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Plücker Relations on Schur Functions

MICHAEL KLEBER* kleber@math.mit.edu
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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Abstract. We present a set of algebraic relations among Schur functions which are a multi-time generalization of the "discrete Hirota relations" known to hold among the Schur functions of rectangular partitions. We prove the relations as an application of a technique for turning Plücker relations into statements about Schur functions and other objects with similar definitions as determinants. We also give a quantum analogue of the relations which incorporates spectral parameters. Our proofs are mostly algebraic, but the relations have a clear combinatorial side, which we discuss.

Keywords: Schur function, Plücker relation, Jacobi-Trudi, quantum, Hirota relation

1. Introduction

Consider the following relationship among the Schur functions s_λ where λ is a rectangular partition:

$$s_{\langle m^\ell\rangle} s_{\langle m^\ell\rangle} = s_{\langle m+1^\ell\rangle} s_{\langle m-1^\ell\rangle} + s_{\langle m^{\ell+1}\rangle} s_{\langle m^{\ell-1}\rangle}. \tag{1}$$

Here $\langle m^\ell\rangle$ is the partition with ℓ parts each of size m, whose Young diagram is an $\ell \times m$ rectangle. A. N. Kirillov noticed this fact as a relation among the characters of finite-dimensional representations of $\mathfrak{sl}_n$ while studying the Bethe Ansatz for a one-dimensional system called the generalized Heisenberg magnet [3].

In later work, Kirillov and Reshetikhin observed that the relations could be viewed as a discrete version of a classical and well-studied dynamical system known to mathematical physics as the discrete Hirota relations [4]. The initial conditions are the characters of the fundamental representations of $\mathfrak{sl}_n$, and expressing the solutions in terms of the initial conditions is precisely the Jacobi-Trudi formula for $s_{\langle m^\ell\rangle}$.

In this paper, we present the natural extension of this set of relations to Schur functions of arbitrary partitions. The relations are all of the form

$$s_\lambda s_\lambda = s_{\lambda+\omega_\ell} s_{\lambda-\omega_\ell} + \text{other terms}. \tag{2}$$

Here we borrow notation from Lie theory: if λ is a partition, then we write $\lambda \pm \omega_\ell$ for the partition obtained by adding or removing a column of height ℓ from the Young diagram of λ; this corresponds to taking the highest weight λ and adding or subtracting the fundamental

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weight ω_ℓ. We have one such relation for every choice of a partition λ and column height ℓ such that λ has a column of height ℓ to begin with (otherwise $\lambda - \omega_\ell$ does not make sense). The various choices of ℓ should be thought of as independent time directions in which we can evolve the dynamical system.

The "other terms" in Eq. (2) are also each products of two Schur functions, and all have coefficients ± 1. The partitions that appear never have more columns or more outside corners than λ does. Thus we get a hierarchy of systems of relations for partitions with up to k corners; when $k = 1$ we are restricted to rectangular partitions, and we recover Eq. (1).

We prove the relations by reducing them to the Plücker relations among minors of a certain matrix, whose construction we define in Section 2. The construction applies not only to Schur functions, which we now view as determinants of the Jacobi-Trudi matrices, but to the determinants of any family of matrices with a similar type of definition. We formalize this notion, giving several other examples and a general version of the construction.

In Section 3 we state and prove the relations. We also prove a generalization of the relations to ones which include "shifts" or "spectral parameters." The generalizations of Schur functions that satisfy this version of the equations are the quantum analogues of characters for finite-dimensional representations for $U_q(\widehat{\mathfrak{sl}_n})$, and the generalized version may be related to the representation theory of quantum affine algebras, which is not yet well understood.

Finally, while most of the earlier proofs are algebraic, in Section 4 we offer a combinatorial interpretation for the relations in terms of the Littlewood-Richardson rule, in which the coefficients of ± 1 in the other terms mentioned above arise from an inclusion-exclusion argument. We give a completely bijective proof for Eq. (1), the rectangular Young diagram version, and we conjecture the existence of bijections with certain properties that would lead to a fully combinatorial proof of Eq. (2) as well.

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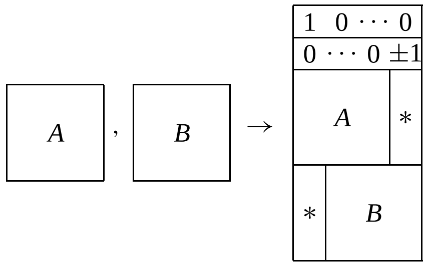
2. Generalized Jacobi-Trudi sets

We will describe a scheme for translating the Plücker relations among minors of a matrix into identities of objects defined by a Jacobi-Trudi style determinantal formula. This general concept is a well-established source for algebraic relations involving Schur functions; see e.g. [7, 8]. A special case of the particular construction we give here was used implicitly in [9] to prove some relations among quantum transfer matrices. Our applications will include Schur functions (characters of representations of SL_n), skew Schur functions, and Schur functions with spectral parameters (quantum characters of $U_q(\widehat{\mathfrak{sl}_n})$).

