Journal of Algebraic Combinatorics, 17, 39–56, 2003

On a Conjecture of R.P. Stanley; Part I—Monomial Ideals

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Received March 9, 2001; Revised April 24, 2002

Abstract. In 1982 Richard P. Stanley conjectured that any finitely generated $\mathbb{Z}^n$-graded module M over a finitely generated $\mathbb{N}^n$-graded $\mathbb{K}$-algebra R can be decomposed in a direct sum $M = \bigoplus_{i=1}^{t} \nu_i S_i$ of finitely many free modules $\nu_i S_i$ which have to satisfy some additional conditions. Besides homogeneity conditions the most important restriction is that the S_i have to be subalgebras of R of dimension at least depth M.

We will study this conjecture for the special case that R is a polynomial ring and M an ideal of R, where we encounter a strong connection to generalized involutive bases. We will derive a criterion which allows us to extract an upper bound on depth M from particular involutive bases. As a corollary we obtain that any monomial ideal M which possesses an involutive basis of this type satisfies Stanley's Conjecture and in this case the involutive decomposition defined by the basis is also a Stanley decomposition of M. Moreover, we will show that the criterion applies, for instance, to any monomial ideal of depth at most 2, to any monomial ideal in at most 3 variables, and to any monomial ideal which is generic with respect to one variable. The theory of involutive bases provides us with the algorithmic part for the computation of Stanley decompositions in these situations.

Keywords: monomial ideal, combinatorial decomposition, involutive basis

1. Introduction

This is the first of two articles studying some aspects of a conjecture formulated by Richard P. Stanley in 1982.

Conjecture 1 ([13], 5.1) Let R be a finitely-generated $\mathbb{N}^n$-graded $\mathbb{K}$-algebra (where $R_0 = \mathbb{K}$ as usual), and let M be a finitely-generated $\mathbb{Z}^n$-graded R-module. Then there exist finitely many subalgebras $S_1, \ldots, S_t$ of R, each generated by algebraically independent $\mathbb{N}^n$-homogeneous elements of R, and there exist $\mathbb{Z}^n$-homogeneous elements $\nu_1, \ldots, \nu_t$ of M, such that

$$M = \bigoplus_{i=1}^{t} \nu_i S_i,$$

where $\dim S_i \geq \operatorname{depth} M$ for all i, and where $\nu_i S_i$ is a free S_i-module (of rank one). Moreover, if $\mathbb{K}$ is infinite and under a given specialization to an $\mathbb{N}$-grading of R is generated by R_1, then we can choose the ($\mathbb{N}^n$-homogeneous) generators of each S_i to lie in R_1.

Definition 1 Let R be a finitely-generated $\mathbb{N}^n$-graded $\mathbb{K}$-algebra and let M be a finitely-generated $\mathbb{Z}^n$-graded R-module. For each non-negative integer d let Δ_d denote the set of all direct sum decompositions $M = \bigoplus_{i=1}^{t} \nu_i S_i$ of M in finitely many free S_i-modules $\nu_i S_i$ ($i \in \{1, \dots, t\}$), where the $S_1, \dots, S_t$ are subalgebras of R each of them generated by at least d algebraically independent $\mathbb{N}^n$-homogeneous elements of R and the $\nu_1, \dots, \nu_t$ are $\mathbb{Z}^n$-homogeneous elements of M.

If not all sets Δ_d are empty then we call the largest d such that $\Delta_d \neq \emptyset$ the *Stanley depth* of M and denote it by Sdepth M.

For $\mathbb{N}^n$-graded polynomial rings $R = \mathbb{K}[x_1, \dots, x_m]$ the existence of Sdepth M, i.e. the existence of a decomposition of M in finitely many direct summands $\nu_i S_i$ satisfying the above conditions, follows e.g. from results of Riquier related to the solution space of systems of partial differential equations [11], see also [14]. Specialized to our studies Stanley's Conjecture reads as follows:

Conjecture 2 Let $R = \mathbb{K}[X]$ be a polynomial ring in the variables $X = \{x_1, \dots, x_n\}$ over a field $\mathbb{K}$. Then for each monomial ideal $I \subset R$ it holds

$$\operatorname{Sdepth} I \geq \operatorname{depth}(XR, I) = \operatorname{depth} I.$$

That is, there exist a finite decomposition (called *Stanley decomposition* throughout this paper) of I of the following type

$$I = \bigoplus_{i=1}^{t} u_i \mathbb{K}[Y_i], \tag{1}$$

where the u_i are (monic) monomials, $Y_i \subseteq X$ and $|Y_i| \geq \operatorname{depth} I$ for all $i = 1, \dots, t$.

The notion depth I could lead to a confusion since it is also a frequently used short cut for depth (I, M), the depth of the ideal I on some R-module M. Therefore, we emphasize that in this paper depth I will always stand for depth (XR, I), the depth of the maximal homogeneous ideal of R on I considered as an R-module.

There is a strong relationship between Stanley decompositions and the Riquier-Janet theory for solving systems of partial differential equations (see [6, 11]) which also produces a decomposition (1), however, without the additional assumption on the dimension of the subalgebras. The classical decompositions due to Janet, Thomas, or Pommaret will violate the crucial condition on the dimension of the subalgebras S_i, in general. Here we will rely on a more general definition of involutive bases introduced in [1]. We will prove a criterion (Corollary 1) showing that a particular type of involutive decomposition satisfies even the harder assumptions of Stanley's Conjecture and, hence, is a Stanley decomposition.

Our definition of involutive bases will turn out to be sufficiently general to allow the application of the above criterion to some large classes of monomial ideals. Among them there are all monomial ideals of depth at most 2 (Corollary 3), all monomial ideals in at most three variables (Theorem 1), and all monomial ideals which are generic with respect to one variable, i.e. whenever two distinct minimal generators have the same degree in some fixed variable then there is another minimal generator of strictly smaller degree in this variable which divides the least common multiple of these minimal generators (Theorem 2). The proofs of the theorems give explicit hints how in these cases a Stanley decomposition can be constructed using the algorithmic methods provided by the theory of involutive bases.

We remark that Example 5 shows that also the notion of involutive bases introduced in [1] is not powerful enough for the construction of Stanley decompositions for arbitrary monomial ideals.

2. Involutive bases of polynomial ideals

Let $X = \{X_1, \dots, X_n\}$ be a finite set of indeterminates and $T = \langle X \rangle$ denote the free commutative monoid generated by X. By an *involutive division* on T we denote a certain type of subrelations of the conventional division which is defined in the following way.

Definition 2 [1, Definition 3.1] Let $(Y_u)_{u \in T}$ be a family of subsets $Y_u \subseteq X$ of indeterminates. The family $\mathcal{M} = (M_u)_{u \in T}$, where $M_u = u\langle Y_u \rangle$ for all $u \in T$, is called the *involutive division* generated by $(Y_u)_{u \in T}$. For $u \in M_v$ we call u an $\mathcal{M}$*-multiple* of v and v an $\mathcal{M}$*-divisor* of u. Furthermore, we say that the variables $x \in Y_u$ are the $\mathcal{M}$*-multipliers* and the variables $y \in X \setminus Y_u$ the $\mathcal{M}$*-nonmultipliers* of $u \in T$. The number $|Y_u|$ of $\mathcal{M}$-multipliers is denoted by $\operatorname{Idim}_{\mathcal{M}} u$ and called the *involutive dimension* of u with respect to $\mathcal{M}$.

Let $V \subseteq T$ be a set of terms and $\sqsubset$ an order on V. The involutive division $\mathcal{M} = (M_u)_{u \in T}$ is called *admissible for* $(V, \sqsubset)$ if for all $v, w \in V$ such that $w \sqsubset v$ it holds either $M_w \subset M_v$ or $M_v \cap wT = \emptyset$. If the admissibility of $\mathcal{N} = (N_u)_{u \in T}$, where $M_u \subseteq N_u$ for all $u \in V$, implies that all inclusions have to be equalities then $\mathcal{M}$ is called a *maximal admissible involutive division for* $(V, \sqsubset)$. The short cut $\mathcal{M}$ *is* (*maximal*) *admissible for* V will express the existence of $\sqsubset$ such that $\mathcal{M}$ is (maximal) admissible for $(V, \sqsubset)$.

