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Spin Models of Index 2 and Hadamard Models

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Abstract. A spin model (for link invariants) is a square matrix W with non-zero complex entries which satisfies certain axioms. Recently it was shown that ${}^{\mathrm{t}}WW^{-1}$ is a permutation matrix (the order of this permutation matrix is called the "index" of W), and a general form was given for spin models of index 2. Moreover, new spin models, called non-symmetric Hadamard models, were constructed. In the present paper, we classify certain spin models of index 2, including non-symmetric Hadamard models.

Keywords: spin model, association scheme, Hadamard matrix, Potts model

1. Introduction

The notion of spin model was introduced by Vaughan Jones [8] to construct invariants of knots and links. A spin model is essentially a square matrix W with nonzero entries which satisfies two conditions (type II and type III). Jones restricted his consideration to symmetric matrices. The notion of a spin model was generalized to the non-symmetric case by Kawagoe et al. [9], and it was further generalized by Bannai and Bannai [1].

Recently, François Jaeger and the author [7] proved that, for every spin model W, its transpose ${}^{\mathrm{t}}W$ is obtained from W by permutation of rows, and called the order of this permutation the "index" of W. Moreover, it was shown that every spin model of index 2 takes the following form:

$$W = \begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ -{}^{\mathrm{t}}B & {}^{\mathrm{t}}B & C & C \\ {}^{\mathrm{t}}B & -{}^{\mathrm{t}}B & C & C \end{pmatrix}, \tag{1}$$

where A, B, C are square matrices of equal sizes. Using this form, a new infinite class of spin models of index 2, called the non-symmetric Hadamard models, was constructed.

Two spin models are said to be equivalent when one is obtained from another by simultaneous permutation of rows and columns. It is clear that equivalent spin models give the same link invariant (see [8]). In the present paper, we classify up to equivalence spin models of index 2 when the matrix A is a Potts model (that is, when A has constant non-diagonal entries). See Sections 2 and 3 for terminology.

Theorem 1.1 *Let W be a spin model having the form* (1) *with A a Potts model. Then W is equivalent to at least one of the following spin models*:

(i) *Non-symmetric Hadamard model.*

(ii) *Tensor product of A with the following spin model*:

$$\begin{pmatrix} 1 & 1 & \eta & -\eta \\ 1 & 1 & -\eta & \eta \\ -\eta & \eta & 1 & 1 \\ \eta & -\eta & 1 & 1 \end{pmatrix}, \quad \textit{where } \eta^4 = -1.$$

(iii) *A spin model of size* 16, *having the form* (1) *with $A = C$ a Potts model*, *and*

$$B = \begin{pmatrix} r & r & ir^{-1} & -ir^{-1} \\ r & r & -ir^{-1} & ir^{-1} \\ ir^{-1} & -ir^{-1} & r & r \\ -ir^{-1} & ir^{-1} & r & r \end{pmatrix},$$

where r is a nonzero complex number, *and $i^2 = -1$.*

Remark 1.2 It is not difficult to verify that for all nonzero complex numbers r, any matrix of the form (1) with A, B, C as in Theorem 1.1(iii) is a spin model.

In Section 2, we review basic terminology for spin models and association schemes. In Section 3, we describe some known facts concerning non-symmetric spin models of index 2. The proof of Theorem 1.1 will be given in Section 4. In Section 5, a symmetric version of Theorem 1.1 is given.

2. Preliminaries

For more details concerning spin models and association schemes, the reader can refer to [6–8] and [2, 3].

Let X be a finite non-empty set. We denote by $Mat_X(\mathbf{C})$ the set of square matrices with complex entries whose rows and columns are indexed by X. For $W \in Mat_X(\mathbf{C})$ and x, $y \in X$, the (x, y)-entry of W is denoted by $W(x, y)$.

$W \in Mat_X(\mathbf{C})$ is said to be of *type II* if W has nonzero entries and satisfies the *type II condition*:

$$\sum_{x \in X} \frac{W(\alpha, x)}{W(\beta, x)} = |X|\delta_{\alpha,\beta} \quad \text{(for all } \alpha, \beta \in X). \tag{2}$$

Let W^- be defined by $W^-(x, y) = W(y, x)^{-1}$ $(x, y \in X)$. Then (2) can be written as $WW^- = |X|I$ (I denotes the identity matrix). Therefore any type II matrix W is non-singular with $W^{-1} = |X|^{-1}W^-$. This implies $(W^-)W = |X|I$. Hence if W is a type II matrix, then its transpose ${}^tW$ is also a type II matrix.