The heart of the construction is an operation $(A, B) \to A \square B$, where A and B are $n \times n$ matrices and $A \square B$ is an $(n+1) \times (2n+2)$ matrix. The operation can be depicted graphically as:



(3)

We will first define the operation for our motivating example, the set of Jacobi-Trudi matrices:

$$\left\{M_\lambda := \left(h_{\lambda_i - i + j}\right)_{i,j=1}^{n} \mid n \in \mathbb{Z}_{\geq 0},\ \lambda \text{ a partition with } n \text{ parts}\right\}.$$

If h_k is the kth homogeneous symmetric function (so $h_0 = 1$ and $h_k = 0$ for $k < 0$), then $\det(M_\lambda)$ is the Schur function s_λ. We permit λ to end with zeros, so if λ is a partition with n parts then we can obtain s_λ as the determinant of such an $m \times m$ matrix for any $m \geq n$.

Construction 2.1 Let λ and μ be partitions with n parts. We define the matrix $M = M_\lambda \Box M_\mu$, with $n + 1$ columns indexed by $\{1, \ldots, n + 1\}$ and $2n + 2$ rows indexed by $\{L, R, 1, \ldots, n, 1', \ldots, n'\}$, as follows:

$$\begin{aligned} M_{Lj} &= \delta(j, 1) \\ M_{Rj} &= (-1)^n \delta(j, n+1) \\ M_{ij} &= h_{\lambda_i - i + j},\ \ i = 1, \ldots, n \\ M_{i'j} &= h_{\mu_i - i + j - 1},\ \ i = 1, \ldots, n \end{aligned}$$

We adopt the notation $[r_1 r_2 \ldots r_k]_M$ for the determinant of the $k \times k$ sub-matrix of a $k \times n$ matrix M consisting of rows with indices $r_1, \ldots, r_k$; when the choice of M is clear from context the subscript will be dropped. Then for $M = M_\lambda \Box M_\mu$, we have $[R12 \ldots n] = s_\lambda$ and $[L1' \ldots n'] = s_\mu$. (The sign of $M_{R,n+1}$ was chosen for convenience precisely to make this happen.) Plücker relations on M will give us relations among Schur functions.

The construction relies on the following property of the set of Jacobi-Trudi matrices $\{M_\lambda\}$: there is a unique way to fill in the "∗" regions in Eq. (3) so that any $n + 1$-row nonzero minor of $M_\lambda \Box M_\mu$ is $\det(M_\nu)$ for some ν. To give a generalization of the construction, we isolate the properties of $\{M_\lambda\}$ which make this happen.

Definition 2.2 Let $\mathcal{M}$ be a set of square matrices. Let $\mathcal{R}_n$ denote the set of n-component vectors that appear as rows in any $n \times n$ matrix $M \in \mathcal{M}$, for each $n \in \mathbb{Z}_+$. We say $\mathcal{M}$ is a *generalized Jacobi-Trudi set* if there exist equivalence relations $\sim_n$ on $\mathcal{R}_n$ such that:

1. Any two rows of an $n \times n$ matrix $M \in \mathcal{M}$ are $\sim_n$ related,
2. If M is an $n \times n$ matrix with nonzero determinant and all of its rows are pairwise $\sim_n$ related, then there is a matrix $M' \in \mathcal{M}$ with the same rows as M (but possibly permuted).

Consider the operators d_L and d_R, which respectively drop the left and right components of a row vector.

3. Take any two rows $r_1, r_2 \in \mathcal{R}_n$ such that $d_L(r_1), d_L(r_2) \in \mathcal{R}_{n-1}$. If $r_1 \sim_n r_2$ then $d_L(r_1) \sim_{n-1} d_L(r_2)$. Furthermore, $d_L(r_1) = d_L(r_2)$ only if $r_1 = r_2$. Thus we can talk about d_L acting on the equivalence classes. Likewise, all this must hold for d_R as well.
4. If A and B are two $\sim_n$ classes such that $d_L(A) = d_R(B)$ then there is a unique $\sim_{n+1}$ class C such that $d_R(C) = A$ and $d_L(C) = B$.

Our archetypical generalized Jacobi-Trudi set of matrices, of course, is the set of Jacobi-Trudi matrices M_λ defined above. In this case there is only one conjugacy class for each $\sim_n$, and it consists of all rows of the form $(h_k, h_{k+1}, \ldots, h_{k+n-1})$ for $k+n-1$ nonnegative. Other examples of generalized Jacobi-Trudi sets include:

Example 1 The matrices $M_{\lambda/\mu} := (h_{\lambda_i - \mu_j - i + j})_{i,j=1}^n$. The determinants of these matrices are the skew Schur functions $s_{\lambda/\mu}$ corresponding to skew Young diagrams λ/μ, with $\mu \subset \lambda$ (*i.e.* $\mu_i \leq \lambda_i$ for all i). In this case, for each n there are infinitely many $\sim_n$ classes, one for each choice of μ: given a row vector $(h_{a_1}, h_{a_2}, \ldots, h_{a_n})$, it can appear in matrices $M_{\lambda/\mu}$ where $\mu_i - \mu_{i+1} = a_{i+1} - a_i - 1$.

The operator d_L (resp. d_R) takes the $\sim_n$ class associated with μ to the $\sim_{n-1}$ class of μ with μ_1 (resp. μ_{n-1}) removed. (Without loss of generality we assume that $\mu_n = 0$.)