Note, the family $\mathcal{M}$ is always indexed by the entire set T of monomials. In some situations, e.g. if involutive divisions admissible on a (finite) subset $V \subset T$ are under consideration, we pass to equivalence classes of families coinciding on V which we simply denote by $\mathcal{M} = (M_u)_{u \in V}$. Using this convention it makes sense to consider the admissibility of an involutive division $(M_u)_{u \in T}$ on arbitrary sets V. In contrast, this would be impossible if we would work with families ranging over smaller index sets.

Example 1 (Part 1) Consider the sequence z^2, xyz, y^2, x^2 of monomials in the variables x, y, z, which is in increasing order with respect to the reverse lexicographical term order $\sqsubset$ extending $z \sqsubset y \sqsubset x$. Let $Y_{z^2} = \{x, y, z\}$, $Y_{xyz} = Y_{y^2} = \{x, y\}$, and $Y_{x^2} = \{x\}$ be the sets of multipliers of the monomials. The cones $z^2\langle Y_{z^2}\rangle$, $xyz\langle Y_{xyz}\rangle$, $y^2\langle Y_{y^2}\rangle$, and $x^2\langle Y_{x^2}\rangle$ are pairwise disjoint and even more is true, $x^2\langle Y_{x^2}\rangle \cap (z^2, xyz, y^2)\langle X\rangle = y^2\langle Y_{y^2}\rangle \cap (z^2, xyz)\langle X\rangle = xyz\langle Y_{xyz}\rangle \cap z^2\langle X\rangle = \emptyset$.

Hence, $\mathcal{M} = \{z^2\langle Y_{z^2}\rangle, xyz\langle Y_{xyz}\rangle, y^2\langle Y_{y^2}\rangle, x^2\langle Y_{x^2}\rangle\}$ describes an equivalence class of involutive divisions which are admissible on the ordered set (z^2, xyz, y^2, x^2). In this case we have $M_v \cap wT = \emptyset$ for all $v, w \in \{z^2, xyz, y^2, x^2\}$ satisfying $w \sqsubset v$. Figure 1 provides an impression of the structure of this equivalence class by displaying the cones formed by the exponent vectors of the $\mathcal{M}$-multiples of the monomials z^2, xyz, y^2, x^2.[1]

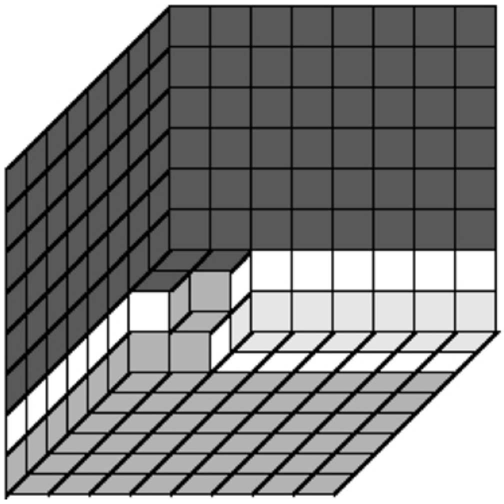

Figure 1. $\mathcal{M} = \{z^2\langle x, y, z\rangle, xyz\langle x, y\rangle, y^2\langle x, y\rangle, x^2\langle x\rangle\}$.

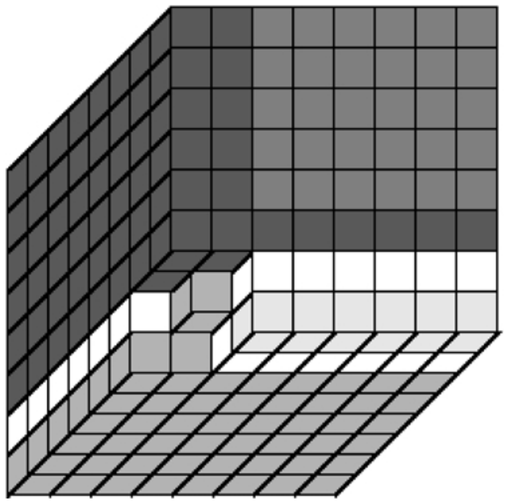

Figure 2. $\mathcal{M} = \{x^2z^3\langle x, z\rangle, z^2\langle x, y, z\rangle, xyz\langle x, y\rangle, y^2\langle x, y\rangle, x^2\langle x\rangle\}$.

The alternative condition an admissible involutive division $\mathcal{M}$ can satisfy for monomials $w \sqsubset v$ means that the cone of $\mathcal{M}$-multiples of w is entirely contained in the cone of $\mathcal{M}$-multiples of v. Such a situation we meet, for instance, in the family $\mathcal{M} = \{x^2z^3\langle x, z\rangle, z^2\langle Y_{z^2}\rangle, xyz\langle Y_{xyz}\rangle, y^2\langle Y_{y^2}\rangle, x^2\langle Y_{x^2}\rangle\}$ which defines an equivalence class of involutive divisions admissible on the ordered set $(x^2z^3, z^2, xyz, y^2, x^2)$ and is displayed in figure 2. Here, we have the cone inclusion $x^2z^3\langle x, z\rangle \subset z^2\langle x, y, z\rangle$.

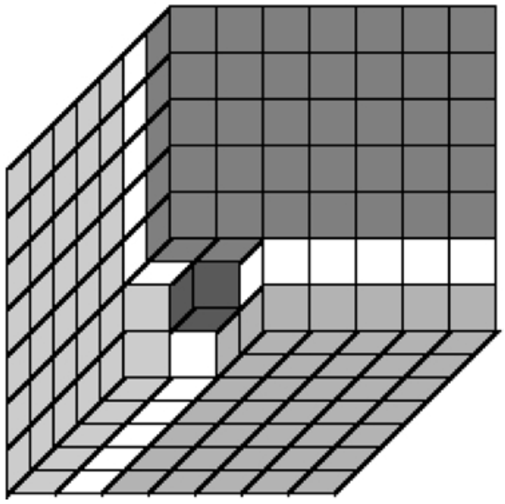

Figure 3. $\mathcal{M} = \{z^2\langle x, z\rangle, xyz\langle x, y, z\rangle, y^2\langle y, z\rangle, x^2\langle x, y\rangle\}$.

Finally, in figure 3 there is displayed the cone arrangement $\mathcal{M} = \{z^2\langle x, z\rangle, xyz\langle x, y, z\rangle, y^2\langle y, z\rangle, x^2\langle x, y\rangle\}$ which is not admissible for any order of the monomials z^2, xyz, y^2, x^2.

Now, let $R = \mathbb{K}[X]$ be the polynomial ring in the variables $X = \{X_1, \ldots, X_n\}$ over a field $\mathbb{K}$. Then the above defined involutive divisions gives rise to the distinction of a particular type of Gröbner bases of ideals of I.

Definition 3 Let $\prec$ be an admissible term order, F a finite set of non-zero polynomials and $\mathcal{M}$ an involutive division admissible for the set $\operatorname{lt} F$ of leading terms of F. If $\operatorname{lt}(I) = \bigcup_{f \in F} M_{\operatorname{lt} f}$, where $I = FR$ denotes the ideal of R generated by F, then F is called an *$\mathcal{M}$-involutive basis of I with respect to* $\prec$. If, in addition, the union on the right hand side is disjoint we call F a *minimal $\mathcal{M}$-involutive basis* of I with respect to $\prec$.

Example 1 (Part 2) Consider the monomial ideal $I = (z^2, xyz, y^2, x^2) \subset \mathbb{K}[x, y, z]$. An ordered $\mathcal{M}$-involutive basis of I (with respect to an arbitrary term order $\prec$) is given by $(z^2, y^2z, xyz, x^2z, y^2, x^2y, x^2)$, where the essential multiplier sets of $\mathcal{M}$ are $Y_{z^2} = \{x, y, z\}$, $Y_{y^2z} = Y_{y^2} = \{x, y\}$, and $Y_{xyz} = Y_{x^2z} = Y_{x^2y} = Y_{x^2} = \{x\}$. Figure 4 shows that the cones M_u, $u \in \{z^2, y^2z, xyz, x^2z, y^2, x^2y, x^2\}$, exhaust the set $\operatorname{lt} I$ of leading terms of I. Moreover, these cones are pairwise disjoint and, therefore, the above $\mathcal{M}$-involutive basis is minimal. A possible completion of the generating set displayed in figure 2 to an $\mathcal{M}$-involutive basis is $(x^2z^3, z^2, y^2z, xyz, x^2z, y^2, x^2y, x^2)$, where $Y_{x^2z^3} = \{x, z\}$ is the additional set of multipliers. However, this basis is not minimal since $x^2z^3\langle x, z\rangle \cap z^2\langle x, y, z\rangle \neq \emptyset$.

