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Lyubeznik's Resolution and Rooted Complexes

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Abstract. We describe a new family of free resolutions for a monomial ideal I, generalizing Lyubeznik's construction. These resolutions are cellular resolutions supported on the rooted complexes of the lcm-lattice of I. Our resolutions are minimal for the matroid ideal of a finite projective space.

Keywords: cellular resolutions, lcm-lattice, geometric lattice, matroid ideal

A basic problem of combinatorial commutative algebra is a construction of explicit (minimal) free resolutions for a monomial ideal I in a polynomial ring R, that is, exact sequences of R-modules:

$$0 \to \Lambda_t \xrightarrow{\partial} \Lambda_{t-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Lambda_0 \xrightarrow{\partial} R \to R/I.$$

One approach involves constructing cellular resolutions, a notion introduced by Bayer and Sturmfels [2]. Examples of cellular resolutions appearing in literature include Taylor resolutions, hull complexes, and Bar resolutions.

In what follows we show that the rooted complexes of the lcm-lattice of a monomial ideal I provide free cellular resolutions of I. Lyubeznik's resolution [5] turns out to be a special case of our construction. We also obtain a sufficient condition under which our resolutions are minimal. A different generalization of Lyubeznik's resolution using discrete Morse theory was recently found by Batzies and Welker [1, Section 3].

We start by reviewing several facts and definitions. Let I be a monomial ideal in a polynomial ring R over a field $\mathbf{k}$, and let $m_1, \ldots, m_n$ be its *minimal* generators (that is, m_i is not a divisor of m_j for any $i \neq j$). The lcm-lattice of I is the set

$$L = L(I) = \{\mathrm{lcm}\{m_{i_1}, \ldots, m_{i_k}\} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

partially ordered by divisibility (see [4]). L is an atomic lattice whose set of atoms is $L_1 = \{m_1, \ldots, m_n\}$; $a \vee b = \mathrm{lcm}\{a, b\}$, $a \wedge b = \gcd\{a, b\}$ for $a, b \in L$; the minimal element of L is $\hat{0} = 1$ and the maximal element is $\hat{1} = \mathrm{lcm}\{m_1, \ldots, m_n\}$. For $\sigma \in L$, the *height* of σ, $h(\sigma)$, is the length of the longest maximal chain from $\hat{0}$ to σ. (In particular, all atoms of L have height 1.) The height of L is defined via $h(L) = h(\hat{1})$.

Let Δ be a simplicial complex on the vertices $m_1, \ldots, m_n$. Label each face $F = \{m_{i_1}, \ldots, m_{i_k}\}$ of Δ by the monomial $m_F = \mathrm{lcm}\{m_{i_1}, \ldots, m_{i_k}\} \in L$. For $\sigma \in L$, denote by $\Delta_{\preceq\sigma}$ the subcomplex of Δ consisting of all faces whose label divides σ. As follows from [2, Proposition 1.2] (see also [6, Proposition 1.1]), a labeled complex Δ provides a free cellular resolution of I if and only if $\Delta_{\preceq\sigma}$ is acyclic (over $\mathbf{k}$) for every $\sigma \in L$. (In this case, the free generators of the i-th module in the resolution are the labels of i-dimensional faces of Δ ($i \geq 0$), and the differential is the (homogenized) boundary map of the chain complex $C_\bullet(\Delta)$. In particular, the length of the resolution is $\dim(\Delta) + 1$.) If, in addition, for every two simplexes $F \subset G \in \Delta$, $m_F \neq m_G$, then Δ provides a minimal free resolution of I.

