Journal of Algebraic Combinatorics **13** (2001), 173–186

Spin Models and Strongly Hyper-Self-Dual Bose-Mesner Algebras

BRIAN CURTIN curtin@math.berkeley.edu
Department of Mathematics, University of California, Berkeley CA 94720

KAZUMASA NOMURA nomura@tmd.ac.jp
College of Liberal Arts and Sciences, Tokyo Medical and Dental University, Kohnodai, Ichikawa, 272-0827 Japan

Received March 23, 1999

Abstract. We introduce the notion of hyper-self-duality for Bose-Mesner algebras as a strengthening of formal self-duality. Let $\mathcal{M}$ denote a Bose-Mesner algebra on a finite nonempty set X. Fix $p \in X$, and let $\mathcal{M}^*$ and $\mathcal{T}$ denote respectively the dual Bose-Mesner algebra and the Terwilliger algebra of $\mathcal{M}$ with respect to p. By a hyper-duality of $\mathcal{M}$, we mean an automorphism ψ of $\mathcal{T}$ such that $\psi(\mathcal{M}) = \mathcal{M}^*$, $\psi(\mathcal{M}^*) = \mathcal{M}$; $\psi^2(A) = {}^{\mathrm{t}}A$ for all $A \in \mathcal{M}$; and $|X \mid \psi\rho$ is a duality of $\mathcal{M}$. $\mathcal{M}$ is said to be hyper-self-dual whenever there exists a hyper-duality of $\mathcal{M}$. We say that $\mathcal{M}$ is strongly hyper-self-dual whenever there exists a hyper-duality of $\mathcal{M}$ which can be expressed as conjugation by an invertible element of $\mathcal{T}$. We show that Bose-Mesner algebras which support a spin model are strongly hyper-self-dual, and we characterize strong hyper-self-duality via the module structure of the associated Terwilliger algebra.

Keywords: Bose-Mesner algebra, Terwilliger algebra, spin model

1. Introduction

The purpose of this paper is to introduce the notion of *hyper-self-duality* for Bose-Mesner algebras (association schemes) as a strengthening of the usual notion of formal self-duality. Our motivation for doing so is the observation that the Bose-Mesner algebras which support spin models have this property.

Let X denote a finite nonempty set of size n, and let M_X denote the C-algebra of matrices with entries in C whose rows and columns are indexed by X. A Bose-Mesner algebra on X is a commutative subalgebra $\mathcal{M}$ of M_X which is closed under entry-wise multiplication $\circ$, which is closed under transposition, and which contains the identity and all ones matrices of M_X. A formal duality of $\mathcal{M}$ is a linear bijection $\Psi : \mathcal{M} \longrightarrow \mathcal{M}$ such that for all A, $B \in \mathcal{M}$: $\Psi(AB) = \Psi(A) \circ \Psi(B)$, $\Psi(A \circ B) = n^{-1}\Psi(A)\Psi(B)$, $\Psi(\Psi(A)) = n\,{}^{\mathrm{t}}A$. A Bose-Mesner algebra $\mathcal{M}$ is said to be *formally self-dual* when there exists a formal duality of $\mathcal{M}$. We review Bose-Mesner algebras and self-duality in Section 2.

A spin model is a matrix which satisfies certain conditions which ensure that it yields an invariant of knots and links via a statistical mechanical construction of V. Jones [21]. Recently it was shown [19, 26, 20] that any spin model W is contained in a formally self-dual Bose-Mesner algebra $\mathcal{N}(W)$. (We discuss spin models in Section 5). While studying these results the authors found that the Bose-Mesner algebra $\mathcal{N}(W)$ is much more than formally self-dual. This led to the introduction of hyper-self-duality for Bose-Mesner algebras.

Hyper-self-duality is defined using Terwilliger algebras. Fix $p \in X$, and for $A \in \mathcal{M}$, let $\rho(A) \in \mathcal{M}_X$ denote the diagonal matrix with (x, x)-entry $\rho(A)(x, x) = A(p, x)$. Set $\mathcal{M}^* = \rho(\mathcal{M})$. The Terwilliger algebra $\mathcal{T} = \mathcal{T}(\mathcal{M}, p)$ associated with $\mathcal{M}$ and p is the subalgebra of M_X generated by $\mathcal{M} \cup \mathcal{M}^*$. We discuss Terwilliger algebras in Section 3.

By a *hyper-duality* of $\mathcal{M}$ (with respect to the base point p), we mean an automorphism ψ of $\mathcal{T}$ such that $\psi(\mathcal{M}) = \mathcal{M}^*$, $\psi(\mathcal{M}^*) = \mathcal{M}$; $\psi^2(A) = {}^tA$ for all $A \in \mathcal{M}$; and $n\psi\rho$ is a formal duality of $\mathcal{M}$. $\mathcal{M}$ is said to be *hyper-self-dual* whenever there exists a hyper-duality of $\mathcal{M}$. We say that $\mathcal{M}$ is *strongly hyper-self-dual* whenever there exists a hyper-duality of $\mathcal{M}$ which can be expressed as conjugation by an invertible element of $\mathcal{T}$.

Our main result concerning spin models is Theorem 5.5, which states that the Bose-Mesner algebra $\mathcal{N}(W)$ supporting the spin model W is strongly hyper-self-dual. Motivated by spin models, we focus on strong hyper-self-duality. In Theorem 4.1 we characterize this property in terms of $\mathcal{T}$-modules.

In Section 6 we present a Bose-Mesner algebra which is hyper-self-dual, but not strongly hyper-self-dual. We conclude with some problems concerning hyper-self-duality.

2. Formally self-dual Bose-Mesner algebras

In this section we review some basic material concerning formally self-dual Bose-Mesner algebras. The reader is referred to [4, 6, 13] for more details. The references [16, 20, 24, 28] also contain material that may be of interest.

Throughout this paper we fix a finite nonempty set X of size n. Let M_X denote the $\mathbf{C}$-algebra of matrices with entries in $\mathbf{C}$ whose rows and columns are indexed by X. For $A \in M_X$ and for $a, b \in X$, let $A(a, b)$ denote the (a, b)-entry of A. For $A, B \in M_X$, let $A \circ B$ denote the Hadamard (entry-wise) product of A and $B : (A \circ B)(x, y) = A(x, y)B(x, y)$. The transpose of A is denoted by ${}^tA$.

A *Bose-Mesner algebra* on X is a commutative subalgebra $\mathcal{M}$ of M_X, which is closed under Hadamard product, which is closed under transposition, and which contains the identity matrix I and the all 1's matrix J.

