



Taut Distance-Regular Graphs of Odd Diameter

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Abstract. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and distinct eigenvalues $\theta_0 > \theta_1 > \dots > \theta_D$. Let M denote the Bose-Mesner algebra of Γ . For $0 \leq i \leq D$, let E_i denote the primitive idempotent of M associated with θ_i . We refer to E_0 and E_D as the *trivial* idempotents of M . Let E, F denote primitive idempotents of M . We say the pair E, F is *taut* whenever (i) E, F are nontrivial, and (ii) the entry-wise product $E \circ F$ is a linear combination of two distinct primitive idempotents of M . We show the pair E, F is taut if and only if there exist real scalars α, β such that

$$\sigma_{i+1}\rho_{i+1} - \sigma_{i-1}\rho_{i-1} = \alpha\sigma_i(\rho_{i+1} - \rho_{i-1}) + \beta\rho_i(\sigma_{i+1} - \sigma_{i-1}) \quad (1 \leq i \leq D-1),$$

where $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the cosine sequences of E, F , respectively. We define Γ to be *taut* whenever Γ has at least one taut pair of primitive idempotents but Γ is not 2-homogeneous in the sense of Nomura and Curtin. Assume Γ is taut and D is odd, and assume the pair E, F is taut. We show

$$\begin{aligned} \frac{\sigma_{i+1} - \alpha\sigma_i}{\sigma\sigma_i - \sigma_{i-1}} &= \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}}, \\ \frac{\rho_{i+1} - \beta\rho_i}{\rho\rho_i - \rho_{i-1}} &= \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} \end{aligned}$$

for $1 \leq i \leq D-1$, where $\sigma = \sigma_1, \rho = \rho_1$. Using these equations, we recursively obtain $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ in terms of the four real scalars $\sigma, \rho, \alpha, \beta$. From this we obtain all intersection numbers of Γ in terms of $\sigma, \rho, \alpha, \beta$. We showed in an earlier paper that the pair E_1, E_d is taut, where $d = (D-1)/2$. Applying our results to this pair, we obtain the intersection numbers of Γ in terms of $k, \mu, \theta_1, \theta_d$, where μ denotes the intersection number c_2 . We show that if Γ is taut and D is odd, then Γ is an antipodal 2-cover.

Keywords: distance-regular graph, association scheme, bipartite graph, tight graph, taut graph

1. Introduction

Let Γ denote a distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let M denote the Bose-Mesner algebra of Γ . It is well-known that M has a basis consisting of primitive idempotents; we refer to these as the primitive idempotents of Γ . M is closed under the entry-wise product, so given primitive idempotents E, F of Γ , the entry-wise product $E \circ F$ is a linear combination of the primitive idempotents of Γ . The coefficients in this linear combination are the Krein parameters of Γ . We are interested in the case where many of these coefficients are zero, so $E \circ F$ is a linear combination of a small number of primitive idempotents. For example, suppose Γ is Q -polynomial relative to E . Then $E \circ F$ is a linear combination of at most three primitive idempotents of Γ . For a related example, suppose

the above Q -polynomial structure is dual bipartite in the sense of Dickie and Terwilliger [4]. Then $E \circ F$ is a linear combination of at most two primitive idempotents of Γ .

Motivated by these examples, we study those pairs E, F where $E \circ F$ is a linear combination of at most two primitive idempotents of Γ . We summarize what is known so far. To do this, we use the following notation. Let $k = \theta_0 > \theta_1 > \cdots > \theta_D$ denote the distinct eigenvalues of Γ , and for $0 \leq i \leq D$, let E_i denote the primitive idempotent of Γ associated with θ_i . A primitive idempotent of Γ will be called *trivial* whenever it has rank 1. If Γ is not bipartite, then E_0 is the unique trivial primitive idempotent of Γ . If Γ is bipartite, then E_0, E_D are the only trivial primitive idempotents of Γ . Let E, F denote primitive idempotents of Γ . By [1, Props. II.3.7, II.3.8], $E \circ F$ is not zero, so $E \circ F$ is a linear combination of at least one primitive idempotent of Γ . In some cases, $E \circ F$ is a scalar multiple of a primitive idempotent of Γ . For example, suppose at least one of E, F is trivial. Then $E \circ F$ is a scalar multiple of a primitive idempotent of Γ . We say the pair E, F is *tight* whenever (i) E, F are nontrivial, and (ii) $E \circ F$ is a scalar multiple of a primitive idempotent of Γ . In [11], Jurišić, Koolen and Terwilliger introduce the notion of a *tight* distance-regular graph. Pascasio proves in [18, Theorem 1.3] that Γ is tight if and only if Γ has at least one tight pair of primitive idempotents. In this paper, we define Γ to be tight whenever Γ has at least one tight pair of primitive idempotents.

Suppose Γ is tight. Then by a result of Pascasio [18], Γ is not bipartite. Moreover, the pair E, F is tight if and only if E, F is a permutation of E_1, E_D . See [10, 11] for a detailed discussion of the tight graphs. For related papers, see [6–9, 15–19].

Now we consider the case where $E \circ F$ is a linear combination of two distinct primitive idempotents of Γ . To keep things simple, we restrict our attention to the case where Γ is bipartite. For the rest of this introduction, assume Γ is bipartite. We define the pair E, F to be *taut* whenever (i) E, F are nontrivial, and (ii) $E \circ F$ is a linear combination of two distinct primitive idempotents of Γ . There are a few ways a pair of primitive idempotents can be taut.

Suppose Γ is 2-homogeneous in the sense of Nomura [13] and Curtin [3]. In this case, we showed in [12] that the pair E, F is taut if and only if E, F are nontrivial with at least one equal to E_1 or E_{D-1} .

We define Γ to be *taut* whenever Γ is not 2-homogeneous and there exists at least one taut pair of primitive idempotents of Γ . Suppose Γ is taut and D is odd. In [12], we showed the pair E, F is taut if and only if the set $\{E, F\}$ is one of $\{E_1, E_d\}, \{E_1, E_{d+1}\}, \{E_{D-1}, E_d\}, \{E_{D-1}, E_{d+1}\}$, where $d = (D - 1)/2$. Now suppose Γ is taut and D is even. In [12], we showed the pair E, F is taut if and only if the set $\{E, F\}$ is one of $\{E_1, E_d\}, \{E_{D-1}, E_d\}$, where $d = D/2$.

We now summarize our results in the present paper. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Let E, F denote nontrivial primitive idempotents of Γ , and let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding cosine sequences. We show the following are equivalent: (i) the pair E, F is taut, and (ii) there exist complex scalars α, β such that

$$\sigma_{i+1}\rho_{i+1} - \sigma_{i-1}\rho_{i-1} = \alpha\sigma_i(\rho_{i+1} - \rho_{i-1}) + \beta\rho_i(\sigma_{i+1} - \sigma_{i-1}) \quad (1)$$

for $1 \leq i \leq D - 1$. Moreover, if (i), (ii) hold, then α, β are real.

We now suppose Γ is taut and D is odd. Further suppose the pair E, F is taut. Using (1), we show

$$\frac{\sigma_{i+1} - \alpha\sigma_i}{\sigma\sigma_i - \sigma_{i-1}} = \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}}, \quad (2)$$

$$\frac{\rho_{i+1} - \beta\rho_i}{\rho\rho_i - \rho_{i-1}} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} \quad (3)$$

for $1 \leq i \leq D-1$, where α, β are from (1), and where $\sigma = \sigma_1, \rho = \rho_1$. Using (2), (3), we recursively obtain $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ in terms of the four real scalars $\sigma, \rho, \alpha, \beta$. From this we obtain all intersection numbers of Γ in terms of $\sigma, \rho, \alpha, \beta$. Applying this result to the taut pair E_1, E_d , we obtain the intersection numbers of Γ in terms of the four parameters $k, \mu, \theta_1, \theta_d$, where μ denotes the intersection number c_2 .

Finally, we show that if Γ is taut and D is odd, then Γ is an antipodal 2-cover.

2. Preliminaries

In this section, we set our notation and review some basic definitions and results. For more information, the reader may consult the books of Bannai and Ito [1], Brouwer et al. [2], and Godsil [5].

