_{i+1}^T + m_{2n-i}^T - m_{2n+1-i}^T\right)\omega_i + \left(m_n^T - m_{n+1}^T\right)\omega_n$.

View A_m inside C_n as the Levi subalgebra whose generators correspond to the m leftmost nodes of the Dynkin diagram for C_n, where $1 \leq m \leq n-1$.

Lemma 5.3 *Let S be an A_{n-1}-maximal column tableau in $L_C^{KN}(n, \omega_k)$ (respectively, $L_C^{DeC}(n, \omega_k)$), and set $\mu := wt_{A_{n-1}}(S)$. Then the A_{n-1}-component containing S is isomorphic to $L_A^{\text{GT-left}}(n-1, \mu)$ (respectively $L_A^{\text{GT-right}}(n-1, \mu)$). Moreover, no other A_{n-1}-maximal column tableau has the same C_n-weight as S.*

Proof: A column S will be A_{n-1}-maximal in $L_C^{KN}(n, \omega_k)$ or $L_C^{DeC}(n, \omega_k)$ if and only if (1) $S = \{1, \ldots, k\}$, (2) $S = \{1, \ldots, i, n+1, \ldots, n+k-i\}$ for $0 < i < k$, or (3) $S = \{n+1, \ldots, n+k\}$. Apply Lemma 5.2 to see that for type (1), μ is ω_k if $k < n$ and ω_0 if $k = n$. For type (2), $\mu = \omega_i + \omega_{n-k+i}$, and for type (3) $\mu = \omega_{n-k}$. One can use this explicit description of the A_{n-1}-maximal elements and their weights to see that distinct A_{n-1}-maximal elements have distinct C_n-weights. The shape corresponding to μ has at most two columns. We will describe a bijection ϕ from the A_{n-1}-component containing S to $L_A^{GT\text{-}left}(n-1, \mu)$. Let R be another column in the A_{n-1}-component of S. Obtain a tableau $\phi(R)$ of shape μ as follows. To get the first column of $\phi(R)$, take the complement of $R \cap \{n+1, n+2, \ldots, 2n\}$, and then subtract each of these elements from $2n+1$. To get the second column of $\phi(R)$, simply take $R \cap \{1, 2, \ldots, n\}$. Now check that this bijection gives an isomorphism of edge-colored posets.

Finally, suppose S is A_{n-1}-maximal in $L_C^{DeC}(n, \omega_k)$. We will describe a bijection ψ from the A_{n-1}-component containing S to $L_A^{GT\text{-}right}(n-1, \mu)$. (If S is of type (1) above, then $\mu^{sym} = \omega_{n-k}$, and for type (2), $\mu^{sym} = \omega_{n-i} + \omega_{k-i}$. For type (3), $\mu^{sym} = \omega_k$ if $k < n$ and ω_0 if $k = n$.) For R in the A_{n-1}-component of S obtain a tableau $\psi(R)$ of shape μ^{sym} as follows. To get the first column of $\psi(R)$, take the complement of $R \cap \{1, \ldots, n\}$ and then subtract each of these elements from $n+1$. To get the second column of $\psi(R)$, take $R \cap \{n+1, \ldots, 2n\}$ and then subtract n from each of these elements. This gives a bijection of edge-colored posets. $\square$

Denote by A'_m $(1 \leq m \leq n-1)$ the subalgebra of C_n whose generators correspond to the nodes $n-m, n-m+1, \ldots, n-1$.

Theorem 5.4 *The KN and De Concini bases for $C_n(\omega_k)$ are solitary and edge-minimal. The KN basis (respectively, De Concini basis) is the unique weight basis for $C_n(\omega_k)$ up to diagonal equivalence which restricts irreducibly for the chain of subalgebras $A_1 \subset \cdots \subset A_{n-1} \subset C_n$ (respectively, $C_n \supset A'_{n-1} \supset A'_{n-2} \supset \cdots \supset A'_2 \supset A'_1$).*

Proof: Follows from Lemma 5.3 and Lemma 3.7.A. $\square$

Corollary 5.5 *The KN basis and the De Concini basis for $C_n(\omega_k)$ are diagonally equivalent if and only if $k = 1$ or $k = n$.*

