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Unipotent Brauer Character Values of $GL(n, \mathbb{F}_q)$ and the Forgotten Basis of the Hall Algebra

JONATHAN BRUNDAN* brundan@darkwing.uoregon.edu
Department of Mathematics, University of Oregon, Eugene, Oregon 97403-1222, U.S.A.

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Abstract. We give a formula for the values of irreducible unipotent p-modular Brauer characters of $GL(n, \mathbb{F}_q)$ at unipotent elements, where p is a prime not dividing q, in terms of (unknown!) weight multiplicities of quantum GL_n and certain generic polynomials $S_{\lambda,\mu}(q)$. These polynomials arise as entries of the transition matrix between the renormalized Hall-Littlewood symmetric functions and the forgotten symmetric functions. We also provide an alternative combinatorial algorithm working in the Hall algebra for computing $S_{\lambda,\mu}(q)$.

Keywords: symmetric function, general linear group, unipotent representation, Brauer character

1. Introduction

In the character theory of the finite general linear group $G_n = GL(n, \mathbb{F}_q)$, the *Gelfand-Graev character* Γ_n plays a fundamental role. By definition [5], Γ_n is the character obtained by inducing a "general position" linear character from a maximal unipotent subgroup. It has support in the set of unipotent elements of G_n and for a unipotent element u of type λ (i.e. the block sizes of the Jordan normal form of u are the parts of the partition λ) Kawanaka [7, 3.2.24] has shown that

$$\Gamma_n(u) = (-1)^n(1-q)(1-q^2)\dots\left(1-q^{h(\lambda)}\right), \tag{1.1}$$

where $h(\lambda)$ is the number of non-zero parts of λ. The starting point for this article is the problem of calculating the operator determined by Harish-Chandra multiplication by Γ_n.

We have restricted our attention throughout to character values at unipotent elements, when it is convenient to work in terms of the *Hall algebra*, that is [13, Section 10.1], the vector space $\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}_n$, where $\mathfrak{g}_n$ denotes the set of unipotent-supported $\mathbb{C}$-valued class functions on G_n, with multiplication coming from the Harish-Chandra induction operator. For a partition λ of n, let $\pi_\lambda \in \mathfrak{g}_n$ denote the class function which is 1 on unipotent elements of type λ and zero on all other conjugacy classes of G_n. Then, $\{\pi_\lambda\}$ is a basis for the Hall algebra labelled by all partitions. Let $\gamma_n : \mathfrak{g} \to \mathfrak{g}$ be the linear operator determined by multiplication in $\mathfrak{g}$ by Γ_n. We describe in Section 2 an explicit recursive algorithm, involving the combinatorics of addable and removable nodes, for calculating the effect of γ_n on the

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basis $\{\pi_\lambda\}$. As an illustration of the algorithm, we rederive Kawanaka's formula (1.1) in 2.12.

Now recall from [13] that $\mathfrak{g}$ is isomorphic to the algebra $\Lambda_{\mathbb{C}}$ of symmetric functions over $\mathbb{C}$, the isomorphism sending the basis element π_λ of $\mathfrak{g}$ to the Hall-Littlewood symmetric function $\tilde{P}_\lambda \in \Lambda_{\mathbb{C}}$ (renormalized as in [9, section II.3, ex. 2]). Consider instead the element $\vartheta_\lambda \in \mathfrak{g}$ which maps under this isomorphism to the *forgotten symmetric function* $f_\lambda \in \Lambda_{\mathbb{C}}$ (see [9, section I.2]). Introduce the renormalized Gelfand-Graev operator $\hat{\gamma}_n = \delta \circ \gamma_n$, where $\delta : \mathfrak{g} \to \mathfrak{g}$ is the linear map with $\delta(\pi_\lambda) = \frac{1}{q^{h(\lambda)}-1}\pi_\lambda$ for all partitions λ. We show in Theorem 3.5 that

$$\vartheta_\lambda = \sum_{(n_1,\dots,n_h)} \hat{\gamma}_{n_1} \circ \hat{\gamma}_{n_2} \circ \cdots \circ \hat{\gamma}_{n_h}\big(\pi_{(0)}\big), \tag{1.2}$$

summing over all $(n_1, \dots, n_h)$ obtained by reordering the non-zero parts $\lambda_1, \dots, \lambda_h$ of λ in all possible ways. Thus, we obtain a direct combinatorial construction of the 'forgotten basis' $\{\vartheta_\lambda\}$ of the Hall algebra.

Let $K = (K_{\lambda,\mu})$ denote the matrix of Kostka numbers [9, I, (6.4)], $\tilde{K} = (\tilde{K}_{\lambda,\mu}(q))$ denote the matrix of Kostka-Foulkes polynomials (renormalized as in [9, III, (7.11)]) and $J = (J_{\lambda,\mu})$ denote the matrix with $J_{\lambda,\mu} = 0$ unless $\mu = \lambda'$ when it is 1, where λ' is the conjugate partition to λ. Consulting [9, section I.6, section III.6], the transition matrix between the bases $\{\pi_\lambda\}$ and $\{\vartheta_\lambda\}$, i.e. the matrix $S = (S_{\lambda,\mu}(q))$ of coefficients such that

$$\vartheta_\lambda = \sum_\mu S_{\lambda,\mu}(q)\pi_\mu, \tag{1.3}$$

is then given by the formula $S = K^{-1}J\tilde{K}$; in particular, this implies that $S_{\lambda,\mu}(q)$ is a polynomial in q with integer coefficients. Our alternative approach to computing ϑ_λ using (1.2) allows explicit computation of the polynomials $S_{\lambda,\mu}(q)$ in some extra cases (e.g. when $\mu = (1^n)$) not easily deduced from the matrix product $K^{-1}J\tilde{K}$.

