



# Tight Distance-Regular Graphs

ALEKSANDAR JURISČIĆ

*IMFM and Nova Gorica Polytechnic, 19, 1000 Ljubljana, Slovenia*

JACK KOOLEN

*FSP Mathematisierung, University of Bielefeld, D-33501 Bielefeld, Germany*

PAUL TERWILLIGER

*Department of Mathematics, University of Wisconsin, 480 Lincoln Drive Madison WI 53706, USA*

**Abstract.** We consider a distance-regular graph  $\Gamma$  with diameter  $d \geq 3$  and eigenvalues  $k = \theta_0 > \theta_1 > \dots > \theta_d$ . We show the intersection numbers  $a_1, b_1$  satisfy

$$\left(\theta_1 + \frac{k}{a_1 + 1}\right)\left(\theta_d + \frac{k}{a_1 + 1}\right) \geq -\frac{ka_1b_1}{(a_1 + 1)^2}.$$

We say  $\Gamma$  is *tight* whenever  $\Gamma$  is not bipartite, and equality holds above. We characterize the tight property in a number of ways. For example, we show  $\Gamma$  is tight if and only if the intersection numbers are given by certain rational expressions involving  $d$  independent parameters. We show  $\Gamma$  is tight if and only if  $a_1 \neq 0$ ,  $a_d = 0$ , and  $\Gamma$  is 1-homogeneous in the sense of Nomura. We show  $\Gamma$  is tight if and only if each local graph is connected strongly-regular, with nontrivial eigenvalues  $-1 - b_1(1 + \theta_1)^{-1}$  and  $-1 - b_1(1 + \theta_d)^{-1}$ . Three infinite families and nine sporadic examples of tight distance-regular graphs are given.

**Keywords:** distance-regular graph, equality, tight graph, homogeneous, locally strongly-regular parameterization

## 1. Introduction

Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $k = \theta_0 > \theta_1 > \dots > \theta_d$  (see Section 2 for definitions). We show the intersection numbers  $a_1, b_1$  satisfy

$$\left(\theta_1 + \frac{k}{a_1 + 1}\right)\left(\theta_d + \frac{k}{a_1 + 1}\right) \geq -\frac{ka_1b_1}{(a_1 + 1)^2}. \quad (1)$$

We define  $\Gamma$  to be *tight* whenever  $\Gamma$  is not bipartite, and equality holds in (1). We characterize the tight condition in the following ways.

Our first characterization is linear algebraic. For all vertices  $x \in X$ , let  $\hat{x}$  denote the vector in  $\mathbb{R}^X$  with a 1 in coordinate  $x$ , and 0 in all other coordinates. Suppose for the moment that  $a_1 \neq 0$ , let  $x, y$  denote adjacent vertices in  $X$ , and write  $w = \sum \hat{z}$ , where the sum is over all vertices  $z \in X$  adjacent to both  $x$  and  $y$ . Let  $\theta$  denote one of  $\theta_1, \theta_2, \dots, \theta_d$ , and let  $E$  denote

the corresponding primitive idempotent of the Bose-Mesner algebra. We say the edge  $xy$  is *tight with respect to*  $\theta$  whenever  $E\hat{x}, E\hat{y}, Ew$  are linearly dependent. We show that if  $xy$  is tight with respect to  $\theta$ , then  $\theta$  is one of  $\theta_1, \theta_d$ . Moreover, we show the following are equivalent: (i)  $\Gamma$  is tight; (ii)  $a_1 \neq 0$  and all edges of  $\Gamma$  are tight with respect to both  $\theta_1, \theta_d$ ; (iii)  $a_1 \neq 0$  and there exists an edge of  $\Gamma$  which is tight with respect to both  $\theta_1, \theta_d$ .

Our second characterization of the tight condition involves the intersection numbers. We show  $\Gamma$  is tight if and only if the intersection numbers are given by certain rational expressions involving  $d$  independent variables.

Our third characterization of the tight condition involves the concept of *1-homogeneous* that appears in the work of Nomura [14–16]. See also Curtin [7]. We show the following are equivalent: (i)  $\Gamma$  is tight; (ii)  $a_1 \neq 0, a_d = 0$ , and  $\Gamma$  is 1-homogeneous; (iii)  $a_1 \neq 0, a_d = 0$ , and  $\Gamma$  is 1-homogeneous with respect to at least one edge.

Our fourth characterization of the tight condition involves the local structure and is reminiscent of some results by Cameron, Goethals and Seidel [5] and Dickie and Terwilliger [8]. For all  $x \in X$ , let  $\Delta(x)$  denote the vertex subgraph of  $\Gamma$  induced on the vertices in  $X$  adjacent to  $x$ . For notational convenience, define  $b^+ := -1 - b_1(1 + \theta_d)^{-1}$  and  $b^- := -1 - b_1(1 + \theta_1)^{-1}$ . We show the following are equivalent: (i)  $\Gamma$  is tight; (ii) for all  $x \in X$ ,  $\Delta(x)$  is connected strongly-regular with nontrivial eigenvalues  $b^+, b^-$ ; (iii) there exists  $x \in X$  such that  $\Delta(x)$  is connected strongly-regular with nontrivial eigenvalues  $b^+, b^-$ .

We present three infinite families and nine sporadic examples of tight distance-regular graphs. These are the Johnson graphs  $J(2d, d)$ , the halved cubes  $\frac{1}{2}H(2d, 2)$ , the Taylor graphs [19], four 3-fold antipodal covers of diameter 4 constructed from the sporadic Fisher groups [3, p. 397], two 3-fold antipodal covers of diameter 4 constructed by Soicher [18], a 2-fold and a 4-fold antipodal cover of diameter 4 constructed by Meixner [13], and the Patterson graph [3, Thm. 13.7.1], which is primitive, distance-transitive and of diameter 4.

## 2. Preliminaries

In this section, we review some definitions and basic concepts. See the books of Bannai and Ito [1] or Brouwer, Cohen, and Neumaier [3] for more background information.

Let  $\Gamma = (X, R)$  denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set  $X$ , edge set  $R$ , path-length distance function  $\partial$ , and diameter  $d := \max\{\partial(x, y) \mid x, y \in X\}$ . For all  $x \in X$  and for all integers  $i$ , we set  $\Gamma_i(x) := \{y \in X \mid \partial(x, y) = i\}$ . We abbreviate  $\Gamma(x) := \Gamma_1(x)$ . By the *valency* of a vertex  $x \in X$ , we mean the cardinality of  $\Gamma(x)$ . Let  $k$  denote a nonnegative integer. Then  $\Gamma$  is said to be *regular*, with *valency*  $k$ , whenever each vertex in  $X$  has valency  $k$ .  $\Gamma$  is said to be *distance-regular* whenever for all integers  $h, i, j$  ( $0 \leq h, i, j \leq d$ ), and for all  $x, y \in X$  with  $\partial(x, y) = h$ , the number

$$p_{ij}^h := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of  $x$  and  $y$ . The constants  $p_{ij}^h$  are known as the *intersection numbers* of  $\Gamma$ . For notational convenience, set  $c_i := p_{1i-1}^i$  ( $1 \leq i \leq d$ ),  $a_i := p_{1i}^i$  ( $0 \leq i \leq d$ ),  $b_i := p_{1i+1}^i$  ( $0 \leq i \leq d-1$ ),  $k_i := p_{ii}^0$  ( $0 \leq i \leq d$ ), and define  $c_0 = 0, b_d = 0$ . We note  $a_0 = 0$  and

$c_1 = 1$ . From now on,  $\Gamma = (X, R)$  will denote a distance-regular graph of diameter  $d \geq 3$ . Observe  $\Gamma$  is regular with valency  $k = k_1 = b_0$ , and that

$$k = c_i + a_i + b_i \quad (0 \leq i \leq d). \quad (2)$$

We now recall the Bose-Mesner algebra. Let  $\text{Mat}_X(\mathbb{R})$  denote the  $\mathbb{R}$ -algebra consisting of all matrices with entries in  $\mathbb{R}$  whose rows and columns are indexed by  $X$ . For each integer  $i$  ( $0 \leq i \leq d$ ), let  $A_i$  denote the matrix in  $\text{Mat}_X(\mathbb{R})$  with  $x, y$  entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i, \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

$A_i$  is known as the *ith distance matrix* of  $\Gamma$ . Observe

$$A_0 = I, \quad (3)$$

$$A_0 + A_1 + \cdots + A_d = J \quad (J = \text{all 1's matrix}), \quad (4)$$

$$A_i^t = A_i \quad (0 \leq i \leq d), \quad (5)$$

$$A_i A_j = \sum_{h=0}^d p_{ij}^h A_h \quad (0 \leq i, j \leq d). \quad (6)$$

We abbreviate  $A := A_1$ , and refer to this as the *adjacency matrix* of  $\Gamma$ . Let  $M$  denote the subalgebra of  $\text{Mat}_X(\mathbb{R})$  generated by  $A$ . We refer to  $M$  as the *Bose-Mesner algebra* of  $\Gamma$ . Using (3)–(6), one can readily show  $A_0, A_1, \dots, A_d$  form a basis for  $M$ . By [1, pp. 59, 64], the algebra  $M$  has a second basis  $E_0, E_1, \dots, E_d$  such that

$$E_0 = |X|^{-1} J, \quad (7)$$

$$E_0 + E_1 + \cdots + E_d = I, \quad (8)$$

$$E_i^t = E_i \quad (0 \leq i \leq d), \quad (9)$$

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d). \quad (10)$$

The  $E_0, E_1, \dots, E_d$  are known as the *primitive idempotents* of  $\Gamma$ . We refer to  $E_0$  as the *trivial idempotent*.

Let  $\theta_0, \theta_1, \dots, \theta_d$  denote the real numbers satisfying  $A = \sum_{i=0}^d \theta_i E_i$ . Observe  $A E_i = E_i A = \theta_i E_i$  for  $0 \leq i \leq d$ , and that  $\theta_0, \theta_1, \dots, \theta_d$  are distinct since  $A$  generates  $M$ . It follows from (7) that  $\theta_0 = k$ , and it is known  $-k \leq \theta_i \leq k$  for  $0 \leq i \leq d$  [1, p. 197]. We refer to  $\theta_i$  as the *eigenvalue* of  $\Gamma$  associated with  $E_i$ , and call  $\theta_0$  the *trivial eigenvalue*. For each integer  $i$  ( $0 \leq i \leq d$ ), let  $m_i$  denote the rank of  $E_i$ . We refer to  $m_i$  as the *multiplicity* of  $E_i$  (or  $\theta_i$ ). We observe  $m_0 = 1$ .

We now recall the cosines. Let  $\theta$  denote an eigenvalue of  $\Gamma$ , and let  $E$  denote the associated primitive idempotent. Let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote the real numbers satisfying

$$E = |X|^{-1} m \sum_{i=0}^d \sigma_i A_i, \quad (11)$$

where  $m$  denotes the multiplicity of  $\theta$ . Taking the trace in (11), we find  $\sigma_0 = 1$ . We often abbreviate  $\sigma = \sigma_1$ . We refer to  $\sigma_i$  as the  $i$ th cosine of  $\Gamma$  with respect to  $\theta$  (or  $E$ ), and call  $\sigma_0, \sigma_1, \dots, \sigma_d$  the cosine sequence of  $\Gamma$  associated with  $\theta$  (or  $E$ ). We interpret the cosines as follows. Let  $\mathbb{R}^X$  denote the vector space consisting of all column vectors with entries in  $\mathbb{R}$  whose coordinates are indexed by  $X$ . We observe  $\text{Mat}_X(\mathbb{R})$  acts on  $\mathbb{R}^X$  by left multiplication. We endow  $\mathbb{R}^X$  with the Euclidean inner product satisfying

$$\langle u, v \rangle = u^t v \quad (u, v \in \mathbb{R}^X), \tag{12}$$

where  $t$  denotes transposition. For each  $x \in X$ , let  $\hat{x}$  denote the element in  $\mathbb{R}^X$  with a 1 in coordinate  $x$ , and 0 in all other coordinates. We note  $\{\hat{x} \mid x \in X\}$  is an orthonormal basis for  $\mathbb{R}^X$ .

**Lemma 2.1** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ . Let  $E$  denote a primitive idempotent of  $\Gamma$ , and let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote the associated cosine sequence. Then for all integers  $i$  ( $0 \leq i \leq d$ ), and for all  $x, y \in X$  such that  $\partial(x, y) = i$ , the following (i)–(iii) hold.*

- (i)  $\langle E\hat{x}, E\hat{y} \rangle = m|X|^{-1}\sigma_i$ , where  $m$  denotes the multiplicity of  $E$ .
- (ii) The cosine of the angle between the vectors  $E\hat{x}$  and  $E\hat{y}$  equals  $\sigma_i$ .
- (iii)  $-1 \leq \sigma_i \leq 1$ .

**Proof:** Line (i) is a routine application of (10), (11), (12). Line (ii) is immediate from (i), and (iii) is immediate from (ii). □

**Lemma 2.2 [3, Sect. 4.1.B]** *Let  $\Gamma$  denote a distance-regular graph with diameter  $d \geq 3$ . Then for any complex numbers  $\theta, \sigma_0, \sigma_1, \dots, \sigma_d$ , the following are equivalent.*

- (i)  $\theta$  is an eigenvalue of  $\Gamma$ , and  $\sigma_0, \sigma_1, \dots, \sigma_d$  is the associated cosine sequence.
- (ii)  $\sigma_0 = 1$ , and

$$c_i\sigma_{i-1} + a_i\sigma_i + b_i\sigma_{i+1} = \theta\sigma_i \quad (0 \leq i \leq d), \tag{13}$$

where  $\sigma_{-1}$  and  $\sigma_{d+1}$  are indeterminates.

