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Modified Stirling Numbers and p-Divisibility in the Universal Typical p^k-Series

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Abstract. The 1-dimensional universal formal group law is a power series (in two variables and with coefficients in Lazard's ring) carrying a lot of geometrical and algebraic properties. For a prime p, we study the corresponding "p-localized" formal group law through its associated p^k-series, $[p^k](x) = \sum_{s\geq 0} a_{k,s}x^{s(p-1)+1}$—the p^k-fold iterated formal sum of a variable x. The coefficients $a_{k,s}$ lie in the Brown-Peterson ring $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ and we describe part of their structure as polynomials in the variables v_i with p-local coefficients. This is achieved by introducing a family of filtrations $\{W_\varphi\}_{\varphi\geq 1}$ in BP_* and studying the value of $a_{k,s}$ in each of the associated (bi)graded rings BP_*/W_φ. This allows us to identify, among monomials in $a_{k,s}$ of minimal W_φ-filtration ($1 \leq \varphi \leq k$), an explicit monomial $m_{\varphi,k,s}$ carrying the lowest possible p-divisibility. The p-local coefficient of $m_{\varphi,k,s}$ is described as a Stirling-type number of the second kind and its actual value is computed up to p-local units. It turns out that $m_{k,k,s}$ not only carries the lowest W_k-filtration but, more importantly, the lowest p-divisibility among *all* other monomials in $a_{k,s}$. In particular, we obtain a complete description of the p-divisibility properties of each $a_{k,s}$.

Keywords: formal group laws, universal typical p^k-series, Stirling numbers

1. Introduction

The theory of formal group laws has shown to be of special importance in mathematics mainly due to the wide variety of connections it has had with other mathematical branches like geometry, algebraic topology, number theory and combinatorics. One of the basic connections arises through the universal example. For the purposes of this paper we localize at a given prime p. Let BP stand for the p-local Brown-Peterson spectrum with homotopy groups $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ (say Araki generators $v_i \in BP_{2(p^i-1)}$) and let $\mu_p = \mu_p(x, y) \in BP^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = BP^*[[x, y]]$ denote the corresponding Euler class for the tensor product of the canonical complex line bundles over the axes of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$. A fundamental theorem of Quillen claims that μ_p is both algebraically and topologically universal (see [2, Theorem 4.6]). This bridge has led to a number of basic developments. For instance, Hopkins-Miller and Hopkins-Mahowald have used a partial converse of Quillen's theorem in constructing higher K-theories related to elliptic curves (see [17] and [34]). One of their resulting spectra has been used by Bruner, Davis and Mahowald [4] in obtaining sharper information for the elusive problem of finding optimal Euclidean immersions for real projective spaces.

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In this paper we concentrate on the algebro-combinatorial properties of μ_p. We follow the idea originally introduced by Johnson in [18] (as modified in [11, 12]) to make an indirect study of μ_p through its associated n-series; that is, the formal power series $[n](x)$ inductively defined by $[n+1](x) = \mu_p([n](x), x)$, with $[0](x) = 0$. Since BP is p-local, the n-series carries the same information as the p^k-series, where $k = \nu(n)$ is the highest power of p dividing n. We will focus on the latter series. By sparseness it takes the form

$$[p^k](x) = \sum_{s \geq 0} a_{k,s} x^{\bar{s}}, \tag{1}$$

where $\bar{s}$ will stand for $s(p-1)+1$ and where $a_{k,s} \in BP_{2s(p-1)}$ is the s-th (nontrivial) coefficient. Thus each $a_{k,s}$ is a polynomial

$$a_{k,s} = \sum c_{k,s,I} v^I \tag{2}$$

with p-local coefficients $c_{k,s,I}$. The summation is over sequences $I = (i_1, i_2, \ldots)$ of non-negative integers, almost all zero, where v^I stands for the monomial $v_1^{i_1} v_2^{i_2} \ldots$.

Remark 1.1 We prove in [12] that each coefficient $c_{k,s,I}$ (and therefore $a_{k,s}$ itself) is divisible by $p^{\mu_{k,s}}$, where

$$\mu_{k,s} = k s_{\langle 0 \rangle} + (k-1) s_{\langle 1 \rangle} + \cdots + s_{\langle k-1 \rangle}. \tag{3}$$

Here and in what follows, for a non-negative integer s we write $s_{\langle i \rangle}$ for the ith coefficient in the p-adic decomposition of $\bar{s}$ and set

$$\alpha_s = s_{\langle 0 \rangle} + s_{\langle 1 \rangle} + \cdots. \tag{4}$$

As an immediate consequence of Theorem 1.2 below, we see that $p^{\mu_{k,s}+1}$ does not divide $a_{k,s}$, that is, $\mu_{k,s}$ is in fact the highest power of p dividing $a_{k,s}$.

Theorem 1.2 *Let $\mu_{k,s}$ be defined as in* (3) *and let $\lambda_{k,s}$ be defined by*

$$(p-1)\lambda_{k,s} = -1 + s_{\langle 0 \rangle} + p s_{\langle 1 \rangle} + \cdots + p^k s_{\langle k \rangle} + p^{k-1}\big(s_{\langle k+1 \rangle} + s_{\langle k+2 \rangle} + \cdots\big). \tag{5}$$

Then, up to p-local units, the monomial $v^{I_{k,s}}$ shows up in (2) *with coefficient $p^{\mu_{k,s}}$. Here $I_{k,s} = (\lambda_{k,s}, i_{2,k,s}, i_{3,k,s}, \ldots)$, where $i_{j,k,s} = p^{k-1} s_{\langle j+k-1 \rangle}$, for $j \geq 2$.*

Remark 1.3 In our notation $1 \equiv \bar{s} \equiv \alpha_s$ modulo $p-1$, so that $\lambda_{k,s}$ above is indeed an integer. We observe that $a_{k,s}$ is never divisible by $v_j^{p^k}$ (in fact by v_j^p in view of the case $\varphi = 1$ in Theorem 1.4 below) if $j \geq 2$; however, any large power of v_1 divides $v^{I_{k,s}}$ for a suitable value of s.