Obviously, any $\mathcal{M}$-involutive basis F of I is also a Gröbner basis of I. Using some well-known facts on Gröbner bases and taking into account the uniqueness of the $\mathcal{M}$-reduction

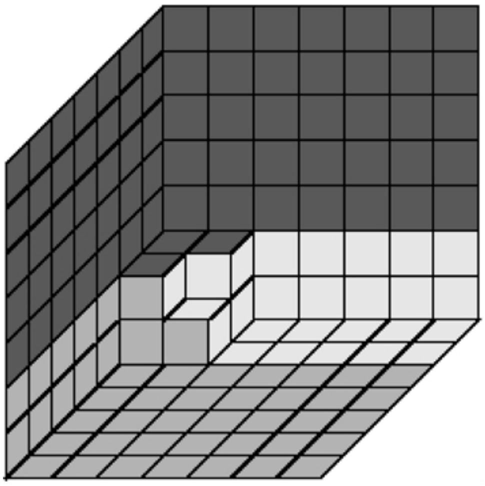

Figure 4. Ordered involutive basis $(z^2, y^2z, xyz, x^2z, y^2, x^2y, x^2)$.

process [1, Definition 5.1] with respect to minimal involutive bases it follows

$$I = \bigoplus_{f \in F} f\mathbb{K}[Y_f] \tag{2}$$

for an arbitrary minimal $\mathcal{M}$-involutive basis F of I, where $\mathcal{M}$ is generated by $(Z_u)_{u \in T}$ and $Y_f = Z_{\operatorname{lt} f}$.

For given I, $\prec$ and $\mathcal{M}$ there need not exist any $\mathcal{M}$-involutive basis of I with respect to $\prec$. However, at least we have:

Proposition 1 *Let F be a finite set of non-zero polynomials generating the ideal $I \subset R$ and $\prec$ be an admissible term order. Then there exists a finite set $G \subset I \setminus \{0\}$, a linear order $\sqsubset$ of* lt G *and a* (*maximal*) *admissible involutive division $\mathcal{M}$ on* (lt $G, \sqsubset$) *such that G is a* (*minimal*) *$\mathcal{M}$-involutive basis of I with respect to $\prec$.*

Proof: Algorithm 4 from [1] was proved to compute in a finite number of steps an involutive division $\mathcal{M}$ and a finite set G such that G is an $\mathcal{M}$-involutive basis of I w.r.t. $\prec$. $\sqsubset$ can be even prescribed as e.g. a reverse lexicographical order of the set T but can be altered during the completion process as well. It is an easy exercise to show that $\mathcal{M}$ can be maximized and G be minimized by possibly enlarging the cones $M_{\operatorname{lt} g}$ and by removing redundant elements from G. □

Example 1 (Part 3) Consider $U = \{z^2, yz^2, xyz, y^2, xy^2, x^2, x^2z\}$ and an involutive division defined by the multiplier sets $Y_{z^2} = \{x, z\}$, $Y_{yz^2} = \{z\}$, $Y_{xyz} = \{x, y, z\}$, $Y_{y^2} = \{y, z\}$, $Y_{xy^2} = \{y\}$, $Y_{x^2} = \{x, y\}$, $Y_{x^2z} = \{x\}$.

The set of involutive cones are pairwise disjoint and exhaust lt I, see figure 5. Nevertheless, U is not an $\mathcal{M}$-involutive basis since, similarly to the example from figure 3, $\mathcal{M}$ is not admissible on $(U, \sqsubset)$ for any linear order $\sqsubset$.

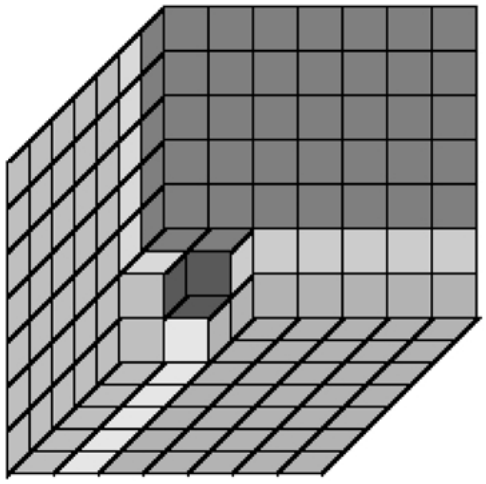

Figure 5. "Non-involutive" basis $\{z^2, yz^2, xyz, y^2, xy^2, x^2, x^2z\}$.

The reason to exclude such situations is that the order $\sqsubset$ is essential for the correctness of the completion algorithm 4 from [1] as well as for the forthcoming Corollary 1 which is one of the main results of this paper.

3. An upper bound criterion for the depth

In the previous section we recalled the existence of minimal $\mathcal{M}$-involutive bases with respect to maximal admissible involutive divisions. Now, we will study the properties of such bases in the case of monomial ideals in view of Conjecture 2. Note, the admissible term order $\prec$ is of no importance in the monomial case. If the reader likes he may assume an arbitrary $\prec$ to be fixed.

If some decomposition (2) arising from an involutive basis satisfies Stanley's Conjecture then, obviously, there is also a minimal $\mathcal{M}$-involutive basis with respect to a maximal admissible involutive division $\mathcal{M}$ having this property. In the rest of the paper by an *$\mathcal{M}$-involutive basis U* of a monomial ideal I we will always mean that U consists of monic monomials, i.e. elements of T, $\mathcal{M}$ is a maximal involutive division admissible on $(U, \sqsubset)$, and U is a minimal $\mathcal{M}$-involutive basis of I. Recall, each monomial ideal I possesses a uniquely determined minimal generating set formed by monic monomials. When the ideal I is clear from the context we will refer to this set by B. Unless stated differently we consider only nontrivial ideals I, i.e. $0 \subset I \subset R$. We start with the definition of some notions which will turn out to be useful during our studies of Stanley's Conjecture.

Definition 4 Let U be an $\mathcal{M}$-involutive basis of I, $u \in U$, and x an $\mathcal{M}$-nonmultiplier of u. An element $v \in U$ is called a *x-witness* for u iff $v \sqsubset u$, $u \nmid v$, $\deg_x v > \deg_x u$ and $\deg_y v \leq \deg_y u$ for all $\mathcal{M}$-nonmultipliers $y \neq x$ of u. If, in addition, $\deg_x v = \deg_x u + 1$ then v is called a *strong x-witness*.

A set $W(u) \subset U$ consisting of a (strong) x-witness for u for each $\mathcal{M}$-nonmultiplier x will be called a (*strong*) *witness set* for u.

Example 1 (Part 4) Recall the ordered $\mathcal{M}$-involutive basis $(z^2, y^2z, xyz, x^2z, y^2, x^2y, x^2)$ with corresponding multiplier sets $Y_{z^2} = \{x, y, z\}$, $Y_{y^2z} = Y_{y^2} = \{x, y\}$ and $Y_{xyz} = Y_{x^2z} = Y_{x^2y} = Y_{x^2} = \{x\}$ displayed in figure 4.

The variables y and z are the $\mathcal{M}$-nonmultipliers of $u = x^2$. The only z-witness for u is the monomial z^2, however, it is not a strong one.

In contrast, x^2z and y^2z are no z-witnesses for u since x^2z is a multiple of u and y^2z has a larger degree than u in both nonmultipliers.

Furthermore, z^2 is a strong z-witness for both $u' = x^2z$ and $u'' = xyz$. The monomial x^2z possesses the strong witness set $W(x^2z) = \{xyz, z^2\}$.