Let D denote one of the square roots of $|X|$. A *spin model* on X with *loop variable* D is a type II matrix $W \in Mat_X(\mathbf{C})$ which satisfies the *type III condition*:

$$\sum_{x\in X} \frac{W(\alpha,x)W(\beta,x)}{W(\gamma,x)} = D\frac{W(\alpha,\beta)}{W(\alpha,\gamma)W(\gamma,\beta)} \quad \text{(for all } \alpha,\beta,\gamma \in X). \tag{3}$$

It is known (see [9]) that, under the type II condition, (3) is equivalent to the following identity:

$$\sum_{x\in X} \frac{W(\gamma,x)}{W(\alpha,x)W(\beta,x)} = D\frac{W(\alpha,\gamma)W(\gamma,\beta)}{W(\alpha,\beta)} \quad \text{(for all } \alpha,\beta,\gamma \in X). \tag{4}$$

Setting $\beta = \gamma$ in (4),

$$\sum_{x\in X} \frac{1}{W(\alpha,x)} = DW(\beta,\beta). \tag{5}$$

Since β does not appear in the left-side of (5), the diagonal entry $W(\beta,\beta)$ is a constant (independent of the choice of β) which is called the *modulus* of W.

Observe that, for any spin models W_i on X_i with loop variables D_i ($i = 1, 2$), their tensor (Kronecker) product $W_1 \otimes W_2$ is a spin model with loop variable $D = D_1 D_2$.

For $W \in Mat_X(\mathbf{C})$ and for a permutation σ of X, let W^σ be defined by $W^\sigma(\alpha,\beta) = W(\sigma(\alpha),\sigma(\beta))$ for $\alpha, \beta \in X$. Observe that if W is a spin model, then W^σ is also a spin model. Two spin models W, W' are said to be *equivalent* if $W' = W^\sigma$ for some permutation σ of X.

A (*class d*) *association scheme* on X is a partition of $X \times X$ with nonempty relations R_0, $R_1, \ldots, R_d$, where $R_0 = \{(x,x) \mid x \in X\}$ which satisfy the following conditions:

(i) For every i in $\{0,1,\ldots,d\}$, there exists i' in $\{0,1,\ldots,d\}$ such that $R_{i'} = \{(y,x) \mid (x,y) \in R_i\}$.
(ii) There exist integers p_{ij}^k $(i,j,k \in \{0,1,\ldots,d\})$ such that for every $(x,y) \in R_k$, there are precisely p_{ij}^k elements z such that $(x,z) \in R_i$ and $(z,y) \in R_j$.
(iii) $p_{ij}^k = p_{ji}^k$ for every i, j in $\{0,1,\ldots,d\}$.

For $x \in X$, let $R_i(x)$ denote the set of y such that $(x,y) \in R_i$. Observe that $|R_i(x)| = p_{ii'}^0$ for all $x \in X$, so $|R_i(x)|$ is a constant which is independent of the choice of $x \in X$. We call $|R_i(x)|$ the *valency* of R_i.

In [5, 6, 11], it was shown that every spin model can be constructed on some association scheme. More precisely, let W be a spin model on X, then there exists an association scheme $R_0, R_1, \ldots, R_d$ on X and constants $t_0, t_1, \ldots, t_d$ such that $W(x,y) = t_i$ for all $(x,y) \in R_i$ $(i = 0, 1, \ldots, d)$.

3. Spin models of index 2

In the present section, we recall some results of [7] which we need in the proof of Theorem 1.1.

Let $W \in Mat_X(\mathbf{C})$ be a spin model. By [7] Proposition 2, ${}^{\mathrm{t}}WW^{-1}$ is a permutation matrix. So, there is a permutation σ of X such that ${}^{\mathrm{t}}W(x, y) = W(\sigma(x), y)$ for all $x, y \in X$. The order of σ is called the *index* of W. By [7] Proposition 7, when W has index 2, X can be ordered and split into 4 blocks Y_1, Y_2, Y_3, Y_4 of equal sizes, so that W takes the following form:

$$W = \begin{array}{c} \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} Y_1 & Y_2 & Y_3 & Y_4 \end{array} \\ \begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ -{}^{\mathrm{t}}B & {}^{\mathrm{t}}B & C & C \\ {}^{\mathrm{t}}B & -{}^{\mathrm{t}}B & C & C \end{pmatrix} \end{array} \quad \text{with } A, C \text{ symmetric.} \tag{6}$$

We may regard Y_i $(i = 1, 2, 3, 4)$ as copies of a set Y, and A, B, C as matrices in $Mat_Y(\mathbf{C})$.