The monomial in $a_{k,s}$ described by Theorem 1.2 is in fact part of a general pattern: our main result, Theorem 1.4 below, generalizes Theorem 1.2 by identifying k monomials in each $a_{k,s}$ (which may not be all different, for instance for small values of s; however, in the typical situation, Theorem 1.4 does detect k different monomials). The following notation helps to describe the new monomials. For $s \geq 0$ and $1 \leq \varphi \leq k$ set

$$\mu_{\varphi,k,s} = ks_{\langle 0\rangle} + (k-1)s_{\langle 1\rangle} + \cdots + (k-\varphi+1)s_{\langle \varphi-1\rangle} + (k-\varphi)\big(s_{\langle\varphi\rangle} + s_{\langle\varphi+1\rangle} + \cdots\big), \tag{6}$$

$$(p-1)\lambda_{\varphi,s} = -1 + s_{\langle 0\rangle} + ps_{\langle 1\rangle} + \cdots + p^{\varphi}s_{\langle\varphi\rangle} + p^{\varphi-1}\big(s_{\langle\varphi+1\rangle} + s_{\langle\varphi+2\rangle} + \cdots\big) \tag{7}$$

and

$$I_{\varphi,s} = (\lambda_{\varphi,s}, i_{2,\varphi,s}, i_{3,\varphi,s}, \ldots), \tag{8}$$

where $i_{j,\varphi,s} = p^{\varphi-1}s_{\langle j+\varphi-1\rangle}$ for $j \geq 2$. Note that for $\varphi = k$ these definitions extend the corresponding ones in (3) and in Theorem 1.2.

Theorem 1.4 *For $s \geq 0$, $1 \leq \varphi \leq k$ and up to p-local units, the monomial $v^{I_{\varphi,s}}$ shows up in* (2) *with coefficient $p^{\mu_{\varphi,k,s}}$.*

It is to be observed that, as suggested by the notation, the monomial $v^{I_{\varphi,s}}$ is independent of k and, therefore, shows up in every $a_{k,s}$ with $k \geq \varphi$. Moreover, while Theorem 1.4 claims that the coefficient of this monomial is of the form $c_{\varphi,k,s}\,p^{\mu_{\varphi,k,s}}$ in $a_{k,s}$, with $c_{\varphi,k,s}$ a p-local unit, Proposition 3.5 below computes the actual mod-p value of $c_{\varphi,k,s}$ which, in particular, turns out to be independent of φ and k.

As for the methods, the proof of Theorem 1.4 requires using suitable "weight" filtrations W_φ $(1 \leq \varphi \leq k)$ in BP_* (Section 2) which are slight variations of the usual filtration by powers of the invariant prime ideal $(p, v_1, v_2, \ldots)$. In more detail, the formal logarithm for μ_p yields an inductive formula for the coefficients of the p^k-series. In the bigraded object BP_*/W_φ associated to W_φ, this gives an expression for $a_{k,s}$ in terms of "highly" p-divisible terms together with the monomial $c_{\varphi,k,s}\,p^{\mu_{\varphi,k,s}}v^{I_{\varphi,s}}$ in Theorem 1.4 (Section 3). Up to this point the methods are purely algebraic. The combinatorics arise in describing the p-local unit $c_{\varphi,k,s}$. This requires a determination of the mod p values of the following combinatorial function (Section 4).

Definition 1.5 For $s \geq 0$ let $\phi(s)$ be the number of ways in which $\bar{s}$ distinct objects can be partitioned into p unlabeled subsets each having size congruent with 1 modulo $p-1$.

Remark 1.6 The function ϕ is a modified version of the usual definition for Stirling numbers of the second kind as the number of equivalence relations defined on a set: we are imposing an extra condition on the size of each class. For instance $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(2) = \binom{2p-1}{p}$. Note that the last two agree modulo p. This is a general fact which will be essential for our work.

Proposition 1.7 *For $s \geq 1$, $\phi(s) \equiv 1 \pmod{p}$.*

This result and Proposition 3.5 solve the two main combinatorial problems left open in [19]. Up to the author's knowledge, Proposition 1.7 has not appeared in the literature before.[1] The author's original proof used a rather involved induction argument which had the advantage of relating Propositions 1.7 and 3.5 to the same combinatorial phenomenon. Later on, Ira Gessel and Martin Klazar independently suggested a proof of Proposition 1.7 based on a direct analysis with the exponential generating function of ϕ. The simple and elegant proof presented here was suggested by one of the reviewers assigned to the original version of this work.

In an appendix we have briefly addressed two points: (potential) applications and (possible) extensions for the theory of formal groups and, in particular, the results in this paper.

2. The weight filtrations

Let $\log(x) = \sum_{s\geq 0} m_s x^{p^s} \in (BP^* \otimes \mathbb{Q})[[x]]$ be the formal logarithm for the universal p-typical formal group μ_p [16]. By expanding both sides of the relation $\log([p^k](x)) = p^k \log(x)$ and equating coefficients we get the inductive relation

$$-a_{k,s} = \sum m_i \binom{p^i}{U} a_k^U - p^k \delta_s \tag{9}$$

where $\delta_s = m_n$ if $\bar{s} = p^n$, and $\delta_s = 0$ otherwise. The sum is taken over $i \geq 1$ and over sequences $U = (u_0, u_1, \ldots)$ of non-negative integers satisfying the two conditions

$$p^i = u_0 + u_1 + u_2 + \cdots, \tag{10}$$

$$\bar{s} = \bar{0}u_0 + \bar{1}u_1 + \bar{2}u_2 + \cdots. \tag{11}$$

We use the short hand a_k^U for the product $a_{k,0}^{u_0} a_{k,1}^{u_1} \cdots$, and $\binom{p^i}{U} = \binom{p^i}{u_0, u_1, \ldots}$ stands for the multinomial coefficient. We distill information from (9) through the following family of filtrations in $BP_* \otimes \mathbb{Q}$.

Definition 2.1 Fix a positive integer φ and let $\nu : \mathbb{Q} \to \mathbb{Z}$ be the usual p-valuation; that is, $\nu(q)$ stands for the highest power of p "dividing" a given rational number q.

(a) The φ-weight of a monomial $qv_1^{\ell_1} v_2^{\ell_2} \cdots v_n^{\ell_n} \in BP_* \otimes \mathbb{Q}$ is $\omega_\varphi(qv_1^{\ell_1} v_2^{\ell_2} \cdots v_n^{\ell_n}) = p^{\varphi-1}\nu(q) + \sum_{i=1}^n \ell_i$. More generally, the φ-weight of an element $v \in BP_* \otimes \mathbb{Q}$, denoted by $\omega_\varphi(v)$, is defined as the smallest of the φ-weights of monomials in v. We agree to set $\omega_\varphi(0) = \infty$.

(b) The φ-weight filtration $W_\varphi = \{W_{\varphi,j}\}_{j\in\mathbb{Z}}$ in $BP_* \otimes \mathbb{Q}$ is defined by $W_{\varphi,j} = \{v \in BP_* \otimes \mathbb{Q} : \omega_\varphi(v) \geq j\}$.