Throughout this paper, let Γ denote a distance-regular graph with vertex set X and diameter D . As usual, we let p_{ij}^h ($0 \leq h, i, j \leq D$) denote the intersection numbers of Γ . It is conventional to abbreviate $c_i := p_{1i-1}^1$ ($1 \leq i \leq D$), $a_i := p_{1i}^1$ ($0 \leq i \leq D$), $b_i := p_{1i+1}^1$ ($0 \leq i \leq D-1$), $k_i := p_{ii}^0$ ($0 \leq i \leq D$), and to define $c_0 := 0, b_D := 0$. For convenience, we write $\mu := c_2$. By [2, p. 127] we have

$$k_i = \frac{b_0 b_1 b_2 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \quad (0 \leq i \leq D). \quad (4)$$

We let $A_0, A_1, \dots, A_D \in \text{Mat}_X(\mathbb{C})$ denote the distance matrices of Γ . The matrix A_1 is the adjacency matrix of Γ ; we frequently abbreviate $A := A_1$. We let M denote the Bose-Mesner algebra of Γ , which is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by A . It is well-known that the distance matrices form a basis for M . We let E_0, E_1, \dots, E_D denote the primitive idempotents of M (we frequently refer to these matrices as the primitive idempotents of Γ), where E_0 is a scalar multiple of the all 1's matrix. The primitive idempotents E_0, E_1, \dots, E_D also form a basis for M . For $0 \leq i \leq D$, we let $m_i := \text{rank}(E_i)$ denote the multiplicity of E_i .

There exist distinct real numbers $\theta_0, \theta_1, \dots, \theta_D$ such that $A = \sum_{i=0}^D \theta_i E_i$. We say that θ_i is the *eigenvalue* of Γ associated with E_i . One can show $\theta_0 = k$ and $-k \leq \theta_i \leq k$ for $0 \leq i \leq D$ [1, Thm. III.1.3].

Let E denote a primitive idempotent of Γ , and let m denote the multiplicity of E . By [1, Section II.3], there exist real scalars $\sigma_0, \sigma_1, \dots, \sigma_D$ such that $\sigma_0 = 1$ and

$$E = |X|^{-1} m \sum_{i=0}^D \sigma_i A_i. \quad (5)$$

The sequence $\sigma_0, \sigma_1, \dots, \sigma_D$ is called the *cosine sequence* of Γ associated with E . We abbreviate $\sigma := \sigma_1$ and refer to σ as the *first cosine* of E . Let $\theta_0 > \theta_1 > \dots > \theta_D$ denote the eigenvalues of Γ , and let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote the cosine sequence associated with θ_1 . Then by [5, Section 13, Lemma 2.1],

$$\sigma_0 > \sigma_1 > \dots > \sigma_D. \quad (6)$$

Since each entry in the distance matrices is either 0 or 1, we have

$$A_i \circ A_j = \delta_{ij} A_i \quad (0 \leq i, j \leq D), \quad (7)$$

where \circ denotes the entry-wise matrix product. It follows that M is closed under \circ . In particular, there exist scalars $q_{ij}^h \in \mathbb{C}$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h \quad (0 \leq i, j \leq D). \quad (8)$$

The scalars q_{ij}^h are known as the *Krein parameters* of Γ . The Krein parameters are real and nonnegative [2, Theorem 2.3.2]. From [2, Lemma 2.3.1], we have

$$q_{ij}^0 = \delta_{ij} m_i \quad (0 \leq i, j \leq D). \quad (9)$$

In this paper we will be considering pairs of primitive idempotents E_i, E_j such that $E_i \circ E_j$ is a linear combination of one or two primitive idempotents. Observe the existence of such a pair E_i, E_j implies that a number of the Krein parameters are zero.

For the rest of this section, we recall some facts about bipartite distance-regular graphs. Let Γ denote a bipartite distance-regular graph with diameter D , valency k , and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_D$. It is well-known that $a_i = 0$ ($0 \leq i \leq D$), and so $c_i + b_i = k$ ($0 \leq i \leq D$) [2, Prop. 4.2.2]. Furthermore, one can show

$$\theta_{D-i} = -\theta_i \quad (0 \leq i \leq D). \quad (10)$$

Lemma 2.1 ([2, p. 128]) *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$. For any complex scalars $\sigma_0, \sigma_1, \dots, \sigma_D$, the following are equivalent.*

- (i) $\sigma_0, \sigma_1, \dots, \sigma_D$ is a cosine sequence of Γ .
- (ii) $\sigma_0 = 1, \sigma_{D-1} = \sigma \sigma_D$, and

$$c_i(\sigma_{i-1} - \sigma_{i+1}) = k(\sigma \sigma_i - \sigma_{i+1}) \quad (1 \leq i \leq D-1). \quad (11)$$

- (iii) $\sigma_0 = 1, \sigma_{D-1} = \sigma \sigma_D$, and

$$b_i(\sigma_{i+1} - \sigma_{i-1}) = k(\sigma \sigma_i - \sigma_{i-1}) \quad (1 \leq i \leq D-1). \quad (12)$$

Furthermore, suppose (i)–(iii) hold. Then $\theta = k\sigma$ is the eigenvalue of Γ associated with the cosine sequence $\sigma_0, \sigma_1, \dots, \sigma_D$.

Corollary 2.2 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$, and let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote a cosine sequence of Γ . Then for any integer i ($1 \leq i \leq D - 1$), the following are equivalent.*

- (i) $\sigma_{i+1} = \sigma_{i-1}$.
- (ii) $\sigma \sigma_i = \sigma_{i-1}$.
- (iii) $\sigma \sigma_i = \sigma_{i+1}$.

Corollary 2.3 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote a cosine sequence of Γ . Then (i), (ii) hold below.*

- (i) $\sigma_i \neq 0$ or $\sigma_{i+1} \neq 0$ ($0 \leq i \leq D - 1$).
- (ii) $\sigma_D \neq 0$.

Lemma 2.4 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$. Let E, F denote primitive idempotents of Γ with cosine sequences $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$, respectively. Then for all integers i ($1 \leq i \leq D - 1$),*

$$\sigma_{i+1}\rho_{i-1} - \sigma_{i-1}\rho_{i+1} = \sigma\sigma_i(\rho_{i-1} - \rho_{i+1}) - \rho\rho_i(\sigma_{i-1} - \sigma_{i+1}). \quad (13)$$

Proof: Observe

$$(\sigma\sigma_i - \sigma_{i+1})(\rho_{i-1} - \rho_{i+1}) = (\rho\rho_i - \rho_{i+1})(\sigma_{i-1} - \sigma_{i+1}) \quad (14)$$

since both sides equal $c_i(\rho_{i-1} - \rho_{i+1})(\sigma_{i-1} - \sigma_{i+1})/k$ in view of (11). Multiplying out (14) and cancelling terms, we obtain (13). \square

Definition 2.5 *Let Γ denote a bipartite distance-regular graph. Let E, F denote primitive idempotents of Γ , and let θ, θ' denote the corresponding eigenvalues. We say E and F are opposites whenever $\theta' = -\theta$.*

Lemma 2.6 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$. Let E, F denote primitive idempotents of Γ with cosine sequences $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$, respectively. Then the following are equivalent:*

- (i) E and F are opposites.
- (ii) $\rho = -\sigma$.
- (iii) $\rho_i = (-1)^i \sigma_i$ ($0 \leq i \leq D$).

Proof: Routine using Lemma 2.1. \square

Lemma 2.7 [2, Prop. 4.4.7] *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_D$. Let E denote a primitive idempotent of Γ with cosine sequence $\sigma_0, \sigma_1, \dots, \sigma_D$.*

- (i) *Suppose $E = E_0$. Then $\sigma_i = 1$ ($0 \leq i \leq D$).*
- (ii) *Suppose $E = E_D$. Then $\sigma_i = (-1)^i$ ($0 \leq i \leq D$).*
- (iii) *Suppose E is one of E_1, E_2, \dots, E_{D-1} . Then $-1 < \sigma_i < 1$ ($1 \leq i \leq D - 1$).*

Definition 2.8 Let Γ denote a bipartite distance-regular graph with valency k . We say that an eigenvalue θ of Γ is *trivial* whenever $\theta = k$ or $\theta = -k$. We say that a primitive idempotent E of Γ is trivial whenever the associated eigenvalue is trivial. One can show that a primitive idempotent E is trivial if and only if $\text{rank}(E) = 1$.

The following result appears in [18], although parts (i), (ii) of this result are from the folklore of the subject of distance-regular graphs.

Lemma 2.9 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_D$. Then (i)–(iii) hold below.

- (i) $E_0 \circ E_i = |X|^{-1} E_i$ ($0 \leq i \leq D$).
- (ii) $E_D \circ E_i = |X|^{-1} E_{D-i}$ ($0 \leq i \leq D$).
- (iii) Let E, F denote primitive idempotents of Γ other than E_0 and E_D . Then $E \circ F$ is not a scalar multiple of a primitive idempotent of Γ .