As is easily verified, any spin model of the form (6) has index 2. In [4], the assertion of [7] Proposition 7 has been generalized to any index.

Now let W be any matrix of the form (6). By [7] Proposition 8, W is a spin model with loop variable $2D$, where $D^2 = |Y|$, if and only if the following (i), (ii) hold.

(i) A, C are spin models with loop variable D and B is a type II matrix.
(ii) The following identities hold for all α, β, γ in Y:

$$\sum_{y \in Y} \frac{A(\alpha, y)B(y, \beta)}{B(y, \gamma)} = D\frac{B(\alpha, \beta)}{C(\beta, \gamma)B(\alpha, \gamma)}, \tag{7}$$

$$\sum_{y \in Y} \frac{C(\alpha, y)B(\beta, y)}{B(\gamma, y)} = D\frac{B(\beta, \alpha)}{A(\beta, \gamma)B(\gamma, \alpha)}, \tag{8}$$

$$\sum_{y \in Y} \frac{B(y, \beta)B(y, \gamma)}{A(\alpha, y)} = -D\frac{C(\beta, \gamma)}{B(\alpha, \beta)B(\alpha, \gamma)}, \tag{9}$$

$$\sum_{y \in Y} \frac{B(\beta, y)B(\gamma, y)}{C(\alpha, y)} = -D\frac{A(\beta, \gamma)}{B(\beta, \alpha)B(\gamma, \alpha)}. \tag{10}$$

A *Potts model* takes the form $aI + b(J - I)$ for some constants a, b, where I denotes the identity and J the all 1's matrix. It is known (and not difficult to see) that $a = -u^3$, $b = u^{-1}$ for some complex number u satisfying $-u^2 - u^{-2} = D$, where D denotes the loop variable.

A non-symmetric *Hadamard model* takes the form (6) with $A = C$ a Potts model and $B = \eta H$, where H is a Hadamard matrix (i.e., a type II matrix with entries ± 1) and $\eta^4 = -1$.

4. Proof of Theorem 1.1

Lemma 4.1 *Let W be a spin model of the form* (6). *If W takes precisely two values on $(Y_1 \cup Y_2) \times (Y_3 \cup Y_4)$, then $B = \eta H$ for some Hadamard matrix H and for some η with $\eta^4 = -1$.*

Proof: Observe that B takes the values in $\{\eta, -\eta\}$ for some η. Hence $H := \eta^{-1} B$ has entries ± 1, so that H is a Hadamard matrix. Setting $\beta = \gamma$ in (9),

$$\sum_{y \in Y} \frac{\eta^2 H(y, \beta)^2}{A(\alpha, y)} = -D \frac{C(\beta, \beta)}{\eta^2 H(\alpha, \beta)^2},$$

and this becomes

$$(-\eta^4) \sum_{y \in Y} \frac{1}{A(\alpha, y)} = DC(\beta, \beta).$$

On the other hand, since A is a spin model, the following identity holds by (5):

$$\sum_{y \in Y} \frac{1}{A(\alpha, y)} = DA(\beta, \beta).$$

The above two identities together with $A(\beta, \beta) = C(\beta, \beta)$ (these are equal to the modulus of W) implies $-\eta^4 = 1$. $\square$

Lemma 4.2 *Let D, u be numbers such that $D^2 = |Y|$, $-u^2 - u^{-2} = D$. Let W be a matrix of the form* (6) *with $A = C = -u^3 I + u^{-1}(J - I)$. Then W is a spin model if and only if the following* (i), (ii) *hold*:

(i) *B is a type II matrix.*

(ii) *The following identities hold for all $\alpha, \beta, \gamma \in Y$:*

$$\sum_{y \in Y} B(y, \beta) B(y, \gamma) = (1 + u^{-4}) \left(B(\alpha, \beta) B(\alpha, \gamma) + \frac{uA(\beta, \gamma)}{B(\alpha, \beta) B(\alpha, \gamma)} \right), \tag{11}$$

$$\sum_{y \in Y} B(\beta, y) B(\gamma, y) = (1 + u^{-4}) \left(B(\beta, \alpha) B(\gamma, \alpha) + \frac{uA(\beta, \gamma)}{B(\beta, \alpha) B(\gamma, \alpha)} \right). \tag{12}$$