Proof: In Corollary 3.4 of [2] we showed that $L_C^{KN}(n, \omega_k)$ and $L_C^{DeC}(n, \omega_k)$ are isomorphic as posets (without regard to edge-coloring) if and only if $k = 1$ or $k = n$. When $k = 1$ or $k = n$, the bijections described in the proof of that corollary also preserve edge-colors. Thus $L_C^{KN}(n, \omega_k)$ and $L_C^{DeC}(n, \omega_k)$ are isomorphic as edge-colored posets if and only if $k = 1$ or $k = n$. By Theorem 5.4 these supports are solitary, so the corresponding bases coincide if and only if $k = 1$ or $k = n$. $\square$

Let C_m be the Levi subalgebra of C_n corresponding to the m rightmost nodes in the Dynkin diagram for C_n, with $C_1 = A_1$.

Theorem 5.6 *The De Concini basis is the unique weight basis for $C_n(\omega_k)$ up to diagonal equivalence which restricts irreducibly for the chain of subalgebras $C_n \supset C_{n-1} \supset \cdots \supset C_2 \supset C_1$.*

Proof: Use an argument similar to Lemma 5.3 so that Lemma 3.7.A can be applied. Let T be a C_{n-1}-maximal column tableau in $L_C^{DeC}(n, \omega_k)$. Let $i = |T \cap \{1, 2n\}|$. Then $wt_{C_{n-1}}(T)$ is the $(k-i)$th fundamental weight for C_{n-1}. From the definitions it follows that the C_{n-1}-component of T is isomorphic to the $(k-i)$th symplectic lattice for C_{n-1}. Moreover, it is not hard to see that distinct C_{n-1}-maximal elements of $L_C^{DeC}(n, \omega_k)$ will have distinct C_n-weights. Now apply Lemma 3.7.A. $\square$

Corollary 5.7 *The De Concini basis is diagonally equivalent to Molev's basis in* [14] *for the fundamental representations of* $sp(2n, \mathbb{C})$.

Proof: It can be seen that Molev's basis restricts irreducibly for the chain of subalgebras $C_n \supset C_{n-1} \supset \cdots \supset C_2 \supset C_1$. $\square$

6. One-dimensional weight space representations

In this section we characterize those irreducible representations which have only one supporting graph (the one-dimensional weight space representations of Propositions 6.2 and 6.3), say how to construct these representations uniformly across type (Theorem 6.4), and re-derive their classification (Theorem 6.7) (cf. Theorem 4.6.3 of Howe [8]). The supporting graphs for these representations enjoy the following extremal properties.

Proposition 6.1 *A one-dimensional weight space representation V has a unique supporting graph which is solitary, edge-minimal, edge-minimizing, and positive integral. If V is irreducible, its unique support is a distributive lattice.*

Proof: Since all weight bases for V are diagonally equivalent, Lemma 3.1.B implies V has only one supporting graph. This unique support is automatically solitary, edge-minimal, and edge-minimizing. The other assertions follow from Theorem 6.4 and Corollary 6.8. $\square$

Proposition 6.2 *A representation V is irreducible and all weight spaces of V are one-dimensional if and only if V has a connected supporting graph P such that the i components of P are all chains, for* $1 \leq i \leq n$.

The proof of Proposition 6.2 appears in Section 7. The assumption of irreducibility is needed only for the assertions $\mathbf{2} \Rightarrow \mathbf{1}$ and $\mathbf{4} \Rightarrow \mathbf{1}$ in the following result.

Proposition 6.3 *Let V be an irreducible representation. Then the following are equivalent*:
1. *All weight spaces of V are one-dimensional.*
2. *The representation V has only one supporting graph.*
3. *The weight diagram for V is a supporting graph for V.*
4. *The crystal graph associated to V is a supporting graph for V.*

