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On Even Generalized Table Algebras

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Abstract. Generalized table algebras were introduced in Arad, Fisman and Muzychuk (*Israel J. Math.* **114** (1999), 29–60) as an axiomatic closure of some algebraic properties of the Bose-Mesner algebras of association schemes. In this note we show that if all non-trivial degrees of a generalized integral table algebra are even, then the number of real basic elements of the algebra is bounded from below (Theorem 2.2). As a consequence we obtain some interesting facts about association schemes the non-trivial valencies of which are even. For example, we proved that if all non-identical relations of an association scheme have the same valency which is even, then the scheme is symmetric.

Keywords: generalized table algebras, association schemes

1. Introduction

Let R be an integral domain. An R-algebra A with a distinguished basis $\mathbf{B}$ is called a *generalized table algebra* (briefly, GT-algebra) *with a distinguished basis* $\mathbf{B}$ if it satisfies the following axioms [2]:

GT0. A is a free left R-module with a basis $\mathbf{B}$.
GT1. A is an R-algebra with unit $\mathbf{1}$, and $\mathbf{1} \in \mathbf{B}$.
GT2. There exists an antiautomorphism $a \to \bar{a}, a \in A$, such that $\overline{(\bar{a})} = a$ holds for all $a \in A$ and $\bar{\mathbf{B}} = \mathbf{B}$.

Let $\lambda_{abc} \in R$ be the structure constants of A in the basis $\mathbf{B}$, i.e.,

$$ab = \sum_{c\in\mathbf{B}} \lambda_{abc} c, a, b \in \mathbf{B}$$

GT3. For each $a, b \in \mathbf{B}$ $\lambda_{ab\mathbf{1}} = \lambda_{ba\mathbf{1}}$, and $\lambda_{ab\mathbf{1}} = 0$ if $a \neq \bar{b}$.

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In what follows the notation $(A, \mathbf{B})$ will mean a GT-algebra A with the distinguished basis $\mathbf{B}$. We also set $\mathbf{B}^{\#} := \mathbf{B}\backslash\{\mathbf{1}\}$.

A GT-algebra will be called *real* if $R \subseteq \mathbb{R}$, $\lambda_{a\bar{a}\mathbf{1}} > 0$ and $\lambda_{abc} \geq 0$ for each triple $a, b, c \in \mathbf{B}$. A real commutative GT-algebra is called *a table algebra*. In what follows we shall consider only real GT-algebras.

Let t: $A \to R$ be the linear function defined by $t(\sum_{b\in\mathbf{B}} x_b b) = x_{\mathbf{1}}$. As a direct consequence of GT3, we obtain that $t(xy) = t(yx)$, $x, y \in A$.

We define a bilinear form $\langle , \rangle$ on A by setting

$$\langle x, y\rangle = t(x\bar{y}).$$

According to GT3, $\langle , \rangle$ is a symmetric bilinear form values of which may be computed by the following formula:

$$\left\langle \sum_{b\in\mathbf{B}} x_b b, \sum_{b\in\mathbf{B}} y_b b \right\rangle = \sum_{b\in\mathbf{B}} x_b y_b \lambda_{b\bar{b}\mathbf{1}}. \tag{1}$$

A subset $\mathbf{D} \subset \mathbf{B}$ is said to be *closed* subset if the R-submodule $\langle c\rangle_{c\in\mathbf{D}}$ is a GT-algebra with distinguished basis $\mathbf{D}$.

An element $b \in \mathbf{B}$ is called *real* (or *symmetric*), if $\bar{b} = b$ [1].

If $(A; \mathbf{B})$ is a real GT-algebra, then, by Theorem 3.14 [2] there exists a unique algebra homomorphism $|\,|$: $A \to \mathbb{R}$ such that $|b| = |\bar{b}| > 0$ for each $b \in \mathbf{B}$. We call it the *degree homomorphism*. The positive real numbers $\{|b|\}_{b\in\mathbf{B}}$ are called the *degrees* of $(A, \mathbf{B})$.

A real GT-algebra such that all its structure constants λ_{abc} and all the degrees $|b|$ are rational integers [4] is called an *integral GT-algebra* (briefly, IGTA). A commutative integral GT-algebra is exactly *integral table algebra* (ITA) as defined in [4].