For $n \geq 0$, $\omega_\varphi(m_n)$ can be inductively computed from the formula $pm_n = \sum_{i=0}^n m_i v_{n-i}^{p^i}$ [31, A.2.2.2]. The properties we need are summarized in the following result. For $p = 2$ the proof is given in [12] and this immediately generalizes to $p > 2$. We omit the details.

Proposition 2.2 *For a positive integer i let* $g(i) = (p^i - 1)/(p - 1)$.

(a) $\omega_\varphi(uv) = \omega_\varphi(u) + \omega_\varphi(v)$, *for* $u, v \in BP_* \otimes \mathbb{Q}$. *In particular, the* φ*-weight filtration is a multiplicative decreasing filtration in* $BP_* \otimes \mathbb{Q}$.

(b) $\omega_\varphi(m_n) = g(n) - np^{\varphi-1}$, *for* $0 \le n \le \varphi - 1$.
(c) $\omega_\varphi(m_n) \ge g(\varphi - 1) - (\varphi - 1)p^{\varphi-1}$, *for* $n \ge \varphi - 1$.

The following is a more explicit statement of Theorem 1.4. The proof is the crux of this paper.

Theorem 2.3 *For* $s \ge 0$ *and* $1 \le \varphi \le k$ *let* $\mu_{\varphi,k,s}$, $\lambda_{\varphi,s}$ *and* $I_{\varphi,s}$ *be as defined in* (6)–(8). *Then the* φ*-weight filtration of* $a_{k,s}$, $\omega_{\varphi,k,s}$ *for short, is given by*

$$\omega_{\varphi,k,s} = p^{\varphi-1}\mu_{\varphi,k,s} + \lambda_{\varphi,s} + p^{\varphi-1}\big(s_{\langle\varphi+1\rangle} + s_{\langle\varphi+2\rangle} + \cdots\big). \tag{12}$$

Furthermore, $a_{k,s} \equiv c_s p^{\mu_{\varphi,k,s}} v^{I_{\varphi,s}}$ *modulo* $(p^{\mu_{\varphi,k,s}+1}) \cap W_{\varphi,\omega_{\varphi,k,s}} + W_{\varphi,\omega_{\varphi,k,s}+1}$, *where* c_s *is a* p*-local unit.*

Thus, the monomial $c_s p^{\mu_{\varphi,k,s}} v^{I_{\varphi,s}}$ above captures both the the p-divisibility and the φ-weight of $a_{k,s}$ in the sense that any other monomial in (2) either has a larger φ-weight or a larger p-divisibility. Note by the way that only the mod-p value of c_s is relevant here. It will be described in Proposition 3.5.

The following alternative expression for $\omega_{\varphi,k,s}$ will be useful in the course of proving Theorem 2.3.

Lemma 2.4 *Set* $A_{\varphi,s} = \sum_{j=0}^{\varphi-1}(p^{\varphi-1}(p-1)(\varphi-j) - p^\varphi + p^j)s_{\langle j\rangle}$ *and* $d_{\varphi,k} = p^\varphi + p^{\varphi-1}(p-1)(k-\varphi)$, *then* $(p-1)\omega_{\varphi,k,s} = -1 + d_{\varphi,k}\alpha_s + A_{\varphi,s}$.

Proof: From (12), (6), (7) and (4) (in that order) we get

$$\begin{aligned}
&(p-1)\omega_{\varphi,k,s}\\
&\quad= (p-1)p^{\varphi-1}\mu_{\varphi,k,s} + (p-1)\lambda_{\varphi,s} + (p-1)p^{\varphi-1}\sum_{j\ge\varphi+1} s_{\langle j\rangle}\\
&\quad= (p-1)p^{\varphi-1}\left(\sum_{j=0}^{\varphi-1}(k-j)s_{\langle j\rangle} + (k-\varphi)\sum_{j\ge\varphi} s_{\langle j\rangle}\right)\\
&\qquad - 1 + \sum_{j=0}^{\varphi} p^j s_{\langle j\rangle} + p^{\varphi-1}\sum_{j\ge\varphi+1} s_{\langle j\rangle} + (p-1)p^{\varphi-1}\sum_{j\ge\varphi+1} s_{\langle j\rangle}\\
&\quad= (p-1)p^{\varphi-1}\left(\sum_{j=0}^{\varphi-1}(k-j)s_{\langle j\rangle} + (k-\varphi)\sum_{j\ge\varphi} s_{\langle j\rangle}\right) - 1 + \sum_{j=0}^{\varphi-1} p^j s_{\langle j\rangle} + p^\varphi \sum_{j\ge\varphi} s_{\langle j\rangle}\\
&\quad= -1 + (p^{\varphi-1}(p-1)(k-\varphi) + p^\varphi)\alpha_s\\
&\qquad + \sum_{j=0}^{\varphi-1}(-p^{\varphi-1}(p-1)(k-\varphi) - p^\varphi + (p-1)p^{\varphi-1}(k-j) + p^j)s_{\langle j\rangle}. \qquad \square
\end{aligned}$$

We close the section by recalling a few auxiliary technical tools. The first one is a well known relation (see for instance [43]); the last two are Lemmas 7 and 9 in [19] and Lemma 2.8 in [12].

Lemma 2.5 *Let m have p-adic decomposition $m = \sum_{i\geq 0} m_i p^i$, then $m = (p-1)\nu(m!) + \alpha(m)$, where $\alpha(m) = \sum_{i\geq 0} m_i$.*

Lemma 2.6 (Johnson [19]) *With the notation of* (4) *and* (11) *we have $\alpha_s \leq \alpha_0\alpha(u_0) + \alpha_1\alpha(u_1) + \cdots$. In the presence of* (a) *below, the above inequality is in fact an equality if and only if condition* (b) *below holds.*
(a) $u_t < p$, *for all $t \geq 0$, and*
(b) $s_{\langle j\rangle} = 0_{\langle j\rangle}u_0 + 1_{\langle j\rangle}u_1 + \cdots$, *for all $j \geq 0$.*

Lemma 2.7 *Let $\ell \in \mathbb{N}$ and assume given $c_0 \geq c_1 \geq \cdots \geq c_{\ell-1} \geq 0$. If $\sum_{j\geq 0} \varepsilon_j p^j = \sum_{j\geq 0} e_j p^j$, where $\varepsilon_j \geq 0$ and $0 \leq e_j \leq p-1$ for $j \geq 0$, and where only finitely many ε_j's and e_j's are non-zero, then $c_0\varepsilon_0 + c_1\varepsilon_1 + \cdots + c_{\ell-1}\varepsilon_{\ell-1} \geq c_0e_0 + c_1e_1 + \cdots + c_{\ell-1}e_{\ell-1}$.*

3. Filtering the coefficients

The calculations in this section are rather technical, they transform the algebraic information in (9) into a combinatorial problem which will be tackled in Section 4.