Let $\mathcal{M}$ be a Bose-Mesner algebra on X of dimension $\dim_{\mathbf{C}} \mathcal{M} = d + 1$. It is well-known that $\mathcal{M}$ has a unique linear basis $\{A_i\}_{i=0}^d$ such that

$$A_0 = I, \quad A_i \circ A_j = \delta_{ij} A_i \quad (0 \le i, j \le d), \quad \sum_{i=0}^{d} A_i = J, \tag{1}$$

where δ_{ij} is the Kronecker symbol. Observe that A_i has entries in $\{0, 1\}$ since $A_i \circ A_i = A_i$. We call $\{A_i\}_{i=0}^d$ the *basis of Hadamard idempotents* of $\mathcal{M}$. For the rest of this paper we fix an ordering $A_0, A_1, \ldots, A_d$ of the Hadamard idempotents. It is also well-known that $\mathcal{M}$ has a unique basis $\{E_i\}_{i=0}^d$ such that

$$E_0 = n^{-1} J, \quad E_i E_j = \delta_{ij} E_i \quad (0 \le i, j \le d), \quad \sum_{i=0}^{d} E_i = I. \tag{2}$$

We call $\{E_i\}_{i=0}^d$ the *basis of primitive idempotents* of $\mathcal{M}$.

By a *formal duality of* $\mathcal{M}$ we mean a linear bijection $\Psi : \mathcal{M} \longrightarrow \mathcal{M}$ such that for all $A, B \in \mathcal{M}$

$$\Psi(AB) = \Psi(A) \circ \Psi(B), \quad \Psi(A \circ B) = n^{-1}\Psi(A)\Psi(B), \quad \Psi(\Psi(A)) = n\,{}^{t}\!A. \quad (3)$$

$\mathcal{M}$ is said to be *formally self-dual* when there exists a formal duality of $\mathcal{M}$.

Note that a formal duality Ψ of $\mathcal{M}$ satisfies

$$\Psi({}^{t}\!A) = {}^{t}\Psi(A) \quad (4)$$

for all $A \in \mathcal{M}$ since

$$\Psi({}^{t}\!A) = \Psi(n^{-1}\Psi^2(A)) = n^{-1}\Psi^2(\Psi(A)) = n^{-1}n\,{}^{t}\Psi(A) = {}^{t}\Psi(A).$$

Lemma 2.1 *A linear map $\Psi : \mathcal{M} \longrightarrow \mathcal{M}$ is a formal duality of $\mathcal{M}$ if and only if there exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents such that*

$$\Psi(E_i) = A_i, \qquad \Psi(A_i) = n\,{}^{t}E_i \quad (0 \le i \le d). \quad (5)$$

Proof: First suppose Ψ is a formal duality of $\mathcal{M}$. Observe that $\{\Psi(E_i)\}_{i=0}^{d}$ is a basis of $\mathcal{M}$ since Ψ is a linear bijection. Also observe that for all $B \in \mathcal{M}$

$$\Psi(E_0)B = n\Psi(E_0 \circ \Psi^{-1}(B)) = \Psi(J \circ \Psi^{-1}(B)) = \Psi(\Psi^{-1}(B)) = B,$$

so $\Psi(E_0) = I = A_0$. Next observe that

$$\Psi(E_i) = \Psi(\delta_{ij}E_iE_j) = \delta_{ij}\Psi(E_i) \circ \Psi(E_j) \quad (0 \le i, j \le d),$$
$$\sum_{i=0}^{d}\Psi(E_i) = \Psi\left(\sum_{i=0}^{d}E_i\right) = \Psi(I) = \Psi(\Psi(E_0)) = n\,{}^{t}E_0 = J.$$

Thus $\{\Psi(E_i)\}_{i=0}^{d}$ is the basis of Hadamard idempotents of $\mathcal{M}$. Now we may order the primitive idempotents such that $\Psi(E_i) = A_i (0 \le i \le d)$. Finally,

$$\Psi(A_i) = \Psi(\Psi(E_i)) = n\,{}^{t}E_i \quad (0 \le i \le d).$$

Next suppose (5) holds for some ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents. We compute for all i, j $(0 \le i, j \le d)$

$$\Psi(A_i \circ A_j) = \delta_{ij}\Psi(A_i) = \delta_{ij}n\,{}^{t}E_i = n^{-1}n\,{}^{t}E_i n\,{}^{t}E_j = n^{-1}\Psi(A_i)\Psi(A_j),$$
$$\Psi(E_iE_j) = \delta_{ij}\Psi(E_i) = \delta_{ij}A_i = A_i \circ A_j = \Psi(E_i) \circ \Psi(E_j),$$
$$\Psi^2(E_i) = \Psi(A_i) = n\,{}^{t}E_i.$$

Thus Ψ is a formal duality of $\mathcal{M}$ since each of the defining conditions holds on a linear basis of $\mathcal{M}$. □

An ordering of primitive idempotents satisfying (5) is called a *standard ordering* of the primitive idempotents for Ψ.

3. Terwilliger algebras and hyper-duality

Let $\mathcal{M}$ be a Bose-Mesner algebra on X of dimension $\dim_{\mathbf{C}}(\mathcal{M}) = d + 1$. Fix a "base point" $p \in X$. For $A \in \mathcal{M}$, let $\rho(A) \in M_X$ denote the diagonal matrix with (x, x)-entry $\rho(A)(x, x) = A(p, x)$. Set $\mathcal{M}^* = \{\rho(A) \mid A \in \mathcal{M}\}$. $\mathcal{M}^*$ is called the *dual Bose-Mesner algebra* of $\mathcal{M}$ (with respect to the base point p). The map $\rho : \mathcal{M} \longrightarrow \mathcal{M}^*$ is a linear bijection. Set $E_i^* = \rho(A_i)$ and $A_i^* = \rho(nE_i)(0 \le i \le d)$. It is known that $\{E_i^*\}_{i=0}^d$ and $\{A_i^*\}_{i=0}^d$ are bases of $\mathcal{M}^*$. The subalgebra $\mathcal{T}$ of M_X generated by $\mathcal{M} \cup \mathcal{M}^*$ is called the *Terwilliger algebra* of $\mathcal{M}$ (with respect to the base point p). Further details on Terwilliger algebras can be found in [30].

Lemma 3.1 *Fix an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents, and let $\psi : \mathcal{M}^* \longrightarrow \mathcal{M}$ be a linear map. Then the following are equivalent.*
(i) $\psi(A_i^*) = A_i$ *and* $\psi(E_i^*) = {}^{t}E_i \quad (0 \le i \le d)$.
(ii) $\Psi := n\psi\rho$ *is a formal duality of* $\mathcal{M}$, *and the ordering* $E_0, E_1, \ldots, E_d$ *of the primitive idempotents is standard for* Ψ.