- (iii)  $\sigma_0 = 1, k\sigma = \theta$ , and

$$c_i(\sigma_{i-1} - \sigma_i) - b_i(\sigma_i - \sigma_{i+1}) = k(\sigma - 1)\sigma_i \quad (1 \leq i \leq d), \tag{14}$$

where  $\sigma_{d+1}$  is an indeterminate.

For later use we record a number of consequences of Lemma 2.2.

**Lemma 2.3** *Let  $\Gamma$  denote a distance-regular graph with diameter  $d \geq 3$ . Let  $\theta$  denote an eigenvalue of  $\Gamma$ , and let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote the associated cosine sequence. Then (i)–(vi) hold below.*

- (i)  $kb_1\sigma_2 = \theta^2 - a_1\theta - k$ .
- (ii)  $kb_1(\sigma - \sigma_2) = (k - \theta)(1 + \theta)$ .

- (iii)  $kb_1(1 - \sigma_2) = (k - \theta)(\theta + k - a_1)$ .
- (iv)  $k^2b_1(\sigma^2 - \sigma_2) = (k - \theta)(k + \theta(a_1 + 1))$ .
- (v)  $c_d(\sigma_{d-1} - \sigma_d) = k(\sigma - 1)\sigma_d$ .
- (vi)  $a_d(\sigma_{d-1} - \sigma_d) = k(\sigma_{d-1} - \sigma\sigma_d)$ .

**Proof:** To get (i), set  $i = 1$  in (13), and solve for  $\sigma_2$ . Lines (ii)–(iv) are routinely verified using (i) above and  $k\sigma = \theta$ . To get (v), set  $i = d$ ,  $b_d = 0$  in Lemma 2.2 (iii). To get (vi), set  $c_d = k - a_d$  in (v) above, and simplify the result.  $\square$

In this article, the second largest and minimal eigenvalue of a distance-regular graph turn out to be of particular interest. In the next several lemmas, we give some basic information on these eigenvalues.

**Lemma 2.4 [9, Lem. 13.2.1]** *Let  $\Gamma$  denote a distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Let  $\theta$  denote one of  $\theta_1, \theta_d$  and let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote the cosine sequence for  $\theta$ .*

- (i) *Suppose  $\theta = \theta_1$ . Then  $\sigma_0 > \sigma_1 > \dots > \sigma_d$ .*
- (ii) *Suppose  $\theta = \theta_d$ . Then  $(-1)^i \sigma_i > 0$  ( $0 \leq i \leq d$ ).*

Recall a distance-regular graph  $\Gamma$  is *bipartite* whenever the intersection numbers satisfy  $a_i = 0$  for  $0 \leq i \leq d$ , where  $d$  denotes the diameter.

**Lemma 2.5** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ . Let  $\theta_d$  denote the minimal eigenvalue of  $\Gamma$ , and let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote the associated cosine sequence. Then the following are equivalent: (i)  $\Gamma$  is bipartite; (ii)  $\theta_d = -k$ ; (iii)  $\sigma_1 = -1$ ; (iv)  $\sigma_2 = 1$ . Moreover, suppose (i)–(iv) hold. Then  $\sigma_i = (-1)^i$  for  $0 \leq i \leq d$ .*

**Proof:** The equivalence of (i), (ii) follows from [3, Prop. 3.2.3]. The equivalence of (ii), (iii) is immediate from  $k\sigma_1 = \theta_d$ . The remaining implications follow from [3, Prop. 4.4.7].  $\square$

**Lemma 2.6** *Let  $\Gamma$  denote a distance-regular graph with diameter  $d \geq 3$  and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Then (i)–(iii) hold below.*

- (i)  $0 < \theta_1 < k$ .
- (ii)  $a_1 - k \leq \theta_d < -1$ .
- (iii) *Suppose  $\Gamma$  is not bipartite. Then  $a_1 - k < \theta_d$ .*

**Proof:** (i) The eigenvalue  $\theta_1$  is positive by [3, Cor. 3.5.4], and we have seen  $\theta_1 < k$ .  
(ii) Let  $\sigma_1, \sigma_2$  denote the first and second cosines for  $\theta_d$ . Then  $\sigma_2 \leq 1$  by Lemma 2.1 (iii), so  $a_1 - k \leq \theta_d$  in view of Lemma 2.3 (iii). Also  $\sigma_1 < \sigma_2$  by Lemma 2.4 (ii), so  $\theta_d < -1$  in view of Lemma 2.3 (ii).  
(iii) Suppose  $\theta_d = a_1 - k$ . Applying Lemma 2.3 (iii), we find  $\sigma_2 = 1$ , where  $\sigma_2$  denotes the second cosine for  $\theta_d$ . Now  $\Gamma$  is bipartite by Lemma 2.5, contradicting our assumptions. Hence  $\theta_d > a_1 - k$ , as desired.  $\square$

We mention a few results on the intersection numbers.

**Lemma 2.7** Let  $\Gamma = (X, R)$  denote a nonbipartite distance-regular graph with diameter  $d \geq 3$ , let  $x, y$  denote adjacent vertices in  $X$ , and let  $E$  denote a nontrivial primitive idempotent of  $\Gamma$ . Then the vectors  $E\hat{x}$  and  $E\hat{y}$  are linearly independent.

**Proof:** Let  $\sigma$  denote the first cosine associated to  $E$ . Then  $\sigma \neq 1$ , since  $E$  is nontrivial, and  $\sigma \neq -1$ , since  $\Gamma$  is not bipartite. Applying Lemma 2.1 (ii), we see  $E\hat{x}$  and  $E\hat{y}$  are linearly independent.  $\square$

**Lemma 2.8 [3, Prop. 5.5.1]** Let  $\Gamma$  denote a distance-regular graph with diameter  $d \geq 3$  and  $a_1 \neq 0$ . Then  $a_i \neq 0$  ( $1 \leq i \leq d - 1$ ).

**Lemma 2.9 [3, Lem. 4.1.7]** Let  $\Gamma$  denote a distance-regular graph with diameter  $d \geq 3$ . Then the intersection numbers satisfy

$$p_{ii}^1 = \frac{b_1 b_2 \dots b_{i-1}}{c_1 c_2 \dots c_i} a_i, \quad p_{i-1,i}^1 = \frac{b_1 b_2 \dots b_{i-1}}{c_1 c_2 \dots c_{i-1}} \quad (1 \leq i \leq d).$$

For the remainder of this section, we describe a point of view we will adopt throughout the paper.

**Definition 2.10** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ , and fix adjacent vertices  $x, y \in X$ . For all integers  $i$  and  $j$  we define  $D_i^j = D_i^j(x, y)$  by

$$D_i^j = \Gamma_i(x) \cap \Gamma_j(y). \tag{15}$$

We observe  $|D_i^j| = p_{ij}^1$  for  $0 \leq i, j \leq d$ , and  $D_i^j = \emptyset$  otherwise. We visualize the  $D_i^j$  as follows (figure 1).

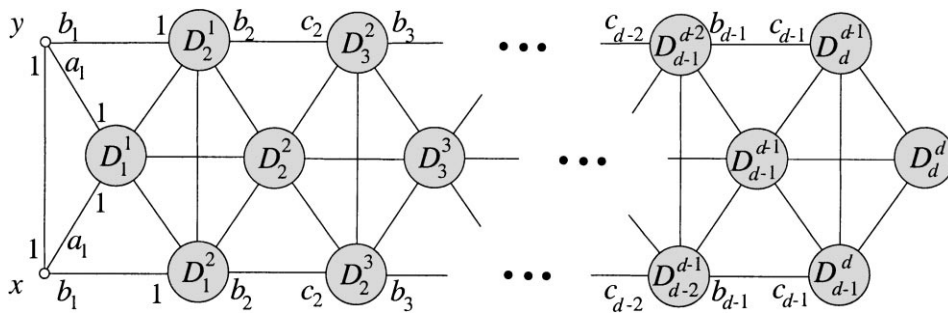


Figure 1. Distance distribution corresponding to an edge. Observe:  $D_i^{i-1} \cup D_i^i \cup D_i^{i+1} = \Gamma_i(x)$  for  $i = 1, \dots, d$ . The number beside edges connecting cells  $D_i^j$  indicate how many neighbours a vertex from the closer cell has in the other cell, see Lemma 2.11.

**Lemma 2.11** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ . Fix adjacent vertices  $x, y \in X$ , and pick any integer  $i$  ( $1 \leq i \leq d$ ). Then with reference to Definition 2.10, the following (i) and (ii) hold.*

- (i) *Each  $z \in D_{i-1}^i$  (resp.  $D_{i-1}^{i-1}$ ) is adjacent to*
- |                      |  |  |   |
|----------------------|--|--|---|
| (a) <i>precisely</i> | $c_{i-1}$  |  | <i>vertices in <math>D_{i-2}^{i-1}</math> (resp. <math>D_{i-1}^{i-2}</math>),</i> |
| (b) <i>precisely</i> | $c_i - c_{i-1} -  \Gamma(z) \cap D_{i-1}^{i-1} $ |  | <i>vertices in <math>D_i^{i-1}</math> (resp. <math>D_{i-1}^i</math>),</i>         |
| (c) <i>precisely</i> | $a_{i-1} -  \Gamma(z) \cap D_{i-1}^{i-1} $       |  | <i>vertices in <math>D_{i-1}^i</math> (resp. <math>D_i^{i-1}</math>),</i>         |
| (d) <i>precisely</i> | $b_i$  |  | <i>vertices in <math>D_i^{i+1}</math> (resp. <math>D_{i+1}^i</math>),</i>         |
| (e) <i>precisely</i> | $a_i - a_{i-1} +  \Gamma(z) \cap D_{i-1}^{i-1} $ |  | <i>vertices in <math>D_i^i</math>.</i>  |
- (ii) *Each  $z \in D_i^i$  is adjacent to*
- |                      |   |  |  |
|----------------------|---|--|--|
| (a) <i>precisely</i> | $c_i -  \Gamma(z) \cap D_{i-1}^{i-1} $  |  | <i>vertices in <math>D_{i-1}^i</math>,</i>     |
| (b) <i>precisely</i> | $c_i -  \Gamma(z) \cap D_{i-1}^{i-1} $  |  | <i>vertices in <math>D_{i-1}^{i-1}</math>,</i> |
| (c) <i>precisely</i> | $b_i -  \Gamma(z) \cap D_{i+1}^{i+1} $  |  | <i>vertices in <math>D_i^{i+1}</math>,</i>     |
| (d) <i>precisely</i> | $b_i -  \Gamma(z) \cap D_{i+1}^{i+1} $  |  | <i>vertices in <math>D_{i+1}^i</math>,</i>     |
| (e) <i>precisely</i> | $a_i - b_i - c_i +  \Gamma(z) \cap D_{i-1}^{i-1}  +  \Gamma(z) \cap D_{i+1}^{i+1} $ |  | <i>vertices in <math>D_i^i</math>.</i>         |

**Proof:** Routine. □

### 3. Edges that are tight with respect to an eigenvalue

Let  $\Gamma = (X, R)$  denote a graph, and let  $\Omega$  denote a nonempty subset of  $X$ . By the *vertex subgraph* of  $\Gamma$  induced on  $\Omega$ , we mean the graph with vertex set  $\Omega$ , and edge set  $\{xy \mid x, y \in \Omega, xy \in R\}$ .

**Definition 3.1** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$  and intersection number  $a_1 \neq 0$ . For each edge  $xy \in R$ , we define the scalar  $f = f(x, y)$  by

$$f := a_1^{-1} |\{(z, w) \in X^2 \mid z, w \in D_1^1, \partial(z, w) = 2\}|, \quad (16)$$

where  $D_1^1 = D_1^1(x, y)$  is from (15). We observe  $f$  is the average valency of the complement of the vertex subgraph induced on  $D_1^1$ .

We begin with some elementary facts about  $f$ .

**Lemma 3.2** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$  and  $a_1 \neq 0$ . Let  $x, y$  denote adjacent vertices in  $X$ . Then with reference to (15), (16), lines (i)–(iv) hold below.*

- (i) *The number of edges in  $R$  connecting a vertex in  $D_1^1$  with a vertex in  $D_1^2$  is equal to  $a_1 f$ .*
- (ii) *The number of edges in the vertex subgraph induced on  $D_1^1$  is equal to  $a_1(a_1 - 1 - f)/2$ .*
- (iii) *The number of edges in the vertex subgraph induced on  $D_1^2$  is equal to  $a_1(b_1 - f)/2$ .*
- (iv)  $0 \leq f, f \leq a_1 - 1, f \leq b_1$ .

**Proof:** Routine. □

The following lemma provides another bound for  $f$ .