A GT-algebra is called *homogeneous* of degree λ if all its non-trivial degrees are equal to λ. A GT-algebra is called *standard* if $|b| = \lambda_{b\bar{b}\mathbf{1}}$ for each $b \in \mathbf{B}$. We say that a GT-algebra $(A, \mathbf{B}')$ is a *rescaling* of $(A, \mathbf{B})$ if there exist non-zero scalars $r_b \in R, b \in \mathbf{B}$ such that $\mathbf{B}' = \{r_b b \mid b \in \mathbf{B}\}$.

Any real GT-algebra may be rescaled to one which is homogeneous and any IGTA can be rescaled to a homogeneous IGTA ([5], Theorem 1). Any real GT-algebra may be rescaled to a standard one by setting $b' := \frac{|b|}{\lambda_{b\bar{b}\mathbf{1}}} b$. The number

$$o(\mathbf{B}) := \sum_{b\in\mathbf{B}} \frac{|b|^2}{\lambda_{b\bar{b}\mathbf{1}}}$$

does not depend on a rescaling of the table algebra $(A; \mathbf{B})$ [5]. It is called *the order of* $(A; \mathbf{B})$. If $(A; \mathbf{B})$ is standard, then $o(\mathbf{B}) = \sum_{b\in\mathbf{B}} |b| = |\mathbf{B}|$. We need the following

Proposition 1.1 ([2]) *Let* $(A, \mathbf{B})$ *be a real standard GT-algebra. Then for all* $a, b, c \in \mathbf{B}$ *the following conditions hold*:

(i)

$$\sum_{t\in\mathbf{B}} \lambda_{abt}\lambda_{tcd} = \sum_{t\in\mathbf{B}} \lambda_{atd}\lambda_{bct};$$

(ii) $|\mathbf{1}| = 1$ *and* $|b| = |\bar{b}|$;
(iii) $\lambda_{abc} = \lambda_{\bar{b}\bar{a}\bar{c}}$;
(iv) $\lambda_{abc}|c| = \lambda_{\bar{c}a\bar{b}}|b| = \lambda_{c\bar{b}a}|a|$, *and* $\lambda_{b\bar{b}c} = \lambda_{b\bar{b}\bar{c}}$;
(v) $\langle ab, ab\rangle = \langle \bar{a}a, b\bar{b}\rangle$ *and*

$$\sum_{c\in\mathbf{B}} \lambda_{abc}^2|c| = \sum_{c\in\mathbf{B}} \lambda_{\bar{a}ac}\lambda_{b\bar{b}c}|c|;$$

(vi) $\lambda_{aba} = \lambda_{a\bar{b}a} = \lambda_{b\bar{a}\bar{a}} = \lambda_{\bar{b}\bar{a}\bar{a}}$;
(vii) $\forall_{a,b\in\mathbf{B}} \sum_{x\in\mathbf{B}} \lambda_{axb} = \sum_{x\in\mathbf{B}} \lambda_{xab} = |a|$.
(viii) $\forall_{a,b\in\mathbf{B}} \sum_{x\in\mathbf{B}} \lambda_{abx}|x| = |a||b|$.

2. Even GT-algebras

Till the end of the paper we consider GT-algebras such that their structure constants and degrees belong to the ring $\mathbb{S} := \mathbb{Z}[\mathbb{O}^{-1}] \subseteq \mathbb{Q}$, where $\mathbb{O} := \mathbb{Z}\backslash 2\mathbb{Z}$. It is easy to see that $\mathbb{S}$ is a local ring with a unique maximal ideal $2\mathbb{S}$. The elements of $2\mathbb{S}$ ($\mathbb{S}\backslash 2\mathbb{S}$) will be called *even* (resp. *odd*) elements of the ring $\mathbb{S}$.