Another notion we will use is the notion of a rooted complex introduced by Björner and Ziegler [3]. A *rooting map* on L is a function

$$\pi : L\backslash\{\hat{0}\} \to L_1 = \{m_1, \dots, m_n\}$$

that assigns to every element $\sigma \in L$ a monomial $\pi(\sigma) \in L_1$ that divides σ, such that $\pi(\sigma)|\tau|\sigma$ implies $\pi(\sigma) = \pi(\tau)$. Given a rooting map π and a nonempty subset S of L_1, we define $\tilde{\pi}(S) := \pi(\text{lcm}\{m \in S\})$. We say that a subset S of L_1 is *unbroken* (with respect to the rooting map π) if $\tilde{\pi}(S) \in S$, and that S is *rooted* if all nonempty subsets of S are unbroken. The collection of all rooted sets for L and π, $\text{RC}(L, \pi)$, is called the *rooted complex* of L. Clearly, $\text{RC}(L, \pi)$ is a simplicial complex whose vertices are $m_1, \dots, m_n$. Moreover, we have the following result.

Theorem 1 *Let I be a monomial ideal whose minimal generators are $m_1, \dots, m_n$. Let L be the lcm-lattice of I, and let π be a rooting map on L. Then*
(1) *The rooted complex* $\text{RC}(L, \pi)$ *provides a free cellular resolution of I.*
(2) *The length of this resolution is $\leq h(L)$.*

Remark First, we note that there exists at least one rooting map on L. Indeed, consider a total order on the elements of L_1, say, $m_1 \sqsubset m_2 \sqsubset \cdots \sqsubset m_n$. Then $\pi : L \setminus \{\hat{0}\} \to L_1$ defined by $\pi(\sigma) = \min_{\sqsubset}\{m_i : m_i \mid \sigma\}$ is a rooting map on L. Moreover, the free cellular resolution given by the rooted complex $\text{RC}(L, \pi)$ coincides with *Lyubeznik's resolution* constructed in [5]. However, we remark that not every rooting map arises this way, that is, from a total order on L_1: although a rooting map π determines a canonical order on every rooted set for π, these orders are not compatible in general (see [3, Lemma 3.5 and Section 4]).

The proof of Theorem 1 is a consequence of the above discussion on cellular resolutions and the following proposition.

Proposition 1 *Let $\Delta = \text{RC}(L, \pi)$ be a rooted complex of L, and let $\sigma \in L \setminus \{\hat{0}\}$. Then*
(1) *$\Delta_{\preceq\sigma}$ is a cone with apex $\pi(\sigma)$. In particular, $\Delta_{\preceq\sigma}$ is acyclic.*
(2) $\dim(\Delta_{\preceq\sigma}) \leq h(\sigma) - 1$.

Proof: The first statement is essentially [3, Theorem 3.2 (2)]: If $F \in \Delta_{\preceq\sigma}$, then any subset G of $\pi(\sigma) \cup F$ is either a subset of F, and hence is unbroken, or contains $\pi(\sigma)$. In the latter case, $\pi(\sigma)|\text{lcm}\{m_i : m_i \in G\}|\sigma$. Thus, by definition of a rooting map, $\tilde{\pi}(G) = \pi(\sigma) \in G$. Hence, such a G is unbroken as well, implying that $\pi(\sigma) \cup F$ is rooted, that is, $\pi(\sigma) \cup F \in \Delta_{\preceq\sigma}$.

The proof of the second statement is by induction on $h(\sigma)$. It clearly holds if $h(\sigma) = 1$. Let $F = \{m_{i_0}, \ldots, m_{i_k}\}$ be a face of $\Delta_{\preceq\sigma}$, where $k \geq 1$. Consider the label of F, $m_F = \mathrm{lcm}\{m_{i_0}, \ldots, m_{i_k}\} \mid \sigma$. Since F is rooted, it follows that $\tilde{\pi}(F) \in F$ and $m_{F\setminus\{\tilde{\pi}(F)\}}$ is a proper divisor of m_F. Therefore, by induction hypothesis,

$$\dim(F) = \dim(F \setminus \{\tilde{\pi}(F)\}) + 1 \leq h\big(m_{F\setminus\{\tilde{\pi}(F)\}}\big) \leq h(\sigma) - 1.$$