Proof: We show $\mathbf{1} \Leftrightarrow \mathbf{2}$, $\mathbf{1} \Leftrightarrow \mathbf{3}$, and $\mathbf{1} \Leftrightarrow \mathbf{4}$. We have already seen that $\mathbf{1} \Rightarrow \mathbf{2}$. To see that $\mathbf{1} \Rightarrow \mathbf{3}$, note that each basis vector for a weight basis for V can be uniquely identified with its weight. Then apply Lemma 3.2.D. For $\mathbf{1} \Rightarrow \mathbf{4}$, we may use the fact that V has $\Pi(V)$ as its unique supporting graph. But now Lemma 3.6 (together with Proposition 3.5) implies that the crystal graph coincides with $\Pi(V)$. For $\mathbf{3} \Rightarrow \mathbf{1}$, observe that $|\Pi(V)| \leq \dim V$, with equality if and only if all weight spaces of V are one-dimensional. Now assume that V

is irreducible. Use Lemma 3.2.F to show that **2** $\Rightarrow$ **1**. Finally, we show that **4** $\Rightarrow$ **1**. All i-components of the crystal graph are chains, so Proposition 6.2 applies, proving that all weight spaces of V are one-dimensional. $\square$

The following theorem presents a uniform construction of Chevalley generator actions for all one-dimensional weight space representations: Its proof does not depend on the type of the Lie algebra or on the classification of one-dimensional weight space representations. An irreducible representation is *minuscule* if every weight in its weight diagram is in the orbit of the highest weight under the action of the Weyl group. Proctor [18] was aware of how to obtain actions for Chevalley generators on weight bases for the minuscule representations. Wildberger [25] uniformly constructs all minuscule representations and explicitly describes the actions of the Lie algebra generators corresponding to *every* root vector.

By Proposition 6.3 we know that the supporting graph of an irreducible one-dimensional weight space representation must be its weight diagram, and its i-components are chains. The choices for coefficients on the edges are therefore limited by Lemma 3.9. The first choice of coefficients in the next theorem agrees with Lemma 7.2 of [9] for irreducible representations of $sl(2, \mathbb{C})$. The x-coefficient (respectively, y-coefficient) on an edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ is the number of steps to $\mathbf{t}$ (resp. $\mathbf{s}$) from the minimal (resp. maximal) element in the i-component of $\mathbf{t}$. To confirm that the coefficients work globally, we must use a fact concerning the local structure of edges that is developed in the proof of Proposition 6.2. We must also consider all the possible interactions between the actions of any two $sl(2, \mathbb{C})$ Levi subalgebras. This result can also be used to construct the portions of the representation diagram corresponding to the one-dimensional weight space regions of any representation.

Theorem 6.4 *Let V be an irreducible one-dimensional weight space representation. Let P be the unique support for V. For an edge $\mathbf{s} \xrightarrow{i} \mathbf{t}$ in P, set $c_{\mathbf{t},\mathbf{s}} := \rho_i(\mathbf{t})$ and $d_{\mathbf{s},\mathbf{t}} := l_i(\mathbf{t}) - \rho_i(\mathbf{t}) + 1$. With these edge coefficients, P is a representation diagram for V. If the formulas for $c_{\mathbf{t},\mathbf{s}}$ and $d_{\mathbf{s},\mathbf{t}}$ are interchanged everywhere, the resulting edge-labelled poset is also a representation diagram for V.*

Keeping the notation of the theorem statement, let $\mathbf{t}_{max}$ be the maximal element in the i-component of $\mathbf{t}$. Let $\mu := wt(\mathbf{t})$ and $\mu_{max} := \mathbf{t}_{max}$. The first choice of edge coefficients in Theorem 6.4 can be expressed in terms of inner products: $c_{\mathbf{t},\mathbf{s}} = \frac{\langle \mu_{max}, \alpha_i^\vee \rangle + \langle \mu, \alpha_i^\vee \rangle}{2}$ and $d_{\mathbf{s},\mathbf{t}} = 1 + \frac{\langle \mu_{max}, \alpha_i^\vee \rangle - \langle \mu, \alpha_i^\vee \rangle}{2}$. The proof of Theorem 6.4 is given in Section 7.