Proof: (i) $\Rightarrow$ (ii): Set $\Psi = n\psi\rho$. We compute for all $i (0 \le i \le d)$

$$\Psi(A_i) = n\psi\rho(A_i) = n\psi(E_i^*) = n\,{}^{t}E_i,$$
$$\Psi(E_i) = n\psi\rho(E_i) = \psi(A_i^*) = A_i.$$

Thus Ψ is a formal duality of $\mathcal{M}$ by Lemma 2.1, and the ordering of the primitive idempotents is standard for Ψ.

(ii)$\Rightarrow$(i): Since $\Psi = n\psi\rho$ satisfies (5), we compute for all $i (0 \le i \le d)$

$$\psi(A_i^*) = n\psi(\rho(E_i)) = \Psi(E_i) = A_i,$$
$$\psi(E_i^*) = \psi(\rho(A_i)) = n^{-1}\Psi(A_i) = {}^{t}E_i.$$

□

By a *hyper-duality of* $\mathcal{M}$ (with respect to the base point p), we mean an automorphism ψ of $\mathcal{T}$ with the following properties (i)–(iii):

(i) $\psi(\mathcal{M}) = \mathcal{M}^*$, $\psi(\mathcal{M}^*) = \mathcal{M}$.
(ii) $\psi^2(A) = {}^{t}A$ for all $A \in \mathcal{M}$.
(iii) $n\psi\rho$ is a formal duality of $\mathcal{M}$.

A hyper-duality ψ is said to be *inner* when it is an inner automorphism of $\mathcal{T}$:

(iv) There is an invertible matrix $K \in \mathcal{T}$ such that $\psi(A) = K^{-1}AK$ for all $A \in \mathcal{T}$.

$\mathcal{M}$ is said to be *hyper-self-dual* (with respect to p) whenever there exists a hyper-duality (with respect to p), and it is said to be *strongly hyper-self-dual* (with respect to p) whenever there exists an inner hyper-duality (with respect to p).

Lemma 3.2 *Let ψ be an automorphism of $\mathcal{T}$. Then the following are equivalent.*
(i) *ψ is a hyper-duality of $\mathcal{M}$ with respect to p.*
(ii) *There exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents such that for all i $(0 \leq i \leq d)$*

$$\psi({}^{t}A_i) = A_i^*, \quad \psi(A_i^*) = A_i, \quad \psi(E_i) = E_i^*, \quad \psi(E_i^*) = {}^{t}E_i. \tag{6}$$

Suppose (i), (ii) *hold. Then $\Psi = n\psi\rho$ is a formal duality of $\mathcal{M}$ and the ordering of the primitive idempotents in* (ii) *is standard for Ψ.*

Proof: (i) $\Rightarrow$ (ii): By the definition of a hyper-duality, $\Psi = n\psi\rho$ is a formal duality of $\mathcal{M}$. Fix an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents which is standard for Ψ. Then by Lemma 3.1, $\Psi(A_i^*) = A_i$ and $\Psi(E_i^*) = {}^{t}E_i (0 \leq i \leq d)$. Using $\psi^2(A) = {}^{t}A$ for $A \in \mathcal{M}$, we compute for all $i (0 \leq i \leq d)$

$$\psi({}^{t}A_i) = \psi^{-1}(\psi^2({}^{t}A_i)) = \psi^{-1}(A_i) = A_i^*,$$
$$\psi(E_i) = \psi^{-1}(\psi^2(E_i)) = \psi^{-1}({}^{t}E_i) = E_i^*.$$

(ii) $\Rightarrow$ (i): Observe that ψ exchanges bases of $\mathcal{M}$ and $\mathcal{M}^*$, so $\psi(\mathcal{M}) = \mathcal{M}^*$ and $\psi(\mathcal{M}^*) = \mathcal{M}$. Now fix an ordering of primitive idempotents such that (6) holds. Observe that for all i $(0 \leq i \leq d)$

$$\psi^2(E_i) = \psi(\psi(E_i)) = \psi(E_i^*) = {}^{t}E_i.$$

Hence $\psi^2(A) = {}^{t}A$ for all A in $\mathcal{M}$ since $\{E_i\}_{i=0}^d$ is a linear basis of $\mathcal{M}$. Finally, we compute for all i $(0 \leq i \leq d)$

$$n\psi\rho(E_i) = \psi\rho(nE_i) = \psi(A_i^*) = A_i,$$
$$n\psi\rho(A_i) = n\psi(E_i^*) = n\,{}^{t}E_i.$$

Hence $n\psi\rho$ is a formal duality of $\mathcal{M}$ by Lemma 3.1 □

4. Strong hyper-self-duality

The purpose of this section is to give a module characterization of strong hyper-self-duality. Since Terwilliger algebras are semisimple, their simple modules determine the structure of the Terwilliger algebra itself. The literature generally presents module descriptions of Terwilliger algebras. Before stating the main result of this section, we recall some basic facts about the simple modules of Terwilliger algebras. The reader is referred to [12] for a general discussion of semisimple algebras.

Let $\mathcal{M}$ be a Bose-Mesner algebra on X of dimension $\dim_{\mathbf{C}}(\mathcal{M}) = d + 1$. Fix a base point $p \in X$, and let $\mathcal{T}$ be the Terwilliger algebra of $\mathcal{M}$ with respect to p. $\mathcal{T}$ is semisimple, and so $\mathcal{T}$ decomposes as $\mathcal{T} = \oplus_{\lambda\in\Lambda}\varphi_\lambda\mathcal{T}$, where $\varphi_\lambda (\lambda \in \Lambda)$ are the primitive central idempotents of $\mathcal{T}$. Let $V = \mathbf{C}^X$ denote the n-dimensional vector space consisting of column vectors whose entries are indexed by X. By a *$\mathcal{T}$-module* we mean a linear subspace U

of V which is closed under the action of $\mathcal{T}: Au \in U$ for all $A \in \mathcal{T}$ and for all $u \in U$. V decomposes as $V = \oplus_{\lambda \in \Lambda} \varphi_\lambda V$, and each $\varphi_\lambda V$ is a direct sum of mutually isomorphic simple $\mathcal{T}$-modules. Moreover, the subalgebra $\varphi_\lambda \mathcal{T}$ is isomorphic to the endomorphism algebra $\mathrm{End}_{\mathbf{C}}(U)$, where U is any simple $\mathcal{T}$-module contained in $\varphi_\lambda V$; the isomorphism maps $L \in \varphi_\lambda \mathcal{T}$ to the endomorphism $u \mapsto Lu$ $(u \in U)$.