Proof: As recalled in Section 3, W is a spin model if and only if B is type II and Eqs. (7)–(10) hold. We begin by showing that Eqs. (7) and (8) necessarily hold. Since $A = C = -u^3 I + u^{-1}(J - I)$ and $D = -u^2 - u^{-2}$, the identity (7) becomes

$$(-u^3)\frac{B(\alpha,\beta)}{B(\alpha,\gamma)} + u^{-1}\sum_{y\in Y-\{\alpha\}}\frac{B(y,\beta)}{B(y,\gamma)} = \frac{(-u^2-u^{-2})}{A(\beta,\gamma)}\frac{B(\alpha,\beta)}{B(\alpha,\gamma)}.$$

Since B is a type II matrix, we have $\sum_{y\in Y} B(y,\beta)B(y,\gamma)^{-1} = |Y|\delta_{\beta,\gamma}$. So the above becomes

$$(-u^3 - u^{-1})\frac{B(\alpha,\beta)}{B(\alpha,\gamma)} + u^{-1}(-u^2-u^{-2})^2\delta_{\beta,\gamma} = \frac{(-u^2-u^{-2})}{A(\beta,\gamma)}\frac{B(\alpha,\beta)}{B(\alpha,\gamma)}.$$

Now verify this equation in each case of $\beta = \gamma$ and $\beta \neq \gamma$. The identity (8) can be verified in a similar way.

Next we show that (9) is equivalent to (11). The identity (9) becomes

$$-u^{-3}B(\alpha,\beta)B(\alpha,\gamma) + u\sum_{y\in Y-\{\alpha\}} B(y,\beta)B(y,\gamma) = (u^2+u^{-2})\frac{A(\beta,\gamma)}{B(\alpha,\beta)B(\alpha,\gamma)},$$

and this becomes

$$(-u^{-3}-u)B(\alpha,\beta)B(\alpha,\gamma) + u\sum_{y\in Y} B(y,\beta)B(y,\gamma) = (u^2+u^{-2})\frac{A(\beta,\gamma)}{B(\alpha,\beta)B(\alpha,\gamma)}.$$

This is equivalent to (11). In a similar way, we can see that (10) is equivalent to (12). □

Remark 4.3 Observe that the identity (12) is obtained from (11) by replacing B with ${}^{t}B$. Therefore if (11) implies some equation, then the equation also holds when B is replaced with ${}^{t}B$.

We fix the following notation for the rest of this section.

Let X be a finite set which is partitioned as $X = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$, where Y_i ($i = 1, 2, 3, 4$) are copies of a set Y with $|Y| = n \geq 2$. Fix complex numbers D and u such that

$$D^2 = n, \quad -u^2 - u^{-2} = D. \tag{13}$$

Let W be a spin model of the form (6) with A a Potts model:

$$A = -u^3 I + u^{-1}(J - I). \tag{14}$$

Lemma 4.4 $A = C$.

Proof: Set $X_1 = Y_1 \cup Y_2$, $X_2 = Y_3 \cup Y_4$, $S_1 = (X_1 \times X_1) \cup (X_2 \times X_2)$ and $S_2 = (X_1 \times X_2) \cup (X_2 \times X_1)$. Since W takes the form (6), $W(x, y) = W(y, x)$ for all $(x, y) \in S_1$, and $W(x, y) = -W(y, x)$ for all $(x, y) \in S_2$.

Let $R_0, R_1, \ldots, R_d$ be an association scheme such that $W(x, y) = t_i$ for all $(x, y) \in R_i$ $(i = 0, 1, \ldots, d)$. Observe that $t_i = W(x, y) = W(y, x) = t_{i'}$ for any $(x, y) \in R_i \cap S_1$, and $t_i = W(x, y) = -W(y, x) = -t_{i'}$ for any $(x, y) \in R_i \cap S_2$ $(i = 0, 1, \ldots, d)$. This means that each relation R_i is contained in S_1 or S_2, so that S_1 is partitioned into disjoint union of some R_i's: say $S_1 = R_0 \cup R_1 \cup \cdots \cup R_\ell$. Since A is given by (14) and since W takes the form (6), $t_0 = t_s = -u^3$ for some $s \in \{1, \ldots, \ell\}$, and $t_i = u^{-1}$ for all $i \in \{1, \ldots, \ell\} - \{s\}$. Observe that R_0 and R_s have valency 1.