Lemma 3.1 *Let $\omega_{\varphi,k,s}$ be as defined in* (12).
(a) *Consider a summand $\sigma = m_i\binom{p^i}{U}a_k^U$ in* (9). *Then $\omega_\varphi(\sigma) \geq \omega_{\varphi,k,s}$. This is a strict inequality provided either one of conditions* (a) *or* (b) *in Lemma* 2.6 *fails.*
(b) $\omega_\varphi(a_{k,s}) \geq \omega_{\varphi,k,s}$.

Proof: When $\bar{s} = p^n$, Proposition 2.2 implies $\omega_\varphi(p^k m_n) \geq \omega_{\varphi,k,s}$; therefore part (b) follows from part (a) and (9). We now prove part (a) by induction over s, the case $s = 0$ being vacuously true. Since $p^i m_i$ lies in the ith power of the ideal $(p, v_1, v_2, \ldots)$ (see [41]) we have $i \leq \omega_\varphi(p^i m_i) = ip^{\varphi-1} + \omega_\varphi(m_i)$, in view of Proposition 2.2(a). Thus $\omega_\varphi(m_i) \geq i(1-p^{\varphi-1})$ and then

$$\begin{aligned}\omega_\varphi\left(m_i\binom{p^i}{U}\right) &\geq i(1-p^{\varphi-1}) + p^{\varphi-1}\left(\nu(p^i!) - \sum_{t\geq 0}\nu(u_t!)\right)\\ &= i(1-p^{\varphi-1}) + \frac{p^{\varphi-1}(p^i-1)}{p-1} - p^{\varphi-1}\sum_{t\geq 0}\nu(u_t!). \qquad (13)\end{aligned}$$

On the other hand, Proposition 2.2 and Lemmas 2.4–2.6, together with (10) and the inductive hypothesis yield

$$\begin{aligned}(p-1)\omega_\varphi\left(a_k^U\right) &\geq (p-1)\sum_{t\geq 0}u_t\omega_{\varphi,k,t} = \sum_{t\geq 0}u_t(-1 + d_{\varphi,k}\alpha_t + A_{\varphi,t})\\ &= -p^i + d_{\varphi,k}\sum_{t\geq 0}((p-1)\nu(u_t!) + \alpha(u_t))\alpha_t + \sum_{t\geq 0}u_tA_{\varphi,t}\\ &\geq -p^i + d_{\varphi,k}\sum_{t\geq 0}(p-1)\nu(u_t!)\alpha_t + d_{\varphi,k}\alpha_s + \sum_{t\geq 0}u_tA_{\varphi,t}. \qquad (14)\end{aligned}$$

But from (11) we see $\sum_{j\geq 0} s_{\langle j\rangle} p^j = \sum_{t\geq 0}(\sum_{j\geq 0} t_{\langle j\rangle} p^j)u_t = \sum_{j\geq 0}(\sum_{t\geq 0} u_t t_{\langle j\rangle})p^j$, so that $A_{\varphi,s} \leq \sum_{j=0}^{\varphi-1}[(p^{\varphi-1}(p-1)(\varphi-j) - p^{\varphi} + p^j)\sum_{t\geq 0} u_t t_{\langle j\rangle}] = \sum_{t\geq 0} u_t A_{\varphi,t}$, in view of Lemma 2.7. Therefore, with σ as in the hypothesis, (13) and (14) now give

$$\begin{aligned}(p-1)\omega_\varphi(\sigma) &\geq i(p-1)(1-p^{\varphi-1}) + p^i(p^{\varphi-1}-1) - p^{\varphi-1} + d_{\varphi,k}\alpha_s + A_{\varphi,s}\\ &\quad + (p-1)(-p^{\varphi-1} + d_{\varphi,k})\sum_{t\geq 0}\nu(u_t!)\\ &\geq i(p-1)(1-p^{\varphi-1}) + p^i(p^{\varphi-1}-1) - p^{\varphi-1} + d_{\varphi,k}\alpha_s + A_{\varphi,s} \qquad (15)\\ &\geq -1 + d_{\varphi,k}\alpha_s + A_{\varphi,s} \;=\; (p-1)\omega_{\varphi,k,s}, \qquad (16)\end{aligned}$$

in view of Lemma 2.4 again. The proof is completed by noticing that inequality (15) is strict when condition (a) in Lemma 2.6 fails; otherwise, inequality (14) is strict when condition (b) in Lemma 2.6 fails. □

Remark 3.2 In view of Lemma 3.1, the proof of Theorem 2.3 depends only on establishing the required congruence *modulo the larger ideal* $(p^{\mu_{\varphi,k,s}+1}) + W_{\varphi,\omega_{\varphi,k,s}+1}$.

Corollary 3.3 *Modulo the ideal in Remark* 3.2, (9) *simplifies to* $-a_{k,s} \equiv \sum m_1\binom{p}{U}a_k^U - p^k\delta_s$, *where the sum is taken over sequences U which satisfy* (10) (*for* $i = 1$) *as well as conditions* (a) *and* (b) *in Lemma* 2.6.

Proof: Lemma 3.1 gives $-a_{k,s} \equiv \sum m_i\binom{p^i}{U}a_k^U - p^k\delta_s$ modulo φ-filtration larger than $\omega_{\varphi,k,s}$, where the summation is over $i \geq 1$ and over sequences U satisfying (10) as well as parts (a) and (b) in Lemma 2.6. Observe that there can not be any such sequence U when $\alpha_s = 1$—implying the desired conclusion in this case. Indeed, with $\bar{s} = p^n$ and if u_ℓ is a positive term in U, then for each $j \geq 0$ with $\ell_{\langle j\rangle} > 0$, the relation $0_{\langle j\rangle}u_0 + 1_{\langle j\rangle}u_1 + \cdots + \ell_{\langle j\rangle}u_\ell + \cdots = s_{\langle j\rangle} = \delta(j,n)$ (Kronecker's delta) implies $\ell_{\langle j\rangle} = u_\ell = 1$ and $j = n$. This means $\bar{\ell} = p^n$, or $\ell = g(n)$ where g is as in Proposition 2.2, so that $u_{g(n)} = 1$ and $u_r = 0$ for $r \neq g(n)$. But this is incompatible with (10). Thus we can assume $\alpha_s > 1$. Then, for $\varphi = 1$ the desired conclusion follows directly from Eq. (37) in [19], whereas for $\varphi \geq 2$ it follows from the proof of Lemma 3.1 with the added observation that the inequality $i(p-1)(1-p^{\varphi-1}) + p^i(p^{\varphi-1}-1) - p^{\varphi-1} \geq -1$ used in (16) is strict for $i \geq 2$. □

Although the congruence in Corollary 3.3 above would seem to be circularly giving some $a_{k,s}$ in terms of *all* other $a_{k,t}$, $t \geq 0$, it is actually inductive (on α_s): in view of condition (b) in Lemma 2.6, the Uth summand in that congruence must satisfy $\alpha_s = \sum_{t\geq 0}\alpha_t u_t$, so that for any nontrivial factor $a_{k,t}^{u_t}$ (that is, one with $u_t \neq 0$) we have $\alpha_t < \alpha_s$ in view of (10). Thus, the next result will ground an inductive proof for Theorem 2.3.