Lemma 6.5 *Each of the following representations has a weight space with dimension exceeding one: $A_n(a_1\omega_1 + a_n\omega_n)$, where $a_1 > 0$, $a_n > 0$, and $n \geq 2$; $A_3(a\omega_2)$ with $a > 1$; $B_n(\omega_2)$ for $n \geq 3$; $B_n(a\omega_1)$ with $a > 1$; $B_n(a\omega_1 + \omega_n)$ with $a > 0$; $C_n(\omega_2)$ with $n \geq 3$; $C_n(\omega_n)$ with $n \geq 4$; $C_n(a\omega_1)$ with $a > 1$; $C_n(a\omega_1 + \omega_n)$ with $a > 0$; $C_2(a_1\omega_1 + a_2\omega_2)$ where $a_1 + a_2 \geq 2$; $D_n(a\omega_1)$ with $a > 1$; $D_n(\omega_2)$; $D_4(a\omega_3)$ where $a > 1$; $D_4(a\omega_4)$ where $a > 1$; $F_4(\omega_1)$; $F_4(\omega_4)$; $F_4(\omega_1 + \omega_4)$; $E_6(\omega_2)$; $E_7(\omega_1)$; $E_8(\omega_8)$; $G_2(\omega_2)$; and $G_2(a\omega_1 + b\omega_2)$ where $a + b \geq 2$.*

Proof: For the classical cases, one can use tableaux as in [12]. The following are adjoint representations: $F_4(\omega_1)$; $E_6(\omega_2)$; $E_7(\omega_1)$; $E_8(\omega_8)$; and $G_2(\omega_2)$. For $F_4(\omega_4)$ and $F_4(\omega_1 + \omega_4)$, compute the character. For the remaining G_2 cases, one can use the tableaux described in [13]. □

Lemma 6.6 *Let V be an irreducible one-dimensional weight space representation with unique supporting graph P. Let $\mathcal{K}$ be a Levi subalgebra of $\mathcal{L}$, and regard V as a $\mathcal{K}$-module via the induced action. Then P restricts irreducibly under the action of $\mathcal{K}$, and each $\mathcal{K}$-irreducible component in the decomposition of V is a one-dimensional weight space representation of $\mathcal{K}$.*

Proof: The i-components of P are chains, and each $\mathcal{K}$-component of P inherits this property. Now apply Proposition 6.2 to each of the $\mathcal{K}$-components of P. □

Our proof of the following theorem uses a restriction method based on Lemma 6.6.

Theorem 6.7 (Classification) *The minuscule representations of the simple Lie algebras are $A_n(\omega_k)$, $B_n(\omega_n)$, $C_n(\omega_1)$, $D_n(\omega_1)$, $D_n(\omega_{n-1})$, $D_n(\omega_n)$, $E_6(\omega_1)$, $E_6(\omega_6)$, and $E_7(\omega_7)$. The representations $A_n(k\omega_1)$ (for $k > 1$), $A_n(k\omega_n)$ (for $k > 1$), $B_n(\omega_1)$, $C_2(\omega_2)$, $C_3(\omega_3)$, and $G_2(\omega_1)$ are the only other irreducible one-dimensional weight space representations of simple Lie algebras.*

The proof is below. Representation diagrams for the representations of Theorem 6.7 are described in Section 4 for the Type A cases, in [18] and [25] for the minuscule cases, in Section 5 for the Type C cases, and in [6] for $B_n(\omega_1)$ and $G_2(\omega_1)$. To use Theorem 6.4 to construct a particular irreducible one-dimensional weight space representation, one would first need to form the weight diagram. The diagrams for the various cases could be found in the references cited above. Then one would locate the strings of color i for each i and assign the prescribed coefficients. For $A_n(k\omega_1)$, the second (first) choice of coefficients of Theorem 6.4 are the coefficients which arise for the (factorial normalized) monomial basis for the kth symmetric power of the defining representation of $sl(n+1,\mathbb{C})$. The same is true for $A_n(k\omega_n)$, the kth symmetric power of the dual of the defining representation. For the other irreducible one-dimensional weight space representations of the simple Lie algebras, the first choice of coefficients of Theorem 6.4 agrees with the coefficients described in the references cited above.