Now pick any $x \in X_2$, and observe that, for each $y \in X_2$, $W(x, y) = t_i$ for some $i \in \{0, 1, \ldots, \ell\}$. So, when y runs over X_2, $W(x, y)$ takes twice the value $-u^{-3}$ and $2n - 2$ times the value u^{-1}. This implies $A = C$. □

Lemma 4.5 *Let $B' \in Mat_Y(\mathbf{C})$ be obtained from B by permutation of columns (or rows). Let W' be the matrix of the form (6) with B replaced by B'. Then W' is a spin model which is equivalent to W.*

Proof: There is a permutation π of Y such that $B'(x, y) = B(x, \pi(y))$ for all $x, y \in Y$. Let σ be a permutation of X such that $\sigma(y) = y$ for $y \in Y_1 \cup Y_2$ and $\sigma(y) = \pi(y)$ for $y \in Y_3 \cup Y_4$. Since A $(=C)$ has the form (14), A is invariant under the action of σ. Moreover permutation of colums of B corresponds to permutation of rows of ${}^tB$. Now we can see that $W^\sigma = W'$, so that W' is a spin model which is equivalent to W. □

Lemma 4.6 *Let B' be obtained by changing signs of each entry of a column (or a row) of B. Let W' be the matrix of the form (6) with B replaced by B'. Then W' is a spin model which is equivalent to W.*

Proof: There is $\beta \in Y$ such that $B'(x, \beta) = -B(x, \beta)$ and $B'(x, y) = B(x, y)$ for all $x \in Y$ and for all $y \in Y - \{\beta\}$. Let σ be a permutation of X which fixes all elements of X but which exchages β in Y_3 with β in Y_4. Since W takes the form (6), C is invariant under σ. Hence $W^\sigma = W'$. □

Remark 4.7 Observe that we used only (6) in the proof of Lemma 4.6. Therefore Lemma 4.6 can be applied to any spin model of index 2.

By Lemmas 4.5 and 4.6, we may freely permute the columns (or the rows) of B, and we may change signs of any column (or row).

Lemma 4.8

(i) *For all $\alpha, \beta \in Y$,*

$$\sum_{y \in Y} B(y, \beta)^2 = (1 + u^{-4}) \left(B(\alpha, \beta)^2 - \frac{u^4}{B(\alpha, \beta)^2} \right). \tag{15}$$

(ii) *For all $\alpha, \beta, \gamma \in Y$ with $\beta \neq \gamma$,*

$$\sum_{y \in Y} B(y, \beta) B(y, \gamma) = (1 + u^{-4}) \left(B(\alpha, \beta) B(\alpha, \gamma) + \frac{1}{B(\alpha, \beta) B(\alpha, \gamma)} \right). \tag{16}$$

Proof: Immediate from (11). □

In the following, let $T \in Mat_Y(\mathbf{C})$ denote the entrywise-product square of B:

$$T(x, y) = B(x, y)^2 \quad (x, y \in Y).$$

Lemma 4.9 *Let t be an entry of T. Then every entry of T is contained in $\{t, -u^4t^{-1}\}$.*

Proof: Since α does not appear in the left-side of (15), the right-side does not depend on the choice of α. Pick any $\alpha, \alpha', \beta, \beta' \in Y$, and set $t = T(\alpha, \beta), t' = T(\alpha', \beta), t'' = T(\alpha', \beta')$. Then $t - u^4t^{-1} = t' - u^4t'^{-1}$, and this becomes $(t - t')(tt' + u^4) = 0$, so $t' \in \{t, -u^4t^{-1}\}$. Using (15) for ${}^{t}B$, we can conclude that $t'' \in \{t', -u^4t'^{-1}\}$. Hence $t'' \in \{t, -u^4t^{-1}\}$. □

Lemma 4.10 *Suppose $t = -u^4t^{-1}$ for some entry t of T. Then W is a non-symmetric Hadamard model.*

Proof: Obtained from Lemmas 4.9 and 4.1. □

In the rest of this section, we assume that, for all entries t of T,

$$t \neq -u^4t^{-1}, \tag{17}$$

and both t and $-u^4t^{-1}$ appear in T.

Lemma 4.11 *Let t be an entry of T. Suppose that t (respectively $-u^4t^{-1}$) appears m times (respectively m' times) in a column or row of T. Then*

$$m = \frac{(1 + u^{-4})(t^2 + u^8)}{t^2 + u^4}, \tag{18}$$

$$m' = \frac{(1 + u^4)(t^2 + 1)}{t^2 + u^4}. \tag{19}$$

Proof: From Lemma 4.9 and (15),

$$mt + m'(-u^4t^{-1}) = (1 + u^{-4})(t - u^4t^{-1}).$$

Use $m + m' = n = (-u^2 - u^{-2})^2$ to get m and m'. □

Lemma 4.12 *Let $\alpha, \alpha', \beta, \beta' \in Y$ such that $\alpha \neq \alpha'$, $\beta \neq \beta'$. Set $p = B(\alpha, \beta)$, $q = B(\alpha, \beta')$, $r = B(\alpha', \beta)$, $s = B(\alpha', \beta')$. Then at least one of the following* (i), (ii) *holds.*

(i) $pqrs = 1$,

(ii) $pq = rs$, $pr = qs$, $p^2 = s^2$, *and* $q^2 = r^2$.