Proposition 3.4 *For* $\bar{s} = p^n$ *the congruence in Theorem* 2.3 (*modulo the ideal in Remark* 3.2) *holds with* $c_s = 1$.

Proof: Assume first $n \leq \varphi$. By definition $\mu_{\varphi,k,s} = k - n$, $\lambda_{\varphi,s} = g(n)$ and $\omega_{\varphi,k,s} = (k-n)p^{\varphi-1} + g(n)$, where $g(n)$ is as in Proposition 2.2. Then the desired congruence follows from $a_{k,s} \equiv p^{k-n}v_1^{g(n)}$ modulo p^{k-n+1}, which is the conclusion of Corollary 2.6 in [12]. Assume now $n > \varphi$, so that $\mu_{\varphi,k,s} = k-\varphi$, $\lambda_{\varphi,s} = g(\varphi-1)$ and $\omega_{\varphi,k,s} = p^{\varphi-1}(k-\varphi+1)+g(\varphi-1)$. Now the desired congruence is

$$a_{k,s} \equiv p^{k-\varphi} v_1^{g(\varphi-1)} v_{n-\varphi+1}^{p^{\varphi-1}} \bmod (p^{k-\varphi+1}) + W_{\varphi,\omega_{\varphi,k,s}+1}. \tag{17}$$

Under our notation, Lemma 11 in [19] translates as $a_{k,s} \equiv p^{k-1}a_{1,s}$ modulo $W_{1,k+1}$ which, for $\varphi = 1$, implies (17) in view of the relations $\omega_{1,k,s} = k$ and $a_{1,s} \equiv v_n$ modulo $W_{1,2}$—the latter being a standard consequence of the formal sum expression [2]

$$[p](x) = \sum_{i\geq 0}^{\mu_p} v_i x^{p^i}. \tag{18}$$

Thus we further assume $\varphi \geq 2$ (in particular $k \geq 2$). Consider the inductive equation

$$[p^k](x) = [p^{k-1}]([p](x)) = \sum_{j\geq 0} a_{k-1,j}\,([p](x))^{j(p-1)+1}. \tag{19}$$

By Remark 1.1, $a_{k-1,j}$ is divisible by $p^{k-\varphi+1}$ unless $j(p-1)+1 = rp^{\varphi-1}$ for some $r > 0$. But in such a case, $\omega_\varphi(a_{k-1,j}([p](x))^{j(p-1)+1}) \geq 1 + rp^{\varphi-1} \geq \omega_{\varphi,k,s}+1$, for $r > k-\varphi+1$, while if $2 \leq r \leq k-\varphi+1$ (so that $\varphi < k$), Lemma 3.1(b) gives

$$\begin{aligned}
&\omega_\varphi\big(a_{k-1,j}([p](x))^{j(p-1)+1}\big)\\
&\quad\geq \omega_{\varphi,k-1,j} + rp^{\varphi-1} = p^{\varphi-1}\big((k-1)j_{\langle 0\rangle} + (k-2)j_{\langle 1\rangle} + \cdots + (k-\varphi)j_{\langle\varphi-1\rangle}\\
&\qquad + (k-\varphi-1)\big(j_{\langle\varphi\rangle} + j_{\langle\varphi+1\rangle} + \cdots\big)\big) + \frac{1}{p-1}\big(-1 + j_{\langle 0\rangle} + pj_{\langle 1\rangle} + \cdots + p^\varphi j_{\langle\varphi\rangle}\\
&\qquad + p^{\varphi-1}\big(j_{\langle\varphi+1\rangle} + j_{\langle\varphi+2\rangle} + \cdots\big)\big) + p^{\varphi-1}\big(j_{\langle\varphi+1\rangle} + j_{\langle\varphi+2\rangle} + \cdots\big) + rp^{\varphi-1}
\end{aligned}$$

which is easily verified to be larger than $\omega_{\varphi,k,s}$ (for this it is convenient to consider the three cases $\nu(r) = 0$, $\nu(r) = 1$ and $\nu(r) \geq 2$). Thus, (19) becomes

$$[p^k](x) \equiv a_{k-1,g(\varphi-1)}([p](x))^{p^{\varphi-1}} \bmod (p^{k-\varphi+1}) + W_{\varphi,\omega_{\varphi,k,s}+1}. \tag{20}$$

Now, a second application of [12, Corollary 2.6] yields $a_{k-1,g(\varphi-1)} \equiv p^{k-\varphi}v_1^{g(\varphi-1)}$ modulo $p^{k-\varphi+1}$; therefore, in terms of the μ_p-formal sum expression (18), the right hand side of (20) transforms as

$$\begin{aligned}
a_{k-1,g(\varphi-1)}\,([p](x))^{p^{\varphi-1}} &\equiv p^{k-\varphi}v_1^{g(\varphi-1)}\left(\sum_{j\geq 0}^{\mu_p} v_j x^{p^j}\right)^{p^{\varphi-1}} && \bmod (p^{k-\varphi+1})\\
&\equiv p^{k-\varphi}v_1^{g(\varphi-1)}\left(\sum_{j\geq 0} v_j x^{p^j}\right)^{p^{\varphi-1}} && \bmod W_{\varphi,\omega_{\varphi,k,s}+1}\\
&\equiv p^{k-\varphi}v_1^{g(\varphi-1)}\sum_{j\geq 0} v_j^{p^{\varphi-1}} x^{p^{j+\varphi-1}} && \bmod (p^{k-\varphi+1}).
\end{aligned}$$

All together gives $[p^k](x) \equiv p^{k-\varphi} v_1^{g(\varphi-1)} \sum_{j\geq 0} v_j^{p^{\varphi-1}} x^{p^{j+\varphi-1}}$ modulo $(p^{k-\varphi+1}) + W_{\varphi,\omega_{\varphi,k,s}+1}$, and (17) follows by comparing coefficients of the p^nth power of x. □

Proof of Theorem 2.3: Let $\alpha_s \geq 2$ and pick a sequence U as in Corollary 3.3. Using (6), (7), (8), (10) with $i = 1$, (12) and condition (b) in Lemma 2.6 we easily get

(i) $\sum_{t\geq 0} u_t \lambda_{\varphi,t} = -1 + \lambda_{\varphi,s}$,
(ii) $\sum_{t\geq 0} u_t i_{j,\varphi,t} = i_{j,\varphi,s}$,
(iii) $\sum_{t\geq 0} u_t \mu_{\varphi,k,t} = \mu_{\varphi,k,s}$ and, therefore,
(iv) $\sum_{t\geq 0} u_t \omega_{\varphi,k,t} = -1 + \omega_{\varphi,k,s}$.