Proof of Theorem 6.7: Use the references mentioned in the previous paragraph to construct the supporting graph in each case. Then observe that for each supporting graph, distinct elements have distinct weights, so all weight spaces for the associated representation are one-dimensional. We now show that the one-dimensional weight space representations listed in the theorem statement are the only possibilities. For A_n ($n \geq 2$) we induct on n. Case $n = 2$ is covered by Lemma 6.5. Now let $n \geq 3$ and assume the theorem statement is true for A_{n-1}. Let $A_n(\lambda)$ be a one-dimensional weight space representation, where $\lambda = a_1\omega_1 + \cdots a_n\omega_n$, and let P be its unique supporting graph. Restrict to the Levi subalgebra A_{n-1} inside A_n whose generators correspond to the $n-1$ leftmost nodes of the Dynkin diagram for A_n. Apply Lemma 6.6 to the A_{n-1}-component which contains the maximal element of P. Then we must have one of these possibilities: (1) $\lambda = a_1\omega_1 + a_n\omega_n$; (2)

$\lambda = \omega_i + a_n\omega_n$ with $2 \leq i \leq n-2$; or (3) $\lambda = a_{n-1}\omega_{n-1} + a_n\omega_n$. Next, restrict to A'_{n-1}, the Levi subalgebra whose generators correspond to the rightmost $n-1$ nodes of the Dynkin diagram for A_n. Lemma 6.6 leaves us with these possibilities for λ: $(1)'$ $\lambda = a_1\omega_1 + a_2\omega_2$; $(2)'$ $\lambda = a_1\omega_1 + \omega_i$, where $3 \leq i \leq n-1$; $(3)'$ $\lambda = a_1\omega_1 + a_n\omega_n$. Combining these facts we are left with: $\lambda = a_1\omega_1$, $\lambda = a_n\omega_n$, $\lambda = \omega_i$, $\lambda = a_1\omega_1 + a_n\omega_n$ (with $a_1 > 0$ and $a_n > 0$), or $\lambda = a_2\omega_2$ (only if $n = 3$). The latter two possibilities are ruled out by Lemma 6.5. Analysis of the one-dimensional weight space representations for the other simple Lie algebras is similar: induct on n (the rank of the Lie algebra) by restricting to the action of simple Levi subalgebras of rank $n-1$. □

Suppose $\mathcal{L} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_m$ with each $\mathcal{L}_i$ simple. A dominant weight λ for $\mathcal{L}$ is written $\lambda_1 + \cdots + \lambda_m$, where each λ_i is dominant for $\mathcal{L}_i$. One can see that all weight spaces for $\mathcal{L}(\lambda)$ are one-dimensional if and only if all weight spaces of $\mathcal{L}_i(\lambda_i)$ are one-dimensional for each i.

Corollary 6.8 *The unique supporting graph for an irreducible one-dimensional weight space representation is a distributive lattice.*

Proof: For the simple Lie algebras, consult the references listed in the paragraph preceding the proof of Theorem 6.7. Now consider the one-dimensional weight space representation $\mathcal{L}(\lambda)$ where $\mathcal{L}$ is non-simple. With $\mathcal{L}(\lambda)$ described as above, Lemma 3.3.B shows that the supporting graph for $\mathcal{L}(\lambda)$ is the product of supports for the one-dimensional weight space representations $\mathcal{L}_i(\lambda_i)$. A finite product of distributive lattices is again a distributive lattice, cf. [22] Section 3.4. □

7. Technical proofs for Section 6

Proof of Proposition 6.2: If all weight spaces of V are one dimensional and V is irreducible, use Lemma 3.1.A and Lemma 3.1.F to see that the i-components in the unique supporting graph P are chains and that P is connected. For the converse, first we show that P has no "open vees." Suppose $\mathbf{s} \xrightarrow{i} \mathbf{u}$ and $\mathbf{t} \xrightarrow{j} \mathbf{u}$ for distinct elements $\mathbf{s}$ and $\mathbf{t}$. This implies that $i \neq j$. Apply Lemma 3.9 to the i-components of P to see that any weight basis $\{v_\mathbf{x}\}_{\mathbf{x}\in P}$ with support P has non-zero edge coefficients. Since $Y_j X_i v_\mathbf{s} = c v_\mathbf{t}$ for some $c \neq 0$, then $X_i Y_j v_\mathbf{s} = c v_\mathbf{t}$, which implies that there exists a unique $\mathbf{r}$ such that $\mathbf{r} \xrightarrow{j} \mathbf{s}$ and $\mathbf{r} \xrightarrow{i} \mathbf{t}$. Similarly, if $\mathbf{r} \xrightarrow{j} \mathbf{s}$ and $\mathbf{r} \xrightarrow{i} \mathbf{t}$ for some $\mathbf{r}$ in P, then there exists a unique $\mathbf{u}$ such that $\mathbf{s} \xrightarrow{i} \mathbf{u}$ and $\mathbf{t} \xrightarrow{j} \mathbf{u}$. Connectedness now implies that P has a unique maximal element and a unique minimal element.