**Lemma 3.3** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$  and  $a_1 \neq 0$ . Let  $x, y$  denote adjacent vertices in  $X$ , and write  $f = f(x, y)$ . Then for each nontrivial eigenvalue  $\theta$  of  $\Gamma$ ,*

$$(k + \theta)(1 + \theta)f \leq b_1(k + \theta(a_1 + 1)). \quad (17)$$

**Proof:** Let  $\sigma_0, \dots, \sigma_d$  denote the cosine sequence of  $\theta$  and let  $E$  denote the corresponding primitive idempotent. Set

$$w := \sum_{z \in D_1^1} \hat{z},$$

where  $D_1^1 = D_1^1(x, y)$  is from (15). Let  $G$  denote the Gram matrix for the vectors  $E\hat{x}, E\hat{y}, Ew$ ; that is

$$G := \begin{pmatrix} \|E\hat{x}\|^2 & \langle E\hat{x}, E\hat{y} \rangle & \langle E\hat{x}, Ew \rangle \\ \langle E\hat{y}, E\hat{x} \rangle & \|E\hat{y}\|^2 & \langle E\hat{y}, Ew \rangle \\ \langle Ew, E\hat{x} \rangle & \langle Ew, E\hat{y} \rangle & \|Ew\|^2 \end{pmatrix}.$$

On one hand, the matrix  $G$  is positive semi-definite, so it has nonnegative determinant. On the other hand, by Lemma 2.1,

$$\begin{aligned} \det(G) &= m^3 |X|^{-3} \det \begin{pmatrix} \sigma_0 & \sigma_1 & a_1 \sigma_1 \\ \sigma_1 & \sigma_0 & a_1 \sigma_1 \\ a_1 \sigma_1 & a_1 \sigma_1 & a_1(\sigma_0 + (a_1 - f - 1)\sigma_1 + f\sigma_2) \end{pmatrix} \\ &= m^3 a_1 |X|^{-3} (\sigma - 1)((\sigma - \sigma_2)(1 + \sigma)f - (1 - \sigma)(a_1\sigma + 1 + \sigma)), \end{aligned}$$

where  $m$  denotes the multiplicity of  $\theta$ . Since  $a_1 > 0$  and  $\sigma < 1$ , we find

$$(\sigma - \sigma_2)(1 + \sigma)f \leq (1 - \sigma)(a_1\sigma + 1 + \sigma). \quad (18)$$

Eliminating  $\sigma, \sigma_2$  in (18) using  $\theta = k\sigma$  and Lemma 2.3(ii), and simplifying the result using  $\theta < k$ , we routinely obtain (17). □

**Corollary 3.4** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$  and  $a_1 \neq 0$ . Let  $x, y$  denote adjacent vertices in  $X$ , and let  $\theta$  denote a nontrivial eigenvalue of  $\Gamma$ . Then with reference to Definition 2.10, the following are equivalent.*

- (i) Equality is attained in (17).
- (ii)  $E\hat{x}, E\hat{y}, \sum_{z \in D_1^1} E\hat{z}$  are linearly dependent.



$$(iii) \sum_{z \in D_1^1} E\hat{z} = \frac{a_1\theta}{k+\theta}(E\hat{x} + E\hat{y}).$$

We say the edge  $xy$  is *tight with respect to  $\theta$*  whenever (i)–(iii) hold above.

**Proof:** (i) $\Leftrightarrow$ (ii) Let the matrix  $G$  be as in the proof of Lemma 3.3. Then we find (i) holds if and only if  $G$  is singular, if and only if (ii) holds.

(ii) $\Rightarrow$ (iii)  $\Gamma$  is not bipartite since  $a_1 \neq 0$ , so  $E\hat{x}$ , and  $E\hat{y}$  are linearly independent by Lemma 2. It follows

$$\sum_{z \in D_1^1} E\hat{z} = \alpha E\hat{x} + \beta E\hat{y} \quad (19)$$

for some  $\alpha, \beta \in \mathbb{R}$ . Taking the inner product of (19) with each of  $E\hat{x}$ ,  $E\hat{y}$  using Lemma 2.1, we readily obtain  $\alpha = \beta = a_1\theta(k + \theta)^{-1}$ .

(iii) $\Rightarrow$ (ii) Clear.  $\square$

Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ ,  $a_1 \neq 0$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Pick adjacent vertices  $x, y \in X$ , and write  $f = f(x, y)$ . Referring to (17), we now consider which of  $\theta_1, \theta_2, \dots, \theta_d$  gives the best bounds for  $f$ . Let  $\theta$  denote one of  $\theta_1, \theta_2, \dots, \theta_d$ . Assume  $\theta \neq -1$ ; otherwise (17) gives no information about  $f$ . If  $\theta > -1$  (resp.  $\theta < -1$ ), line (17) gives an upper (resp. lower) bound for  $f$ . Consider the partial fraction decomposition

$$b_1 \frac{k + \theta(a_1 + 1)}{(k + \theta)(1 + \theta)} = \frac{b_1}{k - 1} \left( \frac{ka_1}{k + \theta} + \frac{b_1}{1 + \theta} \right).$$

Since the map  $F : \mathbb{R} \setminus \{-k, -1\} \rightarrow \mathbb{R}$ , defined by

$$x \mapsto \frac{ka_1}{k + x} + \frac{b_1}{1 + x}$$

is strictly decreasing on the intervals  $(-k, -1)$  and  $(-1, \infty)$ , we find in view of Lemma 2.6 that the least upper bound for  $f$  is obtained at  $\theta = \theta_1$ , and the greatest lower bound is obtained at  $\theta = \theta_d$ .

**Theorem 3.5** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ ,  $a_1 \neq 0$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . For all edges  $xy \in R$ ,*

$$b_1 \frac{k + \theta_d(a_1 + 1)}{(k + \theta_d)(1 + \theta_d)} \leq f(x, y) \leq b_1 \frac{k + \theta_1(a_1 + 1)}{(k + \theta_1)(1 + \theta_1)}. \quad (20)$$

**Proof:** This is immediate from (17) and Lemma 2.6.  $\square$

**Corollary 3.6** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ ,  $a_1 \neq 0$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . For all edges  $xy \in R$ ,*

- (i)  $xy$  is tight with respect to  $\theta_1$  if and only if equality holds in the right inequality of (20),
- (ii)  $xy$  is tight with respect to  $\theta_d$  if and only if equality holds in the left inequality of (20),
- (iii)  $xy$  is not tight with respect to  $\theta_i$  for  $2 \leq i \leq d - 1$ .

**Proof:** (i), (ii) Immediate from (17) and Corollary 3.4.

(iii) First suppose  $\theta_i = -1$ . We do not have equality for  $\theta = \theta_i$  in (17), since the left side equals 0, and the right side equals  $b_1^2$ . In particular,  $xy$  is not tight with respect to  $\theta_i$ . Next suppose  $\theta_i \neq -1$ . Then we do not have equality for  $\theta = \theta_i$  in (17) in view of the above mentioned fact, that the function  $F$  is strictly decreasing on the intervals  $(-k, -1)$  and  $(-1, \infty)$ .  $\square$

#### 4. Tight edges and combinatorial regularity

**Theorem 4.1** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$  and intersection number  $a_1 \neq 0$ . Let  $\theta$  denote a nontrivial eigenvalue of  $\Gamma$ , and let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote its cosine sequence. Let  $x, y$  denote adjacent vertices in  $X$ . Then with reference to Definition 2.10, the following are equivalent.*

- (i)  $xy$  is tight with respect to  $\theta$ .
- (ii) For  $1 \leq i \leq d$ ; both  $\sigma_{i-1} \neq \sigma_i$ , and for all  $z \in D_{i-1}^i$

$$|\Gamma_{i-1}(z) \cap D_1^1| = \frac{a_1}{1 + \sigma} \frac{\sigma \sigma_{i-1} - \sigma_i}{\sigma_{i-1} - \sigma_i}, \tag{21}$$

$$|\Gamma_i(z) \cap D_1^1| = \frac{a_1}{1 + \sigma} \frac{\sigma_{i-1} - \sigma \sigma_i}{\sigma_{i-1} - \sigma_i}. \tag{22}$$

**Proof:** (i) $\Rightarrow$ (ii) Let the integer  $i$  be given. Observe by Corollary 3.6 that  $\theta$  is either the second largest eigenvalue  $\theta_1$  or the least eigenvalue  $\theta_d$ , so  $\sigma_{i-1} \neq \sigma_i$  in view of Lemma 2.4. Pick any  $z \in D_{i-1}^i$ . Observe  $D_1^1$  contains  $a_1$  vertices, and each is at distance  $i - 1$  or  $i$  from  $z$ , so

$$|\Gamma_{i-1}(z) \cap D_1^1| + |\Gamma_i(z) \cap D_1^1| = a_1. \tag{23}$$

Let  $E$  denote the primitive idempotent associated to  $\theta$ . By Corollary 3.4(iii), and since  $xy$  is tight with respect to  $\theta$ ,

$$\sum_{w \in D_1^1} E \hat{w} = \frac{a_1 \sigma}{1 + \sigma} (E \hat{x} + E \hat{y}). \tag{24}$$

Taking the inner product of (24) with  $E \hat{z}$  using Lemma 2.1, we obtain

$$\sigma_{i-1} |\Gamma_{i-1}(z) \cap D_1^1| + \sigma_i |\Gamma_i(z) \cap D_1^1| = \frac{a_1 \sigma}{1 + \sigma} (\sigma_{i-1} + \sigma_i). \tag{25}$$

Solving the system (23), (25), we routinely obtain (21), (22).

(ii) $\Rightarrow$ (i) We show equality holds in (17). Counting the edges between  $D_1^1$  and  $D_1^2$  using (21) (with  $i = 2$ ), we find in view of Lemma 3.2(i) that

$$f(x, y) = b_1 \frac{\sigma^2 - \sigma_2}{(1 + \sigma)(\sigma - \sigma_2)}. \quad (26)$$

Eliminating  $\sigma, \sigma_2$  in (26) using  $\theta = k\sigma$  and Lemma 2.3(ii), (iv), we readily find equality holds in (17). Now  $xy$  is tight with respect to  $\theta$  by Corollary 3.4.  $\square$

**Theorem 4.2** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$  and  $a_1 \neq 0$ . Let  $\theta$  denote a nontrivial eigenvalue of  $\Gamma$ , and let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote its cosine sequence. Let  $x, y$  denote adjacent vertices in  $X$ . Then with reference to Definition 2.10, the following are equivalent.*

- (i)  $xy$  is tight with respect to  $\theta$ ,
- (ii) For  $1 \leq i \leq d - 1$ ; both  $\sigma_i \neq \sigma_{i+1}$ , and for all  $z \in D_i^i$

$$|\Gamma_{i+1}(z) \cap D_1^1| = |\Gamma_{i-1}(z) \cap D_1^1| \frac{\sigma_{i-1} - \sigma_i}{\sigma_i - \sigma_{i+1}} + a_1 \frac{1 - \sigma}{1 + \sigma} \frac{\sigma_i}{\sigma_i - \sigma_{i+1}}, \quad (27)$$

$$\begin{aligned} |\Gamma_i(z) \cap D_1^1| &= -|\Gamma_{i-1}(z) \cap D_1^1| \frac{\sigma_{i-1} - \sigma_{i+1}}{\sigma_i - \sigma_{i+1}} + a_1 \frac{2\sigma}{1 + \sigma} \\ &\quad - a_1 \frac{1 - \sigma}{1 + \sigma} \frac{\sigma_{i+1}}{\sigma_i - \sigma_{i+1}}. \end{aligned} \quad (28)$$

Suppose (i)–(ii) above, and that  $a_d \neq 0$ . Then for all  $z \in D_d^d$

$$|\Gamma_{d-1}(z) \cap D_1^1| = -a_1 \frac{1 - \sigma}{1 + \sigma} \frac{\sigma_d}{\sigma_{d-1} - \sigma_d}, \quad (29)$$

$$|\Gamma_d(z) \cap D_1^1| = a_1 + a_1 \frac{1 - \sigma}{1 + \sigma} \frac{\sigma_d}{\sigma_{d-1} - \sigma_d}. \quad (30)$$

**Proof:** (i) $\Rightarrow$ (ii) Let the integer  $i$  be given. Observe by Corollary 3.6 that  $\theta$  is either the second largest eigenvalue  $\theta_1$  or the least eigenvalue  $\theta_d$ , so  $\sigma_i \neq \sigma_{i+1}$  by Lemma 2.4. Pick any  $z \in D_i^i$ . Proceeding as in the proof of Theorem 4.1(i) $\Rightarrow$ (ii), we find

$$|\Gamma_{i-1}(z) \cap D_1^1| + |\Gamma_i(z) \cap D_1^1| + |\Gamma_{i+1}(z) \cap D_1^1| = a_1, \quad (31)$$

$$\sigma_{i-1} |\Gamma_{i-1}(z) \cap D_1^1| + \sigma_i |\Gamma_i(z) \cap D_1^1| + \sigma_{i+1} |\Gamma_{i+1}(z) \cap D_1^1| = \frac{2\sigma \sigma_i a_1}{1 + \sigma}. \quad (32)$$

Solving (31), (32) for  $|\Gamma_i(z) \cap D_1^1|, |\Gamma_{i+1}(z) \cap D_1^1|$ , we routinely obtain (27) and (28).

(ii) $\Rightarrow$ (i) Setting  $i = 1$  in (27), and evaluating the result using (16), we find

$$f(x, y) = \frac{1 - \sigma}{\sigma - \sigma_2} + a_1 \frac{1 - \sigma}{1 + \sigma} \frac{\sigma}{\sigma - \sigma_2}. \quad (33)$$

Eliminating  $\sigma, \sigma_2$  in (33) using  $\theta = k\sigma$  and Lemma 2.3 (ii), we find equality holds in (17). Now  $xy$  is tight with respect to  $\theta$  by Corollary 3.4.

Now suppose (i)–(ii) hold above, and that  $a_d \neq 0$ . Pick any  $z \in D_d^d$ . Proceeding as in the proof of Theorem 4.1(i)  $\Rightarrow$ (ii), we find

$$|\Gamma_{d-1}(z) \cap D_1^1| + |\Gamma_d(z) \cap D_1^1| = a_1, \quad (34)$$

$$\sigma_{d-1}|\Gamma_{d-1}(z) \cap D_1^1| + \sigma_d|\Gamma_d(z) \cap D_1^1| = \frac{2\sigma_d\sigma a_1}{1+\sigma}. \quad (35)$$

Observe  $\sigma_{d-1} \neq \sigma_d$  by (ii) above, so the linear system (34), (35) has unique solution (29), (30).  $\square$

## 5. The tightness of an edge

**Definition 5.1** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ , intersection number  $a_1 \neq 0$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . For each edge  $xy \in R$ , let  $t = t(x, y)$  denote the number of nontrivial eigenvalues of  $\Gamma$  with respect to which  $xy$  is tight. We call  $t$  the *tightness* of the edge  $xy$ . In view of Corollary 3.6 we have:

- (i)  $t = 2$  if  $xy$  is tight with respect to both  $\theta_1$  and  $\theta_d$ ;
- (ii)  $t = 1$  if  $xy$  is tight with respect to exactly one of  $\theta_1$  and  $\theta_d$ ;
- (iii)  $t = 0$  if  $xy$  is not tight with respect to  $\theta_1$  or  $\theta_d$ .