□

We now turn to the question of when a rooted complex provides a *minimal free cellular resolution*. This will require the following definitions. A finite lattice $(\mathcal{L}, \preceq)$ is *graded* if every maximal chain in $\mathcal{L}$ has the same length. A graded atomic lattice $\mathcal{L}$ is a *geometric* lattice if it satisfies the following condition (called *semimodularity*)

$$h(x) + h(y) \geq h(x \vee y) + h(x \wedge y) \quad \text{for all } x, y \in L. \tag{1}$$

Thus, if a is an atom of a geometric lattice $\mathcal{L}$, then

$$h(a \vee z) = 1 + h(z) \tag{2}$$

for every $z \in \mathcal{L}$ that is incomparable with a. (Indeed, since in this case $a \vee z \succ z$, it follows that $h(a\vee z) \geq 1+h(z)$. On the other hand, by (1), $h(a\vee z) \leq h(a)+h(z)-h(a\wedge z) = 1+h(z)$, implying (2).)

Theorem 2 *Let I be a monomial ideal whose minimal generators are $m_1, \ldots, m_n$, and let L be the lcm-lattice of I. If L is a geometric lattice, then* $\mathrm{RC}(L, \pi)$ *is a minimal free cellular resolution of I for every rooting map π.*

Proof: Let π be a rooting map on L. Consider $S \in \mathrm{RC}(L, \pi)$, its label $m_S = \mathrm{lcm}\{m \in S\}$, and the corresponding atom $\pi(m_S) = \tilde{\pi}(S) \in S$. Since $S \setminus \tilde{\pi}(S)$ is rooted, $\pi(m_S)$ does not divide $m_{S\setminus\tilde{\pi}(S)}$. Therefore, Eq. (2) yields

$$h(m_S) = 1 + h\big(m_{S\setminus\tilde{\pi}(S)}\big),$$

and we infer by induction on the size of S that $h(m_S) = |S|$ for any $S \in RC(L, \pi)$. Thus, if $S' \subset S \in RC(L, \pi)$, then $m_S \neq m_{S'}$, and hence $RC(L, \pi)$ provides a minimal free cellular resolution of I. □

Application to matroid ideals

Let $\mathcal{M}$ be a (simple) matroid on the ground set $\{1, \ldots, p\}$ (we refer the reader to [8] for definitions and facts about matroids and geometric lattices), and let M be its *matroid ideal* (in the sense of [6]), that is,

$$M = \big\langle x_{i_1} x_{i_2} \ldots x_{i_k} : \{i_1, \ldots, i_k\} \text{ is a cocircuit of } \mathcal{M}\big\rangle \subset \mathbf{k}[x_1, \ldots, x_p].$$

Thus, M is the Stanley-Reisner ring of the complex of independent sets of the dual matroid $\mathcal{M}^*$. (Equivalently, a proper square-free monomial ideal M of $\mathbf{k}[x_1, \ldots, x_p]$ is a *matroid ideal* if and only if for every pair of monomials $m_1, m_2 \in M$ and any $i \in \{1, \ldots, p\}$ such that x_i divides both m_1 and m_2, the monomial $\text{lcm}\{m_1, m_2\}/x_i$ is in M as well.) Stanley computed Betti numbers of matroid ideals [7, Theorem 9], and a technique for minimally resolving such ideals was found in [6, Section 3]. However the question of existence of minimal free cellular resolutions for (non-orientable) matroid ideals remained open. Here we settle a special case of this question.