Next, we claim that if $wt(\mathbf{s}) = wt(\mathbf{t})$, then $\mathbf{s} = \mathbf{t}$. Suppose otherwise, so $wt(\mathbf{s}) = wt(\mathbf{t})$ but $\mathbf{s} \neq \mathbf{t}$. Let $\mathbf{r}$ be an element of P for which $\mathbf{r} = \mathbf{s}_0 \xrightarrow{i_1} \mathbf{s}_1 \xrightarrow{i_2} \cdots \xrightarrow{i_r} \mathbf{s}_r = \mathbf{s}$ and $\mathbf{r} = \mathbf{t}_0 \xrightarrow{j_1} \mathbf{t}_1 \xrightarrow{j_2} \cdots \xrightarrow{j_r} \mathbf{t}_r = \mathbf{t}$, where these chains only have the element $\mathbf{r}$ in common. Moreover, choose $\mathbf{r}$ to be the "closest" such element to $\mathbf{s}$ and $\mathbf{t}$; that is, if $\mathbf{r}'$ is an element that is r' steps below both $\mathbf{s}$ and $\mathbf{t}$, then $r \leq r'$. (Such an element $\mathbf{r}$ exists since by Lemma 3.2.B, $\mathbf{s}$ and $\mathbf{t}$ have the same rank and hence are the same number of steps above the minimal element.) Notice that we cannot have $i_1 = j_1$, or otherwise $\mathbf{s}_1 = \mathbf{t}_1$. However,

$wt(\mathbf{s}) = wt(\mathbf{r}) + \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r}$ and $wt(\mathbf{t}) = wt(\mathbf{r}) + \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_r}$, so α_{i_1} must be one of $\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_r}$. Let q be least such that $i_1 = j_q$. Now there is a unique element $\mathbf{x}_2$ with $\mathbf{s}_1 \xrightarrow{j_1} \mathbf{x}_2$ and $\mathbf{t}_1 \xrightarrow{i_1} \mathbf{x}_2$, by the previous paragraph. It follows that there is a unique element $\mathbf{x}_3$ with $\mathbf{x}_2 \xrightarrow{j_2} \mathbf{x}_3$ and $\mathbf{t}_2 \xrightarrow{i_1} \mathbf{x}_3$. Continue this to get a unique element $\mathbf{x}_q$ with $\mathbf{x}_{q-1} \xrightarrow{j_{q-1}} \mathbf{x}_q$ and $\mathbf{t}_{q-1} \xrightarrow{i_1} \mathbf{x}_q$. However, $i_1 = j_q$, and so $\mathbf{t}_{q-1} \xrightarrow{j_q} \mathbf{x}_q$. But we already have $\mathbf{t}_{q-1} \xrightarrow{j_q} \mathbf{t}_q$, and since the j_q-components of P are chains, we have $\mathbf{x}_q = \mathbf{t}_q$. In particular, $\mathbf{s}_1$ is $r-1$ steps below $\mathbf{t}$ and $\mathbf{s}$. This contradicts the choice for $\mathbf{r}$, so we must have $\mathbf{s} = \mathbf{t}$. So the weight spaces of V are one-dimensional. It follows that any maximal vector in V will appear as a maximal element in the unique supporting graph P. Since P has a unique maximal element, there is a unique maximal vector in V (up to scalars), and hence V is irreducible. $\square$

Proof of Theorem 6.4: We must show that the given edge-labelled poset P corresponding to the first choice of edge coefficients satisfies conditions (1), (2), and (3) of Proposition 3.4. By Proposition 6.3, $P = \Pi(V)$ is a supporting graph for V. Then Proposition 3.5 applies to the edge-colored graph P, so condition (3) of Proposition 3.4 is met. Condition (2) is met by applying Proposition 6.2, Lemma 3.9, and then Lemma 3.2.A to the i-components of P.