**Theorem 5.2** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$  and  $a_1 \neq 0$ . For all edges  $xy \in R$ , the tightness  $t = t(x, y)$  is given by

$$t = 3d + 1 - \dim(MH), \quad (36)$$

where  $M$  denotes the Bose-Mesner algebra of  $\Gamma$ , where

$$H = \text{Span} \left\{ \hat{x}, \hat{y}, \sum_{z \in D_1^1(x,y)} \hat{z} \right\}, \quad (37)$$

and where  $MH$  means  $\text{Span}\{mh \mid m \in M, h \in H\}$ .

**Proof:** Since  $E_0, E_1, \dots, E_d$  is a basis for  $M$ , and in view of (10),

$$MH = \sum_{i=0}^d E_i H \quad (\text{direct sum}),$$

and it follows

$$\dim MH = \sum_{i=0}^d \dim E_i H.$$

Note that  $\dim E_0H = 1$ . For  $1 \leq i \leq d$ , we find by Lemma 2.7 and Corollary 3.4(ii) that  $\dim E_iH = 2$  if  $xy$  is tight with respect to  $\theta_i$ , and  $\dim E_iH = 3$  otherwise. The result follows.  $\square$

## 6. Tight graphs and the fundamental bound

In this section, we obtain an inequality involving the second largest and minimal eigenvalue of a distance-regular graph. To obtain it, we need the following lemma.

**Lemma 6.1** *Let  $\Gamma$  denote a nonbipartite distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Then*

$$\frac{k + \theta_1(a_1 + 1)}{(k + \theta_1)(1 + \theta_1)} - \frac{k + \theta_d(a_1 + 1)}{(k + \theta_d)(1 + \theta_d)} \quad (38)$$

$$= \Psi \frac{(a_1 + 1)(\theta_d - \theta_1)}{(1 + \theta_1)(1 + \theta_d)(k + \theta_1)(k + \theta_d)}, \quad (39)$$

where

$$\Psi = \left( \theta_1 + \frac{k}{a_1 + 1} \right) \left( \theta_d + \frac{k}{a_1 + 1} \right) + \frac{ka_1b_1}{(a_1 + 1)^2}. \quad (40)$$

**Proof:** Put (38) over a common denominator, and simplify.  $\square$

We now present our inequality. We give two versions.

**Theorem 6.2** *Let  $\Gamma$  denote a distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Then (i), (ii) hold below.*

(i) *Suppose  $\Gamma$  is not bipartite. Then*

$$\frac{k + \theta_d(a_1 + 1)}{(k + \theta_d)(1 + \theta_d)} \leq \frac{k + \theta_1(a_1 + 1)}{(k + \theta_1)(1 + \theta_1)}. \quad (41)$$

$$(ii) \quad \left( \theta_1 + \frac{k}{a_1 + 1} \right) \left( \theta_d + \frac{k}{a_1 + 1} \right) \geq -\frac{ka_1b_1}{(a_1 + 1)^2}. \quad (42)$$

We refer to (42) as the Fundamental Bound.

**Proof:** (i) First assume  $a_1 = 0$ . Then the left side of (41) equals  $(1 + \theta_d)^{-1}$ , and is therefore negative. The right side of (41) equals  $(1 + \theta_1)^{-1}$ , and is therefore positive. Next assume  $a_1 \neq 0$ . Then (41) is immediate from (20).

(ii) First assume  $\Gamma$  is bipartite. Then  $\theta_d = -k$  and  $a_1 = 0$ , so both sides of (42) equal 0. Next assume  $\Gamma$  is not bipartite. Then (42) is immediate from (i) above, Lemma 6.1, and Lemma 2.6.  $\square$

We now consider when equality is attained in Theorem 6.2. To avoid trivialities, we consider only the nonbipartite case.

**Corollary 6.3** *Let  $\Gamma$  denote a nonbipartite distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Then the following are equivalent.*

- (i) *Equality holds in (41).*
- (ii) *Equality holds in (42).*
- (iii)  *$a_1 \neq 0$  and every edge of  $\Gamma$  is tight with respect to both  $\theta_1$  and  $\theta_d$ .*
- (iv)  *$a_1 \neq 0$  and there exists an edge of  $\Gamma$  which is tight with respect to both  $\theta_1$  and  $\theta_d$ .*

**Proof:** (i) $\Leftrightarrow$ (ii) Immediate from Lemma 6.1.

(i),(ii) $\Rightarrow$ (iii) Suppose  $a_1 = 0$ . We assume (42) holds with equality, so  $(\theta_1 + k)(\theta_d + k) = 0$ , forcing  $\theta_d = -k$ . Now  $\Gamma$  is bipartite by Lemma 2.5, contradicting the assumption. Hence  $a_1 \neq 0$ . Let  $xy$  denote an edge of  $\Gamma$ . Observe the expressions on the left and right in (20) are equal, so they both equal  $f(x, y)$ . Now  $xy$  is tight with respect to both  $\theta_1, \theta_d$  by Corollary 3.6(i),(ii).

(iii) $\Rightarrow$ (iv) Clear.

(iv) $\Rightarrow$ (i) Suppose the edge  $xy$  is tight with respect to both  $\theta_1, \theta_d$ . By Corollary 3.6(i),(ii), the scalar  $f(x, y)$  equals both the expression on the left and the expression on the right in (20), so these expressions are equal.  $\square$

**Definition 6.4** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ . We say  $\Gamma$  is *tight* whenever  $\Gamma$  is not bipartite and the equivalent conditions (i)–(iv) hold in Corollary 6.3.

We wish to emphasize the following fact.

**Proposition 6.5** *Let  $\Gamma$  denote a tight distance-regular graph with diameter  $d \geq 3$ . Then  $a_i \neq 0$  ( $1 \leq i \leq d - 1$ ).*

**Proof:** Observe  $a_1 \neq 0$  by Corollary 6.3(iii) and Definition 6.4. Now  $a_2, \dots, a_{d-1}$  are nonzero by Lemma 2.8.  $\square$

We finish this section with some inequalities involving the eigenvalues of tight graphs.

**Lemma 6.6** *Let  $\Gamma = (X, R)$  denote a tight distance-regular graph with diameter  $d \geq 3$  and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Then (i)–(iv) hold below.*

- (i)  $\theta_d < \frac{-k}{a_1+1}$ .
- (ii) Let  $\rho, \rho_2$  denote the first and second cosines for  $\theta_d$ , respectively. Then  $\rho^2 < \rho_2$ .
- (iii) Let  $\sigma, \sigma_2$  denote the first and second cosines for  $\theta_1$ , respectively. Then  $\sigma^2 > \sigma_2$ .
- (iv) For each edge  $xy$  of  $\Gamma$ , the scalar  $f = f(x, y)$  satisfies  $0 < f < b_1$ .

**Proof:** (i) Observe (42) holds with equality since  $\Gamma$  is tight, and  $a_1 \neq 0$  by Proposition 6.5, so

$$\left(\theta_1 + \frac{k}{a_1 + 1}\right)\left(\theta_d + \frac{k}{a_1 + 1}\right) < 0.$$

Since  $\theta_1 > \theta_d$ , the first factor is positive, and the second is negative. The result follows.

(ii) By Lemma 2.3(iv),

$$k^2 b_1 (\rho^2 - \rho_2) = (k - \theta_d)(k + \theta_d(a_1 + 1)). \quad (43)$$

The right side of (43) is negative in view of (i) above, so  $\rho^2 < \rho_2$ .

(iii) By Lemma 2.3(iv),

$$k^2 b_1 (\sigma^2 - \sigma_2) = (k - \theta_1)(k + \theta_1(a_1 + 1)). \quad (44)$$

The right side of (44) is positive in view of Lemma 2.6(i), so  $\sigma^2 > \sigma_2$ .

(iv) Observe  $f$  equals the expression on the right in (20). This expression is positive and less than  $b_1$ , since  $\theta_1$  is positive.  $\square$

## 7. Two characterizations of tight graphs

**Theorem 7.1** *Let  $\Gamma$  denote a nonbipartite distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Then for all real numbers  $\alpha, \beta$ , the following are equivalent.*

- (i)  $\Gamma$  is tight, and  $\alpha, \beta$  is a permutation of  $\theta_1, \theta_d$ .
- (ii)  $\theta_d \leq \alpha, \beta \leq \theta_1$ , and

$$\left(\alpha + \frac{k}{a_1 + 1}\right)\left(\beta + \frac{k}{a_1 + 1}\right) = -\frac{ka_1 b_1}{(a_1 + 1)^2}. \quad (45)$$

**Proof:** (i) $\Rightarrow$ (ii) Immediate since (42) holds with equality.

(ii) $\Rightarrow$ (i) Interchanging  $\alpha$  and  $\beta$  if necessary, we may assume  $\alpha \geq \beta$ . Since the right side of (45) is nonpositive, we have

$$\begin{aligned} 0 &\leq \alpha + \frac{k}{a_1 + 1} \leq \theta_1 + \frac{k}{a_1 + 1}, \\ 0 &\geq \beta + \frac{k}{a_1 + 1} \geq \theta_d + \frac{k}{a_1 + 1}. \end{aligned}$$

By (45), the above inequalities, and (42), we have

$$-\frac{ka_1 b_1}{(a_1 + 1)^2} = \left(\alpha + \frac{k}{a_1 + 1}\right)\left(\beta + \frac{k}{a_1 + 1}\right)$$

$$\geq \left(\theta_1 + \frac{k}{a_1 + 1}\right) \left(\theta_d + \frac{k}{a_1 + 1}\right) \tag{46}$$

$$\geq -\frac{ka_1b_1}{(a_1 + 1)^2}. \tag{47}$$

Apparently we have equality in (46), (47). In particular (42) holds with equality, so  $\Gamma$  is tight. We mentioned equality holds in (46). Neither side is 0, since  $a_1 \neq 0$  by Proposition 6.5, and it follows  $\alpha = \theta_1, \beta = \theta_d$ .  $\square$

**Theorem 7.2** *Let  $\Gamma = (X, R)$  denote a nonbipartite distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Let  $\theta$  and  $\theta'$  denote distinct eigenvalues of  $\Gamma$ , with respective cosine sequences  $\sigma_0, \sigma_1, \dots, \sigma_d$  and  $\rho_0, \rho_1, \dots, \rho_d$ . The following are equivalent.*

- (i)  $\Gamma$  is tight, and  $\theta, \theta'$  is a permutation of  $\theta_1, \theta_d$ .
- (ii) For  $1 \leq i \leq d$ ,

$$\frac{\sigma\sigma_{i-1} - \sigma_i}{(1 + \sigma)(\sigma_{i-1} - \sigma_i)} = \frac{\rho\rho_{i-1} - \rho_i}{(1 + \rho)(\rho_{i-1} - \rho_i)}, \tag{48}$$

and the denominators in (48) are nonzero.

- (iii) 
$$\frac{\sigma^2 - \sigma_2}{(1 + \sigma)(\sigma - \sigma_2)} = \frac{\rho^2 - \rho_2}{(1 + \rho)(\rho - \rho_2)}, \tag{49}$$

and the denominators in (49) are nonzero.

- (iv)  $\theta$  and  $\theta'$  are both nontrivial, and

$$(\sigma_2\rho_2 - \sigma\rho)(\rho - \sigma) = (\sigma\rho_2 - \sigma_2\rho)(\sigma\rho - 1). \tag{50}$$

**Proof:** (i) $\Rightarrow$ (ii) Recall  $a_1 \neq 0$  by Proposition 6.5. Pick adjacent vertices  $x, y \in X$ , and let  $D_1^1 = D_1^1(x, y)$  be as in Definition 2.10. By Corollary 6.3(iii), the edge  $xy$  is tight with respect to both  $\theta, \theta'$ ; applying (21), we find both sides of (48) equal  $a_1^{-1}|\Gamma_{i-1}(z) \cap D_1^1|$ , where  $z$  denotes any vertex in  $D_{i-1}^i(x, y)$ . In particular, the two sides of (48) are equal. The denominators in (48) are nonzero by Lemma 2.4 and Lemma 2.5.

(ii) $\Rightarrow$ (iii) Set  $i = 2$  in (ii).

(iii) $\Rightarrow$ (iv)  $\theta$  is nontrivial; otherwise  $\sigma = \sigma_2 = 1$ , and a denominator in (49) is zero. Similarly  $\theta'$  is nontrivial. To get (50), put (49) over a common denominator and simplify the result.

(iv) $\Rightarrow$ (i) Eliminating  $\sigma, \sigma_2, \rho, \rho_2$  in (50) using  $\theta = k\sigma, \theta' = k\rho$ , and Lemma 2.3(i), we routinely find (45) holds for  $\alpha = \theta$  and  $\beta = \theta'$ . Applying Theorem 7.1, we find  $\Gamma$  is tight, and that  $\theta, \theta'$  is a permutation of  $\theta_1, \theta_d$ .  $\square$



### 8. The auxiliary parameter

Let  $\Gamma$  denote a tight distance-regular graph with diameter  $d \geq 3$ . We are going to show the intersection numbers of  $\Gamma$  are given by certain rational expressions involving  $d$  independent parameters. We begin by introducing one of these parameters.

