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Rankin-Cohen Brackets and Invariant Theory

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Abstract. Using maps due to Ozeki and Broué-Enguehard between graded spaces of invariants for certain finite groups and the algebra of modular forms of even weight we equip these invariants spaces with a differential operator which gives them the structure of a Rankin-Cohen algebra. A direct interpretation of the Rankin-Cohen bracket in terms of transvectant for the group $SL(2, \mathbf{C})$ is given.

Keywords: Rankin-Cohen brackets, Ozeki and Broué-Enguehard maps, invariants, codes

1. Introduction

Classically, there are many interesting connections between differential operators and the theory of elliptic modular forms and many interesting results with generalizations have been explored (see [3, 11, 15]). For instance, in 1975 H. Cohen constructed certain covariant bilinear operators which he used to obtain modular forms with interesting Fourier coefficients [4]. Later, these operators were called Rankin-Cohen operators by D. Zagier who studied the algebraic relations they satisfy [15]. Futhermore, Rankin-Cohen operators appear as the various terms in the expansion of the composition of two symbols in a certain symbolic calculus associated with $SL(2, \mathbf{R})$ [14]. On the other hand, there are well-known maps [2, 1, 10] between algebras of homogeneous invariants of certain finite groups graded by the degree and the algebra of modular forms for the full modular group graded by the weight; this is related to coding constructions of lattices (see [5, 6, 12]), a binary (resp. ternary) code of length n yielding a modular form of weight $n/2$ (resp. n). In that context it was natural to look for a differential operator acting on polynomials which would attach to a pair of invariant polynomials of respective degree m, n an invariant polynomial of degree $m + n + 4\nu$, in the binary codes case or $m + n + 2\nu$ in the ternary codes case.

This program is achieved in the present article by using an analogue of the derivation of order 2 in [15, Eq. (33)]. In each algebra of invariants considered we introduce a differential

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operator which sends invariants of degree n on invariants of degree $n+4$, or $n+2$ in the ternary case.

Independently of these considerations an explanation of the analogy between Rankin-Cohen brackets and transvectants of invariant theory already noticed in [15] is given. A deep analogy—different from the BE maps—between modular forms and homogeneous polynomials is emphasized. Homogeneous polynomials in two variables transform under the action of $SL(2, \mathbf{C})$ like modular forms of weight $-n$

2. Invariants and modular forms

First we recall some basic facts and notations on modular forms. Denote $E_2(\tau)$, $E_4(\tau)$, $E_6(\tau)$ the Eisenstein series of order 2, 4, 6 and the cusp form $\Delta(\tau)$ of weight 12 as

$$
\begin{aligned}
E_2(\tau) &= 1 - 24\sum_{m=1}^{\infty} \sigma_1(m) q^m,\\
E_4(\tau) &= 1 + 240\sum_{m=1}^{\infty} \sigma_3(m) q^m\\
E_6(\tau) &= 1 - 504\sum_{m=1}^{\infty} \sigma_5(m) q^m,\\
\Delta(\tau) &= q\prod_{m=1}^{\infty} (1-q^m)^{24}
\end{aligned}
$$

where, as usual $q = \exp(2\pi i\tau)$, and the sum of r^{th} power of divisor function is $\sigma_r(m) := \sum_{d|m} d^r$. As is well-known E_4, E_6 are modular forms of weight 4 and 6 but E_2 is not. The following result is due to Hecke.

Theorem 1 *The algebra of modular forms of weight multiple of* 4 *is* $\mathbf{C}[E_4, \Delta]$. *The algebra of modular forms of even weight is* $\mathbf{C}[E_4, E_6]$.