We write that $x \equiv y(\text{mod } 2)$ if $x - y \in 2\mathbb{S}$. A direct check shows that $\mathbb{S}/(2\mathbb{S}) \cong \mathbb{Z}_2$. Therefore

$$\begin{aligned} &r^2 \equiv r(\text{mod } 2) \quad \text{and} \\ &r \equiv 1(\text{mod } 2) \Leftrightarrow r \in \mathbb{S}\backslash(2\mathbb{S}). \end{aligned} \tag{2}$$

We write $\hat{r}$ for the image of $r \in \mathbb{S}$ in $\mathbb{S}/(2\mathbb{S})$. Each non-zero element $r \in \mathbb{S}$ has a unique presentation as the product $r = 2^\alpha s$ with $\alpha \geq 0$ and $s \in \mathbb{S}\backslash 2\mathbb{S}$. We set $\nu_2(r) := \alpha$ and $\nu_2(0) := \infty$.

A GT-algebra is called *even* (*odd*) if all its degrees are even (resp. odd) elements of $\mathbb{S}$.

In what follows we use the following notation

$$\begin{aligned}
&\nu_2(b) := \nu_2(|b|);\\
&\alpha_0 := \min\{\nu_2(b) \mid b \in \mathbf{B}^\#\};\\
&\mathbf{B}^a := \{b \in \mathbf{B} \mid \bar{b} \neq b\};\\
&\mathbf{B}^s := \{b \in \mathbf{B} \mid \bar{b} = b\};\\
&\mathbf{B}_\alpha := \{b \in \mathbf{B} \mid \nu_2(b) = \alpha\};\\
&\mathbf{B}_{\geq\alpha} := \{b \in \mathbf{B} \mid \nu_2(b) \geq \alpha\};\\
&\mathbf{B}_{>\alpha} := \{b \in \mathbf{B} \mid \nu_2(b) > \alpha\};\\
&\mathbf{X}^a := \mathbf{X} \cap \mathbf{B}^a,\ \mathbf{X}^s := \mathbf{X} \cap \mathbf{B}^s \text{ if } \mathbf{X} \subseteq \mathbf{B}.
\end{aligned}$$

Since A is a free $\mathbb{S}$-module with basis $\mathbf{B}$, $(\mathbb{S}\mathbf{B})/(2\mathbb{S}\mathbf{B}) \cong \mathbb{S}/(2\mathbb{S}) \otimes_{\mathbb{S}} \mathbf{B}$. In what follows we shall write $\mathbb{Z}_2\mathbf{B}$ for $\mathbb{S}/(2\mathbb{S}) \otimes_{\mathbb{S}} \mathbf{B}$. We also write b for $1 \otimes b$ unless it leads to a contradiction. If $\mathbf{X} \subseteq \mathbf{B}$, then $\mathbb{Z}_2\mathbf{X}$ denotes $\mathbb{Z}_2$-vector subspace of $\mathbb{Z}_2\mathbf{B}$ spanned by the elements $x \in \mathbf{X}$. If $\mathbf{B}$ is standard, then the $\mathbb{Z}_2$-linear subspace $\mathbb{Z}_2\mathbf{B}_{>\alpha}$ spanned by $\mathbf{B}_{>\alpha}$ is an ideal of the algebra $\mathbb{Z}_2\mathbf{B}$.

The following result was proved in [7] for association schemes, but its proof works also for GT-algebras over $\mathbb{S}$. We give here the proof in order to make the text self-contained.

Proposition 2.1 *Let* $(A, \mathbf{B})$ *be a standard GT-algebra defined over a ring* $R \subseteq \mathbb{S}$. *If all elements of* $\mathbf{B}^\#$ *are non-real*, *then*
(i) $o(\mathbf{B})$ *is odd*;
(ii) $|b|$ *is odd for each* $b \in \mathbf{B}$.

Proof: Part (i) is a direct consequence of $|b| = |\bar{b}|$, $b \in \mathbf{B}$.