Since $\nu\binom{p}{U} = 1$ and $pm_1 \equiv v_1 \bmod p$ [41], we inductively see that the summand $m_1\binom{p}{U} a_k^U$ in the congruence of Corollary 3.3 contributes with $\frac{1}{p}\binom{p}{U}(\prod_{t\geq 0} c_t^{u_t}) p^{\mu_{\varphi,k,s}} v^{I_{\varphi,s}}$ in the expression for $-a_{k,s}$ modulo the ideal in Theorem 2.3. The proof is completed by the next result. □

Proposition 3.5 *Let $c_s \in \mathbb{Z}/p$ be defined (inductively on α_s) by $c_s = 1$ if $\alpha_s = 1$, and $c_s = -\sum \frac{1}{p}\binom{p}{U} \prod_{t\geq 0} c_t^{u_t}$ for $\alpha_s > 1$, where the sum is taken over sequences U as in Corollary 3.3. Then $c_s = (\prod_{j\geq 0} s_{\langle j\rangle}!)^{-1}$.*

The proof of this result is deferred until we had proved Proposition 1.7 in the next and final section.

4. Modified Stirling numbers

For $a \geq 1$ let Σ_a stand for the permutation group (acting on the left) of the set $[a] = \{1, 2, \ldots, a\}$. For an equivalence relation $\sim$ on $[a]$ and a permutation $\sigma \in \Sigma_a$ consider the equivalence relation $\sim_\sigma$ given so that $i \sim_\sigma j$ precisely when $\sigma(i) \sim \sigma(j)$. This produces a (right) action of Σ_a on the set $\mathcal{R}_a$ of equivalence relations on $[a]$. Now, for a prime number p and a positive integer s set $a = \bar{s} = s(p-1)+1$ and consider the set $\Phi(s) \subseteq \mathcal{R}_{\bar{s}}$ consisting of those equivalence relations having exactly p equivalence classes each one of which has size congruent with 1 modulo $p-1$ (thus $\phi(s)$ in Definition 1.5 is the size of $\Phi(s)$). It is clear that $\Phi(s)$ is closed under the action of $\Sigma_{\bar{s}}$. We consider the restricted action

$$\Phi(s) \times \mathbb{Z}/p \to \Phi(s) \tag{21}$$

under the usual group monomorphisms $\mathbb{Z}/p \hookrightarrow \Sigma_p \hookrightarrow \Sigma_{\bar{s}}$. As $\Phi(1)$ consists of a single point, the action of $\mathbb{Z}/p$ on $\Phi(1)$ is trivial; however, the next result (whose straightforward proof is included just for completeness) shows that the situation is certainly different for $s \geq 2$.

Lemma 4.1 *Let $s \geq 2$. The orbit of $\sim\, \in \Phi(s)$ reduces to $\{\sim\}$ if and only if $i \sim j$ for all $i, j \in [p]$.*

Proof: It is clear from the construction that, when $[p]$ is contained in some equivalence class of $\sim$, the relation $\sigma(i) \sim i$ holds for any $i \in [\bar{s}]$ and any $\sigma \in \mathbb{Z}/p$ so that, in particular, $\sim_\sigma \; = \; \sim$. Conversely, assume $\sim_\sigma \; = \; \sim$ for all $\sigma \in \mathbb{Z}/p$, but $i \not\sim i+1$ for $i \in [p-1]$. Then for any $j \in [p-1]$ choose $\sigma_j \in \mathbb{Z}/p$ with $\sigma_j(i) = j$ and observe $i \not\sim i+1 \Rightarrow i \not\sim_{\sigma_j} i+1 \Rightarrow j = \sigma_j(i) \not\sim \sigma_j(i+1) = j+1$. Since $s \geq 2$, there are $j \in [p]$ and $k > p$ with $j \sim k$ and take $\sigma \in \mathbb{Z}/p$ to be the usual generator if $j < p$, but to be the inverse of the usual generator otherwise (so that $j \not\sim \sigma(j)$ as shown above). Now, $j \sim k \Rightarrow j \sim_\sigma k \Rightarrow \sigma(j) \sim \sigma(k) = k$. But the last relation is incompatible with $k \sim j \not\sim \sigma(j)$. □

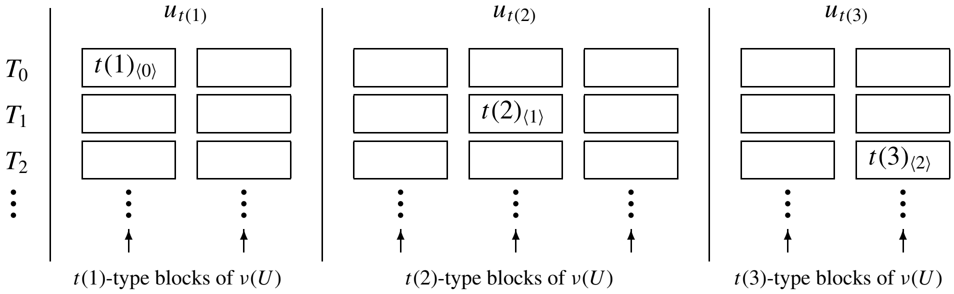
Proof of Proposition 1.7: Let $s \geq 2$ and consider the map $\theta : \mathcal{R}_{\bar{s}} \to \mathcal{R}_{\overline{s-1}}$ given by restriction of equivalence relations under the inclusion $\iota : \left[\,\overline{s-1}\,\right] \to [\bar{s}]$, where $\iota(1) = 1$ and $\iota(\ell) = \ell + p - 1$, for $\ell \geq 2$. With $\Phi'(s)$ standing for the set of elements in $\Phi(s)$ whose orbit under (21) is a singleton, Lemma 4.1 claims that the restriction of θ to $\Phi'(s)$ is a one-to-one map onto $\Phi(s-1)$. But p is prime, so that any $\mathbb{Z}/p$-orbit in $\Phi(s)$ either is a singleton or has size p. Therefore, in a mod-p counting of $\Phi(s)$, we can throw away the latter orbits and obtain $\phi(s) \equiv |\Phi'(s)| = \phi(s-1)$ modulo p. The result follows since, as observed in Remark 1.6, $\phi(1) = 1$. □