Next, we define the invariant counterpart of the preceding situation. If a finite group G acts by linear substitution on $\mathbf{C}[x, y]$ then we shall denote by $\mathbf{C}[x, y]^G$ the algebra of invariant homogeneous polynomials in the variables x, y, and by $\mathbf{C}[x, y]_k^G$ the degree k part of the preceding. Let M_2, N_2 denote the following 2 by 2 matrices

$$
M_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
$$

and

$$
N_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}
$$

These two matrices generate a matrix group H_2 of order 192. There is a subgroup $G_2 \leq H_2$ of index 2 defined as the kernel of the following linear character defined on the generators of H_2 as

$$\chi(M_2) = -1$$
$$\chi(N_2) = 1$$

Define the following invariants for the group $\langle M_2, N_2 \rangle$ of degree 8 and 24, respectively:

$$\psi_8 = x^8 + 14x^4y^4 + y^8,$$

and

$$\nu_{24} = x^4y^4(x^4 - y^4)^4.$$

It is well-known and easy to check by computer that

$$\mathbf{C}[x, y]^{H_2} = \mathbf{C}[\psi_8, \nu_{24}].$$

Similarly, we have

$$\mathbf{C}[x, y]^{G_2} = \mathbf{C}[\psi_8, k_{12}],$$

where k_{12} is an invariant introduced by Klein [6]

$$k_{12} := x^{12} - 33(x^8y^4 + x^4y^8) + y^{12}.$$

The following result is due to Broué-Enguehard [2].

Theorem 2 *The map*

$$\phi_1 : \mathbf{C}[\psi_8, \nu_{24}] \to \mathbf{C}[E_4, \Delta]$$

defined by

$$\phi_1(h(\psi_8, \nu_{24})) = h(E_4, \Delta)$$

is an algebra isomorphism.

The range of this map was extended recently by Ozeki [10] to the ring of even weight modular forms.

Theorem 3 *The map*

$$\phi_2 : \mathbf{C}[\psi_8, k_{12}] \to \mathbf{C}[E_4, E_6]$$

defined by

$$\phi_2(h(\psi_8, k_{12})) = h(E_4, E_6)$$

is an algebra isomorphism.

A similar result also due to Broué-Enguehard [2] uses the group $G_3 := \langle M_3, N_3 \rangle$ where

$$M_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

and

$$N_3 = \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix},$$

with j a complex cubic root of unity. It turns out that G_3 is abstractly isomorphic to $SL_2(\mathbf{F}_3)$ [6]. Primary invariants for G_3 are

$$\psi_4 := x^4 + 8xy^3,$$

as well as

$$k_6 := x^6 - 20x^3y^3 - 8y^6.$$

It is known that

$$\mathbf{C}[x, y]^{G_3} = \mathbf{C}[\psi_4, k_6].$$

Theorem 4 *The map*

$$\phi_3 : \mathbf{C}[\psi_4, k_6] \to \mathbf{C}[E_4, E_6]$$

given by

$$\phi_3(h(\psi_4, k_6)) = h(E_4, E_6)$$

is an algebra isomorphism.

3. Rankin-Cohen brackets

We begin by recalling the setting of [15]. Let $f(\tau)$ (resp. $g(\tau)$) denote a modular form of weight k (resp. l). Let D be the differential operator $q\frac{d}{dq}$. Consider, following [15, (21) p. 63] the homogeneous polynomial of degree n in two variables X, Y

$$H_\nu(k, l; X, Y) := \sum_{r+s=\nu} (-1)^r \binom{\nu + k - 1}{s} \binom{\nu + l - 1}{r} X^r Y^s.$$

The Rankin-Cohen bracket of index ν can then be expressed as

$$[f, g]_\nu := H_\nu(k, l; D_{\tau_1}, D_{\tau_2})(f(\tau_1)g(\tau_2)) \mid_{\tau_1=\tau_2=\tau}$$

In view of the maps defined in the preceding section it is natural, from a categorical standpoint, to define the Rankin-Cohen bracket of two polynomials as the preimage of the RC bracket of their respective images. Specifically, two invariants K, L of degree $2k$ and $2l$ being given we define their RC bracket as

$$\langle K, L\rangle_\nu := \phi_j^{-1}([\phi_j(K), \phi_j(L)]_\nu),$$

for $j = 2, 3$. In order to treat the two groups G_2, G_3 in a unified manner we view the respective invariants as polynomials in two variables Q, R say which will be the two generators of the invariant algebra namely the pair $(\psi_8, k_{12}), (\psi_4, k_6)$.