(ii) Pick an arbitrary $a \in \mathbf{B}$ and denote by $\mathbf{A}$ the set of all $b \in \mathbf{B}$ which appear in the product $a\bar{a}$ with non-zero coefficient. Since $\lambda_{a\bar{a}b} = \lambda_{a\bar{a}\bar{b}}$, $\mathbf{A} = \{a_1, \ldots, a_k, \bar{a}_1, \ldots, \bar{a}_k\}$. It follows from Proposition 1.1, part (viii) that

$$|a| - 1 = \sum_{i=1}^{k} 2\frac{\lambda_{a\bar{a}a_i}|a_i|}{|a|}. \tag{3}$$

By Proposition 1.1, part (iv) $\frac{\lambda_{a\bar{a}a_i}|a_i|}{|a|} = \lambda_{a_i aa} \in R$. Therefore the right-hand side of (3) is even which implies that $|a|$ is odd. □

The above Proposition implies that if $o(\mathbf{B})$ is even, then $\mathbf{B}$ contains a non-identical real element. If $o(\mathbf{B})$ is odd, then we cannot say something definite about the number of real elements in general. Nevertheless, there exists one case when the number of real elements may be bounded from below.

Theorem 2.2 *Let* $(A, \mathbf{B})$ *be a standard GT-algebra the structure constants of which belong to* $\mathbb{S}$. *If* $\alpha_0 > 0$, *then*
(i) *each element of* $\mathbf{B}_{\alpha_0}$ *is real*;
(ii) *the elements* $b^2, b \in B_{\alpha_0}$ *are linearly independent. In particular*, *the elements* $b^2, b \in \mathbf{B}_{\alpha_0}$ *are pairwise distinct*;
(iii) *the factor-algebra* $\mathbb{Z}_2\mathbf{B}/\mathbb{Z}_2\mathbf{B}_{>\alpha_0}$ *is commutative and semisimple.*

Proof: (i) For each $b \in \mathbf{B}_{\alpha_0}$ we define the vector e_b the coordinates of which are labelled by the elements of $\mathbf{B}_{\alpha_0}$ and defined as follows: $e_{bx} := \lambda_{b\bar{b}x}$, $x \in \mathbf{B}_{\alpha_0}$. Consider a $\mathbb{Z}_2$-vector space V spanned by the vectors $\hat{e}_b$, $\hat{e}_{bx} := \hat{\lambda}_{b\bar{b}x}$. Since $\lambda_{b\bar{b}x} = \lambda_{b\bar{b}\bar{x}}$ and $\overline{\mathbf{B}_{\alpha_0}} = \mathbf{B}_{\alpha_0}$, $\dim(V)$ is at most $|\mathbf{B}^s_{\alpha_0}| + |\mathbf{B}^a_{\alpha_0}|/2$. Denote $(x, y) := \sum_{b\in\mathbf{B}_{\alpha_0}} x_b y_b$. Then

$$(\hat{e}_a, \hat{e}_b) = \sum_{x\in\mathbf{B}_{\alpha_0}} \widehat{\lambda_{a\bar{a}x}}\widehat{\lambda_{b\bar{b}x}}.$$

Since $\frac{|x|}{2^{\alpha_0}} \in \mathbb{S}$ for each $x \in \mathbf{B}^\#$ and $\frac{|x|}{2^{\alpha_0}}$ is odd if and only if $x \in \mathbf{B}_{\alpha_0}$,

$$(\hat{e}_a, \hat{e}_b) \equiv \sum_{x\in\mathbf{B}^\#} \lambda_{a\bar{a}x}\lambda_{b\bar{b}x}\frac{|x|}{2^{\alpha_0}} \pmod 2.$$

Since $(A, \mathbf{B})$ is standard, $\nu_2(\lambda_{a\bar{a}\mathbf{1}}\lambda_{b\bar{b}\mathbf{1}}/2^{\alpha_0}) = \nu_2(|a||b|/2^{\alpha_0}) = \alpha_0 > 0$ for each $a, b \in \mathbf{B}_{\alpha_0}$. Therefore

$$(\hat{e}_a, \hat{e}_b) \equiv \sum_{x\in\mathbf{B}} \lambda_{a\bar{a}x}\lambda_{b\bar{b}x}\frac{|x|}{2^{\alpha_0}} \equiv \frac{1}{2^{\alpha_0}}\langle a\bar{a}, b\bar{b}\rangle (\text{mod } 2).$$

By Proposition 1.1, part (v)

$$\frac{1}{2^{\alpha_0}}\langle a\bar{a}, b\bar{b}\rangle = \frac{1}{2^{\alpha_0}}\langle \bar{b}a, \bar{b}a\rangle = \sum_{x\in\mathbf{B}} \lambda_{\bar{b}ax}^2\frac{|x|}{2^{\alpha_0}}.$$