Theorem 4.1 *Let $\mathcal{M}$ be a $(d+1)$-dimensional Bose-Mesner algebra on X. Fix $p \in X$, and let $\mathcal{T}$ be the Terwilliger algebra of $\mathcal{M}$ with respect to p. Then the following are equivalent.*
(i) *$\mathcal{M}$ is strongly hyper-self-dual with respect to p.*
(ii) *There exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of $\mathcal{M}$ such that for each simple $\mathcal{T}$-module U and for any ordered basis Ω_1 of U, there is a second ordered basis Ω_2 of U such that*

$$\begin{array}{ll} [{}^{\mathrm{t}}A_i]_1 = [A_i^*]_2, & [A_i^*]_1 = [A_i]_2, \\ [E_i]_1 = [E_i^*]_2, & [E_i^*]_1 = [{}^{\mathrm{t}}E_i]_2 \end{array} \quad (0 \le i \le d), \tag{7}$$

where $[B]_j$ denotes the matrix representing $B \in \mathcal{T}$ with respect to the basis Ω_j $(j = 1, 2)$.

To prove this theorem we first need some lemmas concerning linear endomorphisms σ of $\mathcal{T}$-modules U whose action on U satisfies

$$\begin{array}{ll} \sigma\, {}^{\mathrm{t}}A_i = A_i^* \sigma, & \sigma A_i^* = A_i \sigma, \\ \sigma E_i = E_i^* \sigma, & \sigma E_i^* = {}^{\mathrm{t}}E_i \sigma \end{array} \quad (0 \le i \le d). \tag{8}$$

Lemma 4.2 *Let ψ be an inner automorphism of $\mathcal{T}$, and let U be a $\mathcal{T}$-module. Then there exists a linear bijection $\sigma : U \longrightarrow U$ such that $\sigma(Au) = \psi(A)\sigma(u)$ for all $A \in \mathcal{T}$ and for all $u \in U$.*

Proof: Since ψ is inner, $\psi(A) = K^{-1}AK$ $(A \in \mathcal{T})$ for some $K \in \mathcal{T}$. Define $\sigma : U \longrightarrow U$ by $\sigma(u) = K^{-1}u$ $(u \in U)$. Then for $A \in \mathcal{T}$, $\sigma(Au) = K^{-1}Au = (K^{-1}AK)(K^{-1}u) = \psi(A)\sigma(u)$. □

Lemma 4.3 *Let U and U' be isomorphic $\mathcal{T}$-modules with isomorphism $f : U \longrightarrow U'$.*
(i) *Let $\sigma : U \longrightarrow U$ be a linear bijection which satisfies* (8) *on U. Then $\sigma' = f\sigma f^{-1} : U' \longrightarrow U'$ also satisfies* (8) *on U'.*
(ii) *Fix $L \in \mathcal{T}$, and define $\sigma : U \longrightarrow U$ by $\sigma(u) = Lu$. Then $f\sigma f^{-1}(u') = Lu'$ for all $u' \in U'$.*

Proof: (i) Pick any $u \in U'$. Since f^{-1} and f are $\mathcal{T}$-module isomorphisms, and since σ satisfies (8), we compute for all $i\,(0 \le i \le d)$

$$\begin{aligned} \sigma'({}^{\mathrm{t}}A_i u) &= f\sigma f^{-1}({}^{\mathrm{t}}A_i u) = f\sigma(f^{-1}({}^{\mathrm{t}}A_i u)) = f\sigma({}^{\mathrm{t}}A_i f^{-1}(u)) \\ &= f(A_i^*\sigma(f^{-1}(u))) = A_i^* f\sigma f^{-1}(u) = A_i^*\sigma'(u). \end{aligned}$$

The other relations of (8) can be verified similarly.

(ii) Pick any $u' \in U'$ and set $u = f^{-1}(u')$. Then $f\sigma f^{-1}(u') = f\sigma(u) = f(Lu) = L f(u) = Lu'$. □

Lemma 4.4 *The following are equivalent.*
(i) *$\mathcal{M}$ is strongly hyper-self-dual with respect to p.*
(ii) *There exists an ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of $\mathcal{M}$ such that, for every simple $\mathcal{T}$-module U, there is a linear bijection $\sigma : U \longrightarrow U$ satisfying* (8) *on U.*

Proof: (i) ⇒ (ii): Clear from Lemmas 3.2 and 4.2.

(ii) ⇒ (i): Recall that $\mathcal{T}$ and V decompose as $\mathcal{T} = \oplus_{\lambda\in\Lambda}\varphi_\lambda\mathcal{T}$ and $V = \oplus_{\lambda\in\Lambda}\varphi_\lambda V$, where $\{\varphi_\lambda \mid \lambda \in \Lambda\}$ denotes the set of primitive central idempotents of $\mathcal{T}$. Fix any $\lambda \in \Lambda$. Note that $\varphi_\lambda V$ is a direct sum of mutually isomorphic simple $\mathcal{T}$-modules: $\varphi_\lambda V = \oplus_{j=1}^m U_j$. For each j $(1 \le j \le m)$, fix a $\mathcal{T}$-module isomorphism $f_j : U_1 \longrightarrow U_j$ with f_1 the identity transformation of U_1. By assumption, there is a linear bijection $\sigma : U_1 \longrightarrow U_1$ such that (8) holds for all $u \in U_1$. Since $\varphi_\lambda\mathcal{T}$ is isomorphic to $\mathrm{End}_{\mathbf{C}}(U_1)$, there is a matrix $L_\lambda \in \varphi_\lambda\mathcal{T}$ such that $\sigma(u) = L_\lambda u$ for all $u \in U_1$. By Lemma 4.3(ii), the map $\sigma_j = f_j\sigma f_j^{-1}$ $(1 \le j \le m)$ satisfies $\sigma_j(u) = L_\lambda u$ for all $u \in U_j$. In addition, by Lemma 4.3(i), $\sigma_j({}^{\mathrm{t}}A_i u) = A_i^*\sigma_j(u)$ holds for all $u \in U_j$ $(0 \le i \le d,\ 1 \le j \le m)$. Hence we obtain

$$L_\lambda {}^{\mathrm{t}}A_i u = A_i^* L_\lambda u \quad (u \in U_j,\ 0 \le i \le d,\ 1 \le j \le m). \tag{9}$$

Note that (9) holds for all $u \in \varphi_\lambda V$ since it holds for all j $(1 \le j \le m)$.