Definition 5 Let U be an $\mathcal{M}$-involutive basis of the ideal I. The *involutive dimension of U with respect to $\mathcal{M}$* is defined by $\operatorname{Idim}_{\mathcal{M}} U := \min_{u \in U} \operatorname{Idim}_{\mathcal{M}} u$. Moreover, the maximum $\operatorname{Idim} I := \max\{\operatorname{Idim}_{\mathcal{M}} U \mid U \text{ is } \mathcal{M}\text{-involutive basis of } I\}$ is called the *involutive dimension of I*.

Remark 1 For arbitrary graded ideals I we have $\operatorname{Idim} I \leq \operatorname{Sdepth} I$.

Recall some obvious facts on involutive bases. Every $\mathcal{M}$-involutive basis U of I contains all minimal generators of I. Any element u of an $\mathcal{M}$-involutive basis U of I possesses a witness set $W(u)$. For the last time let us emphasize our restriction to minimal involutive bases with respect to maximal admissible involutive divisions, both restriction are of course essential in the last statement.

Lemma 1 *Let U be an $\mathcal{M}$-involutive basis of I. Further, assume there exist $u \in U$ and $x \in Y_u$ such that* $\operatorname{Idim}_{\mathcal{M}} u = \operatorname{Idim}_{\mathcal{M}} U$ *and u possesses a strong witness set $W(u)$ satisfying* $\deg_x u > \deg_x v$ *for all $v \in W(u)$.*

Then the $\mathcal{M}$-nonmultipliers of u annihilate a common non-zero element $a \in I/xI$.

Proof: For each $\mathcal{M}$-nonmultiplier y and the corresponding strong y-witness $v_y \in W(u)$ of u define $t_y := \frac{\operatorname{lcm}(u, v_y)}{y}$. By the property of a strong witness it follows that $\deg_y t_y = \deg_y v_y - 1 = \deg_y u$ and $\deg_z t_y = \deg_z u$ for all $\mathcal{M}$-nonmultipliers z of u.

We will show that $a := \operatorname{lcm}_{y \in X \setminus Y_u} t_y$ satisfies the assertions of the lemma. $a \in I$ is obvious, it remains to show $a \notin xI$ and $ya \in xI$ for each $\mathcal{M}$-nonmultiplier y of u. By construction $yt_y = \operatorname{lcm}(u, v_y)$ divides ya. According to our assumptions on x we have $xv_y \mid yt_y$ and, hence, $xv_y \mid ya$. This implies $ya \in xI$, i.e. each $\mathcal{M}$-nonmultiplier y of u annihilates the element a.

Now, assume $a \in xI$, i.e. $\frac{a}{x} \in I$. Since U is an $\mathcal{M}$-involutive basis there exists $w \in U$ such that $\frac{a}{x} \in M_w = w\langle Y_w \rangle$. Hence, $\frac{a}{x} = ws$ for some monomial $s \in \langle Y_w \rangle$. By construction of a we have $\deg_x a = \deg_x u$ and $\deg_y a = \deg_y u = \deg_y v_y - 1$ for all $y \in X \backslash Y_u$. Since $a = wxs$ this implies

$$\deg_x w < \deg_x a = \deg_x u \quad \text{and} \tag{3}$$

$$\deg_y w \leq \deg_y a = \deg_y u = \deg_y v_y - 1 \quad \text{for all } y \in X \setminus Y_u. \tag{4}$$

$\frac{a}{u}$ belongs to the monoid $\langle Y_u \rangle$, i.e. $a \in M_u$, because of (4). Moreover $u \neq w$ according to (3). Consequently, x must be an $\mathcal{M}$-nonmultiplier of w since, otherwise, $a \in M_w \cap M_u$ in contradiction to the assumption that U is a (minimal) $\mathcal{M}$-involutive basis of I. Nevertheless, at least we have $a \in M_u \cap wT$ which in view of the properties of $\mathcal{M}$ implies $u \sqsubset w$ and $v_y \sqsubset u \sqsubset w$ taking into account also the witness property of v_y. Now, if some $\mathcal{M}$-nonmultiplier y of u would be an $\mathcal{M}$-multiplier of w then we had $y\frac{a}{x} \in M_w \cap v_y T$. This is again a contradiction since neither $w \sqsubset v_y$ nor $M_{v_y} \subseteq M_w$ are possible.

In conclusion, $a \in xI$ would imply $\operatorname{Idim}_{\mathcal{M}} w \leq \operatorname{Idim}_{\mathcal{M}} u - 1$ in contradiction to the construction of u. □

Corollary 1 *Let $I \subset R$ be a monomial ideal. If there exist an $\mathcal{M}$-involutive basis U of I, an element $u \in U$ of minimal involutive dimension with respect to $\mathcal{M}$, an $\mathcal{M}$-multiplier $x \in Y_u$ of u, and a strong witness set $W(u)$ for u such that* $\deg_x u > \deg_x v_y$ *for all $v_y \in W(u)$ then it holds* $\operatorname{depth} I \leq \operatorname{Sdepth} I$.

Proof: From [7, Theorem 127] we deduce

$$\begin{aligned}\operatorname{depth}(XR, I) &\leq \operatorname{depth}((X \setminus Y_u)R + xR, I) + |Y_u \setminus \{x\}| \\ &= \operatorname{depth}((X \setminus Y_u)R + xR, I) + \operatorname{Idim}_{\mathcal{M}} u - 1.\end{aligned}$$

Since x is a non-zero-divisor on I and $x \in Y_u$ we have

$$\operatorname{depth}((X \setminus Y_u)R + xR, I) = \operatorname{depth}((X \setminus Y_u)R, I/xI) + 1.$$

The ideal $(X \setminus Y_u)R$ does not contain non-zero-divisors on I/xI according to Lemma 1, consequently, $\operatorname{depth}((X \setminus Y_u)R, I/xI) = 0$. Combining these relations and taking into account Remark 1, finally, yields

$$\operatorname{depth}(XR, I) \leq \operatorname{Idim}_{\mathcal{M}} u = \operatorname{Idim}_{\mathcal{M}} U \leq \operatorname{Idim} I \leq \operatorname{Sdepth} I. \qquad \square$$

In particular, Conjecture 2 holds for any monomial ideal I satisfying the assumptions of Corollary 1. Let us illustrate the statements of the lemma and its corollary by some examples.

Example 2 Any Borel-fixed monomial ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ possesses an involutive basis satisfying the assumptions of Corollary 1 and, hence, satisfies $\operatorname{Sdepth} I \geq \operatorname{Idim} I \geq \operatorname{depth} I$.

Let t denote the largest index such that x_t divides a minimal generator of I. Using [4, Theorem 15.23b] it is easy to show that the monomial ideal I possesses a minimal generator of the form $m_i = x_1^{\alpha_i} x_i^{\beta_i}$, where $\beta_i > 0$, for each $i \in \{1, \ldots, t\}$.

Now, consider an arbitrary $\mathcal{M}$-involutive basis U of I, where $\mathcal{M}$ is admissible for $(U, \sqsubset)$ with respect to an order $\sqsubset$ which is compatible with the pure lexicographical order $\prec$ extending $x_n \prec \cdots \prec x_1$ and U (for the compatibility notion see Definition 7). Obviously, $\{m_1, \ldots, m_t\} \subseteq U$ and $m_t \sqsubset m_{t-1} \sqsubset \cdots \sqsubset m_1$. This implies $M_{m_1} \cap m_i T = \emptyset$ for all $i \in \{2, \ldots, t\}$ and, consequently, each x_i, $i \in \{2, \ldots, t\}$, must be an $\mathcal{M}$-nonmultiplier of m_1. Hence, $\operatorname{Idim}_{\mathcal{M}} U \leq \operatorname{Idim}_{\mathcal{M}} m_1 \leq n - t + 1$. The maximal admissibility of $\mathcal{M}$ for

$(U, \sqsubset)$ ensures that all variables x_j, $t < j \leq n$, are $\mathcal{M}$-multipliers for each element of U. Moreover, $x_1 \in Y_u$ for all $u \in U$ according to the forthcoming Lemma 4, condition 1. In summary we obtain $\mathrm{Idim}_{\mathcal{M}} U = \mathrm{Idim}_{\mathcal{M}} m_1 = n - t + 1$.