Example 2 The set of matrices $T_\lambda(u + c)$, where λ is a partition, u is a formal variable, and $c \in \mathbb{Z}$ is called the shift. We will take the following as a formal definition:

$$T_\lambda(u) := (t_{\lambda_i - i + j}(u + \lambda_1 - \lambda_i + i + j - n - 1))_{i,j=1}^n$$

where λ has n parts, some of which may be zero. Define $s_\lambda^{(u)} := \det(T_\lambda(u))$. The $t_k(u)$ can optionally be specialized to $t_0(u) = 1$, $t_k(u) = 0$ for $k < 0$, as we do with the h_k to get Schur functions.

We will treat the $s_\lambda^{(u)}$ as formal symbols, but see the remarks following Theorem 3.4 for comments and references on the mathematical physics origins of the objects. Essentially, $s_\lambda^{(u)}$ can be regarded as quantum analogues of characters of representations of $U_q(\hat{\mathfrak{g}})$. If we send the entry $t_k(u + c)$ to h_k and therefore ignore the shift (this is letting $u \to \infty$ in the mathematical physics literature) we recover the Jacobi-Trudi matrices M_λ and plain Schur functions s_λ.

To understand the equivalence classes here, note that the rows of any matrix $T_\lambda(u + c)$ are of the form

$$(t_a(u + b), t_{a+1}(u + b + 1), \ldots, t_{a+n-1}(u + b + n - 1))$$

for some choice of integers a and b. The main diagonal of $T_\lambda(u)$ has entries $t_{\lambda_1}(*)$, $t_{\lambda_2}(*)$, $\ldots$, $t_{\lambda_n}(*)$, while the anti-diagonal has $t_*(u)$, $t_*(u + \lambda_1 - \lambda_2), \ldots, t_*(u + \lambda_1 - \lambda_n)$. It is therefore easy to see that if the row beginning with $t_a(u+b)$ appears in the matrix $T_\lambda(u+c)$, we must have $a + b = \lambda_1 - n + 1$. Therefore each $\sim_n$ class contains all rows which share a common value $a + b$.

We remark that given a partition λ with n parts and an $\sim_n$ class A, there is a unique integer c such that the rows of $T_\lambda(u+c)$ are in A.

Now we give a version of Construction 2.1 for any generalized Jacobi-Trudi set of matrices, which we will apply to the examples above. When $\mathcal{M} = \{M_\lambda\}$ this reduces to Construction 2.1.

Construction 2.3 Let $\mathcal{M}$ be a generalized Jacobi-Trudi set of matrices, and take two $n \times n$ matrices $A, B \in \mathcal{M}$. Let $\tilde{A}, \tilde{B}$ denote the $\sim_n$ classes of their respective rows.

We say A and B are compatible if $d_L(\tilde{A}) = d_R(\tilde{B})$. For compatible A, B we can define the $(n+1) \times (2n+2)$ matrix $A \square B$. Let $\tilde{C}$ be the $\sim_{n+1}$ class such that $d_R(\tilde{C}) = \tilde{A}$ and $d_L(\tilde{C}) = \tilde{B}$, whose existence and uniqueness is guaranteed by Definition 2.2. The rows of $A \square B$ are indexed by $\{L, R, 1, \ldots, n, 1', \ldots, n'\}$.

- Row L is $(1, 0, \ldots, 0)$,
- Row R is $(0, \ldots, 0, (-1)^n)$,
- Row i for $i = 1, \ldots, n$ is the (unique) row $r_i \in \tilde{C}$ such that $d_R(r_i)$ is the ith row of A,
- Row i' for $i = 1, \ldots, n$ is the (unique) row $r_{i'} \in \tilde{C}$ such that $d_L(r_{i'})$ is the ith row of B.

We will examine Plücker relations for the matrices $A \square B$. To fix notation, recall the Plücker relations for the $n \times n$ minors of an $n \times 2n$ matrix whose $2n$ rows are indexed by $1, \ldots, n, 1', \ldots, n'$. Pick some integer k, $1 \le k \le n$, and then pick $1 \le r_1 < \cdots < r_k \le n$. The relations state that

$$[12\ldots n][1'2'\ldots n'] = \sum_{1 \le s_1 < \cdots < s_k \le n} \sigma_{RS}([1, 2, \ldots, n][1', 2' \ldots, n'])$$

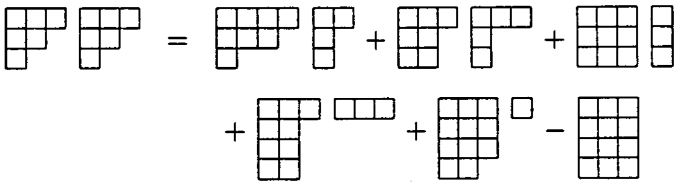
where σ_{RS} exchanges rows r_i with s_i' for $i = 1, \ldots, k$ before evaluating the determinants. We say the rows with labels $1, \ldots, n$ other than $r_1, \ldots, r_k$ are *fixed*.

We are interested in Plücker relations on $A \square B$ in which one of the terms is $[R12\ldots n]$ $[L1'\ldots n'] = \det(A)\det(B)$. To specify an example of this type, we choose matrices A and B from a generalized Jacobi-Trudi set, and we pick some subset of the rows of either A or B (recall that the $\square$ operation is not symmetric) to be the fixed rows in the identity.