Lemma 2.10 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_D$. Then

$$q_{ij}^h = q_{D-i, j}^{D-h} = q_{i, D-j}^{D-h} = q_{D-i, D-j}^h \quad (0 \leq h, i, j \leq D). \quad (15)$$

Proof: To get the equation on the left, we take the entry-wise product of both sides of (8) with E_D and apply Lemma 2.9(ii). The second equation follows since $q_{ij}^h = q_{ji}^h$ ($0 \leq h, i, j \leq D$), and the third equation is a routine consequence of the first one. \square

Corollary 2.11 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_D$. Then

$$q_{ij}^D = \delta_{i, D-j} m_i \quad (0 \leq i, j \leq D). \quad (16)$$

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let E, F denote primitive idempotents of Γ . We mentioned the entry-wise product $E \circ F$ is nonzero, so $E \circ F$ is a linear combination of at least one primitive idempotent of Γ . Recall by Lemma 2.9 that $E \circ F$ is a scalar multiple of a single primitive idempotent of Γ if and only if at least one of E, F is trivial. Suppose E, F are nontrivial, so $E \circ F$ is a linear combination of at least two distinct primitive idempotents of Γ . It is natural to consider when $E \circ F$ is a linear combination of two distinct primitive idempotents of Γ . This is the situation we consider in the following definition.

Definition 2.12 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. We introduce a binary symmetric relation on the set of all primitive idempotents of Γ . We call this the *taut* relation. Let E, F denote primitive idempotents of Γ . We say the pair E, F is *taut* whenever (i) E, F are nontrivial, and (ii) the entry-wise product $E \circ F$ is a linear combination of two distinct primitive idempotents of Γ .

We now recall a class of bipartite distance-regular graphs that contain taut pairs of primitive idempotents.

Definition 2.13 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_D$. Let θ denote a nontrivial eigenvalue of Γ . In [3], Curtin shows

$$(\mu - 1)\theta^2 \leq (k - \mu)(k - 2) \quad (17)$$

and that the set of nontrivial eigenvalues θ of Γ for which equality holds in (17) is either (i) empty, or (ii) $\{\theta_1, \theta_{D-1}\}$. Γ is said to be *2-homogeneous* if (ii) occurs.

Nomura obtains a classification of the 2-homogeneous bipartite distance-regular graphs in [14]. Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$ and valency $k \geq 3$. Nomura shows that if Γ is 2-homogeneous and $D > 5$, then Γ is the D-cube. Furthermore, Curtin proves in [3] that the following are equivalent: (i) Γ is 2-homogeneous, (ii) Γ is an antipodal 2-cover and Q -polynomial, (iii) Γ has a dual bipartite Q -polynomial structure.

Theorem 2.14 ([12]) *Let Γ denote a 2-homogeneous bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_D$. Let E, F denote nontrivial primitive idempotents of Γ . Then the pair E, F is taut if and only if at least one of E, F is equal to E_1 or E_{D-1} .*

Definition 2.15 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. We define Γ to be *taut* whenever Γ is not 2-homogeneous and Γ has at least one taut pair of primitive idempotents.

Taut distance-regular graphs with odd diameter and taut distance-regular graphs with even diameter appear to be fundamentally different objects, and we will handle them separately. In this paper, we will focus on taut distance-regular graphs with odd diameter. Of these, there are three known sporadic examples, each with diameter 5, and one infinite family. The sporadic examples are the Double Hoffman-Singleton graph [2, Section 13.1], Double Gewirtz graph [2, Section 11.4G], and Double 77-graph [2, p. 418]. The infinite family is given below.

Example 2.16 Given an integer $k \geq 3$, let Γ denote the graph $2.O_k$, the double cover of the Odd graph O_k (see [2, Section 9.1D]). Recall Γ is a bipartite distance-regular graph with valency k and diameter $D = 2k - 1$. The intersection numbers of Γ are given by

$$c_{2i} = c_{2i-1} = i \quad (1 \leq i \leq k - 1). \quad (18)$$

The distinct eigenvalues of Γ are $\pm 1, \pm 2, \dots, \pm k$ [2, p. 414]. Moreover, the graph Γ is taut. We will verify this in Example 5.14.

Theorem 2.17 ([12]) *Let Γ denote a taut bipartite distance-regular graph with diameter $D \geq 4$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Let E, F denote nontrivial primitive idempotents of Γ .*

- (i) *Suppose D is odd, and let $d = (D - 1)/2$. Then the pair E, F is taut if and only if the set $\{E, F\}$ is one of $\{E_1, E_d\}, \{E_1, E_{d+1}\}, \{E_{D-1}, E_d\}, \{E_{D-1}, E_{d+1}\}$.*
- (ii) *Suppose D is even, and let $d = D/2$. Then the pair E, F is taut if and only if the set $\{E, F\}$ is one of $\{E_1, E_d\}, \{E_{D-1}, E_d\}$.*

We end this chapter with two results concerning 2-homogeneous bipartite distance-regular graphs.

Theorem 2.18 ([3]) *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Let θ denote a nontrivial eigenvalue of Γ , and let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote the associated cosine sequence. Then the following are equivalent:*

- (i) Γ is 2-homogeneous and $\theta \in \{\theta_1, \theta_{D-1}\}$.
- (ii) There exists a complex scalar λ such that

$$\sigma_{i-1} - \lambda\sigma_i + \sigma_{i+1} = 0 \quad (1 \leq i \leq D - 1). \quad (19)$$

- (iii) There exists a complex scalar λ such that

$$\sigma_{i-1} - \lambda\sigma_i + \sigma_{i+1} = 0 \quad (1 \leq i \leq 2). \quad (20)$$

Furthermore, suppose (i)–(iii) hold. Then λ is real.

Lemma 2.19 ([3, 12]) *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Set*

$$\Delta := (k - 2)(c_3 - 1) - (\mu - 1)p_{22}^2. \quad (21)$$

Then $\Delta \geq 0$. Moreover, Γ is 2-homogeneous if and only if $\Delta = 0$ and Γ has at least one taut pair of primitive idempotents.

3. The Christoffel-Darboux formula

Lemma 3.1 (Christoffel-Darboux formula) [1, Theorem III.1.3] *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote cosine sequences of Γ . Then*

$$k(\sigma - \rho) \sum_{h=0}^i k_h \sigma_h \rho_h = k_i b_i (\sigma_{i+1} \rho_i - \sigma_i \rho_{i+1}) \quad (0 \leq i \leq D), \quad (22)$$

where σ_{D+1}, ρ_{D+1} are indeterminates.

Corollary 3.2 *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote a cosine sequence of Γ . Then*

$$k(\sigma - 1) \sum_{h=0}^i k_h \sigma_h = k_i b_i (\sigma_{i+1} - \sigma_i) \quad (0 \leq i \leq D), \quad (23)$$

where σ_{D+1} is indeterminate.

Proof: By Lemma 2.7(i), the cosine sequence associated with E_0 is $1, 1, \dots, 1$. Let $\rho_j = 1$ ($0 \leq j \leq D$) in (22) to obtain the desired result. \square

We now obtain an equation similar in form to the Christoffel-Darboux formula; this equation is for bipartite distance-regular graphs.

Lemma 3.3 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote cosine sequences of Γ . For $0 \leq i \leq D$,*

$$k^2(\sigma^2 - \rho^2) \sum_{\substack{0 \leq h \leq i-1 \\ i-h \text{ odd}}} k_h \sigma_h \rho_h = k_i c_i b_i (\sigma_{i+1} \rho_{i-1} - \sigma_{i-1} \rho_{i+1}), \quad (24)$$

where $\sigma_{-1}, \rho_{-1}, \sigma_{D+1}, \rho_{D+1}$ are indeterminates.

Proof: Repeatedly applying (11) and using the fact that $b_j = k - c_j$ ($0 \leq j \leq D$), we find that for $0 \leq h \leq D - 1$,

$$k^2 \sigma^2 \sigma_h = c_h c_{h-1} \sigma_{h-2} + (c_h b_{h-1} + b_h c_{h+1}) \sigma_h + b_h b_{h+1} \sigma_{h+2}, \quad (25)$$

where we define $c_{-1} := b_{-1} := 0$, and where σ_{-2} is indeterminate. Similarly,

$$k^2 \rho^2 \rho_h = c_h c_{h-1} \rho_{h-2} + (c_h b_{h-1} + b_h c_{h+1}) \rho_h + b_h b_{h+1} \rho_{h+2}, \quad (26)$$

where ρ_{-2} is indeterminate.

Subtracting $k_h \sigma_h$ times (26) from $k_h \rho_h$ times (25) and using (4), we find

$$\begin{aligned} k^2(\sigma^2 - \rho^2) k_h \sigma_h \rho_h &= k_{h+1} c_{h+1} b_{h+1} (\sigma_{h+2} \rho_h - \sigma_h \rho_{h+2}) \\ &\quad - k_{h-1} c_{h-1} b_{h-1} (\sigma_h \rho_{h-2} - \sigma_{h-2} \rho_h) \end{aligned} \quad (27)$$

for $0 \leq h \leq D - 1$, where we define $k_{-1} := 0$. Fix an integer i ($0 \leq i \leq D$). Summing (27) over all h such that $0 \leq h \leq i - 1$ and such that $i - h$ is odd, we obtain (24). \square

Corollary 3.4 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote a cosine sequence of Γ . For $0 \leq i \leq D$,*

$$k^2(\sigma^2 - 1) \sum_{\substack{0 \leq h \leq i-1 \\ i-h \text{ odd}}} k_h \sigma_h = k_i c_i b_i (\sigma_{i+1} - \sigma_{i-1}), \quad (28)$$

where $\sigma_{-1}, \sigma_{D+1}$ are indeterminates.