Finally, apply Corollary 1 with $u = m_1$ and $x = x_1$ and the first statement of the example follows.

The above example refers to a well-known classical situation. Conjecture 1 is known to be true in the particular case $n = 1$ [13], where decompositions due to [9] or [2] can be used. It is the nature of Borel-fixed ideals that the identity map can serve as the linear variable transformation occurring in Rees' method. So the decomposition obtained for the $\mathbb{N}$-grading satisfies even the stronger conditions posed by the $\mathbb{N}^n$-grading. We remark, that in general Rees' approach requires an infinite field $\mathbb{K}$, but in the particular situation of Borel-fixed ideals it will work even for arbitrary fields $\mathbb{K}$.

Example 3 Let us consider once more the ideal $I = (xyz, x^2, y^2, z^2) \subset \mathbb{K}[x, y, z]$ from Example 1. An involutive basis of I is $\{z^2, y^2z, x^2yz, x^2z, y^2, x^2y, x^2, xyz\}$, where the elements are in increasing order with respect to $\sqsubset$ and the involutive division is defined by the multiplier sets $Y_{z^2} = \{x, y, z\}$, $Y_{y^2z} = Y_{y^2} = \{x, y\}$, $Y_{x^2yz} = Y_{x^2z} = Y_{x^2y} = Y_{x^2} = \{x\}$, and $Y_{xyz} = \emptyset$. Corollary 1 is not applicable to this involutive basis since $\mathrm{Idim}_{\mathcal{M}}(xyz) = 0$.

Recall the involutive basis $\{z^2, y^2z, xyz, x^2z, y^2, x^2y, x^2\}$ of I discussed in parts 2 and 4 of Example 1. Consider $u = x^2z$ and its strong witness set $W(x^2z) = \{xyz, z^2\}$. Since x^2z has larger degree in its $\mathcal{M}$-multiplier x than both witnesses we can apply Corollary 1. Therefore, $\mathrm{depth}\, I = 1$ and $I = z^2\mathbb{K}[x, y, z] \oplus y^2z\mathbb{K}[x, y] \oplus y^2\mathbb{K}[x, y] \oplus xyz\mathbb{K}[x] \oplus x^2z\mathbb{K}[x] \oplus x^2y\mathbb{K}[x] \oplus x^2\mathbb{K}[x]$ is a Stanley decomposition.

The decomposition $I = z^2\mathbb{K}[x, z] \oplus yz^2\mathbb{K}[z] \oplus xyz\mathbb{K}[x, y, z] \oplus y^2\mathbb{K}[y, z] \oplus xy^2\mathbb{K}[y] \oplus x^2\mathbb{K}[x, y] \oplus x^2z\mathbb{K}[x]$ illustrated in figure 5 is a Stanley decomposition of I, too. While the methods developed in this paper do not apply to such decompositions the next two examples will show their importance in view of Stanley's Conjecture.

Example 4 Consider the ideal $I = (yu, xu, yz, xz) \subset \mathbb{K}[x, y, z, u]$. Based on the reverse lexicographical order extending $u \sqsubset z \sqsubset y \sqsubset x$ the minimal basis is already an $\mathcal{M}$-involutive basis, where $Y_{yu} = \{x, y, z, u\}$, $Y_{xu} = \{x, z, u\}$, $Y_{yz} = \{x, y, z\}$, and $Y_{xz} = \{x, z\}$ are the sets of multipliers. xz has minimal involutive dimension but neither its degree in x nor in z is strictly larger than those of both witnesses yz and xu. Note, yu cannot be used as a witness since it has higher degree in both nonmultipliers of xz. Hence, Corollary 1 is not applicable. It is easy to see that the minimal basis is the only (minimal) involutive basis of I. Considering all 24 possible linear orders of the 4 elements we obtain different sets of multipliers. Each time the minimal involutive dimension is 2 and none of the settings fulfills the assumptions of Corollary 1. Since $\mathrm{depth}\, I = 2$ the decomposition $I = yu\mathbb{K}[x, y, z, u] \oplus xu\mathbb{K}[x, z, u] \oplus yz\mathbb{K}[x, y, z] \oplus xz\mathbb{K}[x, z]$ satisfies Stanley's conditions. However, also $I = xyzu\mathbb{K}[x, y, z, u] \oplus yu\mathbb{K}[x, y, u] \oplus xu\mathbb{K}[x, z, u] \oplus yz\mathbb{K}[y, z, u] \oplus xz\mathbb{K}[x, y, z]$ is a Stanley decomposition of I and, hence, $\mathrm{Sdepth}\, I = 3 > \mathrm{Idim}\, I = 2$.

Note, the different natures of Examples 3 and 4. In both cases involutive bases proved to be suitable to construct a decomposition satisfying Stanley's Conjecture. But, while in the first example the decomposition comes along with a proof that it is of Stanley's type this is not so obvious in the second example. One could say, involutive bases are not optimal but still sufficient, i.e. depth $I \le$ Idim I, in Example 4. The next example will show the existence of monomial ideals where depth $I >$ Idim I.

Example 5 [10, Remark 3] Consider $I = (yvw, xvw, zuw, xuw, yzw, zuv, yuv, xzv, xyu, xyz) \subset \mathbb{K}[x, y, z, u, v, w]$. For any order $\sqsubset$ the largest minimal generator of I will have only three multipliers, namely, the three variables occurring in this generator. Hence no involutive basis can provide an involutive dimension higher than 3. An involutive basis yielding exactly 3 can be obtained e.g. by adding the elements $xuvw$, $yuvw$ and $zuvw$ to the minimal basis and ordering using the reverse lexicographical order extending $w \sqsubset \cdots \sqsubset x$. We obtain the decomposition $I = xyz\mathbb{K}[x, y, z] \oplus xyu\mathbb{K}[x, y, z, u] \oplus xzv\mathbb{K}[x, y, z, v] \oplus yuv\mathbb{K}[x, y, u, v] \oplus zuv\mathbb{K}[x, y, z, u, v] \oplus yzw\mathbb{K}[x, y, z, w] \oplus xuw\mathbb{K}[x, y, u, w] \oplus zuw\mathbb{K}[x, y, z, u, w] \oplus xvw\mathbb{K}[x, z, v, w] \oplus yvw\mathbb{K}[x, y, z, v, w] \oplus xuvw\mathbb{K}[x, u, v, w] \oplus yuvw\mathbb{K}[x, y, u, v, w] \oplus zuvw\mathbb{K}[x, y, z, u, v, w]$.

However, depth $I = 4 >$ Idim $I = 3$. But the decomposition $I = xyz\mathbb{K}[x, y, z, w] \oplus xyu\mathbb{K}[x, y, z, u] \oplus xzv\mathbb{K}[x, y, z, v] \oplus yuv\mathbb{K}[x, y, u, v] \oplus zuv\mathbb{K}[y, z, u, v] \oplus yzw\mathbb{K}[y, z, v, w] \oplus xuw\mathbb{K}[x, y, u, w] \oplus zuw\mathbb{K}[x, z, u, w] \oplus xvw\mathbb{K}[x, z, v, w] \oplus yvw\mathbb{K}[x, y, v, w] \oplus xuvw\mathbb{K}[x, u, v, w] \oplus yuvw\mathbb{K}[y, u, v, w] \oplus zuvw\mathbb{K}[z, u, v, w] \oplus yzuw\mathbb{K}[y, z, u, w] \oplus xzuv\mathbb{K}[x, z, u, v] \oplus xzuvw\mathbb{K}[x, z, u, v, w] \oplus xyuvw\mathbb{K}[x, y, u, v, w] \oplus yzuvw\mathbb{K}[x, y, z, u, v, w]$ illustrates the equality Sdepth $I = 4 =$ depth I and Stanley's Conjecture holds.

Example 5 shows that there is no hope to prove Stanley's Conjecture only by means of involutive bases (in the sense of [1]). However, there are at least some interesting subclasses of monomial ideals for which the validity of Stanley's Conjecture follows from Corollary 1. Such situations will be studied now.

4. Ideals of small depth (depth $I \le 2$)

We will prove Stanley's Conjecture for monomial ideals of small depth. First of all, we want to prove that any monomial ideal has an involutive basis such that each basis element has at least one multiplier. In fact, Example 3 shows that not all involutive bases have this property but at least for any reverse lexicographical order $\sqsubset$ this will turn out to be true.