Example 3 Take $\lambda = \langle 2, 1, 1 \rangle$ and $\mu = \langle 4, 3, 1 \rangle$, and consider $T_\lambda(u) \square T_\mu(u)$ (Example 2). Choosing the first two rows of $T_\mu(u)$ as our fixed rows gives us a 7-term Plücker relation. Rearranging the order of the terms (as a precursor to Theorem 3.2), we get:

$$\begin{aligned} s^{(u-1)}_{\langle 3,2,1\rangle} s^{(u+1)}_{\langle 3,2,1\rangle} &= s^{(u)}_{\langle 4,3,1\rangle} s^{(u)}_{\langle 2,1,1\rangle} + s^{(u-1)}_{\langle 3,2,2\rangle} s^{(u+1)}_{\langle 3,1,1\rangle} + s^{(u-1)}_{\langle 3,3,3\rangle} s^{(u+3)}_{\langle 1,1,1\rangle} \\ &\quad + s^{(u)}_{\langle 3,2,2,2\rangle} s^{(u)}_{\langle 3,0\rangle} + s^{(u)}_{\langle 3,3,3,2\rangle} s^{(u+2)}_{\langle 1,0\rangle} - s^{(u)}_{\langle 3,3,3,3\rangle} s^{(u+3)}_{\langle 0,0\rangle} \end{aligned}$$

If we ignore the spectral parameters, we get an identity on plain Schur functions:



In the first version, the zero parts of the partitions are necessary if the identity is to work without setting $t_0(u) = 1$, $t_k(u) = 0$ for $k < 0$. If we are willing to make that specialization, we can drop the zero parts, but we must adjust the shifts at the same time: $s^{(u)}_{\langle\lambda_1\ldots\lambda_n,0\rangle} = s^{(u-1)}_{\langle\lambda_1\ldots\lambda_n\rangle}$. In the second version we have already dropped the information about the zero parts.

3. Main theorem

In this section we present a set of recurrence relations, essentially a discrete dynamical system, to which the Schur functions are a solution. These relations are a generalization of Eq. (1), a system of relations which hold for the Schur functions of partitions with rectangular Young diagrams. We also present the quantum analogue of the relations, in Theorem 3.4 and following comments; this generalizes the relation

$$s^{(u-1)}_{\langle m^\ell\rangle} s^{(u+1)}_{\langle m^\ell\rangle} = s^{(u)}_{\langle m+1^\ell\rangle} s^{(u)}_{\langle m-1^\ell\rangle} + s^{(u)}_{\langle m^{\ell+1}\rangle} s^{(u)}_{\langle m^{\ell-1}\rangle} \tag{4}$$

We prove the relations by reducing them to Plücker relations on $M_\lambda \Box M_\mu$, defined in Section 2. The simple forms in Eqs. (1) and (4) come from the 3-term Plücker relation $[12][34] = [13][24] + [14][32]$.

To state the relations, we first need to define some operations on the partition λ, which we associate with its Young diagram $Y = Y(\lambda)$. Let Y be a Young diagram with n outside corners. That is, we take n points $(x_1, y_1), \ldots, (x_n, y_n)$ in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with $x_1 > \cdots > x_n$ and $y_1 < \cdots < y_n$, and the points in Y are those less than any of the (x_i, y_i) in the product ordering. We identify Y with the partition $\lambda = \langle x_1^{y_1}, x_2^{y_2-y_1}, \ldots, x_n^{y_n-y_{n-1}}\rangle$. We also say that Y has $n+1$ inside corners, numbered from 0 to n; the ith one has coordinates (x_{i+1}, y_i), where $y_0 = x_{n+1} = 0$.

Definition 3.1 Let Y be a Young diagram with n outside corners as above, and pick two integers i, j such that $1 \leq i \leq j \leq n$. We define two Young diagrams by the coordinates of their corners:

$\pi^i_j(Y)$: take the corners of Y, add 1 to each of $x_{i+1}, \ldots, x_j, y_i, \ldots, y_j$
$\mu^i_j(Y)$: take the corners of Y, add -1 to each of $x_{i+1}, \ldots, x_j, y_i, \ldots, y_j$

These operations respectively add and remove a border strip which reaches from the ith outside corner to the jth inside corner.

We will also want to add or remove several nested border strips. Given integers $1 \leq i_1 < \cdots < i_r \leq j_r < \cdots < j_1 \leq n$, we further define

$$\pi^{i_1\cdots i_r}_{j_1\cdots j_r} = \pi^{i_r}_{j_r} \circ \cdots \circ \pi^{i_1}_{j_1}$$
$$\mu^{i_1\cdots i_r}_{j_1\cdots j_r} = \mu^{i_r}_{j_r} \circ \cdots \circ \mu^{i_1}_{j_1}$$

Thus we add or remove border strips reaching from outside corner i_s to inside corner j_s for $1 \leq s \leq r$.

We apply these definitions of $\pi^{i_1\cdots i_r}_{j_1\cdots j_r}$ and $\mu^{i_1\cdots i_r}_{j_1\cdots j_r}$ only considering the coordinates of corners, so the various π^i_j and μ^i_j commute. Note that applying π^i_j, for example, might decrease the number of visible corners of Y (by making y_j the same as y_{j+1}), but we ignore this effect in the latter definitions above. Since the intervals $[i_s, j_s]$ are nested, we will never end up with $x_i < x_{i+1}$ or $y_i > y_{i+1}$.