To explain our interest in this, let χ_λ denote the irreducible unipotent character of G_n labelled by the partition λ, as constructed originally in [12], and let $\sigma_\lambda \in \mathfrak{g}$ denote its projection to unipotent-supported class functions. So, σ_λ is the element of $\mathfrak{g}$ mapping to the Schur function s_λ under the isomorphism $\mathfrak{g} \to \Lambda_{\mathbb{C}}$ (see [13]). Since $\sigma_{\lambda'} = \sum_\mu K_{\lambda,\mu}\vartheta_\mu$ [9, section I.6], we deduce that the value of $\chi_{\lambda'}$ at a unipotent element u of type ν can be expressed in terms of the Kostka numbers $K_{\lambda,\mu}$ and the polynomials $S_{\mu,\nu}(q)$ as

$$\chi_{\lambda'}(u) = \sum_\mu K_{\lambda,\mu}S_{\mu,\nu}(q). \tag{1.4}$$

This is a rather clumsy way of expressing the unipotent character values in the ordinary case, but this point of view turns out to be well-suited to describing the irreducible unipotent *Brauer characters*.

So now suppose that p is a prime not dividing q, $\Bbbk$ is a field of characteristic p and let the multiplicative order of q modulo p be ℓ. In [6], James constructed for each partition λ of n an absolutely irreducible, unipotent $\Bbbk G_n$-module D_λ (denoted $L(1, \lambda)$ in [1]), and showed that the set of all D_λ gives the complete set of non-isomorphic irreducible modules that arise as constituents of the permutation representation of $\Bbbk G_n$ on cosets of a Borel subgroup. Let χ_λ^p denote the Brauer character of the module D_λ, and $\sigma_\lambda^p \in g$ denote the projection of χ_λ^p to unipotent-supported class functions. Then, as a direct consequence of the results of Dipper and James [3], we show in Theorem 4.6 that $\sigma_{\lambda'}^p = \sum_\mu K_{\lambda,\mu}^{p,\ell} \vartheta_\mu$ where $K_{\lambda,\mu}^{p,\ell}$ denotes the weight multiplicity of the μ-weight space in the irreducible high-weight module of high-weight λ for *quantum* GL_n, at an ℓth root of unity over a field of characteristic p. In other words, for a unipotent element u of type ν, we have the modular analogue of (1.4):

$$\chi_{\lambda'}^p(u) = \sum_\mu K_{\lambda,\mu}^{p,\ell} S_{\mu,\nu}(q) \tag{1.5}$$

This formula reduces the problem of calculating the values of the irreducible unipotent Brauer characters at unipotent elements to knowing the modular Kostka numbers $K_{\lambda,\mu}^{p,\ell}$ and the polynomials $S_{\mu,\nu}(q)$.

Most importantly, taking $\nu = (1^n)$ in (1.5), we obtain the degree formula:

$$\chi_{\lambda'}^p(1) = \sum_\mu K_{\lambda,\mu}^{p,\ell} S_{\mu,(1^n)}(q) \tag{1.6}$$

where, as a consequence of (1.2) (see Example 3.7),

$$S_{\mu,(1^n)}(q) = \sum_{(n_1,\ldots,n_h)} \left[\prod_{i=1}^{n} (q^i - 1) \Big/ \prod_{i=1}^{h} (q^{n_1+\cdots+n_i} - 1) \right] \tag{1.7}$$

summing over all $(n_1, \ldots, n_h)$ obtained by reordering the non-zero parts $\mu_1, \ldots, \mu_h$ of μ in all possible ways. This formula was first proved in [1, section 5.5], as a consequence of a result which can be regarded as the modular analogue of Zelevinsky's branching rule [13, section 13.5] involving the affine general linear group. The proof presented here is independent of [1] (excepting some self-contained results from [1, section 5.1]), appealing instead directly to the original characteristic 0 branching rule of Zelevinsky, together with the work of Dipper and James on decomposition matrices. We remark that since all of the integers $S_{\mu,(1^n)}(q)$ are positive, the formula (1.6) can be used to give quite powerful *lower bounds* for the degrees of the irreducible Brauer characters, by exploiting a q-analogue of the Premet-Suprunenko bound for the $K_{\lambda,\mu}^{p,\ell}$. The details can be found in [2].