Lemma 2 *Let $I \subset \mathbb{K}[X]$ be a monomial ideal, $\sqsubset$ a reverse lexicographical order, and U an $\mathcal{M}$-involutive basis of I, where $\mathcal{M}$ is admissible for $(U, \sqsubset)$. Then all sets Y_u, $u \in U$, are non-empty.*

Proof: Assume, there exists $u \in U$ such that $Y_u = \emptyset$. By maximality of $\mathcal{M}$ there exists a x-witness v_x for u, where x is the maximal element of X (w.r.t. $\sqsubset$). The witness property yields on the one hand $\deg_x u < \deg_x v_x$ and $\deg_y u \ge \deg_y v_x$ for all $y \ne x$. But on the other hand it also requires $v_x \sqsubset u$, i.e. $\deg_z u < \deg_z v_x$ for the smallest variable $z \in X$ (w.r.t. $\sqsubset$) for which the degrees of u and v_x are different. This implies $x = z$ and $u \mid v_x$, a contradiction to the witness property. $\square$

Corollary 2 *Any monomial ideal I of depth 1 possesses a Stanley decomposition.*

Definition 6 A monomial ideal I is called *involutively irreducible* if for any involutive division $\mathcal{M}$ and any $\mathcal{M}$-involutive basis U of I the maximal element u of U (w.r.t. the underlying order $\sqsubset$ of $\mathcal{M}$) satisfies $\operatorname{Idim}_{\mathcal{M}} u \leq \operatorname{Idim} I$.

Let $b \in B$ be a minimal generator of I. For any involutive division $\mathcal{M}$ and any $\mathcal{M}$-involutive basis U of I such that b is the maximal element of the minimal generating set B (and, hence, also of the full involutive basis U) with respect to $\sqsubset$ we have $\operatorname{Idim}_{\mathcal{M}} b \leq \dim A_b$, where A_b denotes the ideal $(B \setminus \{b\}) : (b)$ and $\dim A_b$ the Krull-dimension of R/A_b [1, Theorem 3.1]. An immediate consequence is

$$\operatorname{Idim} I \leq \max_{b \in B} \dim A_b \tag{5}$$

and equality holds if and only if I is involutively irreducible. Moreover, we have

$$\operatorname{Idim} I \leq \max_{b \in B'} \dim A'_b \tag{6}$$

for each subset $B' \subseteq B$ containing at least two elements, where $A'_b := (B' \setminus \{b\}) : (b)$. This fact follows easily since the elements of B' have to be placed in some relative order by $\sqsubset$.

Lemma 3 *Any monomial ideal I can be decomposed in a direct sum*

$$I = J \oplus \left(\bigoplus_{i=1}^{t} u_i \mathbb{K}[Y_i] \right), \tag{7}$$

where J is an involutively irreducible ideal, t is a non-negative integer, and $|Y_i| > \operatorname{Idim} I$ for all $i = 1, \ldots, t$.

Proof: For involutively irreducible I the setting $J := I$ and $t := 0$ fulfills the assertion.

So assume, that there exists an $\mathcal{M}$-involutive basis U of I such that $\operatorname{Idim}_{\mathcal{M}} v \geq \operatorname{Idim} I$ for all $v \in U$ and the inequality is strict for the maximal element $u = \max_{\sqsubset} U$. Recall that the maximal element of any $\mathcal{M}$-involutive basis U of I is always a minimal generator of I. I can be written as a direct sum $(U \setminus \{u\})R \oplus u\mathbb{K}[Y_u]$. If the ideal $(U \setminus \{u\})R$ is involutively irreducible we are done. Otherwise, we proceed to decompose $I' := (U \setminus \{u\})R$ in the above way.

There remain two questions. At first we have to show $\operatorname{Idim} I' \geq \operatorname{Idim} I$. But this is obvious, because $U \setminus \{u\}$ is an $\mathcal{M}$-involutive basis of I' since $M_u \cap vT = \emptyset$ for all $v \in U \setminus \{u\}$. At second we have to show that the process terminates. However, all monomials contained in an involutive basis of I divide the least common multiple of the minimal generators of I. Therefore, the number of iterations is bounded by the number of monomials dividing this least common multiple. $\square$

The ideals considered in Examples 3–5 are involutively irreducible.

Example 6 Consider the ideal $I \subset \mathbb{K}[x, y, z, u]$ generated by the set $B = \{yx^2, y^2, xyz^2, u^2, xyzu, yz^2u\}$. While inequality (5) yields $\operatorname{Idim} I \leq \max_{b \in B} \dim A_b = \dim A_{u^2} =$

$\dim(yx^2, y^2, xyz, yz^2) = 3$ the better bound $\mathrm{Idim}\, I \leq \max_{b \in B'} \dim A'_b = \dim A'_{y^2} = \dim(x^2, xz^2, xzu, z^2u) = 2$ can be obtained by application of inequality (6) to the subset $B' = \{yx^2, y^2, xyz^2, xyzu, yz^2u\}$. Hence, I is not involutively irreducible and by the algorithm in the proof of Lemma 3 we get after one step $I = (yx^2, y^2, xyz^2, yu^2, xyzu, yz^2u) \oplus u^2\mathbb{K}[x, z, u]$. Note, that the first summand is not simply the ideal generated by B' but that yu^2 had to be added to the generators. The best bound we can get using (5) and (6) for $J = (yx^2, y^2, xyz^2, yu^2, xyzu, yz^2u)$ is $\mathrm{Idim}\, J \leq 2$

Suppose, $\mathrm{Idim}\, J = 2$. y^2 is the only candidate for the largest (w.r.t. $\sqsubset$) element of an involutive basis of J of involutive dimension 2. In addition, $J' = (yx^2, xy^2, uy^2, xyz^2, yu^2, xyzu, yz^2u)$ has to satisfy $\mathrm{Idim}\, J' = 2$. But, $\mathrm{Idim}\, J' \leq 1$ by (5), a contradiction.

Therefore, we have $\mathrm{Idim}\, I = \mathrm{Idim}\, J = 1$ and J is still not involutively irreducible. Finally, $I = (yx^2, xy^2, uy^2, xyz^2, yu^2, xyzu, yz^2u) \oplus y^2\mathbb{K}[y, z] \oplus u^2\mathbb{K}[x, z, u]$ is a decomposition of type (7).

The importance of reverse lexicographical orders for the theory of involutive bases has been demonstrated in [1]. But, in view of Lemma 1 reverse lexicographical orders have a serious drawback. They do not provide any control to put elements of high degree in a certain variable at the top of an involutive basis. For this purpose we would prefer to use a pure lexicographical order $\sqsubset$. However, if a monoid well-order $\sqsubset$ of T is applied then admissible involutive divisions, and hence involutive bases, will not exist in most cases. We will introduce a special class of linear orders on finite monomial sets which carry the advantages and avoid the drawbacks of the above two order types.

Definition 7 Let $U \subset T$ be a finite set of monomials and $\prec$ a monoid well-order of T. A linear order $\sqsubset$ on U is called *compatible with $\prec$ and U* iff it satisfies the following three conditions

1. $u \prec v \iff u \sqsubset v$ for all elements $u, v \in U$ such that U contains no proper divisor of either of them,
2. $u \sqsubset v$ for all proper divisors $v \in U$ of $u \in U$,
3. whenever $u \sqsubset v$ and $v \prec u$ holds for two elements $u, v \in U$ such that $v \nmid u$ then there exists $w \in U$ such that $w \mid u$, $w \prec v$, and $w \sqsubset v$.

In what follows by using the symbol $\sqsubset_x$ for a linear order on a finite set $U \subset T$ we will indicate that this order is compatible with some pure lexicographical order (always denoted by $\prec_x$ in this context) with largest variable $x \in X$ and the set U. Note, according to this convention the ordered set $(V, \sqsubset_x)$ is not obtained as the restriction of the ordered set $(U, \sqsubset_x)$ for a proper subset V of U. But since the corresponding set will be always clear from the context we use this short cut symbol for the sake of simplicity.