Let $\mathcal{L}$ be a geometric lattice of flats of $\mathcal{M}$. Since the complements of cocircuits of $\mathcal{M}$ are flats of corank 1, it follows that the lcm-lattice of a matroid ideal M, $L(M)$, is isomorphic to the order-dual of the geometric lattice of flats of $\mathcal{M}$: $L(\mathcal{M}) \cong \mathcal{L}^{op}$. Thus, if $\mathcal{M}$ is a *modular matroid*, that is, if

$$h(x) + h(y) = h(x \vee y) + h(x \wedge y) \quad \text{for all } x, y \in L$$

(examples of modular matroids include projective geometries over finite fields), then the lcm-lattice $L(M) \cong \mathcal{L}^{op}$ is a geometric lattice as well. Theorem 2 then implies

Corollary 1 *If $\mathcal{M}$ is a modular matroid, then* RC$(L(M), \pi)$ *is a minimal free cellular resolution of M for every rooting map π on $L(M)$. In particular, Lyubeznik's resolution is a minimal free resolution for the matroid ideal of a finite projective space.*

Example Consider the Fano plane depicted in figure 1(a). Note that the poset $\mathcal{L}$ of rank 3, whose atoms are lines $1, 2, \ldots, 7$ in figure 1(a), whose coatoms are points in figure 1(a), and where the partial order is given by reverse inclusion, is a (modular) geometric lattice. Therefore, it is a lattice of flats of the Fano matroid $\mathcal{M}$ on the ground set $\{1, 2, \ldots, 7\}$. To obtain a list of flats of corank 1 of $\mathcal{M}$ (lines of $\mathcal{M}$), with every point in figure 1(a) associate the set of indices of lines containing it. Thus, the flats of corank 1 of $\mathcal{M}$ (which are also the circuits of $\mathcal{M}$) are

123, 167, 347, 257, 246, 145, 356.

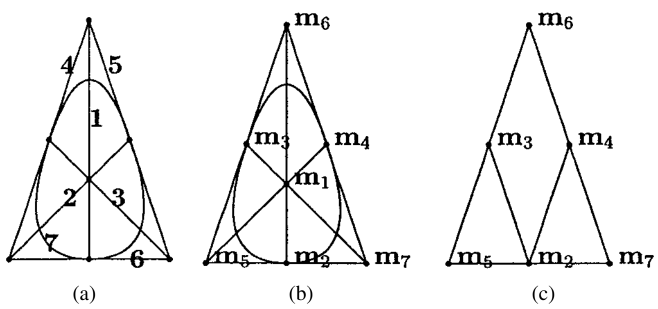

Figure 1. a) Fano plane, b) lcm-lattice of its matroid ideal, c) the corresponding rooted complex.

Hence the cocircuits of $\mathcal{M}$ are also in bijection with the points in figure 1(a): a cocircuit corresponding to a point consists of the indices of lines not passing through this point. In particular, the matroid ideal of $\mathcal{M}$ is

$$M = \langle m_1 = x_4x_5x_6x_7, m_2 = x_2x_3x_4x_5, m_3 = x_1x_2x_5x_6, m_4 = x_1x_3x_4x_6, \\ m_5 = x_1x_3x_5x_7, m_6 = x_2x_3x_6x_7, m_7 = x_1x_2x_4x_7\rangle;$$

its lcm-lattice, $L(M) \cong \mathcal{L}^{op}$, is isomorphic to the lattice of flats of the Fano plane (see figure 1(b)) whose points are points in figure 1(b), and whose lines are lines in figure 1(b).

Using the symmetry of the Fano plane, we define a rooting map π on $L(M)$ by

$$\pi(\hat{1}) = m_1, \quad \pi(m_5 \vee m_2 \vee m_7) = m_2, \quad \pi(m_5 \vee m_3 \vee m_6) = m_3, \\ \pi(m_6 \vee m_4 \vee m_7) = m_4, \quad \pi(m_3 \vee m_2 \vee m_4) = m_2.$$

Equivalently, π arises from the total order $m_1 \sqsubset m_2 \sqsubset \cdots \sqsubset m_7$ on the minimal generators of M (see Remark). The rooted complex $\mathrm{RC}(L(M), \pi)$ is a cone with apex m_1 over a 1-dimensional complex depicted in figure 1(c). It provides a minimal free cellular resolution of M.

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