**Definition 8.1** Let  $\Gamma$  denote a tight distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \cdots > \theta_d$ . Let  $\theta$  denote one of  $\theta_1, \theta_d$ . By the *auxiliary parameter* of  $\Gamma$  associated with  $\theta$ , we mean the scalar

$$\varepsilon = \frac{k^2 - \theta\theta'}{k(\theta - \theta')}, \quad (51)$$

where  $\theta'$  denotes the complement of  $\theta$  in  $\{\theta_1, \theta_d\}$ . We observe the auxiliary parameter for  $\theta_d$  is the opposite of the auxiliary parameter for  $\theta_1$ .

**Lemma 8.2** Let  $\Gamma$  denote a tight distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \cdots > \theta_d$ . Let  $\theta$  denote one of  $\theta_1, \theta_d$ , and let  $\varepsilon$  denote the auxiliary parameter for  $\theta$ . Then (i)–(iv) hold below.

- (i)  $\varepsilon > 0$  if  $\theta = \theta_1$ , and  $\varepsilon < 0$  if  $\theta = \theta_d$ .
- (ii)  $1 < |\varepsilon|$ .
- (iii)  $|\varepsilon| < k\theta_1^{-1}$ .
- (iv)  $|\varepsilon| < -k\theta_d^{-1}$ .

**Proof:** First assume  $\theta = \theta_1$ . By (51),

$$\varepsilon - 1 = (k + \theta_d)(k - \theta_1)(\theta_1 - \theta_d)^{-1}k^{-1} > 0,$$

so  $\varepsilon > 1$ . Recall  $\theta_1 > 0$  and  $\theta_d < 0$ . By this and (51),

$$k\theta_1^{-1} - \varepsilon = \theta_d(k - \theta_1)(k + \theta_1)(\theta_d - \theta_1)^{-1}k^{-1}\theta_1^{-1} > 0,$$

so  $\varepsilon < k\theta_1^{-1}$ . Similarly

$$k\theta_d^{-1} + \varepsilon = \theta_1(k - \theta_d)(k + \theta_d)(\theta_1 - \theta_d)^{-1}k^{-1}\theta_d^{-1} < 0,$$

so  $\varepsilon < -k\theta_d^{-1}$ . We now have the result for  $\theta = \theta_1$ . The result for  $\theta = \theta_d$  follows in view of the last line of Definition 8.1.  $\square$

**Theorem 8.3** Let  $\Gamma$  denote a nonbipartite distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \cdots > \theta_d$ . Let  $\theta$  and  $\theta'$  denote any eigenvalues of  $\Gamma$ , with respective cosine sequences  $\sigma_0, \sigma_1, \dots, \sigma_d$  and  $\rho_0, \rho_1, \dots, \rho_d$ . Let  $\varepsilon$  denote any complex scalar. Then the following are equivalent.

- (i)  $\Gamma$  is tight,  $\theta, \theta'$  is a permutation of  $\theta_1, \theta_d$ , and  $\varepsilon$  is the auxiliary parameter for  $\theta$ .

(ii)  $\theta$  and  $\theta'$  are both nontrivial, and

$$\sigma_i \rho_i - \sigma_{i-1} \rho_{i-1} = \varepsilon(\sigma_{i-1} \rho_i - \rho_{i-1} \sigma_i) \quad (52)$$

for  $1 \leq i \leq d$ .

(iii)  $\theta$  and  $\theta'$  are both nontrivial, and

$$\sigma \rho - 1 = \varepsilon(\rho - \sigma), \quad \sigma_2 \rho_2 - \sigma \rho = \varepsilon(\sigma \rho_2 - \rho \sigma_2). \quad (53)$$

**Proof:** (i) $\Rightarrow$ (ii) It is clear  $\theta, \theta'$  are both nontrivial. To see (52), observe  $\theta, \theta'$  are distinct, so the equivalent statements (i)–(iv) in Theorem 7.2 hold. Putting (48) over a common denominator and simplifying using  $\varepsilon = (1 - \sigma\rho)(\sigma - \rho)^{-1}$ , we get (52).

(ii) $\Rightarrow$ (iii) Set  $i = 1$  and  $i = 2$  in (52).

(iii) $\Rightarrow$ (i) We first show  $\theta \neq \theta'$ . Suppose  $\theta = \theta'$ . Then  $\sigma = \rho$ , so the left equation of (53) becomes  $\sigma^2 = 1$ , forcing  $\sigma = 1$  or  $\sigma = -1$ . But  $\sigma \neq 1$  since  $\theta$  is nontrivial, and  $\sigma \neq -1$  since  $\Gamma$  is not bipartite. We conclude  $\theta \neq \theta'$ . Now  $\sigma \neq \rho$ ; solving the left equation in (53) for  $\varepsilon$ , and eliminating  $\varepsilon$  in the right equation of (53) using the result, we obtain (50). Now Theorem 7.2(iv) holds. Applying Theorem 7.2, we find  $\Gamma$  is tight, and that  $\theta, \theta'$  is a permutation of  $\theta_1, \theta_d$ . Solving the left equation in (53) for  $\varepsilon$ , and simplifying the result, we obtain (51). It follows  $\varepsilon$  is the auxiliary parameter for  $\theta$ .  $\square$

## 9. Feasibility

Let  $\Gamma$  denote a tight distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Let  $\theta, \theta'$  denote a permutation of  $\theta_1, \theta_d$ , with respective cosine sequences  $\sigma_0, \sigma_1, \dots, \sigma_d$  and  $\rho_0, \rho_1, \dots, \rho_d$ . Let  $\varepsilon$  denote the auxiliary parameter for  $\theta$ . Pick any integer  $i$  ( $1 \leq i \leq d$ ), and observe (52) holds. Rearranging terms in that equation, we find

$$\rho_i(\sigma_i - \varepsilon\sigma_{i-1}) = \rho_{i-1}(\sigma_{i-1} - \varepsilon\sigma_i). \quad (54)$$

We would like to solve (54) for  $\rho_i$ , but conceivably  $\sigma_i - \varepsilon\sigma_{i-1} = 0$ . In this section we investigate this possibility.

**Lemma 9.1** *Let  $\Gamma$  denote a tight distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Let  $\theta, \theta'$  denote a permutation of  $\theta_1, \theta_d$ , with respective cosine sequences  $\sigma_0, \sigma_1, \dots, \sigma_d$  and  $\rho_0, \rho_1, \dots, \rho_d$ . Let  $\varepsilon$  denote the auxiliary parameter for  $\theta$ . Then for each integer  $i$  ( $1 \leq i \leq d - 1$ ), the following are equivalent: (i)  $\sigma_{i-1} = \varepsilon\sigma_i$ ; (ii)  $\sigma_{i+1} = \varepsilon\sigma_i$ ; (iii)  $\sigma_{i-1} = \sigma_{i+1}$ ; (iv)  $\rho_i = 0$ . Moreover, suppose (i)–(iv) hold. Then  $\theta = \theta_d$  and  $\theta' = \theta_1$ .*

**Proof:** Observe Theorem 8.3(i) holds, so (52) holds.

(i) $\Rightarrow$ (iv) Replacing  $\sigma_{i-1}$  by  $\varepsilon\sigma_i$  in (52), we find  $\sigma_i \rho_i(1 - \varepsilon^2) = 0$ . Observe  $\varepsilon^2 \neq 1$  by Lemma 8.2(ii). Suppose for the moment that  $\sigma_i = 0$ . We assume  $\sigma_{i-1} = \varepsilon\sigma_i$ , so  $\sigma_{i-1} = 0$ . Now  $\sigma_{i-1} = \sigma_i$ , contradicting Lemma 2.4. Hence  $\sigma_i \neq 0$ , so  $\rho_i = 0$ .

(iv) $\Rightarrow$ (i) Setting  $\rho_i = 0$  in (52), we find  $\rho_{i-1}(\sigma_{i-1} - \varepsilon\sigma_i) = 0$ . Observe  $\rho_{i-1} \neq 0$ , otherwise  $\rho_{i-1} = \rho_i$ , contradicting Lemma 2.4. We conclude  $\sigma_{i-1} = \varepsilon\sigma_i$ , as desired.

(ii) $\Leftrightarrow$ (iv) Similar to the proof of (i) $\Leftrightarrow$ (iv).

(i),(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) We cannot have  $\theta = \theta_1$  by Lemma 2.4(i), so  $\theta = \theta_d, \theta' = \theta_1$ . In particular  $\rho_{i-1} \neq \rho_{i+1}$ . Adding (52) at  $i$  and  $i + 1$ , we obtain

$$\sigma_{i+1}\rho_{i+1} - \sigma_{i-1}\rho_{i-1} = \varepsilon(\sigma_i\rho_{i+1} - \sigma_{i+1}\rho_i + \sigma_{i-1}\rho_i - \sigma_i\rho_{i-1}).$$

Replacing  $\sigma_{i+1}$  by  $\sigma_{i-1}$  in the above line, and simplifying, we obtain

$$(\sigma_{i-1} - \varepsilon\sigma_i)(\rho_{i+1} - \rho_{i-1}) = 0.$$

It follows  $\sigma_{i-1} = \varepsilon\sigma_i$ , as desired.

Now suppose (i)–(iv). Then we saw in the proof of (iii) $\Rightarrow$ (i) that  $\theta = \theta_d, \theta' = \theta_1$ .  $\square$

**Definition 9.2** Let  $\Gamma = (X, R)$  denote a tight distance-regular graph with diameter  $d \geq 3$  and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote any cosine sequence for  $\Gamma$  and let  $\theta$  denote the corresponding eigenvalue. The sequence  $\sigma_0, \sigma_1, \dots, \sigma_d$  (or  $\theta$ ) is said to be *feasible* whenever (i) and (ii) hold below.

- (i)  $\theta$  is one of  $\theta_1, \theta_d$ .
- (ii)  $\sigma_{i-1} \neq \sigma_{i+1}$  for  $1 \leq i \leq d - 1$ .

We observe by Lemma 2.4(i) that  $\theta_1$  is feasible.

We conclude this section with an extension of Theorem 8.3.

**Theorem 9.3** Let  $\Gamma$  denote a nonbipartite distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Let  $\theta$  and  $\theta'$  denote any eigenvalues of  $\Gamma$ , with respective cosine sequences  $\sigma_0, \sigma_1, \dots, \sigma_d$  and  $\rho_0, \rho_1, \dots, \rho_d$ . Let  $\varepsilon$  denote any complex scalar. Then the following are equivalent.

- (i)  $\Gamma$  is tight,  $\theta$  is feasible,  $\varepsilon$  is the auxiliary parameter for  $\theta$ , and  $\theta'$  is the complement of  $\theta$  in  $\{\theta_1, \theta_d\}$ .
- (ii)  $\theta'$  is not trivial,

$$\rho_i = \prod_{j=1}^i \frac{\sigma_{j-1} - \varepsilon\sigma_j}{\sigma_j - \varepsilon\sigma_{j-1}} \quad (0 \leq i \leq d), \quad (55)$$

and denominators in (55) are all nonzero.

**Proof:** (i) $\Rightarrow$ (ii) Clearly  $\theta'$  is nontrivial. To see (55), observe Theorem 8.3(i) holds, so (52) holds. Rearranging terms in (52), we obtain

$$\rho_i(\sigma_i - \varepsilon\sigma_{i-1}) = \rho_{i-1}(\sigma_{i-1} - \varepsilon\sigma_i) \quad (1 \leq i \leq d). \quad (56)$$

Observe  $\sigma_i \neq \varepsilon\sigma_{i-1}$  for  $2 \leq i \leq d$  by Lemma 9.1(ii), and  $\sigma \neq \varepsilon$  by Lemma 8.2(ii), so the coefficient of  $\rho_i$  in (56) is never zero. Solving that equation for  $\rho_i$  and applying induction, we routinely obtain (55).

(ii) $\Rightarrow$ (i) We show Theorem 8.3(iii) holds. Observe  $\theta$  is nontrivial; otherwise  $\sigma = 1$ , forcing  $\rho = 1$  by (55), and contradicting our assumption that  $\theta'$  is nontrivial. One readily verifies (53) by eliminating  $\rho, \rho_2$  using (55). We now have Theorem 8.3(iii). Applying that theorem, we find  $\Gamma$  is tight,  $\theta, \theta'$  is a permutation of  $\theta_1, \theta_d$ , and that  $\varepsilon$  is the auxiliary parameter for  $\theta$ . It remains to show  $\theta$  is feasible. Suppose not. Then there exists an integer  $i$  ( $1 \leq i \leq d-1$ ) such that  $\sigma_{i-1} = \sigma_{i+1}$ . Applying Lemma 9.1, we find  $\sigma_{i+1} = \varepsilon\sigma_i$ . But  $\sigma_{i+1} - \varepsilon\sigma_i$  is a factor in the denominator of (55) (with  $i$  replaced by  $i+1$ ), and hence is not 0. We now have a contradiction, so  $\theta$  is feasible.  $\square$

## 10. A parametrization

In this section, we obtain the intersection numbers of a tight graph as rational functions of a feasible cosine sequence and the associated auxiliary parameter. We begin with a result about arbitrary distance-regular graphs.