Lemma 4 *Let U be an $\mathcal{M}$-involutive basis of the monomial ideal I, where $\mathcal{M}$ is admissible on $(U, \sqsubset_x)$. Then the following three conditions hold*:

1. $x \in Y_v$ *for all* $v \in U$,
2. $\deg_x u \leq \deg_x v$ *for all* $u, v \in U$ *such that* $u \sqsubset_x v$,
3. $\deg_x u = \deg_x v$ *for all* $u, v \in U$ *such that* $v \mid u$.

Proof:

Condition 1. Suppose there exists $u \in U$ such that $x \notin Y_u$ and assume that u is minimal with respect to $\sqsubset_x$ among all monomials of U having this property. By maximality of the involutive division $\mathcal{M}$ there exists a x-witness $v \in U$ for u. Hence, $v \sqsubset_x u$ and $\deg_x u < \deg_x v$. The properties of $\sqsubset_x$ imply the existence of $w \in U$ such that $w \mid v$, $w \prec_x u$, and $w \sqsubset_x u$. We deduce $\deg_x w \leq \deg_x u < \deg_x v$ and, therefore, $\frac{v}{x} \in I$. Since U is an $\mathcal{M}$-involutive basis there exists $t \in U$ such that $\frac{v}{x} \in M_t$. We must have $x \notin Y_t$ since, otherwise, $v \in M_t$ in contradiction to the minimality of the $\mathcal{M}$-involutive basis U. By construction of u this implies $u \sqsubseteq_x t$. Consequently, $w \sqsubset_x t$ and $\frac{v}{x} \in M_t \cap wT$ in contradiction to the admissibility of $\mathcal{M}$ on $(U, \sqsubset_x)$.
Condition 2. Suppose there are monomials $u, v \in U$ such that $u \sqsubset_x v$ and $\deg_x u > \deg_x v$. By the properties of $\sqsubset_x$ it follows $v \prec_x u$ and the existence of $w \in U$ such that w divides u and $w \prec_x v$. We deduce $\deg_x w \leq \deg_x v < \deg_x u$ and, hence, $\frac{u}{x} \in I$. Consequently, there exists $t \in U$ such that $\frac{u}{x} \in M_t$. We have $x \in Y_t$ according to 1 and, therefore, $u \in M_t$, a contradiction to the minimality of the $\mathcal{M}$-involutive basis U.
Condition 3. $v \mid u$ implies $u \sqsubseteq_x v$ and, hence, $\deg_x u \leq \deg_x v$ according to 2. Now, the assertion follows immediately. $\square$

Lemma 5 *Let I be an involutively irreducible monomial ideal $I \subset R$ of* $\operatorname{Idim} I = 1$. *If U is an $\mathcal{M}$-involutive basis of I, where $\mathcal{M}$ is admissible on $(U, \sqsubset_x)$, then the maximal variable $x = \max_{\prec_x} X$ is the only $\mathcal{M}$-multiplier of the maximal element $u = \max_{\sqsubset_x} U$. Moreover, for each variable $y \neq x$ there exists a y-witness v_y for u which is a minimal generator of I and has a strictly smaller degree in x than u.*

Proof: Suppose, $\deg_x v_y = \deg_x u$ for all minimal generators $v_y \in B$ which are a y-witness for u and consider an arbitrary y-witness $v_y \in B$ for u which has maximal degree in y.[2] Then no minimal generator $v \in B \setminus \{v_y\}$ can satisfy $\frac{\operatorname{lcm}(v_y, v)}{v_y} \in \langle x, y\rangle$. Consequently, $\{x, y\}$ is an algebraically independent set for the ideal quotient $A_{v_y} := (B \setminus \{v_y\}) : (v_y)$. Hence, $\dim A_{v_y} \geq 2$ in contradiction to I being involutively irreducible. $\square$

Lemma 5 brings us close to Corollary 1. But there is still a gap, namely, we have not proved yet that there are enough strong witnesses for u.

Lemma 6 *Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an involutively irreducible monomial ideal. Further, let* $\operatorname{Idim} I = 1$ *and U an $\mathcal{M}$-involutive basis of I, where $\mathcal{M}$ is admissible on $(U, \sqsubset_{x_1})$. Then U contains an element u whose only $\mathcal{M}$-multiplier is x_1 and which for each $i \in \{2, \ldots, n\}$ possesses a strong x_i-witness v_i satisfying* $\deg_{x_1} u > \deg_{x_1} v_i$.

Proof: First of all, $x_1 \in Y_v$ for all $v \in U$ according to Lemma 4, condition 1. Let $u \in U$ be the minimal (w.r.t. $\sqsubset_{x_1}$) element such that for all $i \in \{2, \ldots, n\}$ there exists a x_i-witness v_i for u which has the property $\deg_{x_1} u > \deg_{x_1} v_i$. In addition, for each $i \in \{2, \ldots, n\}$ let the x_i-degree of the witness v_i be minimal among all possible choices. According to Lemma 5 such elements u and v_i exist.

Suppose, $k := \deg_{x_i} v_i - \deg_{x_i} u > 1$ for some $i \in \{2, \ldots, n\}$ and consider the element $w \in U$ such that $ux_i^{k-1} \in M_w$. From $M_w \cap uT \neq \emptyset$ we obtain $w \sqsubset u$. Further, we deduce $w \nmid u$ and $\deg_{x_i} v_i > \deg_{x_i} w > \deg_{x_i} u$. We must have $\deg_{x_1} w = \deg_{x_1} u$ since in case $\deg_{x_1} w < \deg_{x_1} u$ one observes easily that w would be a x_i-witness of u which contradicts the choice of v_i. Moreover, $v_j \sqsubset_{x_1} w$ for all $j \in \{2, \ldots, n\}$ according to Lemma 4, condition 2. By admissibility of $\mathcal{M}$ we must have $M_w \cap v_j T = \emptyset$ for all $j = 2, \ldots, n$. In particular, no variable x_j, $j = 2, \ldots, n$, can be an $\mathcal{M}$-multiplier of w since otherwise, $ux_i^{k-1}x_j^l \in M_w \cap v_j T$, where l is the positive integer originating from the equation $\mathrm{lcm}\,(u, v_j) = ux_j^l$. Since $ux_i^{k-1} \in M_w$ requires $\deg_{x_j} u = \deg_{x_j} w$ for all $\mathcal{M}$-nonmultipliers $x_j \neq x_i$ of w, we can deduce that all v_j, $j = 2, \ldots, n$, are also x_j-witnesses for w in contradiction to the choice of u.

In conclusion, we proved $\deg_{x_i} v_i = \deg_{x_i} u + 1$ for all $i \in \{2, \ldots, n\}$ for the chosen u and v_i, $i \in \{2, \ldots, n\}$. □

Corollary 3 *Any monomial ideal $I \subset \mathbb{K}[X]$ of* depth $I \geq 2$ *has at least involutive dimension* 2.

Proof: Assume the contrary, i.e. Idim $I = 1$ for some ideal of depth at least 2. Decompose I in a direct sum $I = J \oplus (\bigoplus_{i=1}^{t} u_i \mathbb{K}[Y_i])$ according to Lemma 3 and apply Lemma 6 to the involutively irreducible part J. This yields an $\mathcal{N}$-involutive basis V of J satisfying the assumptions of Lemma 1. There is an obvious involutive division $\mathcal{M}$ such that the union $U = V \cup \{u_1, \ldots, u_t\}$ becomes an $\mathcal{M}$-involutive basis of the ideal I which satisfies the assumptions of Lemma 1, too. Hence, by Corollary 1 we deduce depth $I \leq 1$ in contradiction to our assumptions on I. □

5. The 3-variate case

Applying our results from the previous section it is now an easy exercise to prove Conjecture 2 for the first non-trivial case $n = 3$.

Theorem 1 *Let $R = \mathbb{K}[x_1, x_2, x_3]$ be a three-variate polynomial ring over a field $\mathbb{K}$. Then for any non-zero monomial ideal $I \subset R$ we have*

$$\mathrm{Sdepth}\, I \geq \mathrm{Idim}\, I \geq \mathrm{depth}\, I =: d,$$

i.e. there exist an involutive division $\mathcal{M}$ generated by a family $Y = (Y_t)_{t \in T}$ and an $\mathcal{M}$-involutive basis U of I such that $|Y_u| \geq d$ for all $u \in U$. Moreover, $I = \bigoplus_{u \in U} u\mathbb{K}[Y_u]$ is a Stanley decomposition of I.