Proof: By Lemma 2.7(i), the cosine sequence associated with E_0 is $1, 1, \dots, 1$. Letting $\rho_j = 1$ ($0 \leq j \leq D$) in (24), we obtain the desired result. \square

4. Some equations involving cosine sequences

Definition 4.1 Throughout this section, we let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let $\theta_0 > \theta_1 > \dots > \theta_D$ denote the distinct eigenvalues of Γ . Furthermore, we let E and F denote nontrivial primitive idempotents of Γ . We let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding cosine sequences. We let G and H denote distinct primitive idempotents of Γ with cosine sequences $\gamma_0, \gamma_1, \dots, \gamma_D$ and $\epsilon_0, \epsilon_1, \dots, \epsilon_D$, respectively.

Lemma 4.2 *With reference to Definition 4.1, the following are equivalent:*

- (i) $E \circ F \in \text{span}\{G, H\}$.
- (ii) *There exist complex scalars a, b such that for $0 \leq i \leq D$,*

$$\sigma_i \rho_i = a \gamma_i + b \epsilon_i. \quad (29)$$

Suppose (i), (ii) hold. Then a, b are given by

$$a = \frac{\sigma \rho - \epsilon}{\gamma - \epsilon}, \quad b = \frac{\gamma - \sigma \rho}{\gamma - \epsilon}. \quad (30)$$

Moreover, a, b are nonzero and real.

Proof: Let $m_\sigma, m_\rho, m_\gamma, m_\epsilon$ denote the multiplicities of E, F, G, H , respectively. Recall

$$E = |X|^{-1} m_\sigma \sum_{i=0}^D \sigma_i A_i, \quad F = |X|^{-1} m_\rho \sum_{i=0}^D \rho_i A_i, \quad (31)$$

$$G = |X|^{-1} m_\gamma \sum_{i=0}^D \gamma_i A_i, \quad H = |X|^{-1} m_\epsilon \sum_{i=0}^D \epsilon_i A_i. \quad (32)$$

(i) \Rightarrow (ii) By assumption, there exist complex scalars ψ, ϕ such that

$$E \circ F = \psi G + \phi H. \quad (33)$$

Eliminating E, F, G, H in (33) using (31), (32), and evaluating the result using (7), we obtain

$$m_\sigma m_\rho \sigma_i \rho_i = |X| (\psi m_\gamma \gamma_i + \phi m_\epsilon \epsilon_i) \quad (0 \leq i \leq D).$$

Apparently (29) holds with

$$a = \frac{|X| \psi m_\gamma}{m_\sigma m_\rho}, \quad b = \frac{|X| \phi m_\epsilon}{m_\sigma m_\rho}. \quad (34)$$

(ii) \Rightarrow (i) By (31), (7), (29), and (32), we have

$$\begin{aligned} E \circ F &= |X|^{-2} m_\sigma m_\rho \sum_{i=0}^D \sigma_i \rho_i A_i \\ &= |X|^{-2} m_\sigma m_\rho \left(a \sum_{i=0}^D \gamma_i A_i + b \sum_{i=0}^D \epsilon_i A_i \right) \\ &= |X|^{-1} m_\sigma m_\rho \left(\frac{a}{m_\gamma} G + \frac{b}{m_\epsilon} H \right), \end{aligned}$$

proving (i).

Now suppose (i), (ii) hold. To obtain (30), set $i = 0, 1$ in (29) and solve the resulting linear equations for a and b . To show a, b are real and nonzero, we refer to the proof of (i) \Rightarrow (ii) above. One may show ψ, ϕ are real and nonzero using Lemma 2.9(iii) and the fact that Krein parameters are real. Now a, b are real and nonzero by (34). \square

With reference to Definition 4.1, suppose for the moment that E, F, G, H satisfy (i), (ii) in Lemma 4.2. In the following lemma, we consider the case when one of G, H is trivial.

Lemma 4.3 *With reference to Definition 4.1, suppose (i), (ii) hold in Lemma 4.2. Then*

- (i) $E = F$ if and only if one of G, H is equal to E_0 .
- (ii) E, F are opposites if and only if one of G, H is equal to E_D .

Furthermore, suppose $E = F$. Then Γ is 2-homogeneous, and $E \in \{E_1, E_{D-1}\}$. Now suppose E, F are opposites. Then Γ is 2-homogeneous, and E, F is a permutation of E_1, E_{D-1} .

Proof:

- (i) Routine consequence of (8), (9), and the linear independence of the primitive idempotents.
- (ii) By Definition 2.5 and (10), E, F are opposites if and only if there exists an integer i ($0 \leq i \leq D$) such that $E = E_i, F = E_{D-i}$. The result is now a routine consequence of (8), (16), and the linear independence of the primitive idempotents.

Now suppose (i) or (ii) holds. By Lemma 4.2(i) and since E, F are nontrivial, we find the pair E, F is taut. Observe Γ is not taut since Theorem 2.17(i), (ii) do not hold; thus Γ is 2-homogeneous by Definition 2.15. If (i) holds, then $E \in \{E_1, E_{D-1}\}$ by Theorem 2.14. If (ii) holds, then E, F is a permutation of E_1, E_{D-1} by Theorem 2.14. \square

Lemma 4.4 *With reference to Definition 4.1, suppose (i), (ii) hold in Lemma 4.2 and that E, F are distinct. Then for any integer i ($0 \leq i \leq D - 1$),*

$$\frac{\sigma_{i+1} \rho_i - \sigma_i \rho_{i+1}}{\sigma - \rho} = a \frac{\gamma_{i+1} - \gamma_i}{\gamma - 1} + b \frac{\epsilon_{i+1} - \epsilon_i}{\epsilon - 1}, \quad (35)$$

where a, b are from (30). Observe the denominators in (35) are nonzero by Lemma 2.7 and Lemma 4.3.

Proof: Using (22), (29), and (23), we observe

$$\begin{aligned} \frac{k_i b_i}{k} \frac{\sigma_{i+1} \rho_i - \sigma_i \rho_{i+1}}{\sigma - \rho} &= \sum_{h=0}^i k_h \sigma_h \rho_h \\ &= a \sum_{h=0}^i k_h \gamma_h + b \sum_{h=0}^i k_h \epsilon_h \\ &= \frac{k_i b_i}{k} \left(a \frac{\gamma_{i+1} - \gamma_i}{\gamma - 1} + b \frac{\epsilon_{i+1} - \epsilon_i}{\epsilon - 1} \right), \end{aligned}$$

and the result follows. \square

Lemma 4.5 *With reference to Definition 4.1, suppose (i), (ii) hold in Lemma 4.2 and that E, F are neither equal nor opposites. Then for any integer i ($1 \leq i \leq D - 1$),*

$$\frac{\sigma_{i+1} \rho_{i-1} - \sigma_{i-1} \rho_{i+1}}{\sigma^2 - \rho^2} = a \frac{\gamma_{i+1} - \gamma_{i-1}}{\gamma^2 - 1} + b \frac{\epsilon_{i+1} - \epsilon_{i-1}}{\epsilon^2 - 1}, \quad (36)$$

where a, b are from (30). Observe the denominators in (36) are nonzero by Lemma 2.6, Lemma 2.7, and Lemma 4.3.

Proof: By (24), (29), and (28), we observe

$$\begin{aligned} \frac{k_i c_i b_i}{k^2} \frac{\sigma_{i+1} \rho_{i-1} - \sigma_{i-1} \rho_{i+1}}{\sigma^2 - \rho^2} &= \sum_{\substack{0 \leq h \leq i-1 \\ i-h \text{ odd}}} k_h \sigma_h \rho_h \\ &= a \sum_{\substack{0 \leq h \leq i-1 \\ i-h \text{ odd}}} k_h \gamma_h + b \sum_{\substack{0 \leq h \leq i-1 \\ i-h \text{ odd}}} k_h \epsilon_h \\ &= \frac{k_i c_i b_i}{k^2} \left(a \frac{\gamma_{i+1} - \gamma_{i-1}}{\gamma^2 - 1} + b \frac{\epsilon_{i+1} - \epsilon_{i-1}}{\epsilon^2 - 1} \right), \end{aligned}$$

and the result follows. \square

5. The main results

In the following theorem, we consider the equation

$$\sigma_{i+1} \rho_{i+1} - \sigma_{i-1} \rho_{i-1} = \alpha \sigma_i (\rho_{i+1} - \rho_{i-1}) + \beta \rho_i (\sigma_{i+1} - \sigma_{i-1}). \quad (37)$$

Theorem 5.1 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let E and F denote nontrivial primitive idempotents of Γ with cosine sequences $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$, respectively. Then the following are equivalent:*

- (i) *E, F is a taut pair.*
- (ii) *There exist complex scalars α, β such that equality holds in (37) for all integers i ($1 \leq i \leq D - 1$).*
- (iii) *There exist complex scalars α, β such that equality holds in (37) for $i = 1, 2, 3$.*

Proof: Let $\theta_0 > \theta_1 > \dots > \theta_D$ denote the distinct eigenvalues of Γ .