First note that, by the modularity of the Rankin-Cohen bracket if f, g are modular forms of even weight then so is $[f, g]_n$ for all integers n. Therefore $[f, g]_n$ will be in $\mathbf{C}[E_4, E_6]$. It is known that on that space the derivation D acts by the formula [15, (32)]

$$Df = \frac{k}{12}E_2 f + \delta(f)$$

where k is the weight of f and $\delta(f)$ is a derivation of order 2 on the algebra $\mathbf{C}[E_4, E_6]$. Define $\mathcal{D}$ the invariant analogue of the derivation δ as

$$\mathcal{D} := -\frac{R}{3}\frac{\partial}{\partial Q} - \frac{Q^2}{2}\frac{\partial}{\partial R}$$

With this notation indeed we have for $h \in \mathbf{C}[x, y]_n^{G_j}$ the relation

$$\delta(\phi_j(h)) = \phi_j(\mathcal{D}(h))$$

for $j = 1, 2$. We are now in a position to state the main result of this section.

Theorem 5 *The Rankin-Cohen bracket for G_2-invariants K, L of respective degrees $2k, 2l$ (or G_3 invariants of degree k, l) is given by*

$$\langle K, L\rangle_\nu = \sum_{r+s=\nu} \binom{\nu + k - 1}{s}\binom{\nu + l - 1}{r} f_r g_s$$

where f_r and g_s are defined recursively by the recurrences

$$f_{r+1} = \mathcal{D}(f_r) - \frac{Q}{144}r(r + k - 1) f_{r-1}$$

and

$$g_{s+1} = \mathcal{D}(g_s) - \frac{Q}{144}s(s+l-1)g_{s-1}$$

with initial conditions $f_0 = K,\ g_0 = L.$

Proof: Follows by the preceding discussion from [15, Prop. 1] with $\phi = E_2/12,\ \Phi = -E_4/144$. □

This somewhat abstract formulation can be made more explicit in each of the two cases at hand by going back to variables x, y. We will give detailed calculations for G_2 and a sketch for G_3.

Corollary 1 *The Rankin-Cohen bracket of order* ν *for* $K \in \mathbf{C}[x, y]_{2k}^{G_2}$*, and* $L \in \mathbf{C}[x, y]_{2l}^{G_2}$ *is a polynomial in* $\mathbf{C}[x, y]_{2k+2l+4\nu}^{G_2}$*, given by the expression in Theorem* 5 *where*

$$\mathcal{D} = \frac{1}{24}\left(x(5y^4 - x^4)\frac{\partial}{\partial x} + y(5x^4 - y^4)\frac{\partial}{\partial y}\right).$$

Proof: Let $h(x, y)$ be an arbitrary invariant in $\mathbf{C}[x, y]^{G_2}$. To express $\mathcal{D}$ in variables x, y we write $h(x, y) = \hat{h}(Q, R)$ and differentiate on both sides to get

$$\partial h/\partial x\ dx + \partial h/\partial\ dy = \partial\hat{h}/\partial Q\ dQ + \partial\hat{h}/\partial R\ dR.$$

After expressing *dx*, *dy* as a function of *dQ*, *dR* and identifying coefficients of *dQ*, *dR* on both sides we get

$$J\partial\hat{h}/\partial Q = \partial h/\partial x\ \partial R/\partial y - \partial h/\partial y\ \partial R/\partial x$$
$$J\partial\hat{h}/\partial R = \partial h/\partial y\ \partial Q/\partial x - \partial h/\partial x\ \partial Q/\partial y$$

where $J := \partial Q/\partial x\ \partial R/\partial y - \partial Q/\partial y\ \partial R/\partial x$, the determinant of the Jacobian matrix for the change of variables. For convenience set $F := Q^3 - R^2$. Then it can be shown, after plugging the preceding expressions into its definition that the derivation operator can be expressed as:

$$\mathcal{D} = (\partial F/\partial y\ \partial/\partial x - \partial F/\partial x\ \partial/\partial y)/6J$$