If $a \neq b$, then

$$\sum_{x\in\mathbf{B}} \lambda_{\bar{b}ax}^2\frac{|x|}{2^{\alpha_0}} = \sum_{x\in\mathbf{B}^\#} \lambda_{\bar{b}ax}^2\frac{|x|}{2^{\alpha_0}} \equiv \sum_{x\in\mathbf{B}^\#} \lambda_{\bar{b}ax}\frac{|x|}{2^{\alpha_0}} \equiv \frac{|a||b|}{2^{\alpha_0}} \equiv 0(\text{mod } 2)$$

Thus we obtain that $(\hat{e}_a, \hat{e}_b) = 0$ if $a \neq b$.

If $a = b$, then

$$(\hat{e}_a, \hat{e}_a) \equiv \sum_{x\in\mathbf{B}_{\alpha_0}} \lambda_{a\bar{a}x}^2 \equiv \sum_{x\in\mathbf{B}_{\alpha_0}} \lambda_{a\bar{a}x} \equiv \sum_{x\in\mathbf{B}^\#} \lambda_{a\bar{a}x}\frac{|x|}{2^{\alpha_0}} = \frac{|a|^2}{2^{\alpha_0}} - \frac{|a|}{2^{\alpha_0}} \equiv 1(\text{mod } 2)$$

Therefore $(\hat{e}_a, \hat{e}_b) = \delta_{ab}$ for $a, b \in \mathbf{B}_{\alpha_0}$. This implies that the vectors $\hat{e}_a$, $a \in \mathbf{B}_{\alpha_0}$ are linearly independent. Hence $\dim(V) = |\mathbf{B}_{\alpha_0}|$. On the other hand, $\dim(V) \leq |\mathbf{B}^s_{\alpha_0}| + |\mathbf{B}^a_{\alpha_0}|/2$. Hence $|\mathbf{B}^a_{\alpha_0}| = 0$.

(ii) Since all elements from $\mathbf{B}_{\alpha_0}$ are real, $\lambda_{a\bar{a}x} = \lambda_{aax}$ and $a^2 = \sum_{b\in\mathbf{B}} \lambda_{a\bar{a}b}b$. It follows from part (i) that the vectors $e_a = (\lambda_{a\bar{a}b})_{b\in\mathbf{B}_{\alpha_0}}$ are linearly independent modulo $2\mathbb{S}$. Therefore, the vectors e_a are linearly independent over $\mathbb{S}$. Hence the elements a^2 are linearly independent.

(iii) Denote $I := \mathbb{Z}_2\mathbf{B}_{>\alpha_0}$ just for a convenience. Since $\mathbb{Z}_2\mathbf{B}_{>\alpha_0}$ is $^-$-invariant, the mapping $x \mapsto \bar{x}$, $x \in \mathbb{Z}_2\mathbf{B}/I$ is well-defined. Since $\mathbb{Z}_2\mathbf{B} = \mathbb{Z}_2\mathbf{B}_{\leq\alpha_0} \oplus I$ and $^-$ acts on $\mathbb{Z}_2\mathbf{B}_{\leq\alpha_0}$ identically (see part (i)), $^-$ acts identically on the factor-algebra $\mathbb{Z}_2\mathbf{B}/I$. On the other hand, $^-$ is an antiautomorphism. Hence $\mathbb{Z}_2\mathbf{B}/I$ is commutative and $^-$ is identical on $\mathbb{Z}_2\mathbf{B}/I$.

A commutative algebra is semisimple if and only if it does not contain nilpotent elements. According to part (i) the vectors $\hat{e}_b$, $b \in \mathbf{B}_{\alpha_0}$ are linearly independent. Since $b^2 \equiv \sum_{c\in\mathbf{B}_{\alpha_0}} \hat{e}_{bc}c(\text{mod } I)$, the elements $\mathbf{1} + I$, $b^2 + I$, $b \in \mathbf{B}_{\alpha_0}$ form a $\mathbb{Z}_2$-basis of $\mathbb{Z}_2\mathbf{B}/I$. Hence $(\mathbb{Z}_2\mathbf{B}/I)^2 = \mathbb{Z}_2\mathbf{B}/I$ and the statement follows. □

We have two immediate corollaries.