(i) \Rightarrow (ii) First assume $E = F$, so $\rho_j = \sigma_j$ for $0 \leq j \leq D$. By Lemma 4.3, we find Γ is 2-homogeneous, and that E is one of E_1, E_{D-1} . By Theorem 2.18, there exists a complex scalar λ such that

$$\sigma_{i+1} + \sigma_{i-1} = \lambda\sigma_i \quad (1 \leq i \leq D - 1). \quad (38)$$

Using (38) and the fact that $\rho_j = \sigma_j$ ($0 \leq j \leq D$), we find that if we set $\alpha = \lambda, \beta = 0$, then (37) holds for $1 \leq i \leq D - 1$.

Now assume E, F are opposites. By Lemma 2.6, $\rho_j = (-1)^j\sigma_j$ for $0 \leq j \leq D$. Applying Lemma 4.3, we find Γ is 2-homogeneous, and E, F is a permutation of E_1, E_{D-1} . By Theorem 2.18, there exists a complex scalar λ such that

$$\sigma_{i+1} + \sigma_{i-1} = \lambda\sigma_i \quad (1 \leq i \leq D - 1). \quad (39)$$

Using (39) and the fact that $\rho_j = (-1)^j\sigma_j$ ($0 \leq j \leq D$), we find that if we set $\alpha = \lambda, \beta = 0$, then (37) holds for $1 \leq i \leq D - 1$.

Now assume E and F are not equal nor opposites. By assumption $E \circ F$ is a linear combination of two distinct primitive idempotents, which we denote by G, H . Let $\gamma_0, \gamma_1, \dots, \gamma_D$ and $\epsilon_0, \epsilon_1, \dots, \epsilon_D$ denote the cosine sequences for G and H , respectively. To obtain (37), we will combine (13), (29), (35), and (36) as follows. Fix an integer i ($1 \leq i \leq D - 1$). By (29),

$$\sigma_{i+1}\rho_{i+1} - \sigma_{i-1}\rho_{i-1} = a(\gamma_{i+1} - \gamma_{i-1}) + b(\epsilon_{i+1} - \epsilon_{i-1}), \quad (40)$$

where a, b are from (30). Adding (35) at i to (35) at $i - 1$, we obtain

$$\frac{\sigma_i(\rho_{i-1} - \rho_{i+1})}{\sigma - \rho} - \frac{\rho_i(\sigma_{i-1} - \sigma_{i+1})}{\sigma - \rho} = \frac{a(\gamma_{i+1} - \gamma_{i-1})}{\gamma - 1} + \frac{b(\epsilon_{i+1} - \epsilon_{i-1})}{\epsilon - 1}. \quad (41)$$

Evaluating the left-hand side of (36) using (13), we find

$$\frac{\sigma\sigma_i(\rho_{i-1} - \rho_{i+1})}{\sigma^2 - \rho^2} - \frac{\rho\rho_i(\sigma_{i-1} - \sigma_{i+1})}{\sigma^2 - \rho^2} = \frac{a(\gamma_{i+1} - \gamma_{i-1})}{\gamma^2 - 1} + \frac{b(\epsilon_{i+1} - \epsilon_{i-1})}{\epsilon^2 - 1}. \quad (42)$$

Consider the equation Eq which is (40) minus the product of u with (41) plus the product of v with (42), where

$$u = \epsilon + \gamma, \quad v = (1 + \epsilon)(1 + \gamma).$$

In the equation Eq, the right-hand side is 0, since

$$1 - \frac{u}{\gamma - 1} + \frac{v}{\gamma^2 - 1} = 0, \quad 1 - \frac{u}{\epsilon - 1} + \frac{v}{\epsilon^2 - 1} = 0.$$

Evaluating the left-hand side of the equation Eq, we obtain (37), where

$$\alpha = \frac{u}{\rho - \sigma} - \frac{v\sigma}{\rho^2 - \sigma^2}, \quad \beta = \frac{u}{\sigma - \rho} - \frac{v\rho}{\sigma^2 - \rho^2}.$$

(ii) \Rightarrow (iii) Immediate.

(iii) \Rightarrow (i) First assume $\rho \neq \pm\sigma$. Consider the matrix

$$B := \begin{pmatrix} \sigma_2\rho_2 - 1 & \sigma(\rho_2 - 1) & \rho(\sigma_2 - 1) \\ \sigma_3\rho_3 - \sigma\rho & \sigma_2(\rho_3 - \rho) & \rho_2(\sigma_3 - \sigma) \\ \sigma_4\rho_4 - \sigma_2\rho_2 & \sigma_3(\rho_4 - \rho_2) & \rho_3(\sigma_4 - \sigma_2) \end{pmatrix}.$$

On one hand, B is singular, so $\det(B) = 0$. On the other hand, recursively eliminating $\sigma_2, \sigma_3, \sigma_4, \rho_2, \rho_3, \rho_4$ in the entries of B using (12), one may verify that $\det(B)$ is equal to

$$k^5 b_1^{-6} b_2^{-4} b_3^{-2} (\sigma^2 - 1)^2 (\rho^2 - 1)^2 (\rho^2 - \sigma^2) \quad (43)$$

times

$$f\sigma^2\rho^2 + g(\sigma^2 + \rho^2) + h, \quad (44)$$

where

$$\begin{aligned} f &= k^4 \mu(\Delta + c_3 - \mu), \\ g &= -k^2 b_2^2 (c_3 - 1), \\ h &= b_2^2 (b_2 (c_3 - 1) + \mu b_3 (k - 2)), \end{aligned}$$

and where Δ is from (21). (To do this, we must use the fact that $p_{22}^2 = \mu^{-1}(b_2(c_3 - 1) + \mu(k - 2))$ [2, Lemma 4.1.7]). Observe factor (43) is not zero, since we assume $\sigma^2 \neq \rho^2$ and since $\sigma^2 \neq 1, \rho^2 \neq 1$ by Lemma 2.7(iii). Apparently factor (44) is zero. Applying [12, Corollary 3.11], we find E, F is a taut pair.

Now assume $\rho = \pm\sigma$. If $\rho = \sigma$, then $\rho_j = \sigma_j$ for $0 \leq j \leq D$. If $\rho = -\sigma$, then $\rho_j = (-1)^j \sigma_j$ for $0 \leq j \leq D$ by Lemma 2.6. We assume (37) holds for $1 \leq i \leq 3$; evaluating these equations using $\rho_j = \sigma_j$ or $\rho_j = (-1)^j \sigma_j$ as appropriate, we find

$$\sigma_{i+1}^2 - \sigma_{i-1}^2 = \lambda \sigma_i (\sigma_{i+1} - \sigma_{i-1}) \quad (1 \leq i \leq 3), \quad (45)$$

where $\lambda = \alpha + \beta$ if $\rho = \sigma$ and $\lambda = \alpha - \beta$ if $\rho = -\sigma$. Letting $i = 1, 2$ in (45), we obtain

$$\sigma_2^2 - 1 = \lambda \sigma (\sigma_2 - 1), \quad (46)$$

$$\sigma_3^2 - \sigma^2 = \lambda \sigma_2 (\sigma_3 - \sigma). \quad (47)$$

Observe $\sigma_2 \neq \pm 1$ by Lemma 2.7(iii). Dividing both sides of (46) by $\sigma_2 - 1$, we obtain

$$\sigma_2 + 1 = \lambda\sigma. \quad (48)$$

Since $\sigma_2 \neq -1$, we see $\sigma \neq 0$ in view of (48). We mentioned $\sigma_2 \neq 1$, so $\sigma_3 \neq \sigma$ by (12). Dividing both sides of (47) by $\sigma_3 - \sigma$, we obtain

$$\sigma_3 + \sigma = \lambda\sigma_2. \quad (49)$$

By (48) and (49), we find condition Theorem 2.18(iii) holds. Applying this theorem, we find Γ is 2-homogeneous and $E \in \{E_1, E_{D-1}\}$. Now E, F is a taut pair by Theorem 2.14. \square

Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Assume Γ has at least one taut pair of primitive idempotents. By Definition 2.15, one of the following holds: (i) Γ is 2-homogeneous, (ii) Γ is taut and D is even, (iii) Γ is taut and D is odd. Case (i) has already been explored by Curtin in [3] and Nomura in [14], so we will not discuss it further here. Cases (ii) and (iii) appear to be fundamentally different, and we will handle them separately. For the rest of this paper, we will focus on case (iii).

Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$ and valency $k \geq 3$. Let E, F denote a taut pair of primitive idempotents of Γ . Let θ, θ' denote the corresponding eigenvalues, and let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding cosine sequences. Let α, β denote complex scalars satisfying (37) for $1 \leq i \leq D - 1$. We use (37) to solve for α, β in terms of k, μ, σ, ρ . Since $\sigma = \theta/k, \rho = \theta'/k$, this gives α, β in terms of k, μ, θ, θ' . Our result is the following.

Corollary 5.2 *Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$ and valency $k \geq 3$. Let E, F denote a taut pair of primitive idempotents of Γ . Let θ, θ' denote the corresponding eigenvalues, and let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding cosine sequences. Let α, β denote complex scalars satisfying (37) for $1 \leq i \leq D - 1$. Then*

$$\alpha = \frac{\theta}{k} + \frac{\theta(k^2 - \theta^2)(b_2(k - 2) - \theta^2(\mu - 1))}{k(\theta^2 - \theta'^2)b_1b_2}, \quad (50)$$

$$\beta = \frac{\theta'}{k} + \frac{\theta'(k^2 - \theta'^2)(b_2(k - 2) - \theta^2(\mu - 1))}{k(\theta'^2 - \theta^2)b_1b_2}. \quad (51)$$

In particular, α, β are real and are uniquely determined by k, μ, θ, θ' .

Proof: Setting $i = 1, 2$ in (37), we obtain the equations

$$\sigma_2\rho_2 - 1 = \alpha\sigma(\rho_2 - 1) + \beta\rho(\sigma_2 - 1), \quad (52)$$

$$\sigma_3\rho_3 - \sigma\rho = \alpha\sigma_2(\rho_3 - \rho) + \beta\rho_2(\sigma_3 - \sigma), \quad (53)$$

where $\sigma = \sigma_1$, $\rho = \rho_1$. Observe these equations are linear in α, β . Recursively eliminating $\sigma_2, \sigma_3, \rho_2, \rho_3$ in these equations using (12) and recalling $\sigma = \theta/k$, $\rho = \theta'/k$, we routinely find that the system (52), (53) has a unique solution for α, β which satisfies (50), (51). \square

Definition 5.3 Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Let $d = (D - 1)/2$, and consider the set consisting of the following four eigenvalues:

$$\theta_1, \quad \theta_d, \quad \theta_{d+1}, \quad \theta_{D-1}. \quad (54)$$

Recall $\theta_{D-1} = -\theta_1$ and $\theta_{d+1} = -\theta_d$. Let θ denote an eigenvalue in (54), and let E denote the corresponding primitive idempotent. By the *weight* of θ (or E), we mean the real scalar

$$\frac{\theta}{k} + \frac{\theta(k^2 - \theta^2)(b_2(k - 2) - \theta^2(\mu - 1))}{k(\theta^2 - \theta'^2)b_1b_2}, \quad (55)$$

where θ' is any eigenvalue in (54) other than $\theta, -\theta$.

We introduced the notion of a weight in Definition 5.3 so that we may easily discuss the scalars α, β appearing in (37). To clarify how we use this notation, we give the following corollary.

Corollary 5.4 Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$ and valency $k \geq 3$. Let E, F denote a taut pair of primitive idempotents with cosine sequences $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$, respectively. Let α, β denote complex scalars. Then the following are equivalent:

- (i) The scalars α, β satisfy (37) for $1 \leq i \leq D - 1$.
- (ii) The scalars α, β are the weights of E, F , respectively.

Proof: Immediate using Theorem 2.17(i), Corollary 5.2, and Definition 5.3. \square

Lemma 5.5 Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Let $d = (D - 1)/2$, and let E denote one of $E_1, E_d, E_{d+1}, E_{D-1}$. Let α denote the weight of E , and let σ denote the first cosine of E . Then (i), (ii) hold below.

- (i) Suppose $E = E_1$ or $E = E_{d+1}$. Then $\alpha > \sigma$.
- (ii) Suppose $E = E_d$ or $E = E_{D-1}$. Then $\alpha < \sigma$.

Proof: Let θ denote the eigenvalue associated with E , and let θ' denote one of $\theta_1, \theta_d, \theta_{d+1}, \theta_{D-1}$ not equal to θ or $-\theta$. Recalling $\theta = k\sigma$ and using Definition 5.3, we find $\alpha - \sigma$ equals

$$(k^2 - \theta^2)(b_2(k - 2) - \theta^2(\mu - 1))k^{-1}b_1^{-1}b_2^{-1} \quad (56)$$

times $\theta(\theta^2 - \theta'^2)^{-1}$. In (56) the factor on the left is positive since $\theta < k$. The factor in the middle is positive by (17) and since Γ is not 2-homogeneous. The remaining factors in

(56) are also positive, so (56) is positive. Using $\theta_1 > \theta_d > \theta_{d+1} > \theta_{D-1}$ and $\theta_1 = -\theta_{D-1}$, $\theta_d = -\theta_{d+1}$, we find $\theta(\theta^2 - \theta^2)^{-1}$ is positive in case (i) and negative in case (ii). The result follows. \square

Lemma 5.6 *Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_D$. Let $d = (D - 1)/2$, and let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote the cosine sequence associated with any one of $\theta_1, \theta_d, \theta_{d+1}, \theta_{D-1}$. Then the expressions*

$$\sigma_{i+1} - \sigma_{i-1}, \quad \sigma\sigma_i - \sigma_{i-1}, \quad \sigma\sigma_i - \sigma_{i+1}$$

are nonzero for $1 \leq i \leq D - 1$.

Proof: By Corollary 2.2, it suffices to show $\sigma_{i+1} - \sigma_{i-1}$ is nonzero. Let θ denote the eigenvalue associated with the cosine sequence $\sigma_0, \sigma_1, \dots, \sigma_D$. The result is true for $\theta = \theta_1$ and $\theta = \theta_{D-1}$ by (6) and Lemma 2.6. Now assume θ is one of θ_d, θ_{d+1} . Fix an integer i ($1 \leq i \leq D - 1$) and assume $\sigma_{i+1} = \sigma_{i-1}$. We obtain a contradiction. To do this, we first observe by Corollary 2.2 that $\sigma_{i-1}, \sigma_{i+1}, \sigma\sigma_i$ are equal. Let E denote the primitive idempotent of Γ associated with θ . Let F denote the primitive idempotent of Γ associated with θ_1 , and let $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding cosine sequence. By Theorem 2.17(i), E, F is a taut pair. Applying Corollary 5.4, we find

$$\sigma_{i+1}\rho_{i+1} - \sigma_{i-1}\rho_{i-1} = \alpha\sigma_i(\rho_{i+1} - \rho_{i-1}) + \beta\rho_i(\sigma_{i+1} - \sigma_{i-1}), \quad (57)$$

where α, β denote the weights of E, F , respectively. Setting $\sigma_{i+1} = \sigma\sigma_i$, $\sigma_{i-1} = \sigma\sigma_i$ in (57), we find

$$(\sigma - \alpha)\sigma_i(\rho_{i+1} - \rho_{i-1}) = 0. \quad (58)$$

If $\sigma_i = 0$ then $\sigma_{i+1} = 0$, violating Corollary 2.3(i); thus $\sigma_i \neq 0$. Recall $\rho_0, \rho_1, \dots, \rho_D$ is the cosine sequence for θ_1 , so $\rho_{i+1} \neq \rho_{i-1}$ by (6). Thus $\sigma = \alpha$, contradicting Lemma 5.5. We conclude $\sigma_{i+1} - \sigma_{i-1}$ is nonzero, as desired. \square

Theorem 5.7 *Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$ and valency $k \geq 3$. Let E, F denote a taut pair of primitive idempotents. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding cosine sequences, respectively, and let α, β denote the corresponding weights. Then*

$$\frac{\sigma_{i+1} - \alpha\sigma_i}{\sigma\sigma_i - \sigma_{i-1}} = \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}} \quad (1 \leq i \leq D - 1), \quad (59)$$

$$\frac{\rho_{i+1} - \beta\rho_i}{\rho\rho_i - \rho_{i-1}} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} \quad (1 \leq i \leq D - 1). \quad (60)$$

Observe the denominators in (59), (60) are nonzero by Lemma 5.6.

Proof: Multiplying (37) by $\sigma\sigma_i - \sigma_{i-1}$, multiplying (13) by $\sigma_{i+1} - \alpha\sigma_i$, and taking the difference between the two products, we find $\sigma_{i-1} - \sigma_{i+1}$ times

$$(\sigma_{i+1} - \alpha\sigma_i)(\rho\rho_i - \rho_{i-1}) - (\beta\rho_i - \rho_{i-1})(\sigma\sigma_i - \sigma_{i-1}) \quad (61)$$

is zero. Observe $\sigma_{i-1} - \sigma_{i+1}$ is nonzero by Lemma 5.6, so (61) is zero. Line (59) follows.