Now set $L = \sum_{\lambda\in\Lambda} L_\lambda$. Clearly L is invertible since L_λ acts invertibly on $\varphi_\lambda V$ for all $\lambda \in \Lambda$.

We claim that

$$L\,{}^{\mathrm{t}}A_i = A_i^* L \quad (0 \le i \le d). \tag{10}$$

To see this, pick any $v \in V$ and write $v = \sum_{\lambda\in\Lambda} v_\lambda$ with $v_\lambda \in \varphi_\lambda V$ $(\lambda \in \Lambda)$. Fix i $(0 \le i \le d)$. From (9), we have $L_\lambda^{\mathrm{t}} A_i v_\lambda = A_i^* L_\lambda v_\lambda$ $(\lambda \in \Lambda)$. Using this relation and $L_\lambda v_\mu = 0$ for $\lambda \ne \mu$, we compute as follows.

$$\begin{aligned}
L^{\mathrm{t}}A_i v &= \left(\sum_{\lambda\in\Lambda} L_\lambda\right){}^{\mathrm{t}}A_i\left(\sum_{\mu\in\Lambda} v_\mu\right) = \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} L_\lambda{}^{\mathrm{t}}A_i v_\mu \\
&= \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda}\delta_{\lambda\mu} L_\lambda{}^{\mathrm{t}}A_i v_\mu = \sum_{\lambda\in\Lambda} L_\lambda{}^{\mathrm{t}}A_i v_\lambda \\
&= \sum_{\lambda\in\Lambda} A_i^* L_\lambda v_\lambda = A_i^*\sum_{\lambda\in\Lambda} L_\lambda v_\lambda \\
&= A_i^*\left(\sum_{\lambda\in\Lambda} L_\lambda\right)\left(\sum_{\mu\in\Lambda} v_\mu\right) = A_i^* L v.
\end{aligned}$$

Hence (10) holds, and so $K^{-1\mathrm{t}}A_i K = A_i^*$ with $K = L^{-1}$. The remaining relations of (6) can be shown similarly. □

Observe that the matrices L_λ produced in the proof of Lemma 4.4 are unique up to a non-zero scalar multiple. We may represent the rescaling of each L_λ as multiplication of the matrix K by $\sum_{\lambda\in\Lambda}\alpha_\lambda\varphi_\lambda$, where α_λ is a nonzero scalar ($\lambda\in\Lambda$). Since $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ is a basis of the center of $\mathcal{T}$, this is exactly the same as multiplication of K by any invertible central element C of $\mathcal{T}$. Clearly the hyper-dualities $A\mapsto K^{-1}AK$ and $A\mapsto (CK)^{-1}A(CK)$ are identical.

Proof of Theorem 4.1: (i) ⇒ (ii): For an arbitrary given basis $\Omega_1=\{u_1,u_2,\dots,u_s\}$ of U, set $\Omega_2=\{\sigma(u_1),\sigma(u_2),\dots,\sigma(u_s)\}$, where σ is as in Lemma 4.4. Then Ω_2 is a basis of U and (ii) holds.

(ii) ⇒ (i): Write $\Omega_1=\{u_1,u_2,\dots,u_s\}$ and $\Omega_2=\{u_1',u_2',\dots,u_s'\}$. Let $\sigma: U\longrightarrow U$ denote the linear map such that $\sigma(u_i)=u_i'$ $(1\le i\le s)$. Then σ satisfies (8). Thus $\mathcal{M}$ is strongly hyper-self-dual by Lemma 4.4. □

5. Spin models

Spin models are square matrices satisfying certain conditions which ensure that they yield an invariant of links via a statistical mechanical construction of V. Jones [21]. Here we consider the nonsymmetric generalization of Jones' spin models which were presented in [22]. One may construct formally self-dual Bose-Mesner algebras from spin models [20, 26 19]. The purpose of this section is to extend this result by showing that such Bose-Mesner algebras are strongly hyper-self-dual. Some specific examples of spin models have been presented in [1, 3, 5, 17, 21, 23, 25]. Let us begin by recalling precisely some facts about spin models.

A *spin model* on X is a matrix $W\in M_X$ with non-zero entries which satisfies the following conditions (for all $a,b,c\in X$):

$$\sum_{x\in X}\frac{W(a,x)}{W(b,x)}=n\delta_{ab}, \tag{11}$$

$$\sum_{x\in X}\frac{W(a,x)W(b,x)}{W(c,x)}=\sqrt{n}\frac{W(a,b)}{W(a,c)W(c,b)}. \tag{12}$$

Setting $b=c$ in (12) shows that every diagonal entry of W is the same; we refer the constant diagonal entry of W as the *modulus* of W.

Let W be a spin model on X. For every pair $(b,c)\in X$, we consider the column vector Y_{bc} in $V=\mathbf{C}^X$ whose x-entry is given by

$$Y_{bc}(x)=\frac{W(x,b)}{W(x,c)}\quad (x\in X).$$

Let $\mathcal{N}(W)$ be the set of matrices A in M_X such that Y_{bc} is an eigenvector for all $b,c\in X$:

$$\mathcal{N}(W)=\{A\in M_X \mid AY_{bc}\in\mathbf{C}Y_{bc}\text{ for all }b,c\in X\}.$$

For $A \in \mathcal{N}(W)$, let $\Psi(A) \in M_X$ be defined by

$$AY_{bc} = \Psi(A)(b, c)Y_{bc} \quad (b, c \in X). \tag{13}$$

For $A \in M_X$ with nonzero entries, let A^- denote the matrix in M_X whose (x, y) entry is $A(y, x)^{-1}$. Observe that $WW^- = nI$ by (11), and so W and W^- are invertible.

Theorem 5.1 ([20, 26]) *Let W be a spin model on X with modulus α.*
(i) *$\mathcal{N}(W)$ is a Bose-Mesner algebra on X containing W.*
(ii) *$\Psi(A) \in \mathcal{N}(W)$ for all $A \in \mathcal{N}(W)$, and the map $A \mapsto \Psi(A)$ is a formal duality of $\mathcal{N}(W)$.*
(iii) *$\Psi(A) = \alpha^{-1} W \circ ({}^tW^-({}^tW \circ A))$ for all $A \in \mathcal{N}(W)$.*

Let $\mathcal{M}$ be a Bose-Mesner algebra such that $W \in \mathcal{M} \subseteq \mathcal{N}(W)$. Such a Bose-Mesner algebra $\mathcal{M}$ is said to *support* W.