Finally, we borrow notation from Lie theory: given a partition λ, we let $\lambda \pm \omega_\ell$ denote the partition obtained from λ by adding or removing a column of height ℓ to $Y(\lambda)$. If $\lambda = \langle \lambda_1, \ldots, \lambda_m \rangle$ and $\mu = \lambda \pm \omega_\ell$, then $\mu_i = \lambda_i \pm 1$ for $1 \leq i \leq \ell$ and $\mu_i = \lambda_i$ for $i > \ell$. Of course, we cannot take $\lambda - \omega_\ell$ if $\lambda_\ell = \lambda_{\ell+1}$, that is, if $Y(\lambda)$ does not have a column of height ℓ to begin with.

Theorem 3.2 (*Main Theorem*) *Take a partition λ whose Young diagram $Y(\lambda)$ has n outside corners. Pick an integer k, $1 \leq k \leq n$, and let ℓ be the kth-shortest column height in $Y(\lambda)$, so $\ell = y_k$ in the coordinates above. Then*

$$s_\lambda s_\lambda = s_{\lambda+\omega_\ell} s_{\lambda-\omega_\ell} + \sum_{r=1}^{\min(k,n-k+1)} \sum_{\substack{1 \leq i_1 < \cdots < i_r \leq k \\ k \leq j_r < \cdots < j_1 \leq n}} (-1)^{r-1} s_{\pi^{i_1\cdots i_r}_{j_1\cdots j_r}(\lambda)} s_{\mu^{i_1\cdots i_r}_{j_1\cdots j_r}(\lambda)}$$

That is, we take a signed double sum over all chains of properly nested intervals $[i_1, j_1] \supset \cdots \supset [i_r, j_r] \ni k$. For each such chain we have the product of two Schur functions, obtained by adding or removing all the corresponding border strips.

Remark 3.3 The recurrence relations can be viewed as defining the multi-time flow of a discrete dynamical system. We think of s_λ as being associated with the lattice point whose ith coordinate is the numberof columns in λ of height i. If we allow arbitrary partitions λ, the system is infinite-dimensional; if we restrict ourselves to representations of $\mathfrak{sl}_{n+1}$ it has dimension n.

First, we note that that no partition appearing in Theorem 3.2 has more outside corners than λ does. Second, we observe that the only partition with more columns than λ is $\lambda + \omega_\ell$. Therefore we can solve for $s_{\lambda+\omega_\ell}$ to get a recurrence relation $s_{\lambda+\omega_\ell} = (s_\lambda^2 - \sum \pm s_\pi s_\mu)/s_{\lambda-\omega_\ell}$, expressing $s_{\lambda+\omega_\ell}$ in terms of Schur functions of partition with strictly fewer columns and no more corners. The only initial conditions that need to be specified are for s_λ when λ has no two columns of the same height.

Example 4 Take λ to be the staircase partition $\langle 3, 2, 1\rangle$ with $n = 3$ corners, and pick $k = 2$. This instance of Theorem 3.2 is the Schur function part of Example 3. The order in which the terms appear there corresponds to taking the double sum over all sets of nested intervals in the order:

$$\underbrace{\{[2,2]\}\quad\{[1,2]\}\quad\{[2,3]\}\quad\{[1,3]\}}_{r=1}\quad\underbrace{\{[1,3]\supset[2,2]\}}_{r=2}$$

We will address the version with spectral parameters in Theorem 3.4.

Proof: The formula is the Plücker relation on $M_{\lambda-\omega_\ell}\square M_{\lambda+\omega_\ell}$ in which we fix rows $1', \ldots, \ell'$. We index the rows by $\{L, R, 1, \ldots, m, 1', \ldots, m'\}$ as in Construction 2.1, where m is the number of parts of λ. The fixed rows are therefore those corresponding to rows of $\lambda + \omega_\ell$ which got longer when the column of height ℓ was added.

First we locate the two pieces of Theorem 3.2 outside the double sum. The term $s_{\lambda-\omega_\ell}s_{\lambda+\omega_\ell}$, of course, is the Plücker term $[R12\ldots m][L1'2'\ldots m']$, as we have pointed out several times before. The $s_\lambda s_\lambda$ term is obtained from the Plücker term $[L1\ldots\ell(\ell+1)'\ldots m'][R1'\ldots\ell'(\ell+1)\ldots m]$, in which we swap L with R and every row of $M_{\lambda+\omega_\ell}$ other than the fixed ones with the corresponding row of $M_{\lambda-\omega_\ell}$. This leaves rows $\ell+1$ through m of the two partitions unchanged in length. The exchange of L and R increases by one the lengths of rows 1 through ℓ of $\lambda-\omega_\ell$ and decreases by one the lengths of rows $1'$ through ℓ' of $\lambda+\omega_\ell$, giving λ in both cases.