To conclude this introduction, we list in the table below the polynomials $S_{\lambda,\mu}(q)$ for $n \leq 4$:

	1
1	1

	2	1^2
2	-1	$q-1$
1^2	1	1

	3	21	1^3
3	1	$1-q$	$(q^2-1)(q-1)$
21	-2	$q-2$	$(q-1)(q+2)$
1^3	1	1	1

	4	31	2^2	21^2	1^4
4	-1	$q-1$	$q-1$	$(1-q)(q^2-1)$	$(q^3-1)(q^2-1)(q-1)$
31	2	$2-q$	$(1-q)(q+2)$	$(q^2-1)(q-2)$	$(q^2-1)(q^3+q^2-2)$
2^2	1	$1-q$	q^2-q+1	$1-q$	$(q^3-1)(q-1)$
21^2	-3	$q-3$	$q-3$	q^2+q-3	q^3+q^2+q-3
1^4	1	1	1	1	1

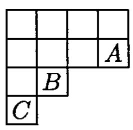
Finally, I would like to thank Alexander Kleshchev for several helpful discussions about this work.

2. An algorithm for computing γ_n

We will write $\lambda \vdash n$ to indicate that λ is a partition of n, that is, a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ of non-negative integers summing to n. Given $\lambda \vdash n$, we denote its *Young diagram* by $[\lambda]$; this is the set of *nodes*

$$[\lambda] = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq j \leq \lambda_i\}.$$

By an *addable node* (for λ), we mean a node $A \in \mathbb{N} \times \mathbb{N}$ such that $[\lambda] \cup \{A\}$ is the diagram of a partition; we denote the new partition obtained by adding the node A to λ by $\lambda \cup A$. By a *removable node* (for λ) we mean a node $B \in [\lambda]$ such that $[\lambda] \backslash \{B\}$ is the diagram of a partition; we denote the new partition obtained by removing B from λ by $\lambda \backslash B$. The *depth* $d(B)$ of the node $B = (i, j) \in \mathbb{N} \times \mathbb{N}$ is the row number i. If B is removable for λ, it will also be convenient to define $e(B)$ (depending also on λ!) to be the depth of the next removable node above B in the partition λ, or 0 if no such node exists. For example consider the partition $\lambda = (4, 4, 2, 1)$, and let A, B, C be the removable nodes in order of increasing depth:



Then, $e(A) = 0, e(B) = d(A) = 2, e(C) = d(B) = 3, d(C) = 4$.

Now fix a prime power q and let G_n denote the finite general linear group $GL(n, \mathbb{F}_q)$ as in the introduction. Let $V_n = \mathbb{F}_q^n$ denote the natural n-dimensional left G_n-module, with standard basis $v_1, \dots, v_n$. Let H_n denote the *affine general linear group* $AGL(n, \mathbb{F}_q)$. This is the semidirect product $H_n = V_n G_n$ of G_n acting on the elementary Abelian group V_n. We always work with the standard embedding $H_n \hookrightarrow G_{n+1}$ that identifies H_n with the subgroup of G_{n+1} consisting of all matrices of the form:

$$\left[\begin{array}{ccc|c} & * & & * \\ \hline 0 & \dots & 0 & 1 \end{array}\right].$$

Thus, we have a chain of subgroups $1 = H_0 \subset G_1 \subset H_1 \subset G_2 \subset H_2 \subset \dots$ (by convention, we allow the notations G_0, H_0 and V_0, all of which denote groups with one element.)

For $\lambda \vdash n$, let $u_\lambda \in G_n \subset H_n$ denote the upper uni-triangular matrix consisting of Jordan blocks of sizes $\lambda_1, \lambda_2, \dots$ down the diagonal. As is well-known, $\{u_\lambda \mid \lambda \vdash n\}$ is a set of representatives of the unipotent conjugacy classes in G_n. We wish to describe instead the unipotent classes in H_n. These were determined in [10, section 1], but the notation here will be somewhat different. For $\lambda \vdash n$ and an addable node A for λ, define the upper uni-triangular $(n+1) \times (n+1)$ matrix $u_{\lambda,A} \in H_n \subset G_{n+1}$ by

$$u_{\lambda,A} = \begin{cases} u_\lambda & \text{if } A \text{ is the deepest addable node,} \\ v_{\lambda_1+\cdots+\lambda_{d(A)}} u_\lambda & \text{otherwise.} \end{cases}$$

If instead $\lambda \vdash (n+1)$ and B is removable for λ (hence addable for $\lambda \backslash B$) define $u_{\lambda,B}$ to be a shorthand for $u_{\lambda\backslash B,B} \in H_n$. To aid translation between our notation and that of [10], we note that $u_{\lambda,B}$ is conjugate to the element denoted $c_{n+1}(1^{(k)}, \mu)$ there, where $k = \lambda_{d(B)}$ and μ is the partition obtained from λ by removing the $d(B)$th row. Then:

Lemma 2.1
(i) *The set*

$$\{u_{\lambda,A} \mid \lambda \vdash n, A \textit{ addable for } \lambda\} = \{u_{\lambda,B} \mid \lambda \vdash (n+1), B \textit{ removable for } \lambda\}$$

is a set of representatives of the unipotent conjugacy classes of H_n.