Lemma 5.1 *$W^- \in \mathcal{M}$ and $\Psi(\mathcal{M}) = \mathcal{M}$. In particular, $\mathcal{M}$ is formally self-dual with formal duality $\Psi\,|_{\mathcal{M}}$.*

Proof: Let $\{A_i\}_{i=0}^d$ denote the basis of Hadamard idempotents of $\mathcal{M}$. Since $W \in \mathcal{M}$, there exist scalars $t_0, t_1, \ldots, t_d$ such that $W = \sum_{i=0}^d t_i A_i$. It follows from definition that $W^- = \sum_{i=0}^d t_i^{-1}\, {}^tA_i$, so $W^- \in \mathcal{M}$. Now Theorem 5.1 (iii) implies that $\Psi(A) \in \mathcal{M}$ for all $A \in \mathcal{M}$ since $\mathcal{M}$ is closed under transposition, Hadamard product, and matrix product. This shows that $\Psi(\mathcal{M}) \subseteq \mathcal{M}$, so $\Psi(\mathcal{M}) = \mathcal{M}$ since Ψ is a linear bijection. □

Now fix a base point $p \in X$, and let $\mathcal{T}$ denote the Terwilliger algebra of $\mathcal{M}$ with respect to p. Define $K \in M_X$ by

$$K(x, y) = \frac{W(x, p)W(p, y)}{W(x, y)} \quad (x, y \in X). \tag{14}$$

Lemma 5.3 *$K = \rho({}^tW){}^tW^-\rho(W)$. In particular K is invertible and $K \in \mathcal{M}^*\mathcal{M}\mathcal{M}^* \subseteq \mathcal{T}$.*

Proof: For any $x, y \in X$,

$$\begin{aligned} K(x, y) &= W(x, p)W(x, y)^{-1}W(p, y) = {}^tW(p, x){}^tW^-(x, y)W(p, y) \\ &= \rho({}^tW)(x, x){}^tW^-(x, y)\rho(W)(y, y) = (\rho({}^tW){}^tW^-\rho(W))(x, y). \end{aligned}$$

Observe that $\rho(W)$ and $\rho({}^tW)$ are invertible since W has non-zero entries. Hence K is invertible. □

Lemma 5.4 *For all $A \in \mathcal{M}$, $AK = K\rho(\Psi(A))$.*

Proof: For any $x,\ y \in X$,

$$\begin{aligned}(AK)(x,y) &= \sum_{z\in X} A(x,z)K(z,y) = \sum_{z\in X} A(x,z)\frac{W(z,p)W(p,y)}{W(z,y)}\\ &= W(p,y)\sum_{z\in X} A(x,z)Y_{py}(z) = W(p,y)(AY_{py})(x)\\ &= W(p,y)(\Psi(A)(p,y)Y_{py})(x) = W(p,y)\Psi(A)(p,y)\frac{W(x,p)}{W(x,y)}\\ &= \Psi(A)(p,y)\frac{W(x,p)W(p,y)}{W(x,y)} = K(x,y)\Psi(A)(p,y)\\ &= K(x,y)\rho(\Psi(A))(y,y) = (K\rho(\Psi(A)))(x,y).\end{aligned}$$

□

Theorem 5.5 *Let W be a spin model on X and let $\mathcal{M}$ be a Bose-Mesner algebra such that $W \in \mathcal{M} \subseteq \mathcal{N}(W)$. Fix a base point $p \in X$, and let $\mathcal{T}$ denote the Terwilliger algebra of $\mathcal{M}$ with respect to p. Let K be as in* (14)*. Then $K \in \mathcal{T}$, and the map $\psi : A \mapsto K^{-1}AK$ $(A \in \mathcal{T})$ is a hyper-duality of $\mathcal{M}$. In particular, $\mathcal{M}$ is strongly hyper-self-dual with respect to p.*

Proof: By Lemma 5.3, K is contained in $\mathcal{T}$. Let Ψ be the formal duality of $\mathcal{M}$ given by Theorem 5.1 (ii) and Lemma 5.2. Fix a standard ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents of $\mathcal{M}$ for Ψ. By (4) and (5), $\Psi({}^{t}A_i) = nE_i$ $(0 \le i \le d)$. Using this relation and Lemma 5.4, we compute for all i $(0 \le i \le d)$

$${}^{t}A_i K = K\rho(\Psi({}^{t}A_i)) = K\rho(nE_i) = KA_i^*.$$

Hence $\psi({}^{t}A_i) = A_i^*$ $(0 \le i \le d)$. The remaining equations of (6) can be shown similarly. Thus ψ is a hyper-duality by Lemma 3.2. □

6. Example

In this section, we present a Bose-Mesner algebra which is hyper-self-dual but not strongly hyper-self-dual, namely the Bose-Mesner algebra of the Shrikhande graph [27].

Let $X = \{(ij) \mid i, j = 1, 2, 3, 4\}$, and define A_1 to be the matrix in M_X with $((ij), (i'j'))$-entry:

$$A_1((ij),(i'j')) = \begin{cases} 1 & \text{if } i \ne i',\ j \ne j' \text{ and } i - j \not\equiv i' - j' \pmod 4,\\ 0 & \text{otherwise.}\end{cases}$$

Set $A_0 = I$, $A_2 = J - A_0 - A_1$, $E_0 = (1/16)J$, $E_1 = (1/16)(6A_0 + 2A_1 - 2A_2)$, $E_2 = I - E_0 - E_1$. Then

$$A_1^2 = 6A_0 + A_1 + 2A_2,\quad A_1A_2 = A_2A_1 = 4A_1 + 4A_2,\quad A_2^2 = 9A_0 + 5A_1 + 4A_2,$$
$$E_1^{\circ 2} = 6E_0 + E_1 + 2E_2,\quad E_1 \circ E_2 = E_2 \circ E_1 = 4E_1 + 4E_2,$$
$$E_2^{\circ 2} = 9E_0 + 5E_1 + 4E_2,$$
$$A_i \circ A_j = \delta_{ij}A_i,\quad E_iE_j = \delta_{ij}E_i \quad (0 \le i,\ j \le 2).$$

Let $\mathcal{M}$ denote the linear span of $\{A_0, A_1, A_2\}$. It follows from its definition and the above computations that $\mathcal{M}$ is a formally self-dual Bose-Mesner algebra with Hadamard idempotents $\{A_0, A_1, A_2\}$ and primitive idempotents $\{E_0, E_1, E_2\}$.