All other Plücker terms can be obtained from the $s_\lambda s_\lambda$ term by exchanging some subset of $\{1, \ldots, \ell\}$ from the first determinant with a subset of the same size drawn from $\{R, (\ell+1), \ldots, m\}$ from the second determinant. What is the effect of exchanging a for b, with $1 \le a \le \ell < b \le m$?

If $\lambda_a = \lambda_{a+1}$ or $\lambda_b = \lambda_{b-1}$, two identical rows (rows a and $(a+1)'$ or rows $(b-1)'$ and b, respectively) now appear in the same determinant, and we get zero contribution. Otherwise, reading down the main diagonals of the resulting matrices reveals that the effect is precisely to change the two minors into those for $\mu^i_j(\lambda)$ and $\pi^i_j(\lambda)$ respectively, where $Y(\lambda)$ has corner coordinates $y_i = a$ and $y_j = b-1$, and to flip the sign, owing to the need to reorder the rows. If instead of b we swap the row labelled R, the exchange has the effect of π^i_n and μ^i_n.

Exchanging subsets larger than a single element is easily seen to mimic the definition of $\pi^{i_1\cdots i_r}_{j_1\cdots j_r}$ and $\mu^{i_1\cdots i_r}_{j_1\cdots j_r}$; the nesting of the intervals arises because the "push" of $Y(\lambda)$ at outside corner a and the "pull" at inside corner b are completely independent. Each swap flips the sign of the resulting term, explaining the coefficient $(-1)^{r-1}$. □

There is a quantum analogue of Theorem 3.2 for the Schur functions with spectral parameters defined in Example 2.

Theorem 3.4 *For any partition λ, we can add spectral parameters to the statement of Theorem 3.2 to get*

$$s^{(u-1)}_\lambda s^{(u+1)}_\lambda = s^{(u)}_{\lambda+\omega_\ell}s^{(u)}_{\lambda-\omega_\ell} + \sum\sum \pm s^{(u+*)}_{\pi(\lambda)}s^{(u+*)}_{\mu(\lambda)}$$

where the parameters inside the sum are as follows: given nested intervals $1 \le i_1 < \cdots < j_1 \le n$, *set* $\alpha = \pi^{i_1\cdots i_r}_{j_1\cdots j_r}(\lambda)$ *and* $\beta = \mu^{i_1\cdots i_r}_{j_1\cdots j_r}(\lambda)$. *Then the corresponding term in the sum is*

$$\begin{array}{ll} s_\alpha^{(u)}\, s_\beta^{(u+\lambda_1-\beta_1)} & \text{if } j_1 = n \\ s_\alpha^{(u-1)}\, s_\beta^{(u+\lambda_1-\beta_1+1)} & \text{if } j_1 < n \end{array}$$

The case when $k = n$ is the subject of [9], where it is proved, as here, by reducing to Plücker relations. Note that for $k = 1$ or n, the double sum is actually a single sum and no negative terms appear.

Proof: As pointed out in Example 2, by appropriate choice of a shift c, we can lift the matrix M_λ to a matrix $T_\lambda(u+c)$ whose rows are in whatever equivalence class we choose. Thus all we will do is pick some equivalence class, lift rows $1, \ldots, m, 1', \ldots, m'$ of $M_{\lambda-\omega_\ell} \Box M_{\lambda+\omega_\ell}$ to that class, and read off the necessary shifts for each minor of our matrix to appear in the Plücker relations. Our choice of equivalence class is almost irrelevant; a different choice would just correspond to adding a constant to u in the final relation.

We follow convention by choosing our equivalence class so that we are dealing with minors of the matrix $M_{\lambda-\omega_\ell}(u) \Box M_{\lambda+\omega_\ell}(u)$, whose $2m$ rows other than L and R all look like

$$(t_{\lambda_1-c}(u-m+c), t_{\lambda_1+1-c}(u-m+1+c), \ldots, t_{\lambda_1+m-c}(u+c))$$

The row with label $1'$ has this form with $c = 0$, while the row with label 1 has $c = 1$. When we drop the left or right components of these rows, respectively, we get the top rows of the matrices $M_{\lambda+\omega_\ell}(u)$ and $M_{\lambda-\omega_\ell}(u)$, as desired.

Given a minor corresponding to $s_*^{(u+c)}$, to identify the shift c, recall that the top right entry in the matrix is $t_*(u+c)$. Thus we can easily see that the $[R1' \ldots \ell'(\ell+1) \ldots m][L1 \ldots \ell(\ell+1)' \ldots m']$ term of the Plücker relation corresponds to $s_\lambda^{(u-1)} s_\lambda^{(u+1)}$, again by looking at the rows 1 and $1'$ examined above.

Using the same reasoning, we see that for any $\alpha = \pi^{i_1\cdots i_r}_{j_1\cdots j_r}(\lambda)$, the associated minor is either $[R1' \ldots]$ (if row R was not swapped away) or $[1' \ldots]$ (if row R was traded). In the first case, we again end up with $s_\alpha^{(u-1)}$; in the second case, we get $s_\alpha^{(u)}$. Row R is swapped if and only if $j_1 = n$, of course: this is the same as saying the partition α has one more part than λ if and only if we added a border strip that reached the bottom row.