Hence we only need to check that $X_i Y_j v_\nu = Y_j X_i v_\nu$ for each ν in the edge-labelled poset P whenever $i \neq j$. The proof of Proposition 6.2 shows that for a given pair of weights ν and π in P, there exists a weight σ such that $\nu \xrightarrow{i} \sigma$ and $\pi \xrightarrow{j} \sigma$ if and only if there exists μ such that $\mu \xrightarrow{j} \nu$ and $\mu \xrightarrow{i} \pi$. Suppose that we have a diamond

$$\begin{tikzcd} & \sigma & \\ \nu \arrow[ur, "i"] & & \pi \arrow[ul, "j"'] \\ & \mu \arrow[ul, "j"] \arrow[ur, "i"'] & \end{tikzcd}$$

in P. We want to know how $l_i(\nu)$ and $l_i(\pi)$ are related. Let ν_{min} and ν_{max} be the min and max elements in the i-component containing ν, and define π_{min} and π_{max} similarly. It now follows that π_{max} is covered by an element in the i-component of ν, so we can write $\nu_{max} = \pi_{max} + \alpha_j + p_i\alpha_i$ where $p_i \geq 0$. Similarly one can see that ν_{min} covers an element in the i-component of π, so we can write $\nu_{min} = \pi_{min} + \alpha_j + q_i\alpha_i$ for some $q_i \geq 0$. So the i-components of ν and of π might appear as in figure 2. From this it follows that $l_i(\nu) = \langle \nu_{max}, \alpha_i^\vee \rangle = \langle \pi_{max} + \alpha_j + p_i\alpha_i, \alpha_i^\vee \rangle = l_i(\pi) + 2p_i + \langle \alpha_j, \alpha_i^\vee \rangle$ and $-l_i(\nu) = \langle \nu_{min}, \alpha_i^\vee \rangle = \langle \pi_{min} + \alpha_j + q_i\alpha_i, \alpha_i^\vee \rangle = -l_i(\pi) + 2q_i + \langle \alpha_j, \alpha_i^\vee \rangle$. It follows that

$$p_i + q_i = -\langle \alpha_j, \alpha_i^\vee \rangle \tag{1}$$

$$l_i(\nu) - l_i(\pi) = p_i - q_i. \tag{2}$$

For brevity, let $l := l_i(\nu)$, $l' := l_i(\pi)$, $k := l_j(\nu)$, and $k' := l_j(\pi)$. Let r (respectively, s) be the rank of σ in its i-component (respectively, j-component). Let r' be the rank of π in its i-component, and let s' be the rank of ν in its j-component. To check condition (1) of Proposition 3.4 for P, we must check that $X_i Y_j v_\nu = Y_j X_i v_\nu$ and $X_j Y_i v_\pi = Y_i X_j v_\pi$ in any diamond such as figure 3. The eight edge coefficients in figure 3 are obtained from the first definitions of $c_{\mathbf{t},\mathbf{s}}$ and $d_{\mathbf{s},\mathbf{t}}$ in the theorem statement. Let Q be any representation diagram for V with support $\Pi(V)$. In the representation diagram Q, the products of the coefficients on the corresponding edges are pictured in figure 4. From figure 4 we get the following relation:

$$r(l+1-r) \cdot s(k+1-s) = r'(l'+1-r') \cdot s'(k'+1-s') \tag{3}$$

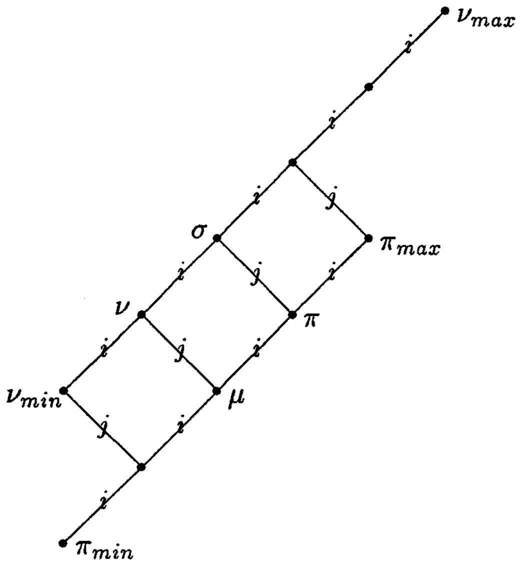

Figure 2. A configuration of i-components for weights μ and ν.