Corollary 2.3 *Let $(A, \mathbf{B})$ be a standard GT-algebra the structure constants of which belong to $\mathbb{S}$. If $\nu_2(b) = \alpha > 0$ for each $b \in \mathbf{B}^\#$, then*

(i) *A is commutative and real*;

(ii) *The elements b^2, $b \in \mathbf{B}$ are linearly independent*;

(iii) $(\mathbb{S}/(2\mathbb{S})) \otimes_{\mathbb{S}} A$ *is semisimple.*

Remark 2.4 If $\alpha = 0$ it might happen that there are symmetric and non-symmetric relations. For example, the S-ring over the group $\mathbb{Z}_4 \times \mathbb{Z}_4$ induced by a fixed-point-free automorphism of order 3 contains one symmetric and four non-symmetric basic elements.

Corollary 2.5 *Let $(A, \mathbf{B})$ be a homogeneous GT-algebra of degree $k \in \mathbb{S}$, $\nu_2(k) > 0$ such that its structure constants belong to $\mathbb{S}$. If $\nu_2(\lambda_{b\bar{b}\mathbf{1}}) = \beta$ for each $b \in \mathbf{B}^\#$ and $\beta \leq \nu_2(k)$, then*
(i) *A is commutative and real*;
(ii) *The elements b^2, $b \in \mathbf{B}$ are linearly independent.*

Proof: Consider the algebra A with a rescaled basis $\mathbf{1}' := \mathbf{1}$, $b' := \frac{k}{\lambda_{b\bar{b}\mathbf{1}}} b$, $b \in \mathbf{B}^\#$. It is well-known that $(A, \mathbf{B}')$ is a standard GT-algebra.

We have that

$$\nu_2(b') = \nu_2\left(\frac{k^2}{\lambda_{b\bar{b}\mathbf{1}}}\right) = 2\nu_2(k) - \beta > 0$$

and

$$\lambda_{a'b'c'} = \frac{k}{\lambda_{a\bar{a}\mathbf{1}}} \frac{k}{\lambda_{b\bar{b}\mathbf{1}}} \frac{\lambda_{c\bar{c}\mathbf{1}}}{k} \lambda_{abc} = \frac{k\lambda_{c\bar{c}\mathbf{1}}}{\lambda_{a\bar{a}\mathbf{1}}\lambda_{b\bar{b}\mathbf{1}}} \lambda_{abc}.$$

Since $\nu_2(\frac{k\lambda_{c\bar{c}\mathbf{1}}}{\lambda_{a\bar{a}\mathbf{1}}\lambda_{b\bar{b}\mathbf{1}}}) = \nu_2(k) - \beta \geq 0$, $\frac{k\lambda_{c\bar{c}\mathbf{1}}}{\lambda_{a\bar{a}\mathbf{1}}\lambda_{b\bar{b}\mathbf{1}}} \in \mathbb{S}$, and, consequently, $\lambda_{a'b'c'} \in \mathbb{S}$. Now Corollary 2.3 yields the claim. □

3. Some applications

Let (X, F) be a finite association scheme in a sense of [8]. The Bose-Mesner algebra A of F is a standard integral table algebra the table basis of which is formed by the adjacency matrices $A(f)$, $f \in F$. The degree of $A(f)$ concides with a valency of the relation f and will be denoted by n_f. We say that a scheme F is even if all its valencies are even. As before we set

$$\nu_2(f) := \nu_2(n_f);$$
$$\alpha_0 := \min\{\nu_2(f) \mid f \in F^\#\};$$
$$F_\alpha := \{f \in F \mid \nu_2(f) = \alpha\};$$
$$F_{\geq\alpha} := \{f \in F \mid \nu_2(f) \geq \alpha\};$$
$$F_{>\alpha} := \{f \in F \mid \nu_2(f) > \alpha\};$$