**Lemma 10.1** *Let  $\Gamma$  denote a distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Let  $\theta, \theta'$  denote a permutation of  $\theta_1, \theta_d$ , with respective cosine sequences  $\sigma_0, \sigma_1, \dots, \sigma_d$  and  $\rho_0, \rho_1, \dots, \rho_d$ . Then*

$$k = \frac{(\sigma - \sigma_2)(1 - \rho) - (\rho - \rho_2)(1 - \sigma)}{(\rho - \rho_2)(1 - \sigma)\sigma - (\sigma - \sigma_2)(1 - \rho)\rho}, \quad (57)$$

$$b_i = k \frac{(\sigma_{i-1} - \sigma_i)(1 - \rho)\rho_i - (\rho_{i-1} - \rho_i)(1 - \sigma)\sigma_i}{(\rho_i - \rho_{i+1})(\sigma_{i-1} - \sigma_i) - (\sigma_i - \sigma_{i+1})(\rho_{i-1} - \rho_i)} \quad (1 \leq i \leq d-1), \quad (58)$$

$$c_i = k \frac{(\sigma_i - \sigma_{i+1})(1 - \rho)\rho_i - (\rho_i - \rho_{i+1})(1 - \sigma)\sigma_i}{(\rho_i - \rho_{i+1})(\sigma_{i-1} - \sigma_i) - (\sigma_i - \sigma_{i+1})(\rho_{i-1} - \rho_i)} \quad (1 \leq i \leq d-1), \quad (59)$$

$$c_d = k\sigma_d \frac{\sigma - 1}{\sigma_{d-1} - \sigma_d} = k\rho_d \frac{\rho - 1}{\rho_{d-1} - \rho_d}, \quad (60)$$

and the denominators in (57)–(60) are never zero.

**Proof:** Line (60) is immediate from Lemma 2.3(v), and the denominators in that line are nonzero by Lemma 2.4. To obtain (58), (59), pick any integer  $i$  ( $1 \leq i \leq d-1$ ), and recall by Lemma 2.2(iii) that

$$c_i(\sigma_{i-1} - \sigma_i) - b_i(\sigma_i - \sigma_{i+1}) = k(\sigma - 1)\sigma_i, \quad (61)$$

$$c_i(\rho_{i-1} - \rho_i) - b_i(\rho_i - \rho_{i+1}) = k(\rho - 1)\rho_i. \quad (62)$$

To solve this linear system for  $c_i$  and  $b_i$ , consider the determinant

$$D_i := \det \begin{pmatrix} \sigma_{i-1} - \sigma_i & \sigma_i - \sigma_{i+1} \\ \rho_{i-1} - \rho_i & \rho_i - \rho_{i+1} \end{pmatrix}.$$

Using Lemma 2.4, we routinely find  $D_i \neq 0$ . Now (61), (62) has the unique solution (58), (59) by elementary linear algebra. The denominators in (58), (59) both equal  $D_i$ ; in particular they are not zero. To get (57), set  $i = 1$  and  $c_1 = 1$  in (59), and solve for  $k$ .  $\square$

**Theorem 10.2** *Let  $\Gamma$  denote a nonbipartite distance-regular graph with diameter  $d \geq 3$ , and let  $\sigma_0, \sigma_1, \dots, \sigma_d, \varepsilon, h$  denote complex scalars. Then the following are equivalent.*

- (i)  $\Gamma$  is tight,  $\sigma_0, \sigma_1, \dots, \sigma_d$  is a feasible cosine sequence for  $\Gamma$ ,  $\varepsilon$  is the associated auxiliary parameter from (51), and

$$h = \frac{(1 - \sigma)(1 - \sigma_2)}{(\sigma^2 - \sigma_2)(1 - \varepsilon\sigma)}. \quad (63)$$

- (ii)  $\sigma_0 = 1, \sigma_{d-1} = \sigma\sigma_d, \varepsilon \neq -1$ ,

$$k = h \frac{\sigma - \varepsilon}{\sigma - 1}, \quad (64)$$

$$b_i = h \frac{(\sigma_{i-1} - \sigma\sigma_i)(\sigma_{i+1} - \varepsilon\sigma_i)}{(\sigma_{i-1} - \sigma_{i+1})(\sigma_{i+1} - \sigma_i)} \quad (1 \leq i \leq d-1), \quad (65)$$

$$c_i = h \frac{(\sigma_{i+1} - \sigma\sigma_i)(\sigma_{i-1} - \varepsilon\sigma_i)}{(\sigma_{i+1} - \sigma_{i-1})(\sigma_{i-1} - \sigma_i)} \quad (1 \leq i \leq d-1), \quad (66)$$

$$c_d = h \frac{\sigma - \varepsilon}{\sigma - 1}, \quad (67)$$

and denominators in (64)–(67) are all nonzero.

**Proof:** Let  $\theta_0 > \theta_1 > \dots > \theta_d$  denote the eigenvalues of  $\Gamma$ .

(i) $\Rightarrow$ (ii) Observe  $\sigma_0 = 1$  by Lemma 2.2(ii), and  $\varepsilon \neq -1$  by Lemma 8.2(ii). Let  $\theta$  denote the eigenvalue associated with  $\sigma_0, \sigma_1, \dots, \sigma_d$ , and observe by Definition 9.2 that  $\theta$  is one of  $\theta_1, \theta_d$ . Let  $\theta'$  denote the complement of  $\theta$  in  $\{\theta_1, \theta_d\}$ , and let  $\rho_0, \rho_1, \dots, \rho_d$  denote the cosine sequence for  $\theta'$ . Observe Theorem 9.3(i) holds. Applying that theorem, we obtain (55). Eliminating  $\rho_0, \rho_1, \dots, \rho_d$  in (57)–(60) using (55), we routinely obtain (64)–(67), and that  $\sigma_{d-1} = \sigma\sigma_d$ .

(ii) $\Rightarrow$ (i) One readily checks

$$c_i(\sigma_{i-1} - \sigma_i) - b_i(\sigma_i - \sigma_{i+1}) = k(\sigma - 1)\sigma_i \quad (1 \leq i \leq d),$$

where  $\sigma_{d+1}$  is an indeterminate. Applying Lemma 2.2(i),(iii), we find  $\sigma_0, \sigma_1, \dots, \sigma_d$  is a cosine sequence for  $\Gamma$ , with associated eigenvalue  $\theta := k\sigma$ . By (64), (65), and since

$k, b_1, \dots, b_{d-1}$  are nonzero,

$$\sigma_j \neq \varepsilon \sigma_{j-1} \quad (1 \leq j \leq d).$$

Set

$$\rho_i := \prod_{j=1}^i \frac{\sigma_{j-1} - \varepsilon \sigma_j}{\sigma_j - \varepsilon \sigma_{j-1}} \quad (0 \leq i \leq d). \quad (68)$$

One readily checks  $\rho_0 = 1$ , and that

$$c_i(\rho_{i-1} - \rho_i) - b_i(\rho_i - \rho_{i+1}) = k(\rho - 1)\rho_i \quad (1 \leq i \leq d),$$

where  $\rho_{d+1}$  is an indeterminant. Applying Lemma 2.2(i),(iii), we find  $\rho_0, \rho_1, \dots, \rho_d$  is a cosine sequence for  $\Gamma$ , with associated eigenvalue  $\theta' := k\rho$ . We claim  $\theta'$  is not trivial. Suppose  $\theta'$  is trivial. Then  $\rho = 1$ . Setting  $i = 1$  and  $\rho = 1$  in (68) we find  $\sigma - \varepsilon = 1 - \varepsilon\sigma$ , forcing  $(1 - \sigma)(1 + \varepsilon) = 0$ . Observe  $\sigma \neq 1$  since the denominator in (67) is not zero, and we assume  $\varepsilon \neq -1$ , so we have a contradiction. We have now shown  $\theta'$  is nontrivial, so Theorem 9.3(ii) holds. Applying that theorem, we find  $\Gamma$  is tight,  $\theta$  is feasible, and that  $\varepsilon$  is the auxiliary parameter of  $\theta$ . To see (63), set  $i = 1$  and  $c_1 = 1$  in (66), and solve for  $h$ .  $\square$

**Proposition 10.3** *With the notation of Theorem 10.2, suppose (i), (ii) hold, and let  $\theta_0 > \theta_1 > \dots > \theta_d$  denote the eigenvalues of  $\Gamma$ . If  $\varepsilon > 0$ , then*

$$\theta_1 = \frac{\sigma(\sigma - \varepsilon)(1 - \sigma_2)}{(1 - \varepsilon\sigma)(\sigma_2 - \sigma^2)}, \quad \theta_d = \frac{1 - \sigma_2}{\sigma_2 - \sigma^2}. \quad (69)$$

If  $\varepsilon < 0$ , then

$$\theta_1 = \frac{1 - \sigma_2}{\sigma_2 - \sigma^2}, \quad \theta_d = \frac{\sigma(\sigma - \varepsilon)(1 - \sigma_2)}{(1 - \varepsilon\sigma)(\sigma_2 - \sigma^2)}. \quad (70)$$

We remark that the denominators in (69), (70) are nonzero.

**Proof:** Let  $\theta$  denote the eigenvalue of  $\Gamma$  associated with  $\sigma_0, \sigma_1, \dots, \sigma_d$ . By Lemma 2.2(iii) and (64), we obtain

$$\begin{aligned} \theta &= k\sigma \\ &= \frac{\sigma(\sigma - \varepsilon)(1 - \sigma_2)}{(1 - \varepsilon\sigma)(\sigma_2 - \sigma^2)}. \end{aligned} \quad (71)$$

Observe  $\theta \in \{\theta_1, \theta_d\}$  since  $\sigma_0, \sigma_1, \dots, \sigma_d$  is feasible. Let  $\theta'$  denote the complement of  $\theta$  in  $\{\theta_1, \theta_d\}$ , and let  $\rho$  denote the first cosine associated with  $\theta'$ . Observe condition (i) holds in Theorem 9.3, so (55) holds. Setting  $i = 1$  in that equation, we find

$$\rho = \frac{1 - \varepsilon\sigma}{\sigma - \varepsilon}. \quad (72)$$

By Lemma 2.2(iii), (64), and (72), we obtain

$$\begin{aligned}\theta' &= k\rho \\ &= \frac{1 - \sigma_2}{\sigma_2 - \sigma^2}.\end{aligned}\tag{73}$$

To finish the proof, we observe by Lemma 8.2(i) that  $\theta = \theta_1$ ,  $\theta' = \theta_d$  if  $\varepsilon > 0$ , and  $\theta = \theta_d$ ,  $\theta' = \theta_1$  if  $\varepsilon < 0$ .  $\square$

**Theorem 10.4** *Let  $\Gamma$  denote a tight distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \cdots > \theta_d$ . Then (i) and (ii) hold below.*

(i)  $a_d = 0$ .

(ii) *Let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote the cosine sequence for  $\theta_1$  or  $\theta_d$ , and let  $\varepsilon$  denote the associated auxiliary parameter from (51).*

*Then*

$$a_i = g \frac{(\sigma_{i+1} - \sigma\sigma_i)(\sigma_{i-1} - \sigma\sigma_i)}{(\sigma_{i+1} - \sigma_i)(\sigma_{i-1} - \sigma_i)} \quad (1 \leq i \leq d-1),\tag{74}$$

where

$$g = \frac{(\varepsilon - 1)(1 - \sigma_2)}{(\sigma^2 - \sigma_2)(1 - \varepsilon\sigma)}.\tag{75}$$

**Proof:** (i) Comparing (64), (67), we see  $k = c_d$ , and it follows  $a_d = 0$ .

(ii) First assume  $\sigma_0, \sigma_1, \dots, \sigma_d$  is the cosine sequence for  $\theta_1$ , and recall this sequence is feasible. Let  $h$  be as in (63). Then Theorem 10.2(i) holds, so Theorem 10.2(ii) holds. Evaluating the right side of  $a_i = k - b_i - c_i$  using (64)–(66), and simplifying the result using (63), we obtain (74), (75). To finish the proof, let  $\rho_0, \rho_1, \dots, \rho_d$  denote the cosine sequence for  $\theta_d$ , and recall by Definition 8.1 that the associated auxiliary parameter is  $\varepsilon' = -\varepsilon$ . We show

$$a_i = \frac{(\varepsilon' - 1)(1 - \rho_2)}{(\rho^2 - \rho_2)(1 - \varepsilon'\rho)} \frac{(\rho_{i+1} - \rho\rho_i)(\rho_{i-1} - \rho\rho_i)}{(\rho_{i+1} - \rho_i)(\rho_{i-1} - \rho_i)}.\tag{76}$$

By Theorem 7.2(ii) (with  $i$  replaced by  $i + 1$ ),

$$\frac{1}{1 + \sigma} \frac{\sigma_{i+1} - \sigma\sigma_i}{\sigma_{i+1} - \sigma_i} = \frac{1}{1 + \rho} \frac{\rho_{i+1} - \rho\rho_i}{\rho_{i+1} - \rho_i}.\tag{77}$$

Subtracting 1 from both sides of Theorem 7.2(ii), and simplifying, we obtain

$$\frac{1}{1 + \sigma} \frac{\sigma_{i-1} - \sigma\sigma_i}{\sigma_{i-1} - \sigma_i} = \frac{1}{1 + \rho} \frac{\rho_{i-1} - \rho\rho_i}{\rho_{i-1} - \rho_i}.\tag{78}$$

By (53),

$$\frac{(\varepsilon - 1)(1 - \sigma_2)(1 + \sigma)^2}{(\sigma^2 - \sigma_2)(1 - \varepsilon\sigma)} = \frac{(\varepsilon' - 1)(1 - \rho_2)(1 + \rho)^2}{(\rho^2 - \rho_2)(1 - \varepsilon'\rho)}. \tag{79}$$

Multiplying together (77)–(79) and simplifying, we obtain (76), as desired.  $\square$

We end this section with some inequalities.

**Lemma 10.5** *Let  $\Gamma$  denote a tight distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Let  $\theta$  denote one of  $\theta_1, \theta_d$ , and let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote the cosine sequence for  $\theta$ . Suppose  $\theta = \theta_1$ . Then*

- (i)  $\sigma_{i-1} > \sigma\sigma_i$  ( $1 \leq i \leq d - 1$ ),
- (ii)  $\sigma\sigma_{i-1} > \sigma_i$  ( $2 \leq i \leq d$ ).