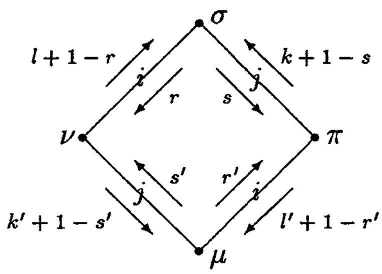

Figure 3. A typical diamond in P.

There are four cases to consider: For Case 0 we have $\langle\alpha_j, \alpha_i^\vee\rangle = 0$ and $\langle\alpha_i, \alpha_j^\vee\rangle = 0$; for Case 1, $\langle\alpha_j, \alpha_i^\vee\rangle = -1$ and $\langle\alpha_i, \alpha_j^\vee\rangle = -1$; for Case 2, $\langle\alpha_j, \alpha_i^\vee\rangle = -2$ and $\langle\alpha_i, \alpha_j^\vee\rangle = -1$; and for Case 3, $\langle\alpha_j, \alpha_i^\vee\rangle = -3$ and $\langle\alpha_i, \alpha_j^\vee\rangle = -1$. For Case 0, Eq. (1) above implies that $p_i = q_i = 0 = p_j = q_j$. It follows that $l = l'$, $k = k'$, $r = r'$, and $s = s'$. Then clearly $X_iY_jv_\nu = Y_jX_iv_\nu$ and $X_jY_iv_\pi = Y_iX_jv_\pi$ in figure 3.

For Case 1, Eq. (1) gives two possibilities for p_i and q_i: $p_i = 1$, $q_i = 0$ or $p_i = 0$, $q_i = 1$. First suppose that $p_i = 1$ and $q_i = 0$. Then we have $l - l' = 1$ and $r = r'$. If $p_j = 1$ and $q_j = 0$, then $k - k' = 1$ and $s = s'$. Substitute this into Eq. (3) to get $(l+1-r)(k+1-s) = (l-r)(k-s)$. But this cannot happen, since the left-hand side of this equation is bigger than the right-hand side. So we must have $p_j = 0$ and $q_j = 1$. Then $k - k' = -1$ and $s' = s + 1$. Substitute into Eq. (3) to get $(l+1-r)s = (l-r)(s+1)$. Put this information into figure 3 to see that $X_iY_jv_\nu = Y_jX_iv_\nu$ and $X_jY_iv_\pi = Y_iX_jv_\pi$. Analysis of the case $p_i = 0$ and $q_i = 1$ is similar, in which case $p_j = 1$ and $q_j = 0$. Cases 2 and 3 can be

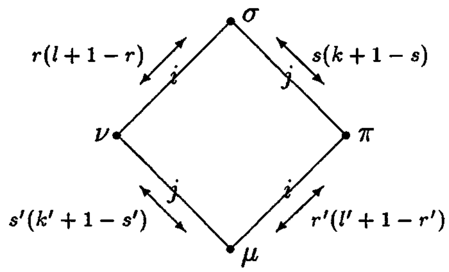

Figure 4. Edge products on the corresponding diamond in Q.

understood with similar arguments. In Case 2, one can show that it is not possible to have $p_i = 1$ and $q_i = 1$. If $p_i = 2$ and $q_i = 0$, it can be shown that $p_j = 0$ and $q_j = 1$. Also, if $p_i = 0$ and $q_i = 2$, then $p_j = 1$ and $q_j = 0$. For Case 3, neither $p_i = 2$, $q_i = 1$ nor $p_i = 1$, $q_i = 2$ is possible. If $p_i = 3$ and $q_i = 0$, then we must have $p_j = 0$ and $q_j = 1$. If $p_i = 0$ and $q_i = 3$, then $p_j = 1$ and $q_j = 0$. This argument applies to the second choice of coefficients in the theorem statement by interchanging the role of $c_{\mathbf{t},\mathbf{s}}$ and $d_{\mathbf{s},\mathbf{t}}$ everywhere in the proof. □

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