Clearly this part of the calculation does not depend on the special BE map under consideration. Now we specialize to $Q = \psi_8$, $R = k_{12}$ to get

$$J = -1728x^3y^3(x^4 - y^4)^3,$$

and

$$F = 108x^4y^4(x^4 - y^4)^4.$$

Therefrom we obtain

$$\partial F/\partial x = 432x^3y^4(5x^4 - y^4)(x^4 - y^4)^3$$
$$\partial F/\partial y = 432x^4y^3(x^4 - 5y^4)(x^4 - y^4)^3. \qquad \square$$

For instance this yields $\mathcal{D}(\psi_8) = -k_{12}/3$ and $\mathcal{D}(k_{12}) = -\frac{\psi_8^2}{2}$. The analogue of the preceding for G_3 is:

Corollary 2 *The Rankin-Cohen bracket of order* ν *for* $K \in \mathbf{C}[x, y]_k^{G_3}$*, and* $L \in \mathbf{C}[x, y]_l^{G_3}$ *is a polynomial in* $\mathbf{C}[x, y]_{k+l+2\nu}^{G_3}$*, given by the expression in Theorem* 5 *where*

$$\mathcal{D} = \frac{1}{12}\left((-x^3 + 4y^3)\frac{\partial}{\partial x} + 3x^2y\,\frac{\partial}{\partial y}\right).$$

Proof: We specialize the preceding to $Q = \psi_4$ and $R = k_6$ to get

$$J = -384y^2(x^3 - y^3)^2,$$

and

$$F = 64y^3(x^3 - y^3)^3.$$

Therefore we obtain

$$\partial F/\partial x = 576x^2y^3(x^3 - y^3)^2$$
$$\partial F/\partial y = 192y^2(x^3 - 4y^3)(x^3 - y^3)^2. \qquad \square$$

For instance this yields $\mathcal{D}(\psi_4) = -k_6/3$.

4. Transvectants

The relationship between transvectants and Rankin Cohen brackets is stated as an open problem in [15, §7] and as immediate in [7, p. 102]. In this section we make this connection explicit. Recall from [7, p. 99] Cayley's so-called Ω process:

$$\Omega := \det\begin{pmatrix} \partial_{x_1} & \partial_{y_1} \\ \partial_{x_2} & \partial_{y_2} \end{pmatrix}$$

a second order differential operator in 4 variables x_1, y_1, x_2, y_2. To quote [13] "this operator plays the role for $GL(n, \mathbf{C})$ of the Reynolds operator for finite groups" (in our case $n = 2$). The r^{th} transvectant of two *functions* U and V is defined in [7, (3.99),p. 99] as

$$(U, V)_r := \Omega^r(U(x_1, y_1)V(x_2, y_2)) \mid_{x_i=x,\, y_i=y}$$

According to [15, p. 63] the polynomials $H_n(k, l; X, Y)$ admit the alternative expression

$$H_n(k, l; X, Y) := \frac{1}{n!}\left(\det\begin{pmatrix}\partial_\xi & X\\ \partial_\eta & Y\end{pmatrix}\right)^n (\xi^{n+k-1}\eta^{n+l-1})\,|_{\,\xi=\eta=1}$$

Plugging these two expressions into the definition of the Rankin-Cohen brackets we obtain

$$[f, g]_n = \frac{1}{n!}(U, V)_n\Big|_{x=\tau, y=1},$$

where U, V are defined as a function of f, g as

$$U(x, y) := y^{n+k-1} f(x)$$
$$V(x, y) := y^{n+l-1} g(x)$$

With these notations in mind we obtain a new proof of the modularity of the Rankin-Cohen bracket.

Theorem 6 *Let f, g be modular forms of weight k, l for some group $\Gamma \subseteq SL(2, \mathbf{Z})$. Then for any integer $n \geq 0$ we have that $[f, g]_n$ is a modular form of weight $k + l + 2n$ for Γ.*

Proof: The relative invariance by Γ follows by noticing like in [7, Proof of Thm 3.45] the equivariance of Ω under linear change of variables. The computation of weight of the forms comes from the fact that the operator D increases the weight by 2. □