(ii) *For $\lambda \vdash (n+1)$ and a removable node B, $|C_{G_{n+1}}(u_\lambda)|/|C_{H_n}(u_{\lambda,B})| = q^{d(B)} - q^{e(B)}$.*

Proof: Part (i) is a special case of [10, 1.3(i)], where all conjugacy classes of the group H_n are described. For (ii), combine the formula for $|C_{G_{n+1}}(u_\lambda)|$ from [12, 2.2] with [10, 1.3(ii)], or calculate directly. □

For any group G, we write $C(G)$ for the set of $\mathbb{C}$-valued class functions on G. Let $\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}_n \subset \bigoplus_{n\geq 0} C(G_n)$ denote the Hall algebra as in the introduction. We recall that $\mathfrak{g}$ is a graded Hopf algebra in the sense of [13], with multiplication $\diamond : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ arising from Harish-Chandra induction and comultiplication $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ arising from Harish-Chandra restriction, see [13, section 10.1] for fuller details. Also defined in the introduction, $\mathfrak{g}$ has the natural 'characteristic function' basis $\{\pi_\lambda\}$ labelled by all partitions.

By analogy, we introduce an extended version of the Hall algebra corresponding to the affine general linear group. This is the vector space $\mathfrak{h} = \bigoplus_{n\geq 0} \mathfrak{h}_n$ where $\mathfrak{h}_n$ is the subspace of $C(H_n)$ consisting of all class functions with support in the set of unipotent elements of H_n, with algebra structure to be explained below. To describe a basis for $\mathfrak{h}$, given $\lambda \vdash n$ and an addable node A, define $\pi_{\lambda,A} \in C(H_n)$ to be the class function which takes value 1 on $u_{\lambda,A}$ and is zero on all other conjugacy classes of H_n. Given $\lambda \vdash (n+1)$ and a removable B, set $\pi_{\lambda,B} = \pi_{\lambda\backslash B,B}$. Then, in view of Lemma 2.1(i), $\{\pi_{\lambda,A} \mid \lambda$ a partition, A addable for $\lambda\} = \{\pi_{\lambda,B} \mid \lambda$ a partition, B removable for $\lambda\}$ is a basis for $\mathfrak{h}$.

Now we introduce various operators as in [13, section 13.1] (but take notation instead from [1, section 5.1]). First, for $n \geq 0$, we have the inflation operator

$$e_0^n : C(G_n) \to C(H_n)$$

defined by $(e_0^n \chi)(vg) = \chi(g)$ for $\chi \in C(G_n)$, $v \in V_n$, $g \in G_n$. Next, fix a non-trivial additive character $\chi : \mathbb{F}_q \to \mathbb{C}^\times$ and let $\chi_n : V_n \to \mathbb{C}^\times$ be the character defined by $\chi_n(\sum_{i=1}^n c_i v_i) = \chi(c_n)$. The group G_n acts naturally on the characters $C(V_n)$ and one easily checks that the subgroup $H_{n-1} < G_n$ centralizes χ_n. In view of this, it makes sense to define for each $n \geq 1$ the operator

$$e_+^n : C(H_{n-1}) \to C(H_n),$$

namely, the composite of inflation from H_{n-1} to $V_n H_{n-1}$ with the action of V_n being via the character χ_n, followed by ordinary induction from $V_n H_{n-1}$ to H_n. Finally, for $n \geq 1$ and $1 \leq i \leq n$, we have the operator

$$e_i^n : C(G_{n-i}) \to C(H_n)$$

defined inductively by $e_i^n = e_+^n \circ e_{i-1}^{n-1}$. The significance of these operators is due to the following lemma [13, section 13.2]:

Lemma 2.2 *The operator* $e_0^n \oplus e_1^n \oplus \cdots \oplus e_n^n : C(G_n) \oplus C(G_{n-1}) \oplus \cdots \oplus C(G_0) \to C(H_n)$ *is an isometry.*

We also have the usual restriction and induction operators

$$\operatorname{res}^{G_n}_{H_{n-1}} : C(G_n) \to C(H_{n-1}) \qquad \operatorname{ind}^{G_n}_{H_{n-1}} : C(H_{n-1}) \to C(G_n).$$

One checks that all of the operators e_0^n, e_+^n, $\operatorname{res}^{G_n}_{H_{n-1}}$ and $\operatorname{ind}^{G_n}_{H_{n-1}}$ send class functions with unipotent support to class functions with unipotent support. So, we can define the following operators between $\mathfrak{g}$ and $\mathfrak{h}$, by restricting the operators listed to unipotent-supported class functions:

$$e_+ : \mathfrak{h} \to \mathfrak{h}, \quad e_+ = \bigoplus_{n \geq 1} e_+^n; \tag{2.3}$$

$$e_i : \mathfrak{g} \to \mathfrak{h}, \quad e_i = \bigoplus_{n \geq i} e_i^n = (e_+)^i \circ e_0; \tag{2.4}$$

$$\operatorname{ind} : \mathfrak{h} \to \mathfrak{g}, \quad \operatorname{ind} = \bigoplus_{n \geq 1} \operatorname{ind}^{G_n}_{H_{n-1}}; \tag{2.5}$$

$$\operatorname{res} : \mathfrak{g} \to \mathfrak{h}, \quad \operatorname{res} = \bigoplus_{n \geq 0} \operatorname{res}^{G_n}_{H_{n-1}} \tag{2.6}$$

where for the last definition, $\operatorname{res}^{G_0}_{H_{-1}}$ should be interpreted as the zero map.