Let $\mathcal{T}$ denote the Terwilliger algebra of $\mathcal{M}$ with respect to the base point (11). For all $x \in X$, the (x, x)-entry of $A_1^* = \rho(16E_1) = \rho(6A_0 + 2A_1 - 2A_2)$ is

$$A_1^*(x,x) = \begin{cases} 6 & \text{if } x = (11), \\ 2 & \text{if } x \in \{(23), (24), (32), (34), (41), (44)\}, \\ -2 & \text{otherwise.} \end{cases}$$

We now present a decomposition of $V = \mathbf{C}^X$ into simple $\mathcal{T}$-modules $V = \oplus_{\ell=0}^{9} U_\ell$ via an ordered basis of each. For each of these simple $\mathcal{T}$-modules we present the matrices representing A_1 and A_1^* with respect to the given ordered basis and a matrix K_ℓ (except for $\ell = 5,\ 6$) which we explain below. A similar decomposition appears in [29]. The action and simplicity of each module can be deduced from the definitions A_1 and A_1^*. Let $e_{ij} \in V$ denote the characteristic vector of (ij) $(i,\ j = 1, 2, 3, 4)$.

$$U_0 : \begin{array}{l} e_{11}, \\ e_{23} + e_{24} + e_{32} + e_{34} + e_{42} + e_{43}, \\ e_{12} + e_{13} + e_{14} + e_{21} + e_{22} + e_{31} + e_{33} + e_{41} + e_{44}. \end{array}$$

$$[A_1] = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 4 \end{pmatrix}, \quad [A_1^*] = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 1 & 6 & 9 \\ 1 & 2 & -3 \\ 1 & -2 & 1 \end{pmatrix}.$$

$$U_1 : \begin{array}{l} e_{23} - e_{32} - e_{34} + e_{43}, \\ -e_{12} + 2e_{13} - e_{14} + e_{21} - 2e_{31} + e_{41}. \end{array}$$

$$[A_1] = \begin{pmatrix} -1 & -3 \\ -1 & 1 \end{pmatrix}, \quad [A_1^*] = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 1 & -3 \\ -1 & -1 \end{pmatrix}.$$

$$U_2 : \begin{array}{l} -e_{23} + 2e_{24} - e_{32} - e_{34} + 2e_{42} - e_{43}, \\ -e_{12} + 2e_{13} - e_{14} - e_{21} + 2e_{22} + 2e_{31} - 4e_{33} - e_{41} + 2e_{44}. \end{array}$$

$$[A_1] = \begin{pmatrix} -1 & -3 \\ -1 & 1 \end{pmatrix}, \quad [A_1^*] = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & -3 \\ -1 & -1 \end{pmatrix}.$$

$$U_3 : \begin{array}{l} -e_{23} - e_{32} + e_{34} + e_{43}, \\ e_{12} - e_{14} + e_{21} + 2e_{22} - e_{41} - 2e_{44}. \end{array}$$

$$[A_1] = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}, \quad [A_1^*] = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 1 & 1 \\ 1/3 & -1 \end{pmatrix}.$$

$$U_4 : \begin{array}{l} e_{23} - 2e_{24} - e_{32} + e_{34} + 2e_{42} - e_{43}, \\ -3e_{12} + 3e_{14} + 3e_{21} - 3e_{41}. \end{array}$$

$$[A_1] = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}, \quad [A_1^*] = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad K_4 = \begin{pmatrix} 1 & 1 \\ 1/3 & -1 \end{pmatrix}.$$

$U_5 : -e_{23} - e_{24} + e_{32} - e_{34} + e_{42} + e_{43}.$
$[A_1] = (-2), \quad [A_1^*] = (2).$

$U_6 : e_{12} - e_{14} + e_{21} - e_{22} - e_{41} + e_{44}.$
$[A_1] = (2), \quad [A_1^*] = (-2).$

$U_7 : -e_{13} + e_{22} - e_{31} + e_{44}.$
$[A_1] = (-2), \quad [A_1^*] = (-2), \quad K_7 = (1).$

$U_8 : -e_{12} - e_{13} - e_{14} + e_{21} + e_{31} + e_{41}.$
$[A_1] = (-2), \quad [A_1^*] = (-2), \quad K_8 = (1).$

$U_9 : -2e_{12} + e_{13} - 2e_{14} - 2e_{21} + e_{22} + e_{31} + 4e_{33} - 2e_{41} + e_{44}.$
$[A_1] = (-2), \quad [A_1^*] = (-2), \quad K_9 = (1).$

Now let $K \in M_X$ denote the matrix whose action on U_ℓ coincides with K_ℓ for all $\ell \neq 5, 6$ and which exchange the given bases of U_5 and U_6. Then $KA_1 = A_1^* K$ and $KA_1^* = A_1 K$. Observe that $\mathcal{M}$ is generated by A_1 and $\mathcal{M}^*$ is generated by A_1^*. It follows that the automorphism $\psi : A \mapsto K^{-1}AK$ $(A \in \mathcal{T})$ of $\mathcal{T}$ satisfies (6), and hence $\mathcal{M}$ is hyper-self-dual by Lemma 3.2. However, the action of A_1 and A_1^* on U_5 (or U_6) shows that $\mathcal{M}$ is not strongly hyper-self-dual by Theorem 4.1.

Remark For all integers $a \geq 1$ and $b \geq 0$, the Doob graph $D(a, b)$ is defined as the direct product of a copies of the Shrikhande graph and b copies of the complete graph on 4 vertices. The Doob graphs are distance-regular, so there is a Bose-Mesner algebra associated to each $D(a, b)$ (see [6]). An induction based upon this direct product construction has allowed the authors to show that the Doob graphs are hyper-self-dual. The induction begins with the above observation that the Shrikhande graph is hyper-self-dual and the easy observation that the complete graph on 4 vertices is strongly hyper-self-dual. The authors have further shown that the Doob graphs are not strongly hyper-self-dual. As above, this was done by producing a $\mathcal{T}$-module which does not have a "self-dual" action.

7. Problems

See [6] for terminology used in this section.

Problem 7.1 Study the known examples of formally self-dual Bose-Mesner algebras (association schemes) to determine which are not hyper-self-dual, which are hyper-self-dual but not strongly hyper-self-dual, and which are strongly hyper-self-dual. While doing so, describe the Terwilliger algebras of these examples so that the type of duality is clear.

We have shown that all self-dual translation Bose-Mesner algebras are hyper-self-dual [9]. Many distance-regular graphs give rise to translation Bose-Mesner algebras, including Hamming graphs, bilinear forms graphs, alternating forms graphs, Hermitian forms graphs, affine $E_6(q)$ graphs, the extended ternary Golay code graph, and Paley graphs. Except for the Hamming graphs (which support spin models), it is open to decide which are strongly hyper-self-dual and to give a complete description of their Terwilliger algebras.