The fact that the weight of the Rankin-Cohen bracket of two forms is $k + l + 2n$ instead of $k + l - 2n$ for the transvectant of order n of two homogeneous polynomials of degrees k, l is simply explained if one compares the functional equation of modular forms and the action of $\left(\begin{smallmatrix}a b\\ c d\end{smallmatrix}\right) \in SL(2, \mathbf{C})$ on an homogeneous polynomial H of degree n in variables x, y given by [7, Eq. 3.9]

$$H((ap + b)/(cp + d)) = (cp + d)^{-n} H(p),$$

with $p = x/y$. Formally this is the transformation law of a modular form of weight $-n$. The weight of a RC bracket is therefore $-k - l - 2r = -(k + l + 2r)$ as it should. This suggests an alternative formula for the Rankin-Cohen bracket: the projective formula for transvectant as in [7, Thm 3.46].

Theorem 7 *Let f, g be modular forms of weights $-m, -n$. Then the Rankin-Cohen bracket of order r is*

$$[f, g]_r = \frac{1}{r!}\sum_{k=0}^{r}(-1)^k\binom{r}{k}\frac{(m-k)!}{(m-r)!}\frac{(n-r+k)!}{(n-r)!}D^{r-k} f D^k g,$$

where for an integer N we set $(-N)! = (-1)^N N!$.

Proof: The proof follows after some algebra from [15, (1)]:

$$[f, g]_\nu = \sum_{r+s=\nu} (-1)^r \binom{\nu+k-1}{s}\binom{\nu+l-1}{r} D^s f D^r g,$$

by letting $n = r,\ r = k,\ s = r - k,\ k = -m,\ l = -n.$ □

5. Conclusion

Since the times of Klein [6] the analogies between invariant theory and modular forms have emerged. We explore these similarities for Rankin-Cohen brackets. Klein's approach was developped in modern times by Broué and Enguehard [2] in relation with coding theory. This aspect is reflected in the first part of the paper. In §4 we develop another approach where a polynomial of degree d transforms under the action of $SL(2, \mathbf{C})$ like a modular form of weight $-d$. This explains the similarity between Rankin-Cohen brackets and transvectants as observed in [15].

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References

1. E. Bannai and M. Ozeki, "Construction of Jacobi forms from certain combinatorial polynomials," *Proc. Japan Acad. Ser. A Math. Sci.* **72** (1996), 12–15.
2. M. Broué and M. Enguehard, "Polynômes des poids de certains codes et fonctions theta de certains réseaux," *Ann. Sc. ENS.* **5** (1972), 157–181.
3. Y. Choie and W. Eholzer, "Rankin-Cohen operators of Jacobi and Siegel forms," *Journal of Number Theory* **68**(2) (1998), 160–177.
4. H. Cohen, "Sums involving the values at negative integers of L-functions of quadratic characters," *Math. Ann.* **217** (1975), 271–285.
5. J.H. Conway and N.J.A. Sloane, *Sphere Packings Lattices and Groups*, 3rd ed., Springer Verlag, New York, 1999.
6. W. Ebeling, *Lattices and Codes*, Vieweg & John, Braunschweig, 1994.
7. P.J. Olver, *Equivalence, Invariants and Symmetry*, Cambridge, 1995.
8. P.J. Olver, *Classical Invariant Theory*, Cambridge, 1999.
9. P.J. Olver and J.A. Sanders, Transvectants, Modular forms and the Heisenberg Algebra, Preprint, 1999.
10. M. Ozeki, "On the notion of Jacobi polynomials for codes," *Math. Proc. Cambridge Phil. Soc.* **121** (1997), 15–30.
11. R.A. Rankin, "The construction of automorphic forms from the derivatives of given forms," *Michigan Math. Jour.* **4** (1957), 181–186.
12. P. Solé, "Codes and modular forms," *proceeding of KIAS number theory conference*, submitted.
13. B. Sturmfels, *Algorithms in Invariant Theory*, Springer, 1993.
14. A. Unterberger and J. Unterberger, "Algebras of symbols and modular forms," *Jour. d'Analyse Math.* **68** (1996), 121–143.
15. D. Zagier, "Modular forms and differential operators," *Proc. Indian Acad. Sci.* (*Math. Sci.*) **104**(1) (1994), 57–75.