We obtain (60) by interchanging the roles of σ_j, ρ_j ($0 \leq j \leq D$) and the roles of α, β in the above argument. \square

Theorem 5.8 *Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$ and valency $k \geq 3$. Let E, F denote a taut pair of primitive idempotents. Let σ, ρ denote the corresponding first cosines, respectively, and let α, β denote the corresponding weights. Then the intersection numbers of Γ are determined by the four scalars $\sigma, \rho, \alpha, \beta$.*

Proof: Given the scalars $\sigma, \rho, \alpha, \beta$, we may use (59), (60) to recursively obtain σ_i and ρ_i for $2 \leq i \leq D$. Using this information and Lemma 5.6, we set $i = 1$ and $c_1 = 1$ in (11) to obtain k . Using (11) and Lemma 5.6, we find c_2, c_3, \dots, c_{D-1} . Using $b_i = k - c_i$ ($1 \leq i \leq D - 1$), we find b_1, b_2, \dots, b_{D-1} . \square

Corollary 5.9 *Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_D$. Let $d = (D - 1)/2$. Then the intersection numbers of Γ are determined by the four scalars $k, \mu, \theta_1, \theta_d$.*

Proof: Recall E_1, E_d is a taut pair of primitive idempotents. Let σ, ρ denote the corresponding first cosines, and let α, β denote the corresponding weights. By Theorem 5.8, we obtain the intersection numbers in terms of $\sigma, \rho, \alpha, \beta$. Using $\sigma = \theta_1/k, \rho = \theta_d/k$, and Corollary 5.2 (with $\theta = \theta_1, \theta' = \theta_d$), we obtain $\sigma, \rho, \alpha, \beta$ in terms of $k, \mu, \theta_1, \theta_d$. \square

In the following lemmas, we obtain some other equations involving the cosines of a taut distance-regular graph of odd diameter.

Lemma 5.10 *Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$ and valency $k \geq 3$. Let E, F denote a taut pair of primitive idempotents. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding cosine sequences, respectively, and let α, β denote the corresponding weights. Then*

$$\frac{\sigma_{i-1} - \alpha\sigma_i}{\sigma\sigma_i - \sigma_{i+1}} = \frac{\beta\rho_i - \rho_{i+1}}{\rho\rho_i - \rho_{i+1}} \quad (1 \leq i \leq D - 1), \quad (62)$$

$$\frac{\rho_{i-1} - \beta\rho_i}{\rho\rho_i - \rho_{i+1}} = \frac{\alpha\sigma_i - \sigma_{i+1}}{\sigma\sigma_i - \sigma_{i+1}} \quad (1 \leq i \leq D - 1). \quad (63)$$

Observe the denominators in (62), (63) are nonzero by Lemma 5.6.

Proof: In the proof of Theorem 5.7, interchange the roles of $\sigma_{i-1}, \sigma_{i+1}$ and also the roles of ρ_{i-1}, ρ_{i+1} . \square

Lemma 5.11 *Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$ and valency $k \geq 3$. Let E, F denote a taut pair of primitive idempotents. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding cosine sequences, respectively, and let α, β denote the corresponding weights. Then*

$$\frac{k}{b_i} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} + \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}} \quad (1 \leq i \leq D-1), \quad (64)$$

$$\frac{k}{c_i} = \frac{\alpha\sigma_i - \sigma_{i+1}}{\sigma\sigma_i - \sigma_{i+1}} + \frac{\beta\rho_i - \rho_{i+1}}{\rho\rho_i - \rho_{i+1}} \quad (1 \leq i \leq D-1). \quad (65)$$

Observe the denominators in (64), (65) are nonzero by Lemma 5.6.

Proof: From (12), we find

$$\frac{k}{b_i} = \frac{\sigma_{i+1} - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} \quad (66)$$

$$= \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} + \frac{\sigma_{i+1} - \alpha\sigma_i}{\sigma\sigma_i - \sigma_{i-1}}. \quad (67)$$

Evaluating the expression on the right in (67) using (59), we obtain (64). We similarly obtain (65) by using (11), (62) in place of (12), (59) in the above argument. \square

Definition 5.12 Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. We say a pair of cosine sequences of Γ is *taut* whenever the associated pair of primitive idempotents is taut.

Theorem 5.13 *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 4$ and valency $k \geq 3$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote two sequences of complex scalars, and abbreviate $\sigma = \sigma_1, \rho = \rho_1$. Let α, β denote complex scalars. Then the following are equivalent:*

- (i) Γ is taut and D is odd. Moreover, $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ form a taut pair of cosine sequences, and α, β are the weights of the corresponding primitive idempotents.
- (ii) $\sigma_0 = 1, \rho_0 = 1, \sigma_{D-1} = \sigma\sigma_D, \rho_{D-1} = \rho\rho_D, \sigma \neq \alpha, \rho \neq \beta, \sigma \neq \pm\rho$, and

$$\frac{\sigma_{i+1} - \alpha\sigma_i}{\sigma\sigma_i - \sigma_{i-1}} = \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}} \quad (1 \leq i \leq D-1), \quad (68)$$

$$\frac{\rho_{i+1} - \beta\rho_i}{\rho\rho_i - \rho_{i-1}} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} \quad (1 \leq i \leq D-1), \quad (69)$$

$$\frac{k}{b_i} = \frac{\alpha\sigma_i - \sigma_{i-1}}{\sigma\sigma_i - \sigma_{i-1}} + \frac{\beta\rho_i - \rho_{i-1}}{\rho\rho_i - \rho_{i-1}} \quad (1 \leq i \leq D-1), \quad (70)$$

and the denominators in (68)–(70) are nonzero.

Proof: (i) \Rightarrow (ii). We assume $\sigma_0, \sigma_1, \dots, \sigma_D$ is a cosine sequence, so $\sigma_0 = 1$ and $\sigma_{D-1} = \sigma\sigma_D$ by Lemma 2.1(ii). Similarly, $\rho_0 = 1$ and $\rho_{D-1} = \rho\rho_D$. Observe $\sigma \neq \pm\rho$ by Lemma 2.6

and Theorem 2.17(i). Observe $\sigma \neq \alpha, \rho \neq \beta$ by Lemma 5.5. Equations (68)–(70) follow from Theorem 5.7 and Lemma 5.11.

(ii) \Rightarrow (i). Let $\theta_0 > \theta_1 > \dots > \theta_D$ denote the eigenvalues of Γ . Replacing the rightmost fraction in (70) using (68) and simplifying, we obtain

$$b_i(\sigma_{i+1} - \sigma_{i-1}) = k(\sigma\sigma_i - \sigma_{i-1}) \quad (1 \leq i \leq D-1). \quad (71)$$

We assume $\sigma_0 = 1, \sigma_{D-1} = \sigma\sigma_D$, so $\sigma_0, \sigma_1, \dots, \sigma_D$ is a cosine sequence of Γ by Lemma 2.1. Similarly, $\rho_0, \rho_1, \dots, \rho_D$ is a cosine sequence of Γ . Let E (respectively F) denote the primitive idempotent of Γ associated with $\sigma_0, \sigma_1, \dots, \sigma_D$ (respectively $\rho_0, \rho_1, \dots, \rho_D$). Observe E, F are nontrivial, since otherwise we would have a zero denominator in (68) at $i = 1$ by Lemma 2.7. We show E, F is a taut pair. To do this, we show (37) holds for $1 \leq i \leq D-1$. Fixing an integer i ($1 \leq i \leq D-1$), we multiply the Eqs. (68), (69) together and simplify to obtain

$$(\sigma_{i+1} - \alpha\sigma_i)(\rho_{i+1} - \beta\rho_i) = (\beta\rho_i - \rho_{i-1})(\alpha\sigma_i - \sigma_{i-1}). \quad (72)$$

Expanding (72), we routinely obtain (37). We have now shown (37) holds for $1 \leq i \leq D-1$. Combining this with Theorem 5.1, we find E, F is a taut pair. Now the cosine sequences $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ form a taut pair by Definition 5.12.