The 2-homogeneous bipartite distance-regular graphs have been shown to be strongly hyper-self-dual, and their Terwilliger algebras have been completely described [8, 15]. We have shown that the Doob graphs are hyper-self-dual, but not strongly hyper-self-dual. The Terwilliger algebras of the Doob graphs are described in [29]; however, it is now natural to ask for an extension of this description which makes it clear which simple modules are dual to one another. The quadratic forms graphs by Egawa are another interesting family of formally self-dual distance-regular graphs. It is open to decide if they are hyper-self-dual.

We have examined some Latin square graphs and found families which are strongly hyper-self-dual and other families which are hyper-self-dual but not strongly hyper-self-dual. We note that no Latin square graph supports a spin model, so strong hyper-self-duality is not just a property arising from spin models. It is open to decide if all Latin square graphs are hyper-self-dual. A partial description of the Terwilliger algebra of any given example can be deduced from [31].

A few formally self-dual Bose-Mesner algebras (association schemes) not related to distance-regular graph are known. Some are translation Bose-Mesner algebras. Other examples include the Bose-Mesner algebras arise from nonsymmetric P- and Q-polynomial association schemes and amorphous association schemes (association schemes of (negative) Latin square type). These examples have not been studied from the perspective of hyper-self-duality. We have shown that the Bose-Mesner algebra constructed from a symmetrizable type II matrix using the construction of [20] is hyper-self-dual [9].

Problem 7.2 Study the distance-regular graphs which support a spin model.

In [10] it was shown that the distance-regular graphs which support a spin model are described by just two parameters, and in [7] it was shown that the simple modules of their Terwilliger algebras are well behaved. Using these results, together with those of [30] and the present paper, it may be possible to completely describe their Terwilliger algebras. In [11] it was shown that the non-Hamming 2-homogeneous bipartite distance-regular graphs of the distance-regular graphs have Terwilliger algebras which are homomorphic images of the quantum universal enveloping algebra of $sl(2)$. The Hamming graphs are related to the classical universal enveloping algebra of $sl(2)$ [15]. Perhaps the Terwilliger algebras of any further examples (if they exist) are related to some 2 parameter deformation of the universal enveloping algebra of $sl(2)$. It may be also be possible to show that the only distance-regular graphs with diameter at least 4 which support a spin model are either triangle-free or Hamming graphs.

References

1. E. Bannai and Et. Bannai, "Spin models on finite cyclic groups," *J. Alg. Combin.* **3** (1994), 243–259.
2. E. Bannai, Et. Bannai, T. Ikuta, and K. Kawagoe, "Spin models constructed from the Hamming association schemes," in *Proceedings of the 10th Algebraic Combinatorics Symposium at Gifu University*, 1992.
3. E. Bannai, Et. Bannai, and F. Jaeger, "On spin models, modular invariance, and duality," *J. Alg. Combin.* **6** (1997), 203–228.

4. E. Bannai and T. Ito, *Algebraic Combinatorics I*, Benjamin/Cummings, Menlo Park, 1984.
5. E. Bannai, F. Jaeger, and A. Sali, "Classification of small spin models," *Kyushu J. of Math.* **48** (1994), 185–200.
6. A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer, New York, 1989.
7. B. Curtin, "Distance-regular graphs which support a spin model are thin," *Discr. Math.* **197/198** (1999), 205–216.
8. B. Curtin, "The Terwilliger algebra of a 2-homogeneous bipartite distance-regular graph," *J. Combin. Theory Ser. B,* to appear.
9. B. Curtin, "Hyper-dual pairs of Bose-Mesner algebras, I, II," preprint.
10. B. Curtin and K. Nomura, "Some formulas for spin models on distance-regular graphs," *J. Comb. Theory Ser. B.* **75** (1999), 206–236.
11. B. Curtin and K. Nomura, "Distance-regular graphs related to the quantum enveloping algebra of $sl(2)$, *J. Alg. Comb.* **12** (2000), 25–36.
12. C.W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, 1962.
13. P. Delsarte, "An algebraic approach to the association schemes of coding theory," *Philips Research Reports Supplements* **10** (1973).
14. M. Doob, "On graph products and association schemes," *Utilitas Math.* **1** (1972), 291–302.
15. J. Go, "The Terwilliger algebra of the hypercube," preprint.
16. C.D. Godsil, *Algebraic Combinatorics*, Chapman & Hall, New York, 1993.
17. F. Jaeger, "Strongly regular graphs and spin models for the Kauffman polynomial," *Geom. Dedicata* **44** (1992), 23–52.
18. F. Jaeger, "On spin models, triply regular association schemes, and duality," *J. Alg. Combin.* **4** (1995), 103–144.
19. F. Jaeger, "Towards a classification of spin models in terms of association schemes," *Advanced Studies in Pure Math.* **24** (1996), 197–225.
20. F. Jaeger, M. Matsumoto, and K. Nomura, "Bose-Mesner algebras related to type II matrices and spin models," *J. Alg. Combin.* **8** (1998), 39–72.
21. V.F.R. Jones, "On knot invariants related to some statistical mechanical models," *Pac. J. Math.* **137** (1989), 311–224.
22. K. Kawagoe, A. Munemasa, and Y. Watatani, "Generalized spin models," *J. Knot Th. Ramif.* **3** (1995), 465–475.
23. A. Munemasa, private communication.
24. A. Neumaier, "Duality in coherent configurations," *Combinatorica* **9** (1989), 59–67.
25. K. Nomura, "Spin models constructed from Hadamard matrices," *J. Combin.Theory Ser. A* **68** (1994), 251–261.
26. K. Nomura, "An algebra associated with a spin model," *J. Alg. Combin.* **6** (1997), 53–58.
27. S.S. Shrikhande, "The uniqueness of the L_2 association scheme," *Ann. Math. Statist.* **30** (1959), 781–798.
28. N.J.A. Sloane, "An introduction to association schemes and coding theory," *Theory and Application of Special Functions* **35** (1975), 225–260.
29. K. Tanabe, "The irreducible modules of the Terwilliger algebras of Doob schemes," *J. Alg.Combin.* **6** (1997), 173–195.
30. P. Terwilliger, "The subconstituent algebra of an association scheme," *J. Alg.Combin.* Part I: **1** (1992), 363–388; Part II: **2** (1993), 73–103; Part III: **2** (1993), 177–210.
31. M. Tomiyama and N. Yamazaki, "The Terwilliger algebra of a strongly regular graph," *Kyushu J. Math* **48** (1994), 323–334.