Proof: Since α does not appear in the left-side of (16), the right-side of (16) does not depend on the choice of α. So

$$B(\alpha,\beta)B(\alpha,\beta')+\frac{1}{B(\alpha,\beta)B(\alpha,\beta')}=B(\alpha',\beta)B(\alpha',\beta')+\frac{1}{B(\alpha',\beta)B(\alpha',\beta')},$$

and this implies

$$(B(\alpha,\beta)B(\alpha,\beta')-B(\alpha',\beta)B(\alpha',\beta'))(B(\alpha,\beta)B(\alpha,\beta')B(\alpha',\beta)B(\alpha',\beta')-1)=0.$$

Hence either $pq=rs$ or $pqrs=1$ holds. Applying (16) for ${}^{t}B$, we get $pr=qs$ or $prqs=1$. Hence if $pqrs\neq 1$, then both $pq=rs$ and $pr=qs$ hold, and these imply $p^2=s^2$ and $q^2=r^2$. □

Lemma 4.13 *Suppose $n=2$. Then, up to permutation and sign change, $B=\eta A$ for some η with $\eta^4=-1$.*

Proof: We have $u^8=-1$ by our assumption $2=n=(-u^2-u^{-2})^2$. Set $T(1,1)=t$. Since both t and $t'=-u^4t^{-1}$ appear in T, and since t appears m times in each column (or row) of T by Lemma 4.11 (where m is given by (18)), T takes the following form:

$$T=\begin{pmatrix} t & -u^4t^{-1}\\ -u^4t^{-1} & t\end{pmatrix}.$$

From (15), we have $t+(-u^4t^{-1})=(1+u^{-4})(t-u^4t^{-1})$. This implies $t^2=u^4$, and so $t=\pm u^2$. If $t=-u^2$, then $t'=-u^4(-u^2)^{-1}=u^2$. So we may assume $t=u^2$ by permuting columns of T if necessary. Hence

$$T=\begin{pmatrix} u^2 & u^{-6}\\ u^{-6} & u^2\end{pmatrix}.$$

This implies $B(1,1)=\pm u$, $B(1,2)=\pm u^{-3}$, $B(2,1)=\pm u^{-3}$, and $B(2,2)=\pm u$. By changing signs of columns if necessary, we may assume that $B(1,1)=u$ and $B(1,2)=-u^{-3}$. By changing signs of the second row if necessary, we may assume that $B(2,2)=u$. Using $u^8=-1$ and Lemma 4.12, we get $B(2,1)=-u^{-3}$. Thus

$$B=\begin{pmatrix} u & -u^{-3}\\ -u^{-3} & u\end{pmatrix}.$$

Setting $\eta=-u^{-2}$, we get $B=\eta A$. □

We assume $n\geq 3$ in the rest of this section.

By a 2-*block* we mean a 2×2 submatrix of T.

Lemma 4.14 *Let t be an entry of T. Suppose that some 2-block contains three t's and one $t' = -u^4t^{-1}$. Then, up to permutation,*

$$T = iu^6 I + iu^{-2}(J - I) \qquad (i^2 = -1).$$

Proof: Let p, q, r, s be the entries of B which appear in the same position of the 2-block. We apply Lemma 4.12 for these p, q, r, s. We may assume $p^2 = q^2 = r^2 = t$ and $s^2 = t'$. Since $p^2 \neq s^2$, we have $pqrs = 1$ by Lemma 4.12. Hence $1 = p^2q^2r^2s^2 = t^3(-u^4t^{-1})$, so that $t^2 = -u^{-4}$, and so $t = iu^{-2}$ for some i with $i^2 = -1$. Thus we get $t' = -u^4t^{-1} = iu^6$.

By (19), the multiplicity m' of t' in a column of T is given by

$$m' = \frac{(1 + u^4)(-u^{-4} + 1)}{(-u^{-4} + u^4)} = 1.$$

Hence $t' = iu^6$ appears precisely once in each column of T, and all the other entries are equal to $t = iu^{-2}$. Now permute columns so that each t' comes to the diagonal position. □

Lemma 4.15 *Let t be an entry of T. Suppose some 2-block contains three t's and one $t' = -u^4t^{-1}$. Then, up to permutation and sign change,*

$$B = -\eta u^3 I + \eta u^{-1}(J - I) = \eta A,$$

for some η with $\eta^4 = -1$.