We show Γ is taut by applying Definition 2.15. We just showed E, F is a taut pair of primitive idempotents. To show Γ is taut, we show Γ is not 2-homogeneous. To do this, we assume Γ is 2-homogeneous and obtain a contradiction. First observe $\sigma \neq 0, \rho \neq 0$; otherwise, we have a zero denominator in (68) for $i = 2$. By Theorem 2.14, at least one of E, F is in the set $\{E_1, E_{D-1}\}$. Without loss of generality, we assume E is one of E_1, E_{D-1} . By Theorem 2.18, there exists a real scalar λ such that

$$\sigma_{i-1} - \lambda\sigma_i + \sigma_{i+1} = 0 \quad (1 \leq i \leq D-1). \quad (73)$$

Combining (73) at $i = 1, 2$, we find

$$\sigma_2(\sigma_2 + 1) = \sigma(\sigma_3 + \sigma). \quad (74)$$

Consider the expression

$$e = \frac{\sigma_2(\sigma_2 + 1) - \sigma(\sigma_3 + \sigma)}{\sigma^2 - \sigma_2}. \quad (75)$$

We remark that the denominator in (75) is nonzero; otherwise $\sigma^2 - 1$ is zero by Corollary 2.2, forcing a zero denominator in (68) at $i = 1$. We evaluate e in two ways. First, by (74), we find $e = 0$. Second, applying (68), (69), we solve for σ_2, σ_3 in terms of $\sigma, \rho, \alpha, \beta$. Eliminating σ_2, σ_3 in e using this, we obtain

$$e = \frac{(\rho - \beta)(\sigma + \rho)(\sigma - \rho)}{\rho(\rho - 1)(\rho + 1)}. \quad (76)$$

We assumed $\rho \neq \beta$, $\sigma \neq \pm\rho$, so $e \neq 0$. We now have a contradiction. We conclude Γ is not 2-homogeneous. Now Γ is taut by Definition 2.15.

We now show D is odd. Suppose D is even. By Theorem 2.17(ii), we find one of E, F is E_d , where $d = D/2$. Since the eigenvalue associated with E_d is 0, we find $\sigma = 0$ or $\rho = 0$, a contradiction. We conclude D is odd. Finally, since we proved (37) holds for $1 \leq i \leq D - 1$, we conclude by Corollary 5.4 that α, β are the weights for E, F , respectively. \square

In Example 2.16, we mentioned that the doubled Odd graphs are taut. Using the above theorem, we can readily verify this is the case. Our result is the following.

Example 5.14 Given an integer $k \geq 3$, let Γ denote the graph $2.O_k$ from Example 2.16. Recall Γ is a bipartite distance-regular graph with valency k and diameter $D = 2k - 1$. Define a sequence of scalars $\sigma_0, \sigma_1, \dots, \sigma_D$ by

$$\sigma_i = \frac{(-1)^i + (2k - 1)(2i - 2k + 1)}{4k(1 - k)} \quad (0 \leq i \leq D). \quad (77)$$

Let $\rho_0, \rho_1, \dots, \rho_D$ denote the sequence $y_0, -y_1, y_1, -y_2, y_2, \dots, -y_k$, where

$$y_i = (-1)^i \binom{k}{i}^{-1} \quad (0 \leq i \leq k). \quad (78)$$

Now define scalars α, β by

$$\alpha = 1 + (k - 1)k^{-2} \quad (79)$$

$$\beta = -k^{-2}. \quad (80)$$

Using (18), we may verify that the scalars α, β and the sequences $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ satisfy all the conditions of Theorem 5.13(ii). Applying Theorem 5.13, we find Γ is taut. Moreover, $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ form a taut pair of cosine sequences, and α, β are the corresponding weights. We remark that the eigenvalues of Γ associated with $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ are $k - 1$ and 1, respectively. \square

6. Antipodal 2-covers

In this section we prove that any taut bipartite distance-regular graph of odd diameter $D \geq 5$ is an antipodal 2-cover. We begin with the definition of antipodal.

Definition 6.1 Let Γ denote a distance-regular graph with vertex set X and diameter $D \geq 2$. Let Γ_D denote the undirected graph with vertex set X and edge set $\{(x, y) | x, y \in X, \partial(x, y) = D\}$, where ∂ is the path-length distance function of Γ . Γ is said to be *antipodal* whenever Γ_D is a disjoint union of cliques. In this case each clique has cardinality $1 + k_D$, where k_D is from (4). We refer to Γ as an r -cover, where $r = 1 + k_D$.

Lemma 6.2 [2, Prop. 4.2.2] *Let Γ denote a bipartite distance-regular graph of odd diameter $D \geq 3$. Then Γ is antipodal if and only if $b_i = c_{D-i}$ for $0 \leq i \leq D$. In this case, Γ is a 2-cover.*

We mention a result about the cosine sequences of an antipodal 2-cover.

Lemma 6.3 [2, pp. 142–143] *Let Γ denote a bipartite distance-regular graph with diameter $D \geq 3$, valency $k \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_D$. Assume Γ is an antipodal 2-cover. Choose an integer j ($0 \leq j \leq D$), and let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote the cosine sequence of Γ associated with θ_j . If j is even, then $\sigma_i = \sigma_{D-i}$ for $0 \leq i \leq D$. If j is odd, then $\sigma_i = -\sigma_{D-i}$ for $0 \leq i \leq D$.*

Theorem 6.4 *Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$ and valency $k \geq 3$. Then Γ is an antipodal 2-cover.*

Proof: Let $\theta_0 > \theta_1 > \cdots > \theta_D$ denote the eigenvalues of Γ , and let $\sigma_0, \sigma_1, \dots, \sigma_D$ denote the cosine sequence associated with θ_1 . We first show

$$\sigma_{D-j} = \sigma_j \sigma_D \quad (0 \leq j \leq D). \quad (81)$$

To show (81), it is convenient to simultaneously consider the cosine sequence for θ_d , where $d = (D-1)/2$. Let $\rho_0, \rho_1, \dots, \rho_D$ denote the cosine sequence for θ_d . For $0 \leq j \leq D$, let $\sigma'_j := \sigma_{D-j}/\sigma_D$ and $\rho'_j := \rho_{D-j}/\rho_D$. We recall σ_D, ρ_D are nonzero by Corollary 2.3(ii). We show $\sigma'_j = \sigma_j$, $\rho'_j = \rho_j$ for $0 \leq j \leq D$. Our proof is by induction on j . Observe $\sigma'_0 = \sigma_0$, $\rho'_0 = \rho_0$ by construction and since $\sigma_0 = \rho_0 = 1$. Observe $\sigma'_1 = \sigma_1$, $\rho'_1 = \rho_1$ by Lemma 2.1(ii). Now fix an integer i ($1 \leq i \leq D-1$), and assume by induction that $\sigma'_j = \sigma_j$, $\rho'_j = \rho_j$ for $0 \leq j \leq i$. We show $\sigma'_{i+1} = \sigma_{i+1}$, $\rho'_{i+1} = \rho_{i+1}$. Replacing i by $D-i$ in (62), we obtain

$$\frac{\sigma'_{i+1} - \alpha \sigma'_i}{\sigma \sigma'_i - \sigma'_{i-1}} = \frac{\beta \rho'_i - \rho'_{i-1}}{\rho \rho'_i - \rho'_{i-1}}, \quad (82)$$

where α, β denote the weights of θ_1, θ_d , respectively. Comparing (82), (59), and using $\sigma'_i = \sigma_i$, $\sigma'_{i-1} = \sigma_{i-1}$, $\rho'_i = \rho_i$, $\rho'_{i-1} = \rho_{i-1}$, we find $\sigma'_{i+1} = \sigma_{i+1}$. Interchanging the roles of $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ in the above argument, we similarly find $\rho'_{i+1} = \rho_{i+1}$.

We have now shown $\sigma'_j = \sigma_j$ and $\rho'_j = \rho_j$ for $0 \leq j \leq D$. In particular, we have (81). We now show $b_i = c_{D-i}$ for $0 \leq i \leq D$. Assume $1 \leq i \leq D-1$; otherwise the result is trivial. Replacing i by $D-i$ in (11), and evaluating the result using (81), we find

$$c_{D-i}(\sigma_{i+1} - \sigma_{i-1}) = k(\sigma \sigma_i - \sigma_{i-1}). \quad (83)$$

Comparing (83), (12), and using Lemma 5.6, we find $b_i = c_{D-i}$. \square

We finish this paper with a comment on Theorem 5.8. Let Γ denote a taut bipartite distance-regular graph with odd diameter $D \geq 5$, valency $k \geq 3$, and eigenvalues $\theta_0 >$

$\theta_1 > \cdots > \theta_D$. Consider the taut pair E_1, E_d , where $d = (D - 1)/2$. Let $\sigma_0, \sigma_1, \dots, \sigma_D$ and $\rho_0, \rho_1, \dots, \rho_D$ denote the corresponding cosine sequences, and let α, β denote the corresponding weights. Combining Lemma 6.3 and Theorem 6.4, we find $\sigma_{D-i} = -\sigma_i$ and $\rho_{D-i} = (-1)^d \rho_i$ for $0 \leq i \leq D$. In particular,

$$\sigma_{d+1} = -\sigma_d, \quad \rho_{d+1} = (-1)^d \rho_d. \quad (84)$$

In view of Theorem 5.8, the equations in (84) imply two polynomial equations involving $\sigma, \rho, \alpha, \beta$, where we abbreviate $\sigma = \sigma_1, \rho = \rho_1$. For the cases $D = 5$ and $D = 7$, we have obtained these equations explicitly, and they seem rather complicated. In particular, it is not clear how to use them to solve for two of $\sigma, \rho, \alpha, \beta$ in terms of the other two. We will pursue this matter in a future paper.

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