Determining the shift of $\beta = \mu^{i_1\cdots i_r}_{j_1\cdots j_r}(\lambda)$ is more difficult because its top row, other than L and possibly R, might be any of $1, 2, \ldots, \ell, \ell+1$. (Indeed, in Example 2, each of these occurs.) To sidestep this difficulty, we note that the top row of the minor giving rise to β begins $t_{\beta_1}(*)$. Assume that row R was not traded. Since we already know the top row must look like $(t_{\lambda_1+1-c}(*), \ldots, t_*(u+c))$, we conclude that $\beta_1 = \lambda_1 + 1 - c$, so $c = \lambda_1 - \beta + 1$. Likewise, if row R was traded, the top row is one term shorter and ends with $t_*(u+c-1)$, and the shift decreases by one, to $\lambda_1 - \beta_1$. □

We conclude this section with a few comments on the relevance of the quantum version of the theorem.

Remark 3.5 When we restrict λ to being a partition with one corner, *i.e.* a rectangle, we are dealing with the 2-dimensional discrete dynamical system

$$Q^{\ell}_{m+1}(u) = \frac{Q^{\ell}_{m}(u-1)Q^{\ell}_{m}(u+1) - Q^{\ell-1}_{m}(u)Q^{\ell+1}_{m}(u)}{Q^{\ell}_{m-1}(u)} \tag{5}$$

for $\ell = 1, \ldots, n$ and $m \in \mathbb{Z}_+$. Theorem 3.4 states that this system has a solution in which $Q^{\ell}_{m}(u)$ is set to $s^{(u)}_{\langle m^{\ell}\rangle}$, an object which reduces to $s_{\langle m^{\ell}\rangle}$ if we ignore the spectral parameter.

The objects $s^{(u)}_{\lambda}$ themselves have a representation-theoretic interpretation. The body of work on spectra of transfer matrices of certain integrable systems using the Bethe Ansatz (from the mathematical physics point of view; see *e.g.* [1, 6, 9]) has given rise to a notion of q-deformed characters for finite-dimensional representations of Yangians and quantum affine algebras [2]. In this picture, the $s^{(u)}_{\lambda}$ we worked with here correspond to the q-characters of evaluation models, and dropping the spectral parameter corresponds to throwing away some of the structure of $U_q(\widehat{\mathfrak{sl}_n})$ and retaining only the action of the embedded subalgebra $U_q(\mathfrak{sl}_n)$.

Remark 3.6 Attempts to generalize this picture to Lie algebras of types other than A_n began in [4, 6]. In these cases, it appears that the characters of $U_q(\hat{\mathfrak{g}})$ do satisfy a generalized version of Eq. (5). Dropping the spectral parameters, though, no longer gives statements about the fundamental representations of $U_q(\mathfrak{g})$, but about certain non-irreducible representations which are solutions to the discrete Hirota equations. While [4] conjectured character formulas for the analogs of rectangles in types B, C, D, written as sums over "rigged configurations," further exploration is hard because there is currently no general character formula for representations of $U_q(\hat{\mathfrak{g}})$.

Remark 3.7 Recent work of the author ([5]) has shown a stronger result about the generalized discrete Hirota relations, in an attempt to sidestep the lack of a $U_q(\hat{\mathfrak{g}})$ character formula. For each Lie algebra $\mathfrak{g}$, there is a *unique* solution to the recurrence relations in which Q^{ℓ}_{m} is the character of a representation of $U_q(\mathfrak{g})$ all of whose weights lie under $m\omega_\ell$ in the weight lattice. That is, we require that Q^{ℓ}_{m} is a sum of irreducible characters whose highest weights lie under $m\omega_\ell$, each occurring with nonnegative integer coefficients. This positivity constraint on all of the infinitely many characters Q^{ℓ}_{m} is quite rigid.

Theorem 3.4 is the first step in extending this picture from the rectangular case to a full n-dimensional system of relations among a much larger set of $U_q(\widehat{\mathfrak{sl}_n})$ characters. Generalizing these new recurrence relations to other Lie algebras may give us information on irreducible characters of $U_q(\hat{\mathfrak{g}})$ for which we do not yet even have conjectural values.

4. Combinatorial considerations

In this section, we look at the preceding formulas for Schur functions purely combinatorially. We offer a simple combinatorial proof of the rectangle version of the formula, and indicate why we believe that the subtraction that appears in Theorem 3.2 arises from inclusion-exclusion of sets labeled by single intervals.

We will multiply Schur functions using the following reformulation of the Littlewood-Richardson rule, taken from [10], where the technology of crystal bases is used to give an analogue for Lie algebras of type B, C, D as well.

Construction 4.1 We wish to find the multiset S of partitions such that $s_\lambda s_\mu = \sum_{\nu \in S} s_\nu$. To do this, let SSYT$(\mu)$ be the set of all semi-standard Young tableaux of shape μ. For any tableau $T \in$ SSYT(μ), we obtain its reverse column word $rcw(T) = i_1 i_2 \dots i_m$ by reading off the numbers in T, reading each column from top to bottom, beginning with the rightmost column and ending with the leftmost.

Now we let the number k act on the Young diagram $Y = Y(\lambda)$ by adding one box to the kth row, provided $\lambda_k < \lambda_{k-1}$. If $\lambda_k = \lambda_{k-1}$ then the action is illegal. Denote the resulting Young diagram by $Y \leftarrow k$. Then

$$S = \{(((Y \leftarrow i_1) \leftarrow i_2) \cdots \leftarrow i_m) \mid i_1 i_2 \dots i_m = rcw(T)\}$$

where T ranges over all tableaux in SSYT(μ) such that each action is legal.