Proof: The only possible values for d are 1, 2 or 3. For $d = 1$ and $d = 2$ the assertion follows from Corollary 2 and Corollary 3, respectively. The case $d = 3$ is trivial, since I has to be a principal ideal for which each $\mathcal{M}$-involutive basis is a singleton. □

For the most interesting case of monomial ideals I in three variables, i.e. depth $I = 2$, we only proved the existence of a "good" involutive decomposition. But since the given proof is indirect and not at all constructive, unless we have further a-priori information we are left with the brute force algorithm which consists in applying Algorithm ([1], figure 4) using different strategies and orders until an involutive basis of involutive dimension greater or equal 2 or allowing the application of Corollary 1 is found. At least, termination of this method is clear since there is only a finite number of involutive bases of I.

6. Generic monomial ideals and more

Definition 8 Let $I \subset \mathbb{K}[X]$ be a monomial ideal with minimal generators $m_1, \ldots, m_l$. I is called *generic* if the following condition holds: if two distinct minimal generators m_i and m_j have the same positive degree in some variable $x \in X$, there is a third minimal generator m_k which strictly divides $\operatorname{lcm}(m_i, m_j)$, i.e. $\deg_y m_k \leq \deg_y \operatorname{lcm}(m_i, m_j)$ for all $y \in X$ and the inequality is strict for all variables y such that $\deg_y \operatorname{lcm}(m_i, m_j) > 0$.

Here, we follow the notion of generic monomial ideals due to [8] which refines the original notion introduced in [3]. Typically, the so-defined generic monomial ideals are far from being (their own) generic initial ideals [4]. The latter are covered by our Example 2.

Lemma 7 *Let U be an $\mathcal{M}$-involutive basis of the monomial ideal I, where $\mathcal{M}$ is admissible on $(U, \sqsubset_x)$. Furthermore, assume that I has the following property: for any two distinct minimal generators m_i and m_j such that $\deg_x m_i = \deg_x m_j =: d$ there is a third minimal generator m_k which divides $\operatorname{lcm}(m_i, m_j)$ and satisfies $\deg_x m_k < d$.*

Then the minimal element $u \in U$ with respect to $\sqsubset_x$ which satisfies $\operatorname{Idim}_{\mathcal{M}} u = \operatorname{Idim}_{\mathcal{M}} U$ possesses a strong set of witness $W(u)$ such that $\deg_x u > \deg_x v_y$ for all $v_y \in W(u)$.

Proof: First of all, we will show that for each $\mathcal{M}$-nonmultiplier y of u there exists a y-witnesses v_y for u which has smaller x-degree than u. According to Lemma 4, condition 2 we have $\deg_x v_y \leq \deg_x u$ for all $\mathcal{M}$-nonmultipliers y and all y-witnesses v_y for u.

Consider an arbitrary y-witness v_y for u and assume $\deg_x v_y = \deg_x u$. Then there exist minimal generators m_i, m_j of I such that $m_i \mid u$ and $m_j \mid v_y$. According to Lemma 4, condition 3 we have $\deg_x m_i = \deg_x m_j = \deg_x v_y = \deg_x u =: d_x$. The assumptions on I ensure the existence of a minimal generator m_k of I which divides $\operatorname{lcm}(m_i, m_j)$ and satisfies $\deg_x m_k < d_x$. From Lemma 4, condition 2 we deduce $m_k \sqsubset_x u$. Moreover, $\deg_z m_k \leq \deg_z \operatorname{lcm}(m_i, m_j) \leq \deg_z \operatorname{lcm}(u, v_y) = \deg_z u$ for all $\mathcal{M}$-nonmultipliers $z \neq y$ of u. Furthermore, $\deg_y u < \deg_y m_k$ since, otherwise, $M_u \cap m_k T \neq \emptyset$. In summary, we deduce that m_k is a y-witness for u whose x-degree is smaller than that of u.

Hence, for each $\mathcal{M}$-nonmultiplier y of u there exists a y-witness v_y for u such that $\deg_x u > \deg_x v_y$. What remains to show is that there is even a strong one. Let $W(u)$ be a set of witness for u which for each $\mathcal{M}$-nonmultiplier y of u contains a y-witness v_y for u satisfying $\deg_x u > \deg_x v_y$ and having minimal y-degree among all these y-witnesses.

Suppose, $r := \deg_y v_y - \deg_y u > 1$ for some $v_y \in W(u)$. Let $t \in \langle Y_u \rangle$ be an arbitrary monomial such that $\deg_\xi ut \geq \deg_\xi v_z$ for all $\mathcal{M}$-multipliers $\xi \in Y_u$ of u and all witnesses $v_z \in W(u)$ and consider $w \in U$ such that $uty^{r-1} \in M_w$. From $M_w \cap uT \neq \emptyset$ we deduce $w \sqsubset_x u$. Moreover, $\deg_y w > \deg_y u$ since, otherwise, $M_u \cap wT \neq \emptyset$. Hence, the case $w \mid u$

is impossible. Next, we will show that the supposition $w \nmid u$ leads to a contradiction as well. In this case w would be a y-witness for u and, therefore, $\deg_x w = \deg_x u$ by the choice of v_y. Application of Lemma 4, condition 2 yields $v_z \sqsubset_x w$ for all $\mathcal{M}$-nonmultipliers z of u. No $\mathcal{M}$-nonmultiplier z of u can be an $\mathcal{M}$-multiplier of w since, otherwise, we had the contradiction $M_w \cap v_z T \neq \emptyset$. Therefore, $\operatorname{Idim}_{\mathcal{M}} w \leq \operatorname{Idim}_{\mathcal{M}} u$ and $w \sqsubset_x u$ in contradiction to the construction of u.
In conclusion, we observed $\deg_y v_y - \deg_y u = 1$ for all $v_y \in W(u)$. □

Theorem 2 *Let $I \subset \mathbb{K}[X]$ be a monomial ideal. Furthermore, assume that for some variable $x \in X$ the ideal I has the following property: for any two distinct minimal generators m_i and m_j such that* $\deg_x m_i = \deg_x m_j =: d$ *there is a third minimal generator m_k which divides* $\operatorname{lcm}(m_i, m_j)$ *and satisfies* $\deg_x m_k < d$.

Then it holds $\operatorname{depth} I \leq \operatorname{Idim} I \leq \operatorname{Sdepth} I$.

Proof: Immediate consequence of Corollary 1 and Lemma 7. □

In particular, the statement of this theorem applies to all generic monomial ideals. Let us draw the attention to a surprising fact. We never had to assume that U has maximal involutive dimension. Hence, any ordered $\mathcal{M}$-involutive basis $(U, \sqsubset_x)$ of a monomial ideal I which is generic with respect to the variable $x \in X$ provides a Stanley decomposition of I. So, in spite of our purely existential proofs, we are in a very comfortable position when the algorithmic construction of Stanley decompositions is concerned. We just need to run Algorithm [1, figure 4] once by ordering the intermediate bases always by an order $\sqsubset_x$ which is compatible with a reverse lexicographical term order $\prec_x$ in the sense of Definition 7.

Acknowledgment

I thank Bernd Sturmfels for drawing my attention to this subject and for helpful comments on it. Moreover, I am thankful to Ralf Hemmecke for his support on preparing the figures in this paper and to the anonymous referees for lots of helpful remarks and hints.

I am thankful to the Mathematical Sciences Research Institute[3] and the Mathematics Department of the University of California at Berkeley for their hospitality during the one year research visit in which the work on this project was done. Last but not least I am grateful to the Deutsche Forschungsgemeinschaft (DFG) for supporting the visit by a research fellowship.

Notes

1. All figures are produced using the package Calix authored by Ralf Hemmecke. Note, equal gray values mark equally directed cones. White cubes correspond to ordinary multiples of the monomials which are not contained in any of the involutive cones.
2. Note, the assumption $u = \max_{\sqsubset_x} U$ is essential at this point since, otherwise, all y-witnesses may belong to $U \setminus B$.
3. Research at MSRI is supported in part by NSF grant DMS-9701755.

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