As a direct consequence of Theorem 2.2 we obtain the following

Proposition 3.1 *Let (X, F) be an even association scheme. Denote $I := \langle A(f) \mid f \in F_{>\alpha_0}\rangle$. Then*
(i) *each $f \in F_{\alpha_0}$ is symmetric*;
(ii) *the elements $A(f)^2$, $f \in F_{\alpha_0}$ are linearly independent*;
(iii) *the factor-algebra $(\mathbb{Z}_2 \otimes_{\mathbb{Z}} A)/(\mathbb{Z}_2 \otimes_{\mathbb{Z}} I)$ is symmetric, commutative and semisimple.*

Theorem 3.2 *Let* (X, F) *be an even association scheme such that* $\nu_2(f) = \alpha > 0$ *for each* $f \in F^{\#}$. *Then*
(i) (X, F) *is symmetric and commutative*;
(ii) *the elements* $A(f)^2$, $f \in F$ *form a basis of* A;
(iii) *the algebra* $(\mathbb{Z}_2 \otimes_{\mathbb{Z}} A)$ *is semisimple*;
(iv) $\nu_2(m) = \alpha$ *for each non-principal multiplicity* m *of* F;
(v) *if* $f = 2^{\alpha}$ *for each* $f \in F^{\#}$, *then each nontrivial multiplicity of* F *is equal to* 2^{α}.

Proof: The parts (i)–(iii) are direct consequences of the previous statement. Let $n_0 = 1, n_1, \ldots, n_r$ and $m_0 = 1, m_1, \ldots, m_r$ be the valencies and the multiplicities of F.

(iv) Since $(\mathbb{Z}_2 \otimes_{\mathbb{Z}} A)$ is semisimple, the Frame number $|X|^{r+1} \frac{\prod_{i=0}^{r} n_i}{\prod_{i=0}^{r} m_i}$ is odd (Theorem 1.1 [2]). Therefore

$$\sum_{i=1}^{r} \nu_2(m_i) = \sum_{i=1}^{r} \nu_2(n_i) = r\alpha$$

By Theorem 4.2, part (iii) [3] $\nu_2(m_i) \leq \alpha$. Hence $\nu_2(m_i) = \alpha$, as desired.

(v) We have that $m_i \geq 2^{\alpha}$ for $i > 0$, since m_i is divisible by 2^{α}. Now the equality

$$\sum_{i=1}^{r} m_i = |X| - 1 = r2^{\alpha}$$

implies the claim. □

Remark 3.3 If $\nu_p(f)$, $f \in F^{\#}$ is constant for some odd prime p, then the algebra $\mathbb{Z}_p \otimes_{\mathbb{Z}} A$ may not be semisimple. The Johnson scheme [3] with two classes on 7 points is such an example with $p = 5$.

Let G be a finite group, then each subgroup $H \leq G$ gives rise to an association scheme $(G/H, G//H)$ where G/H and $G//H$ are the sets of right and double H-cosets respectively: two points Hg_1, Hg_2 are related via HgH if $Hg_1g_2^{-1}H = HgH$. Following [8] we denote this scheme as $(G/H, G//H)$. The valency of the relation corresponding to the double coset HgH is equal to $[H : H \cap H^g]$. The GT-algebra corresponding to the association scheme $(G/H, G//H)$ is exactly isomorphic to the Hecke algebra of double cosets of the subgroup H.

If $H \leq G$ is such that $\nu_2([H : H \cap H^g]) = \alpha > 0$ holds[1] for each $g \in G\backslash H$, then the association scheme $(G/H, G//H)$ satisfies the conditions of Proposition 3.2 which implies the following

Corollary 3.4 *Let* $H \leq G$ *be finite groups such that* $\nu_2([H : H \cap H^g]) = \alpha > 0$ *for each* $g \in G\backslash H$. *Then*
(i) $HgH = Hg^{-1}H$ *for each* $g \in G$;
(ii) *the character* 1_H^G *is multiplicity-free and* $\nu_2(\chi(1)) = 2^{\alpha}$ *for each non-trivial* $\chi \in \mathbf{Irr}(G)$ *which appears in* 1_H^G.

Note

1. Such a situation happens, for example, if H is a strongly embedded subgroup of G.

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