Now we indicate briefly how to make $\mathfrak{h}$ into a graded Hopf algebra. In view of Lemma 2.2, there are unique linear maps $\diamond : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}$ and $\Delta : \mathfrak{h} \to \mathfrak{h} \otimes \mathfrak{h}$ such that

$$(e_i \chi) \diamond (e_j \tau) = e_{i+j}(\chi \diamond \tau), \tag{2.7}$$

$$\Delta(e_k \psi) = \sum_{i+j=k} (e_i \otimes e_j)\Delta(\psi), \tag{2.8}$$

for all $i, j, k \geq 0$ and $\chi, \tau, \psi \in \mathfrak{g}$. One can check, using Lemma 2.2 and the fact that $\mathfrak{g}$ is a graded Hopf algebra, that these operations endow $\mathfrak{h}$ with the structure of a graded Hopf algebra (the unit element is $e_0\pi_{(0)}$, and the counit is the map $e_i \chi \mapsto \delta_{i,0}\varepsilon(\chi)$ where $\varepsilon : \mathfrak{g} \to \mathbb{C}$ is the counit of $\mathfrak{g}$). Moreover, the map $e_0 : \mathfrak{g} \to \mathfrak{h}$ is a Hopf algebra embedding. Unlike for the operations of $\mathfrak{g}$, we do not know of a natural representation theoretic interpretation for these operations on $\mathfrak{h}$ except in special cases, see [1, section 5.2].

The effect of the operators (2.3)–(2.6) on our characteristic function bases is described explicitly by the following lemma:

Lemma 2.9 *Let $\lambda \vdash n$ and label the addable nodes (resp. removable nodes) of λ as $A_1, A_2, \ldots, A_s$ (resp. $B_1, B_2, \ldots, B_{s-1}$) in order of increasing depth. Also let $B = B_r$ be some fixed removable node. Then,*

(i) $e_0\pi_\lambda = \sum_{i=1}^{s} \pi_{\lambda,A_i}$;
(ii) $e_+\pi_{\lambda,B} = q^{d(B)} \sum_{i=r+1}^{s} \pi_{\lambda,A_i} - q^{e(B)} \sum_{i=r}^{s} \pi_{\lambda,A_i}$.
(iii) $\operatorname{res} \pi_\lambda = \sum_{i=1}^{s-1} \pi_{\lambda,B_i}$;
(iv) $\operatorname{ind} \pi_{\lambda,B} = (q^{d(B)} - q^{e(B)})\pi_\lambda$;
(v) $\operatorname{ind} \circ \operatorname{res} \pi_\lambda = (q^{h(\lambda)} - 1)\pi_\lambda$.

Proof:

(i) For $\mu \vdash n$ and A addable, we have by definition that $(e_0\pi_\lambda)(u_{\mu,A}) = \pi_\lambda(u_\mu) = \delta_{\lambda,\mu}$. Hence, $e_0\pi_\lambda = \sum_A \pi_{\lambda,A}$, summing over all addable nodes A for λ.
(ii) This is a special case of [10, 2.4] translated into our notation.
(iii) For $\mu \vdash n$ and B removable, $(\operatorname{res} \pi_\lambda)(u_{\mu,B})$ is zero unless $u_{\mu,B}$ is conjugate in G_n to u_λ, when it is one. So the result follows on observing that $u_{\mu,B}$ is conjugate in G_n to u_λ if and only if $\mu = \lambda$.
(iv) We can write $\operatorname{ind} \pi_{\lambda,B} = \sum_{\mu\vdash n} c_\mu\pi_\mu$. To calculate the coefficient c_μ for fixed $\mu \vdash n$, we use (iii), Lemma 2.1(ii) and Frobenius reciprocity.
(v) This follows at once from (iii) and (iv) since $\sum_B (q^{d(B)} - q^{e(B)}) = q^{h(\lambda)} - 1$, summing over all removable nodes B for λ. □

Lemma 2.9(i), (ii) give explicit formulae for computing the operator $e_n = (e_+)^n \circ e_0$. The connection between e_n and the Gelfand-Graev operator γ_n defined in the introduction comes from the following result:

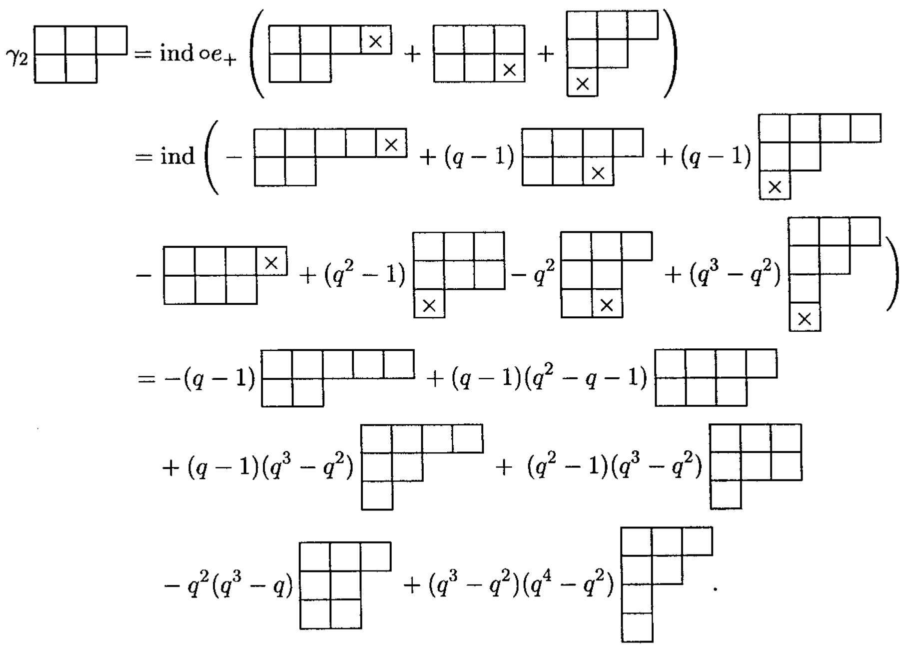
Theorem 2.10 *For $n \geq 1$, $\gamma_n = \operatorname{ind} \circ e_{n-1}$.*

Proof: In [1, Theorem 5.1e], we showed directly from the definitions that for any $\chi \in C(G_m)$ and any $n \geq 1$, the class function $\chi.\Gamma_n \in C(G_{m+n})$ obtained by Harish-Chandra induction from $(\chi, \Gamma_n) \in C(G_m) \times C(G_n)$ is equal to $\mathrm{res}^{H_{m+n}}_{G_{m+n}}(e^{m+n}_n \chi)$. Moreover, by [1, Lemma 5.1c(iii)], we have that $\mathrm{res}^{H_{m+n}}_{G_{m+n}} \circ e^{m+n}_+ = \mathrm{ind}^{G_{m+n}}_{H_{m+n-1}}$. Hence,

$$\chi.\Gamma_n = \mathrm{ind}^{G_{m+n}}_{H_{m+n-1}}\left(e^{m+n-1}_{n-1}\chi\right).$$

The theorem is just a restatement of this formula at the level of unipotent-supported class functions. □

Example 2.11 We show how to calculate $\gamma_2 \pi_{(3,2)}$ using Lemma 2.9 and the theorem. We omit the label π in denoting basis elements, and in the case of the intermediate basis elements of $\mathfrak{h}$, we mark removable nodes with $\times$.



Example 2.12 We apply Theorem 2.10 to rederive the explicit formula (1.1) for the Gelfand-Graev character Γ_n itself. Of course, by Theorem 2.10, $\Gamma_n = \mathrm{ind} \circ e_{n-1}(\pi_{(0)})$. We will in fact prove that

$$e_{n-1}\pi_{(0)} = (-1)^{n-1} \sum_{\lambda \vdash n} \sum_{B \text{ removable}} (1-q)(1-q^2)\dots\left(1-q^{h(\lambda)-1}\right)\pi_{\lambda,B} \qquad (2.13)$$

Then (1.1) follows easily on applying ind using Lemma 2.9(iv) and the calculation in the proof of Lemma 2.9(v).

To prove (2.13), use induction on n, $n = 1$ being immediate from Lemma 2.9(i). For $n > 1$, fix some $\lambda \vdash n$, label the addable and removable nodes of λ as in Lemma 2.9 and take $1 \le r \le s$. Thanks to Lemma 2.9(ii), the π_{λ,A_r}-coefficient of $e_n(\pi_{(0)}) = e_+ \circ e_{n-1}(\pi_{(0)})$ only depends on the π_{λ,B_i}-coefficients of $e_{n-1}(\pi_{(0)})$ for $1 \le i \le \min(r, s-1)$. So by the induction hypothesis the π_{λ,A_r}-coefficient of $e_n(\pi_{(0)})$ is the same as the π_{λ,A_r}-coefficient of

$$(-1)^{n-1}(1-q)\dots\left(1-q^{h(\lambda)-1}\right)\sum_{i=1}^{\min(r,s-1)} e_+\pi_{\lambda,B_i},$$

which using Lemma 2.9(ii) equals

$$(-1)^{n-1}(1-q)\dots\left(1-q^{h(\lambda)-1}\right)\sum_{i=1}^{\min(r,s-1)} \left(\delta_{r,i}q^{d(B_i)} - q^{e(B_i)}\right).$$

This simplifies to $(-1)^n(1-q)\dots(1-q^{h(\lambda)-1})$ if $r < s$ and $(-1)^n(1-q)\dots(1-q^{h(\lambda)})$ if $r = s$, as required to prove the induction step.