Our approach to Proposition 3.5 requires a generalized version of the above combinatorial situation: we want the same sort of partitions, however, now the objects to be partitioned admit repetitions. We set up the situation in detail. For $s \geq 0$ and $j \geq 0$ assume given a set T_j consisting of $s_{\langle j \rangle}$ distinct elements, which we refer to as having "type" j, so that $T_i \cap T_j = \emptyset$ for $i \neq j$. (We keep the conventions in Remark 1.1, so that $T_i = \emptyset$ for almost all i; moreover α_s, the size of $T = \bigcup_{j \geq 0} T_j$, is congruent with 1 modulo $p-1$ in view of Remark 1.3). Let us identify the set $\Phi(\frac{\alpha_s - 1}{p-1})$, denoted by $\Phi(T)$ for simplicity, with the set of partitions of T into p unlabeled sets each one having size congruent with 1 modulo $p-1$. Likewise, let $\Phi_R(T)$ stand for the set of partitions as above but where we do not distinguish among elements of T having the same type. There is an obvious surjective function $\pi : \Phi(T) \to \Phi_R(T)$ obtained by neglecting any distinction among objects of the same type. As we shall see, the proof of Proposition 3.5 demands, on the one hand, knowing the size of $\pi^{-1}(\{y\})$ for each $y \in \Phi_R(T)$, and on the other, using the sequences U in Corollary 3.3 as a way to identify elements in $\Phi(T)$ with the same π-image. We start with the latter task, and for that matter, we denote by $\mathcal{S}$ the set of sequences U satisfying (10) for $i = 1$, as well as conditions (a) and (b) in Lemma 2.6.

Let τ be a block of an element in $\Phi(T)$. For $j \geq 0$ let $\tau_{\langle j \rangle}$ be the number of elements in τ of type j. By construction, $\sum_{j \geq 0} \tau_{\langle j \rangle}$—the size of τ—is congruent with 1 modulo $p-1$, thus there is a unique number $t \geq 0$, called the "type distribution" of τ, such that $\bar{t} = \sum_{j \geq 0} \tau_{\langle j \rangle} p^j$. Note that $t_{\langle j \rangle} = \tau_{\langle j \rangle}$ as $\tau_{\langle j \rangle} \leq s_{\langle j \rangle} < p$. It is clear that two elements $x_1, x_2 \in \Phi(T)$ have the same π-image if and only if the blocks of x_1 can be set into a one-to-one correspondence with those of x_2 so that corresponding blocks have the same type distribution (such a situation will be referred as "x_1 and x_2 having the same type distribution"). Now for $x \in \Phi(T)$ set $U_x = (u_0, u_1, \ldots)$, where u_t is the number of blocks in x having type t. In these conditions (10) for $i = 1$ and condition (b) in Lemma 2.6 clearly hold, as well as the fact that $0 \leq u_t \leq p$ for $t \geq 0$. Moreover, if $u_{t_0} = p$ for some $t_0 \geq 0$, there would be a type $j \geq 0$ repeating at least p times, in contradiction to the fact that $s_{\langle j\rangle} < p$. Therefore the correspondence $x \mapsto U_x$ defines a map $\rho : \Phi(T) \to \mathcal{S}$ which, by construction, is constant on each $\pi^{-1}(\{y\})$. In particular we get an induced map $\bar{\rho} : \Phi_R(T) \to \mathcal{S}$.

Lemma 4.2 *$\bar{\rho}$ is bijective.*

Proof: It suffices to construct a map $\nu : \mathcal{S} \to \Phi(T)$ such that $\pi\nu\rho = \pi$ and $\bar{\rho}\pi\nu$ is the identity on $\mathcal{S}$ (the former condition implies that $\pi\nu$ is surjective; the latter that $\pi\nu$ is injective and, therefore, that $(\pi\nu)^{-1} = \bar{\rho}$). Let $U \in \mathcal{S}$ have nonzero terms $u_{t(1)}, u_{t(2)}, \ldots, u_{t(r)}$. The formulæ $p = \sum_{i=1}^{r} u_{t(i)}$ and $s_{\langle j\rangle} = \sum_{i=1}^{r} u_{t(i)} t(i)_{\langle j\rangle}$ mean we can choose a (numbered) partition of T_j into p blocks in such a way that the first $u_{t(1)}$ blocks have size $t(1)_{\langle j\rangle}$, the second $u_{t(2)}$ blocks have size $t(2)_{\langle j\rangle}, \ldots$, and the last $u_{t(r)}$ blocks have size $t(r)_{\langle j\rangle}$ (some of the blocks may be empty, but we count them anyway). Then $\nu(U)$ is formed by the partition whose ℓth block ($1 \le \ell \le p$) consists of the elements in the ℓth block of each T_j for $j \ge 0$—the typical combinatorial situation ($p = 7$, $r = 3$, $u_{t(1)} = 2$, $u_{t(2)} = 3$ and $u_{t(3)} = 2$) is illustrated in the picture below where the boxes on the jth row represent the chosen partition of T_j, the union of the boxes on a given column form a block of $\nu(U)$ and the number inside boxes exemplifies their size.



As $1 \equiv \bar{t} = \sum_{j\ge 0} t_{\langle j\rangle} p^j \equiv \sum_{j\ge 0} t_{\langle j\rangle}$ modulo $p-1$, $\nu(U)$ is indeed an element in $\Phi(T)$. Then the relation $\rho\nu = 1_{\mathcal{S}}$ is immediate, while the relation $\pi\nu\rho = \pi$ follows from the observation that, for any $x \in \Phi(T)$, $\nu\rho(x)$ has been constructed so to have the same type distribution as x. $\square$

Proof of Proposition 3.5: We proceed by induction on α_s, the result being obvious for $\alpha_s = 1$. For $\alpha_s > 1$ Wilson's theorem gives

$$c_s = \sum_{U\in\mathcal{S}} \left(\prod_{t\ge 0} u_t! \right)^{-1} \prod_{t\ge 0} c_t^{u_t}. \tag{22}$$