Proof: By Lemma 4.14, the diagonal entries of B are $\pm\eta u^3$ for some η with $\eta^2 = i$. By changing signs of columns if necessary, we may assume that $B(1, 1) = -\eta u^3$ and $B(1, y) = \eta u^{-1}$ for all $y \in Y - \{1\}$. By changing signs of rows if necessary, we may assume that $B(y, y) = -\eta u^3$ for all $y \in Y$.

Now pick $\alpha, \beta \in Y$ with $1 \neq \alpha \neq \beta$, and set $B(\alpha, \beta) = \epsilon\eta u^{-1}$ ($\epsilon = \pm 1$). It is enough to show that $\epsilon = 1$. Observe that $u^8 \neq 1$, since $iu^6 \neq iu^{-2}$ by our assumption (17).

First we consider the case $\beta = 1$. We have $B(1, 1) = B(\alpha, \alpha) = -\eta u^3$ and $B(1, \alpha) = \eta u^{-1}$. If $B(1, 1)B(1, \alpha)B(\alpha, 1)B(\alpha, \alpha) = 1$, then we get

$$1 = (-\eta u^3)(\eta u^{-1})(\epsilon\eta u^{-1})(-\eta u^3) = \epsilon\eta^4 u^4 = -\epsilon u^4,$$

and this implies $u^8 = 1$, a contradiction. Hence, by Lemma 4.12, we must have $B(1, 1) B(1, \alpha) = B(\alpha, 1)B(\alpha, \alpha)$, and this implies $\epsilon = 1$.

Next we consider the case $\beta \neq 1$. We have $B(1, \alpha) = B(1, \beta) = \eta u^{-1}$ and $B(\alpha, \alpha) = -\eta u^3$. If $B(1, \alpha)B(1, \beta) = B(\alpha, \alpha)B(\alpha, \beta)$, then we get $(\eta u^{-1})^2 = (-\eta u^3)(\epsilon\eta u^{-1})$, and this implies $u^8 = 1$, a contradiction. Hence, by Lemma 4.12, we must have

$$1 = B(1, \alpha)B(1, \beta)B(\alpha, \alpha)B(\alpha, \beta) = (\eta u^{-1})^2(-\eta u^3)(\epsilon\eta u^{-1}),$$

and this implies $\epsilon = 1$. □

Lemma 4.16 *Let t be an entry of T. Suppose that every 2-block contains an even number of t's. Then $n = 4$, $u^4 = 1$, and T takes the following form up to permutation*:

$$T = \begin{pmatrix} t & t & -t^{-1} & -t^{-1} \\ t & t & -t^{-1} & -t^{-1} \\ -t^{-1} & -t^{-1} & t & t \\ -t^{-1} & -t^{-1} & t & t \end{pmatrix}.$$

Proof: We may assume $T(1,1) = T(2,1) = t$ from our assumption $n \geq 3$. Since every row of T contains $t' = -u^4 t^{-1}$, we may assume $T(1,2) = t'$. Since every block contains an even number of t', we have $T(2,2) = t'$. Since $t \neq t'$, we get $t^2 t'^2 = B(1,1)B(1,2)B(2,1)B(2,2) = 1$ by Lemma 4.12. Hence $u^8 = 1$, so that $u^4 = \pm 1$. Since $n = (-u^2 - u^{-2})^2 = u^4 + u^{-4} + 2$, we must have $u^4 = 1$ and $n = 4$. We also have $t' = -t^{-1}$. From (18), t appears precisely

$$m = \frac{(1 + u^{-4})(t^2 + u^8)}{t^2 + u^4} = \frac{(1+1)(t^2+1)}{t^2+1} = 2$$

times in every column (or row) of T. Now it is clear that T takes the above form after permuting columns (or rows). □

Lemma 4.17 *Let t be an entry of T. Suppose that every 2-block contains even number of t. Then B takes the following form up to permutation and sign change*:

$$B = \begin{pmatrix} r & r & ir^{-1} & -ir^{-1} \\ r & r & -ir^{-1} & ir^{-1} \\ ir^{-1} & -ir^{-1} & r & r \\ -ir^{-1} & ir^{-1} & r & r \end{pmatrix},$$

where $i^2 = -1$, and r is a nonzero complex number.