3. The forgotten basis

Recall from the introduction that for $\lambda \vdash n$, $\chi_\lambda \in C(G_n)$ denotes the irreducible unipotent character parametrized by the partition λ, and $\sigma_\lambda \in \mathfrak{g}$ is its projection to unipotent-supported class functions. Also, $\{\vartheta_\lambda\}$ denotes the 'forgotten' basis of $\mathfrak{g}$, which can be *defined* as the unique basis of $\mathfrak{g}$ such that for each n and each $\lambda \vdash n$,

$$\sigma_{\lambda'} = \sum_{\mu \vdash n} K_{\lambda,\mu}\vartheta_\mu. \tag{3.1}$$

Given $\lambda \vdash n$, we write $\mu \perp_j \lambda$ if $\mu \vdash (n-j)$ and $\lambda_{i+1} \le \mu_i \le \lambda_i$ for all $i = 1, 2, \dots$. This definition arises in the following well-known inductive formula for the Kostka number $K_{\lambda,\mu}$, i.e. the number of standard λ-tableaux of weight μ [9, section I(6.4)]:

Lemma 3.2 *For $\lambda \vdash n$ and any composition $\nu \models n$, $K_{\lambda,\nu} = \sum_{\mu \perp_j \lambda} K_{\mu,\bar{\nu}}$, where j is the last non-zero part of ν and $\bar{\nu}$ is the composition obtained from ν by replacing this last non-zero part by zero.*

We will need the following special case of Zelevinsky's branching rule [13, section 13.5] (see also [1, Corollary 5.4d(ii)] for its modular analogue):

Theorem 3.3 (*Zelevinsky*) *For $\lambda \vdash n$, $\mathrm{res}^{G_n}_{H_{n-1}} \chi_{\lambda'} = \sum_{j \ge 1} \sum_{\mu \perp_j \lambda} e_{j-1}\chi_{\mu'}$.*

Now define the map $\delta : \mathfrak{g} \to \mathfrak{g}$ as in the introduction by setting $\delta(\pi_\lambda) = \frac{1}{q^{h(\lambda)}-1}\pi_\lambda$ for each partition λ and extending linearly to all of $\mathfrak{g}$. The significance of δ is that by Lemma 2.9(v),

$\delta \circ \text{ind} \circ \text{res}(\pi_\lambda) = \pi_\lambda$ for all λ. Set $\hat{\gamma}_n = \delta \circ \gamma_n$ and for a partition λ, define

$$\hat{\gamma}_\lambda = \sum_{(n_1,\dots,n_h)} \hat{\gamma}_{n_h} \circ \cdots \circ \hat{\gamma}_{n_2} \circ \hat{\gamma}_{n_1} \tag{3.4}$$

summing over all compositions $(n_1, \dots, n_h)$ obtained by reordering the $h = h(\lambda)$ non-zero parts of λ in all possible ways.

Theorem 3.5 *For any* $\lambda \vdash n$, $\vartheta_\lambda = \hat{\gamma}_\lambda(\pi_{(0)})$.

Proof: We will show by induction on n that

$$\sigma_{\lambda'} = \sum_{\mu \vdash n} K_{\lambda,\mu} \hat{\gamma}_\mu\big(\pi_{(0)}\big). \tag{3.6}$$

The theorem then follows immediately in view of the definition (3.1) of ϑ_λ. Our induction starts trivially with the case $n = 0$. So now suppose that $n > 0$ and that (3.6) holds for all smaller n. By Theorem 3.3,

$$\text{res } \sigma_{\lambda'} = \sum_{j \geq 1} \sum_{\mu \perp_j \lambda} e_{j-1} \sigma_{\mu'}.$$

Applying the operator $\delta \circ \text{ind}$ to both sides, we deduce that

$$\sigma_{\lambda'} = \sum_{j \geq 1} \sum_{\mu \perp_j \lambda} \hat{\gamma}_j(\sigma_{\mu'}) = \sum_{j \geq 1} \sum_{\mu \perp_j \lambda} \sum_{\nu \vdash (n-j)} K_{\mu,\nu} \hat{\gamma}_j \circ \hat{\gamma}_\nu\big(\pi_{(0)}\big)$$

(we have applied the induction hypothesis)

$$= \sum_{j \geq 1} \sum_{\mu \perp_j \lambda} \sum_{\nu \vdash (n-j)} \sum_{(n_1,\dots,n_h)} K_{\mu,(n_1,\dots,n_h)} \hat{\gamma}_j \circ \hat{\gamma}_{n_h} \circ \cdots \circ \hat{\gamma}_{n_2} \circ \hat{\gamma}_{n_1}\big(\pi_{(0)}\big)$$

(summing over $(n_1, \dots, n_h)$ obtained by reordering the non-zero parts ν in all possible ways)

$$= \sum_{j \geq 1} \sum_{\nu \vdash (n-j)} \sum_{(n_1,\dots,n_h)} K_{\lambda,(n_1,\dots,n_h,j)} \hat{\gamma}_j \circ \hat{\gamma}_{n_h} \circ \cdots \circ \hat{\gamma}_{n_2} \circ \hat{\gamma}_{n_1}\big(\pi_{(0)}\big)$$

(we have applied Lemma 3.2)

$$= \sum_{\eta \vdash n} \sum_{(m_1,\dots,m_k)} K_{\lambda,(m_1,\dots,m_k)} \hat{\gamma}_{m_k} \circ \cdots \circ \hat{\gamma}_{m_2} \circ \hat{\gamma}_{m_1}\big(\pi_{(0)}\big)$$