On the other hand, for $U \in \mathcal{S}$ and using the notation in the proof of Lemma 4.2, a straightforward counting shows that $|\rho^{-1}(U)|$, the size of $\rho^{-1}(U)$, is given by

$$\frac{1}{u_{t(1)}! \cdots u_{t(r)}!} \prod_{j\ge 0} \binom{s_{\langle j\rangle}}{t(1)_{\langle j\rangle}, t(1)_{\langle j\rangle}, \ldots, t(1)_{\langle j\rangle}, \ldots\ldots, t(r)_{\langle j\rangle}, t(r)_{\langle j\rangle}, \ldots, t(r)_{\langle j\rangle}}, \tag{23}$$

where $t(i)_{\langle j\rangle}$ repeats $u_{t(i)}$ times in the jth multinomial coefficient. Since $u_t = 0$ for $t \neq t(i)$, the product of the multinomial coefficients in (23) can be rewritten as

$$\left(\prod_{j\geq 0} s_{\langle j\rangle}!\right)\left(\prod_{t\geq 0}\left(\prod_{j\geq 0} t_{\langle j\rangle}!\right)^{-u_t}\right),$$

whose mod-p values agrees with

$$\left(\prod_{j\geq 0} s_{\langle j\rangle}!\right)\left(\prod_{t\geq 0} c_t^{u_t}\right), \tag{24}$$

in view of the inductive hypothesis and the observations in the paragraph just before Proposition 3.4. Thus (22)–(24) yield

$$c_s = \left(\sum_{U\in\mathcal{S}} |\rho^{-1}(U)|\right)\left(\prod_{j\geq 0} s_{\langle j\rangle}!\right)^{-1}.$$

The conclusion follows since, by Lemma 4.2 and Proposition 1.7,

$$\sum_{U\in\mathcal{S}} |\rho^{-1}(U)| = |\Phi(T)| \equiv 1 \quad \text{modulo } p. \qquad \square$$

Appendix

It should be stressed how just a simple modification (Definition 2.1) of the standard filtration by powers of the ideal $(p, v_1, v_2, \ldots)$ allows us to get so much information on the polynomial structure of the coefficients in the p^k-series. Yet, there are many more obvious modified filtrations worth trying on. We believe that further modifications will eventually shed considerable light toward a true global understanding of these algebraic objects. The next lines are intended to stress the importance of such a goal; indeed, we briefly sample areas which could (or even have) benefit(ted) from the results in this paper. Far from making an exhaustive list of applications, our intention is just to pinpoint explicit situations directly related to our work.

Number theory is perhaps the most natural area linked to the theory of formal groups, and the relations have become abundant over the time. The text [16] gives an excellent revision for known applications (up to the mid 70's) of formal group theory into number theory as well as into arithmetical and algebriac geometry. More recent applications to cryptography, where it is important to have methods for computing the cardinality of the group of rational points of elliptic curves defined over a finite field F, can be derived from the results in [8]. In that work, formal group laws associated to elliptic curves are used to give effective methods to compute isogenies (see also [3, 23] for further developments in this direction). More generally, it is possible to associate formal group laws to algebraic varieties. Most interesting cases seem to be one-dimensional formal groups arising from Calabi-Yau varieties. For instance, the p-series—and in particular the pth coefficient—contains information about the number of rational points on the variety over the field with

p elements [14, 36, 37, 44]. Formal group theory has also proven to have close connections to class field theory, offering alternative approaches which reveal remarkable properties of number and local fields [10].

In combinatorics it is worth noticing the interrelation of formal group theory with umbral calculus [6, 33]. As for applications in other areas of mathematics, the theory of formal groups has played, in fact, a sort of unifying role. Since the 1986 conference at the IAS in Princeton ([22], see also [35, 38])—whose original aim centered at (that time) recent developments of elliptic genera and elliptic cohomology—it has became clear that geometry and physics enter prominently into the subject [1, 30, 42]. In particular, and as already noticed in the introduction, algebraic topology has seen deep connections to those areas via the formal-group-grounds it shares with number theory.

We finish this brief survey with a bit more thorough revision of some aspects in algebraic topology directly related to the results in this paper.

Right from the original work of Johnson [18] it was known that “half” the coefficients in the 2-series were even (but not divisible by 4). It turns out that this information was the key to compute in [9] BP-Euler theoretic obstructions for the existence of Euclidean immersions of real projective spaces. The calculation led to what could be the most general and strongest result known to date on this problem of differential topology and, consequently, was a motivation for the development of this paper. Indeed, the 2-divisibility properties for the 2^k-series obtained here (or in [12]) were used in [13] to compute the corresponding obstructions for the existence of Euclidean immersions of 2-torsion lens spaces, extending in part the main result in [9].

Another (far reaching) connection with algebraic and differential topology starts with the study of bordism classes of free $(\mathbb{Z}/p)^n$-actions on oriented manifolds. This problem led Conner and Floyd [7] to consider the oriented bordism (MSO-homology) of $(B\mathbb{Z}/p)^{\wedge n}$, the iterated n-fold smash product of the classifying space for $\mathbb{Z}/p$ with itself. As they noticed, the bottom “toral” class in these groups plays a fundamental role in the problem, for its MSO_* annihilator ideal I_n is generated by those bordism classes of oriented manifolds admitting a free $(\mathbb{Z}/p)^n$-action. Conner and Floyd’s main geometric results can be recovered provided a conjectured description of I_n holds (the so-called Conner-Floyd conjecture). For this problem one can replace the Thom spectrum MSO by the Brown-Peterson spectrum BP and, in these terms, the iterated n-fold tensor product (over BP_*) of $BP_*(B\mathbb{Z}/p)$ with itself —where the p-series plays a major role—yields a first approximation to I_n. The Conner-Floyd conjecture was proved in the early 80’s: $I_n = (p, v_1, v_2, \ldots, v_{n-1})$ [29, 32], and it turns out that, together with detailed information about the p-series, the above description of I_n leads in fact to a full description of the (additive) structure of the Brown-Peterson homology of $(B\mathbb{Z}/p)^{\wedge n}$ [20, 21]. The relevance of such a calculation has been confirmed by Minami’s work [25–28] on the possible existence of framed manifolds of Kervaire invariant 1 (that is, on the basic problem of understanding stable homotopy classes of spheres detected in the 2-line of the classical Adams spectral sequence). Now, in view of the basic role the p-series played in the above development, it would be interesting to see to what extent the information in this paper for the p^k-series can be used in a calculation of $BP_*(B\mathbb{Z}/p^{k_1} \times \cdots \times B\mathbb{Z}/p^{k_n})$, as well as its implications in the stable homotopy groups of spheres.

Note

1. Some algebraic-combinatoric aspects of the poset of partitions with restricted block size has been studied in [5, 15, 24, 39, 40].

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