Now we will give a purely bijective proof of Eq. (1), the recurrence relation for rectangular Young diagrams. A proof was given in [3] which did not mention the 3-term Plücker relation, but which made use of information from Lie theory about the dimensions of associated $\mathfrak{sl}_n$ representations.

Theorem 4.2 $s_{\langle m^\ell \rangle} s_{\langle m^\ell \rangle} = s_{\langle m^{\ell+1} \rangle} s_{\langle m^{\ell-1} \rangle} + s_{\langle m+1^\ell \rangle} s_{\langle m-1^\ell \rangle}$

Proof: Consider a tableau $T \in$ SSYT$(\langle m^\ell \rangle)$ such that the action of $rcw(T)$ on the Young diagram of shape $\langle m^\ell \rangle$, as in Construction 4.1, is legal. We consider two cases, based on whether or not the leftmost column of T consists exactly of the numbers $1, 2, \dots, \ell$.

If so, consider the tableau T' obtained by removing the leftmost column of T. Observe that $T' \in$ SSYT$(\langle m - 1^\ell \rangle)$, and the action of $rcw(T')$ on $Y(\langle m + 1^\ell \rangle)$ is legal and yields the same Young diagram as the action of $rcw(T)$ on $Y(\langle m^\ell \rangle)$. Furthermore, all elements T' of SSYT$(\langle m - 1^\ell \rangle)$ whose actions are legal arise in this way; we need only note that $rcw(T')$ never tries to build on column $m + 1$ of $Y(\langle m + 1^\ell \rangle)$.

Otherwise, the leftmost column of T contains an entry strictlylarger than ℓ, and therefore so does every column, as rows of T are weakly increasing. Now note that in any column of T, the smallest number greater than ℓ that appears must be $\ell + 1$. This is clear for the rightmost column, since $rcw(T)$ acts legally on $Y(\langle m^\ell \rangle)$, and can be seen inductively working to the left, again because rows weakly increase. Therefore we can consider the tableau T' obtained by removing the $\ell + 1$ from each column and pushing up all the numbers below it; clearly $T' \in$ SSYT$(\langle m^{\ell-1} \rangle)$. As in the first case, this operation gives a bijection between T acting legally on $Y(\langle m^\ell \rangle)$ and T' acting legally on $Y(\langle m^{\ell+1} \rangle)$. □

We are currently unable to provide a generalization of this argument to arbitrary partitions λ, but we strongly believe that one does exist. Based on computational examples, we conjecture the following form for a bijective proof of Theorem 3.2.

Conjecture 4.3 Let λ be a partition with n outside corners, choose a corner k and corresponding weight ω_ℓ, and retain the notions of Theorem 3.2. Let L be the set of SSYT(λ) acting legally on $Y(\lambda)$.

1. The tableaux in SSYT($\lambda - \omega_\ell$) which act legally on $Y(\lambda + \omega_\ell)$ can be put in bijection with a subset A of L.
2. There are subsets $B^i_j \subseteq L \backslash A$, for each $1 \le i \le k \le j \le n$, such that B^i_j is in bijection with SSYT($\mu^i_j(\lambda)$) acting legally on $Y(\pi^i_j(\lambda))$.
3. $L = A \cup \bigcup B^i_j$.
4. The intersection $B^{i_1}_{j_1} \cap \cdots \cap B^{i_r}_{j_r}$ is nonempty if and only if we can reorder the terms to get $1 \le i_1 < \cdots < i_r \le k \le j_r < \cdots < j_1 \le n$, and in that case it is in bijection with SSYT($\mu^{i_1 \cdots i_r}_{j_1 \cdots j_r}(\lambda)$) acting legally on $Y(\pi^{i_1 \cdots i_r}_{j_1 \cdots j_r}(\lambda))$.

All of the bijections between SSYT(λ) acting on $Y(\lambda)$ and SSYT(α) acting on $Y(\beta)$ should respect the Young diagrams produced by the two actions.

The conjecture implies Theorem 3.2, using inclusion-exclusion to take the union $\bigcup B^i_j$. We presently do not know the bijections or even how to identify the sets A, B^i_j in L.

Example 5 Taking the Schur function part of Example 3 once again, the only subtraction that takes place is of the term $s_{\langle 3,3,3,3\rangle} s_{\langle 0,0\rangle}$, corresponding to the nested intervals $[1, 3] \supset [2, 2]$. There is one tableau (the empty tableau) whose shape is the partition of zero. To verify this instance of the conjecture, we need to check that the Young diagram $Y(\langle 3, 3, 3, 3\rangle)$, appears in the terms corresponding to intervals [1, 3] and [2, 2] once each.

This does happen: the element of SSYT($\langle 3, 1, 1\rangle$) whose column word is 44234 acts on $Y(\langle 3, 2, 2\rangle)$, and the element of SSYT($\langle 1, 0\rangle$) whose column word is 4 acts on $Y(\langle 3, 3, 3, 2\rangle)$, both producing $Y(\langle 3, 3, 3, 3\rangle)$.

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