(now summing over $(m_1, \dots, m_k)$ obtained by reordering the non-zero parts of η in all possible ways)

$$= \sum_{\eta \vdash n} K_{\lambda,\eta} \hat{\gamma}_\eta\big(\pi_{(0)}\big)$$

which completes the proof. $\square$

Example 3.7 For $\chi \in \mathfrak{g}_n$, write $\deg \chi$ for its value at the identity element of G_n. We wish to derive the formula (1.7) for $\deg \vartheta_\lambda = S_{\lambda,(1^n)}(q)$ using Theorem 3.5. So, fix $\lambda \vdash n$. Then, by Theorem 3.5,

$$\deg \vartheta_\lambda = \sum_{(n_1,\ldots,n_h)} \deg \left[\hat{\gamma}_{n_h} \circ \cdots \circ \hat{\gamma}_{n_1}\left(\pi_{(0)}\right)\right]. \tag{3.8}$$

We will show that given $\chi \in \mathfrak{g}_m$,

$$\deg \hat{\gamma}_n(\chi) = (q^{m+n-1} - 1)(q^{m+n-2} - 1) \ldots (q^{m+1} - 1) \deg \chi; \tag{3.9}$$

then the formula (1.7) follows easily from (3.8). Now γ_n is Harish-Chandra multiplication by Γ_n, so

$$\deg \gamma_n(\chi) = \deg \Gamma_n \deg \chi \cdot \frac{(q^{m+n} - 1) \ldots (q^{m+1} - 1)}{(q^n - 1) \ldots (q - 1)},$$

the last term being the index in G_{m+n} of the standard parabolic subgroup with Levi factor $G_m \times G_n$. This simplifies using (1.1) to $(q^{m+n} - 1) \ldots (q^{m+1} - 1) \deg \chi$. Finally, to calculate $\deg \hat{\gamma}_n(\chi)$, we need to rescale using δ, which divides this expression by $(q^{m+n} - 1)$.

4. Brauer character values

Finally, we derive the formula (1.5) for the unipotent Brauer character values. So let p be a prime not dividing q and χ_λ^p be the irreducible unipotent p-modular Brauer character labelled by λ as in the introduction. Writing $C^p(G_n)$ for the $\mathbb{C}$-valued class functions on G_n with support in the set of p'-elements of G_n, we view χ_λ^p as an element of $C^p(G_n)$. Let $\dot{\chi}_\lambda$ denote the projection of the ordinary unipotent character χ_λ to $C^p(G_n)$. Then, by [6], we can write

$$\dot{\chi}_\lambda = \sum_{\mu \vdash n} D_{\lambda,\mu} \chi_\mu^p, \tag{4.1}$$

and the resulting matrix $D = (D_{\lambda,\mu})$ is the *unipotent part* of the p-modular decomposition matrix of G_n. One of the main achievements of the Dipper-James theory from [3] (see e.g. [1, (3.5a)]) relates these decomposition numbers to the decomposition numbers of quantum GL_n.

To recall some definitions, let $\Bbbk$ be a field of characteristic p and $v \in \Bbbk$ be a square root of the image of q in $\mathbb{K}$. Let U_n denote the divided power version of the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ specialized over $\Bbbk$ at the parameter v, as defined originally by Lusztig [8] and Du [4, section 2] (who extended Lusztig's construction from $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$). For each partition $\lambda \vdash n$, there is an associated irreducible polynomial representation of U_n of high-weight λ, which we denote by $L(\lambda)$. Also let $V(\lambda)$ denote the standard (or Weyl) module of high-weight λ. Write

$$\text{ch } V(\lambda) = \sum_{\mu \vdash n} D'_{\lambda,\mu} \text{ch } L(\mu), \tag{4.2}$$

so $D' = (D'_{\lambda,\mu})$ is the decomposition matrix for the polynomial representations of quantum GL_n of degree n. Then, by [3]:

Theorem 4.3 (*Dipper and James*) $D'_{\lambda,\mu} = D_{\lambda',\mu'}$.

Let σ_λ^p denote the projection of χ_λ^p to unipotent-supported class functions. The $\{\sigma_\lambda^p\}$ also give a basis for the Hall algebra $\mathfrak{g}$. Inverting (4.1) and using (3.1),

$$\sigma_{\lambda'}^p = \sum_{\mu \vdash n} D^{-1}_{\lambda',\mu'} \sigma_{\mu'} = \sum_{\mu,\nu \vdash n} D^{-1}_{\lambda',\mu'} K_{\mu,\nu} \vartheta_\nu \tag{4.4}$$

where $D^{-1} = (D^{-1}_{\lambda,\mu})$ is the inverse of the matrix D. On the other hand, writing $K^{p,\ell}_{\lambda,\mu}$ for the multiplicity of the μ-weight space of $L(\lambda)$, and recalling that $K_{\lambda,\mu}$ is the multiplicity of the μ-weight space of $V(\lambda)$, we have by (4.2) that

$$K_{\mu,\nu} = \sum_{\eta \vdash n} D'_{\mu,\eta} K^{p,\ell}_{\eta,\nu}. \tag{4.5}$$

Substituting (4.5) into (4.4) and applying Theorem 4.3, we deduce:

Theorem 4.6 $\sigma_{\lambda'}^p = \sum_{\mu \vdash n} K^{p,\ell}_{\lambda,\mu} \vartheta_\mu$.

Now (1.5) and (1.6) follow at once. This completes the proof of the formulae stated in the introduction.

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