Suppose  $\theta = \theta_d$ . Then

- (iii)  $(-1)^i(\sigma\sigma_i - \sigma_{i-1}) > 0$  ( $1 \leq i \leq d - 1$ ),
- (iv)  $(-1)^i(\sigma_i - \sigma\sigma_{i-1}) > 0$  ( $2 \leq i \leq d$ ).

**Proof:** (i) We first show  $\sigma_{i-1} - \sigma\sigma_i$  is nonnegative. Recall  $a_1 \neq 0$  by Proposition 6.5, so Theorem 4.1 applies. Let  $x, y$  denote adjacent vertices in  $X$ , and recall by Corollary 6.3 that the edge  $xy$  is tight with respect to  $\theta$ . Now Theorem 4.1(i) holds, so (22) holds. Observe the left side of (22) is nonnegative, so the right side is nonnegative. In that expression on the right, the factors  $1 + \sigma$  and  $\sigma_{i-1} - \sigma_i$  are positive, so the remaining factor  $\sigma_{i-1} - \sigma\sigma_i$  is nonnegative, as desired. To finish the proof, observe  $\sigma_{i-1} - \sigma\sigma_i$  is a factor on the right in (74), so it is not zero in view of Proposition 6.5.

(ii)–(iv) Similar to the proof of (i) above.  $\square$

### 11. The 1-homogeneous property

In this section, we show the concept of tight is closely related to the concept of 1-homogeneous that appears in the work of Nomura [14–16].

**Theorem 11.1** *Let  $\Gamma = (X, R)$  denote a tight distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote the cosine sequence associated with  $\theta_1$  or  $\theta_d$ . Fix adjacent vertices  $x, y \in X$ . Then with the notation of Definition 2.10 we have the following: For all integers  $i$  ( $1 \leq i \leq d - 1$ ), and for all vertices  $z \in D_i^i$ ,*

$$|\Gamma_{i-1}(z) \cap D_1^1| = c_i \frac{(\sigma^2 - \sigma_2)(\sigma_i - \sigma_{i+1})}{(\sigma - \sigma_2)(\sigma\sigma_i - \sigma_{i+1})}, \tag{80}$$

$$|\Gamma_{i+1}(z) \cap D_1^1| = b_i \frac{(\sigma^2 - \sigma_2)(\sigma_{i-1} - \sigma_i)}{(\sigma - \sigma_2)(\sigma_{i-1} - \sigma\sigma_i)}. \tag{81}$$



**Proof:** First assume  $\sigma_0, \sigma_1, \dots, \sigma_d$  is the cosine sequence for  $\theta_1$ , and let  $\rho_0, \rho_1, \dots, \rho_d$  denote the cosine sequence for  $\theta_d$ . The edge  $xy$  is tight with respect to both  $\theta_1, \theta_d$ , so by Theorem 4.2(ii),

$$|\Gamma_{i+1}(z) \cap D_1^1| = |\Gamma_{i-1}(z) \cap D_1^1| \frac{\sigma_{i-1} - \sigma_i}{\sigma_i - \sigma_{i+1}} + a_1 \frac{1 - \sigma}{1 + \sigma} \frac{\sigma_i}{\sigma_i - \sigma_{i+1}}, \quad (82)$$

$$|\Gamma_{i+1}(z) \cap D_1^1| = |\Gamma_{i-1}(z) \cap D_1^1| \frac{\rho_{i-1} - \rho_i}{\rho_i - \rho_{i+1}} + a_1 \frac{1 - \rho}{1 + \rho} \frac{\rho_i}{\rho_i - \rho_{i+1}}. \quad (83)$$

Eliminating  $\rho_0, \rho_1, \dots, \rho_d$  in (83) using (55), we obtain

$$\begin{aligned} |\Gamma_{i+1}(z) \cap D_1^1| &= |\Gamma_{i-1}(z) \cap D_1^1| \frac{\sigma_{i-1} - \sigma_i}{\sigma_i - \sigma_{i+1}} \frac{\sigma_{i+1} - \varepsilon \sigma_i}{\sigma_{i-1} - \varepsilon \sigma_i} \\ &\quad + a_1 \frac{(1 - \sigma)(\sigma_{i+1} - \varepsilon \sigma_i)}{(1 + \sigma)(1 - \varepsilon)(\sigma_i - \sigma_{i+1})}, \end{aligned} \quad (84)$$

where  $\varepsilon$  denotes the auxiliary parameter associated with  $\theta_1$ . Solving (82), (84) for  $|\Gamma_{i+1}(z) \cap D_1^1|$  and  $|\Gamma_{i-1}(z) \cap D_1^1|$ , and evaluating the result using (63), (65), (66), (74), we get (80), (81), as desired. To finish the proof observe by Theorem 7.2(ii), (iii) that

$$\frac{(\sigma^2 - \sigma_2)(\sigma_i - \sigma_{i+1})}{(\sigma - \sigma_2)(\sigma \sigma_i - \sigma_{i+1})} = \frac{(\rho^2 - \rho_2)(\rho_i - \rho_{i+1})}{(\rho - \rho_2)(\rho \rho_i - \rho_{i+1})}, \quad (85)$$

$$\frac{(\sigma^2 - \sigma_2)(\sigma_{i-1} - \sigma_i)}{(\sigma - \sigma_2)(\sigma_{i-1} - \sigma \sigma_i)} = \frac{(\rho^2 - \rho_2)(\rho_{i-1} - \rho_i)}{(\rho - \rho_2)(\rho_{i-1} - \rho \rho_i)}. \quad (86)$$

□

**Theorem 11.2** *Let  $\Gamma = (X, R)$  denote a tight distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Let  $\sigma_0, \sigma_1, \dots, \sigma_d$  denote the cosine sequence for  $\theta_1$  or  $\theta_d$ . Fix adjacent vertices  $x, y \in X$ . Then with the notation of Definition 2.10 we have the following (i), (ii).*

(i) *For all integers  $i$  ( $1 \leq i \leq d - 1$ ), and for all  $z \in D_i^i$ ,*

$$|\Gamma(z) \cap D_{i-1}^{i-1}| = c_i \frac{(\sigma_i - \sigma_{i+1})(\sigma \sigma_{i-1} - \sigma_i)}{(\sigma_{i-1} - \sigma_i)(\sigma \sigma_i - \sigma_{i+1})}, \quad (87)$$

$$|\Gamma(z) \cap D_{i+1}^{i+1}| = b_i \frac{(\sigma_{i-1} - \sigma_i)(\sigma_i - \sigma \sigma_{i+1})}{(\sigma_i - \sigma_{i+1})(\sigma_{i-1} - \sigma \sigma_i)}. \quad (88)$$

(ii) *For all integers  $i$  ( $2 \leq i \leq d$ ), and for all  $z \in D_{i-1}^i \cup D_i^{i-1}$ ,*

$$|\Gamma(z) \cap D_{i-1}^{i-1}| = a_{i-1} \frac{(1 - \sigma)(\sigma_{i-1}^2 - \sigma_{i-2} \sigma_i)}{(\sigma_{i-1} - \sigma_i)(\sigma_{i-2} - \sigma \sigma_{i-1})}. \quad (89)$$

**Proof:** (i) To prove (87), we assume  $i \geq 2$ ; otherwise both sides are zero. Let  $\alpha_i$  denote the expression on the right in (80). Let  $N$  denote the number of ordered pairs  $uv$  such that

$$u \in \Gamma_{i-1}(z) \cap D_1^1, \quad v \in \Gamma(z) \cap D_{i-1}^{i-1}, \quad \partial(u, v) = i - 2.$$

We compute  $N$  in two ways. On one hand, by (80), there are precisely  $\alpha_i$  choices for  $u$ , and given  $u$ , there are precisely  $c_{i-1}$  choices for  $v$ , so

$$N = \alpha_i c_{i-1}. \tag{90}$$

On the other hand, there are precisely  $|\Gamma(z) \cap D_{i-1}^{i-1}|$  choices for  $v$ , and given  $v$ , there are precisely  $\alpha_{i-1}$  choices for  $u$ , so

$$N = |\Gamma(z) \cap D_{i-1}^{i-1}| \alpha_{i-1}. \tag{91}$$

Observe by Lemma 2.4, Lemma 6.6, and (80) that  $\alpha_{i-1} \neq 0$ ; combining this with (90), (91), we find

$$|\Gamma(z) \cap D_{i-1}^{i-1}| = c_{i-1} \alpha_i \alpha_{i-1}^{-1}.$$

Eliminating  $\alpha_{i-1}$ ,  $\alpha_i$  in the above line using (80), we obtain (87), as desired. Concerning (88), first assume  $i = d - 1$ . We show both sides of (88) are zero. To see the left side is zero, recall  $a_d = 0$  by Theorem 10.4, forcing  $p_{dd}^1 = 0$  by Lemma 2.9, so  $D_d^d = \emptyset$  by the last line in Definition 2.10. The right side of (88) is zero since the factor  $\sigma_{d-1} - \sigma_d$  in the numerator is zero by Lemma 2.3(vi). We now show (88) for  $i \leq d - 2$ . Let  $\beta_i$  denote the expression on the right in (81). Let  $N'$  denote the number of ordered pairs  $uv$  such that

$$u \in \Gamma_{i+1}(z) \cap D_1^1, \quad v \in \Gamma(z) \cap D_{i+1}^{i+1}, \quad \partial(u, v) = i + 2.$$

We compute  $N'$  in two ways. On one hand, by (81), there are precisely  $\beta_i$  choices for  $u$ , and given  $u$ , there are precisely  $b_{i+1}$  choices for  $v$ , so

$$N' = \beta_i b_{i+1}. \tag{92}$$

On the other hand, there are precisely  $|\Gamma(z) \cap D_{i+1}^{i+1}|$  choices for  $v$ , and given  $v$ , there are precisely  $\beta_{i+1}$  choices for  $u$ , so

$$N' = |\Gamma(z) \cap D_{i+1}^{i+1}| \beta_{i+1}. \tag{93}$$

Observe by Lemma 2.4, Lemma 6.6, and (81) that  $\beta_{i+1} \neq 0$ ; combining this with (92), (93), we find

$$|\Gamma(z) \cap D_{i+1}^{i+1}| = b_{i+1} \beta_i \beta_{i+1}^{-1}.$$

Eliminating  $\beta_i$ ,  $\beta_{i+1}$  in the above line using (81), we obtain (88), as desired.

(ii) Let  $\gamma_i$  denote the expression on the right in (21), and let  $\delta_i$  denote the expression on the right in (87). Let  $N''$  denote the number of ordered pairs  $uv$  such that

$$u \in \Gamma_{i-1}(z) \cap D_1^1, \quad v \in \Gamma(z) \cap D_{i-1}^{i-1}, \quad \partial(u, v) = i - 2.$$

We compute  $N''$  in two ways. On one hand, by Theorem 4.1(ii), there are precisely  $\gamma_i$  choices for  $u$ . Given  $u$ , we find by (87) (with  $x$  and  $i$  replaced by  $u$  and  $i - 1$ , respectively) that there are precisely  $c_{i-1} - \delta_{i-1}$  choices for  $v$ ; consequently

$$N'' = \gamma_i(c_{i-1} - \delta_{i-1}). \quad (94)$$

On the other hand, there are precisely  $|\Gamma(z) \cap D_{i-1}^{i-1}|$  choices for  $v$ , and given  $v$ , there are precisely  $\alpha_{i-1}$  choices for  $u$ , where  $\alpha_{i-1}$  is from the proof of (i) above. Hence

$$N'' = |\Gamma(z) \cap D_{i-1}^{i-1}| \alpha_{i-1}. \quad (95)$$

Combining (94), (95),

$$|\Gamma(z) \cap D_{i-1}^{i-1}| = \gamma_i(c_{i-1} - \delta_{i-1})\alpha_{i-1}^{-1}.$$

Eliminating  $\alpha_{i-1}$ ,  $\gamma_i$ ,  $\delta_{i-1}$  in the above line using (80), (21), (87), respectively, and simplifying the result using Theorem 10.4(ii), we obtain (89), as desired.  $\square$

**Definition 11.3** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ , and fix adjacent vertices  $x, y \in X$ .

(i) For all integers  $i, j$  we define the vector  $w_{ij} = w_{ij}(x, y)$  by

$$w_{ij} = \sum_{z \in D_i^j} \hat{z}, \quad (96)$$

where  $D_i^j = D_i^j(x, y)$  is from (15).

(ii) Let  $\mathcal{L}$  denote the set of ordered pairs

$$\mathcal{L} = \{ij \mid 0 \leq i, j \leq d, p_{ij}^1 \neq 0\}. \quad (97)$$

We observe that for all integers  $i, j$ ,  $w_{ij} \neq 0$  if and only if  $ij \in \mathcal{L}$ .

(iii) We define the vector space  $W = W(x, y)$  by

$$W = \text{Span}\{w_{ij} \mid ij \in \mathcal{L}\}. \quad (98)$$

**Lemma 11.4** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ , and assume  $a_1 \neq 0$ . Then

(i)  $\mathcal{L} = \{i-1, i \mid 1 \leq i \leq d\} \cup \{i, i-1 \mid 1 \leq i \leq d\} \cup \{ii \mid 1 \leq i \leq e\}$ ,  
where  $e = d - 1$  if  $a_d = 0$  and  $e = d$  if  $a_d \neq 0$ .

$$(ii) \quad |\mathcal{L}| = \begin{cases} 3d & \text{if } a_d \neq 0, \\ 3d - 1 & \text{if } a_d = 0. \end{cases} \quad (99)$$

(iii) Let  $x, y$  denote adjacent vertices in  $X$ , and let  $W = W(x, y)$  be as in (98). Then

$$\dim W = \begin{cases} 3d & \text{if } a_d \neq 0, \\ 3d - 1 & \text{if } a_d = 0. \end{cases} \quad (100)$$

**Proof:** Routine application of Lemma 2.8 and Lemma 2.9.  $\square$

**Lemma 11.5** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ , fix adjacent vertices  $x, y \in X$ , and let the vector space  $W = W(x, y)$  be as in (98). Then the following are equivalent.