Proof: We may assume that T takes the form given in Lemma 4.16. Set $B(1,1) = r$. Then the entries of B corresponding to t in T are contained in $\{r, -r\}$, and the entries of B corresponding to $-t^{-1}$ in T are contained in $\{r', -r'\}$, where $r' = ir^{-1}$ $(i^2 = -1)$. By changing signs of columns (or rows) if necessary, we may assume that $B(1,2) = r$, $B(1,3) = r'$, $B(1,4) = -r'$, $B(2,1) = r$, $B(3,1) = r'$, $B(4,1) = -r'$.

If $B(2,2) = -r$, then we get $1 = B(1,1)B(1,2)B(2,1)B(2,2) = -r^4$, so that $t = r^2 = -r^{-2} = -t^{-1}$, contradicting $t \neq t'$. Hence $B(2,2) = r$. Using the type II condition (for ${}^{\mathrm{t}}B$): $\sum_{y=1}^{4} B(y,1)B(y,2)^{-1} = 0$, we get $B(3,2) = -r'$ and $B(4,2) = r'$. We get also $B(2,3) = -r'$ and $B(2,4) = r'$ by type II condition.

By the type II condition (2), $\sum_{y=1}^{4} B(y,3)B(y,4)^{-1} = 0$, so $B(3,3) = B(3,4)$ and $B(4,3) = B(4,4)$. Also by the type II condition, $\sum_{y=1}^{4} B(3,y)B(4,y)^{-1} = 0$, so $B(3,3) = B(4,3)$. Hence $B(3,3) = B(3,4) = B(4,3) = B(4,4)$. When $B(3,3) = -r$, change signs of the third row and the fourth row, and permute these two rows. □

This completes the proof of Theorem 1.1.

5. Symmetric version

By [7] Proposition 9, we have the following correspondence between non-symmetric spin models of index 2 and some symmetric spin models.

Lemma 5.1 *Let A, B, C be three matrices in $Mat_Y(\mathbf{C})$ with A, C symmetric, and let η, D be numbers such that $\eta^4 = -1$, $D^2 = |Y|$. Then*

$$\begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ {}^{\mathrm{t}}B & -{}^{\mathrm{t}}B & C & C \\ -{}^{\mathrm{t}}B & {}^{\mathrm{t}}B & C & C \end{pmatrix} \tag{20}$$

is a (symmetric) spin model with loop variable $2D$ if and only if

$$\begin{pmatrix} A & A & \eta B & -\eta B \\ A & A & -\eta B & \eta B \\ -\eta{}^{\mathrm{t}}B & \eta{}^{\mathrm{t}}B & C & C \\ \eta{}^{\mathrm{t}}B & -\eta{}^{\mathrm{t}}B & C & C \end{pmatrix}$$

is a (non-symmetric) spin model with loop variable $2D$.

In [10], the author constructed symmetric Hadamard models, which takes the form (20), where $A = C$ is a Potts model and $B = \omega H$ for some Hadamard matrix H and for some ω with $\omega^4 = 1$.

From Theorem 1.1 and Lemma 5.1, we obtain:

Corollary 5.2 *Let W be a spin model having the form* (20) *with A a Potts model, and C symmetric. Then W is equivalent to at least one of the following spin models:*

(i) *Symmetric Hadamard model.*

(ii) *Tensor product of A with the following spin model:*

$$\begin{pmatrix} 1 & 1 & \omega & -\omega \\ 1 & 1 & -\omega & \omega \\ \omega & -\omega & 1 & 1 \\ -\omega & \omega & 1 & 1 \end{pmatrix}, \quad \text{where } \omega^4 = 1.$$

(iii) *A spin model of size* 16, *having the form* (20) *with* $A = C$ *a Potts model*, *and*

$$B = \begin{pmatrix} r & r & r^{-1} & -r^{-1} \\ r & r & -r^{-1} & r^{-1} \\ r^{-1} & -r^{-1} & r & r \\ -r^{-1} & r^{-1} & r & r \end{pmatrix},$$

where r *is a nonzero complex number.*

Remark 5.3 It is not difficult to verify that for all nonzero complex numbers r, any matrix of the form (20) with A, B, C as in Corollary 5.2(iii) is a spin model.

6. Conclusion

Theorem 1.1 will be useful in the classification of non-symmetric spin models on a d-class association scheme with small d. It will be of great interest to classify spin models of the form (6) when B is a scalar multiple of an Hadamard matrix. Another interesting case is when A is constructed on some association scheme of class $d = 3$.

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