- (i) The vector space  $W$  is  $A$ -invariant.
- (ii) For all integers  $i, j, r, s$  ( $ij \in \mathcal{L}$  and  $rs \in \mathcal{L}$ ), and for all  $z \in D_i^j$ , the scalar  $|\Gamma(z) \cap D_r^s|$  is a constant independent of  $z$ .
- (iii) The following conditions hold.
  - (a) For all integers  $i$  ( $1 \leq i \leq d$ ), and for all  $z \in D_i^i$ , the scalars  $|\Gamma(z) \cap D_{i-1}^{i-1}|$  and  $|\Gamma(z) \cap D_{i+1}^{i+1}|$  are constants independent of  $z$ .
  - (b) For all integers  $i$  ( $2 \leq i \leq d$ ), and for all  $z \in D_{i-1}^i \cup D_i^{i-1}$ , the scalar  $|\Gamma(z) \cap D_{i-1}^{i-1}|$  is a constant independent of  $z$ .

**Proof:** (i) $\Leftrightarrow$ (ii) Routine.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (ii) Follows directly from Lemma 2.11.  $\square$

**Definition 11.6** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ . For each edge  $xy \in R$ , the graph  $\Gamma$  is said to be *1-homogeneous with respect to  $xy$*  whenever (i)–(iii) hold in Lemma 11.5. The graph  $\Gamma$  is said to be *1-homogeneous* whenever it is 1-homogeneous with respect to all edges in  $R$ .

**Theorem 11.7** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ . Then the following are equivalent.

- (i)  $\Gamma$  is tight,
- (ii)  $a_1 \neq 0$ ,  $a_d = 0$ , and  $\Gamma$  is 1-homogeneous,
- (iii)  $a_1 \neq 0$ ,  $a_d = 0$ , and  $\Gamma$  is 1-homogeneous with respect to at least one edge.

**Proof:** (i) $\Rightarrow$ (ii) Observe  $a_1 \neq 0$  by Proposition 6.5, and  $a_d = 0$  by Theorem 10.4. Pick any edge  $xy \in R$ . By Theorem 11.2, we find conditions (iii)(a), (iii)(b) hold in Lemma 11.5, so  $\Gamma$  is 1-homogeneous with respect to  $xy$  by Definition 11.6. Apparently  $\Gamma$  is 1-homogeneous with respect to every edge, so  $\Gamma$  is 1-homogeneous.

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Suppose  $\Gamma$  is 1-homogeneous with respect to the edge  $xy \in R$ . We show  $xy$  is tight with respect to both  $\theta_1, \theta_d$ . To do this, we show the tightness  $t = t(x, y)$  from

Definition 5.1 equals 2. Consider the vector space  $W = W(x, y)$  from (98), and the vector space  $H$  from (37). Observe  $W$  is  $A$ -invariant by Lemma 11.5, and  $W$  contains  $H$ , so it contains  $MH$ , where  $M$  denotes the Bose-Mesner algebra of  $\Gamma$ . The space  $W$  has dimension  $3d - 1$  by (100), so  $MH$  has dimension at most  $3d - 1$ . Applying (36), we find  $t \geq 2$ . From the discussion at the end of Definition 5.1, we observe  $t = 2$ , and that  $xy$  is tight with respect to both  $\theta_1, \theta_d$ . Now  $\Gamma$  is tight in view of Corollary 6.3(iv) and Definition 6.4.  $\square$

## 12. The local graph

**Definition 12.1** Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ . For each vertex  $x \in X$ , we let  $\Delta = \Delta(x)$  denote the vertex subgraph of  $\Gamma$  induced on  $\Gamma(x)$ . We refer to  $\Delta$  as the *local graph* associated with  $x$ . We observe  $\Delta$  has  $k$  vertices, and is regular with valency  $a_1$ . We further observe  $\Delta$  is not a clique.

In this section, we show the local graphs of tight distance-regular graphs are strongly-regular. We begin by recalling the definition and some basic properties of strongly-regular graphs.

**Definition 12.2 [3, p. 3]** A graph  $\Delta$  is said to be *strongly-regular* with parameters  $(v, \kappa, \lambda, \mu)$  whenever  $\Delta$  has  $v$  vertices and is regular with valency  $\kappa$ , adjacent vertices of  $\Delta$  have precisely  $\lambda$  common neighbors, and distinct non-adjacent vertices of  $\Delta$  have precisely  $\mu$  common neighbors.

**Lemma 12.3 [3, Thm. 1.3.1]** Let  $\Delta$  denote a connected strongly-regular graph with parameters  $(v, \kappa, \lambda, \mu)$ , and assume  $\Delta$  is not a clique. Then  $\Delta$  has precisely three distinct eigenvalues, one of which is  $\kappa$ . Denoting the others by  $r, s$ ,

$$v = \frac{(\kappa - r)(\kappa - s)}{\kappa + rs}, \quad \lambda = \kappa + r + s + rs, \quad \mu = \kappa + rs. \quad (101)$$

The multiplicity of  $\kappa$  as an eigenvalue of  $\Delta$  equals 1. The multiplicities with which  $r, s$  appear as eigenvalues of  $\Delta$  are given by

$$\text{mult}_r = \frac{\kappa(s+1)(\kappa-s)}{\mu(s-r)}, \quad \text{mult}_s = \frac{\kappa(r+1)(\kappa-r)}{\mu(r-s)}. \quad (102)$$

**Theorem 12.4** Let  $\Gamma = (X, R)$  denote a tight distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . Pick  $\theta \in \{\theta_1, \theta_d\}$ , let  $\sigma, \sigma_2$  denote the first and second cosines for  $\theta$ , respectively, and let  $\varepsilon$  denote the associated auxiliary parameter from (51). Then for any vertex  $x \in X$ , the local graph  $\Delta = \Delta(x)$  satisfies (i)–(iv) below.

(i)  $\Delta$  is strongly-regular with parameters  $(k, a_1, \lambda, \mu)$ , where  $k$  is the valency of  $\Gamma$ , and

$$a_1 = -\frac{(1 - \sigma_2)(1 + \sigma)(1 - \varepsilon)}{(\sigma - \sigma_2)(1 - \varepsilon\sigma)}, \quad (103)$$

$$\lambda = a_1 \frac{2\sigma}{1+\sigma} - a_1 \frac{1-\sigma}{1+\sigma} \frac{\sigma_2}{\sigma-\sigma_2} - \frac{1-\sigma_2}{\sigma-\sigma_2}, \quad (104)$$

$$\mu = \frac{a_1}{1+\sigma} \frac{\sigma^2 - \sigma_2}{\sigma - \sigma_2}. \quad (105)$$

- (ii)  $\Delta$  is connected and not a clique.  
 (iii) The distinct eigenvalues of  $\Delta$  are  $a_1, r, s$ , where

$$r = \frac{a_1\sigma}{1+\sigma}, \quad s = -\frac{1-\sigma_2}{\sigma-\sigma_2}. \quad (106)$$

- (iv) The multiplicities of  $r, s$  are given by

$$\text{mult}_r = \frac{(1+\sigma)(\sigma-\varepsilon)}{\sigma_2-\sigma^2}, \quad \text{mult}_s = -\frac{(1-\varepsilon)(1+\sigma)(\sigma_2-\varepsilon\sigma)}{(\sigma_2-\sigma^2)(1-\varepsilon\sigma)}. \quad (107)$$

**Proof:** (i) Clearly  $\Delta$  has  $k$  vertices and is regular with valency  $a_1$ . The formula (103) is from Theorem 10.4(ii). Pick distinct vertices  $y, z \in \Delta$ . We count the number of common neighbors of  $y, z$  in  $\Delta$ . First suppose  $y, z$  are adjacent. By (28) (with  $i = 1$ ) we find  $y, z$  have precisely  $\lambda$  common neighbors in  $\Delta$ , where  $\lambda$  is given in (104). Next suppose  $y, z$  are not adjacent. By (21) (with  $i = 2$ ), we find  $y, z$  have precisely  $\mu$  common neighbors in  $\Delta$ , where  $\mu$  is given in (105). The result now follows in view of Definition 12.2.

(ii) We saw in Definition 12.1 that  $\Delta$  is not a clique. Observe the scalar  $\mu$  in (105) is not zero, since  $a_1 \neq 0$  by Proposition 6.5, and since  $\sigma^2 \neq \sigma_2$  by Lemma 6.6(ii),(iii). It follows  $\Delta$  is connected.

(iii) The scalar  $a_1$  is an eigenvalue of  $\Delta$  by Lemma 12.3. Using (104), (105), we find the scalars  $r, s$  in (106) satisfy

$$\lambda = a_1 + r + s + rs, \quad \mu = a_1 + rs.$$

Comparing this with the two equations on the right in (101), we find the scalars  $r, s$  in (106) are the remaining eigenvalues of  $\Delta$ .

- (iv) By (102) and (i) above,

$$\text{mult}_r = \frac{a_1(s+1)(a_1-s)}{\mu(s-r)}, \quad \text{mult}_s = \frac{a_1(r+1)(a_1-r)}{\mu(r-s)}.$$

Eliminating  $a_1, \mu, r, s$  in the above equations using (103), (105), (106), we routinely obtain (107).  $\square$

**Definition 12.5** Let  $\Gamma$  denote a distance-regular graph with diameter  $d \geq 3$ , and eigenvalues  $\theta_0 > \theta_1 > \dots > \theta_d$ . We define

$$b^- := -1 - \frac{b_1}{1+\theta_1}, \quad b^+ := -1 - \frac{b_1}{1+\theta_d}.$$

We recall  $a_1 - k \leq \theta_d < -1 < \theta_1$  by Lemma 2.6, so  $b^- < -1, b^+ \geq 0$ .

**Theorem 12.6** *Let  $\Gamma = (X, R)$  denote a distance-regular graph with diameter  $d \geq 3$ . Then the following are equivalent.*

- (i)  $\Gamma$  is tight.
- (ii) For all  $x \in X$ , the local graph  $\Delta(x)$  is connected strongly-regular with eigenvalues  $a_1, b^+, b^-$ .
- (iii) There exists  $x \in X$  for which the local graph  $\Delta(x)$  is connected strongly-regular with eigenvalues  $a_1, b^+, b^-$ .

**Proof:** (i) $\Rightarrow$ (ii) Pick any  $x \in X$ , and let  $\Delta = \Delta(x)$  denote the local graph. By Theorem 12.4, the graph  $\Delta$  is connected and strongly-regular. The eigenvalues of  $\Delta$  other than  $a_1$  are given by (106), where for convenience we take the eigenvalue  $\theta$  involved to be  $\theta_1$ . Eliminating  $\sigma, \sigma_2$  in (106) using  $\theta_1 = k\sigma$  and Lemma 2.3(i), and simplifying the results using equality in the fundamental bound (42), we routinely find  $r = b^+, s = b^-$ .

(ii) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (i) Since  $\Delta = \Delta(x)$  is connected, its valency  $a_1$  is not zero. In particular  $\Gamma$  is not bipartite. The graph  $\Delta$  is not a clique, so (101) holds for  $\Delta$ . Applying the equation on the left in that line, we obtain

$$k(a_1 + b^+b^-) = (a_1 - b^+)(a_1 - b^-). \quad (108)$$

Eliminating  $b^+, b^-$  in (108) using Definition 12.5, and simplifying the result, we routinely obtain equality in the fundamental bound (42). Now  $\Gamma$  is tight, as desired.  $\square$

### 13. Examples of tight distance-regular graphs

The following examples (i)–(xii) are tight distance-regular graphs with diameter at least 3. In each case we give the intersection array, the second largest eigenvalue  $\theta_1$ , and the least eigenvalue  $\theta_d$ , together with their respective cosine sequences  $\{\sigma_i\}, \{\rho_i\}$ , and the auxiliary parameter  $\varepsilon$  for  $\theta_1$ . Also, we give the parameters and nontrivial eigenvalues of the local graphs.

- (i) The *Johnson graph*  $J(2d, d)$  has diameter  $d$  and intersection numbers  $a_i = 2i(d-i)$ ,  $b_i = (d-i)^2$ ,  $c_i = i^2$  for  $i = 0, \dots, d$ , cf. [3, p. 255]. It is distance-transitive, an antipodal double-cover, and  $Q$ -polynomial with respect to  $\theta_1$ .  
Each local graph is a *lattice graph*  $K_d \times K_d$ , with parameters  $(d^2, 2(d-1), d-2, 2)$  and nontrivial eigenvalues  $r = d-2, s = -2$ , cf. [3, p. 256].
- (ii) The *halved cube*  $\frac{1}{2}H(2d, 2)$  has diameter  $d$  and intersection numbers  $a_i = 4i(d-i)$ ,  $b_i = (d-i)(2d-2i-1)$ ,  $c_i = i(2i-1)$  for  $i = 0, \dots, d$ , cf. [3, p. 264]. It is distance-transitive, an antipodal double-cover, and  $Q$ -polynomial with respect to  $\theta_1$ .  
Each local graph is a Johnson graph  $J(2d, 2)$ , with parameters  $(d(2d-1), 4(d-1), 2(d-1), 4)$  and nontrivial eigenvalues  $r = 2d-4, s = -2$ , cf. [3, p. 267].
- (iii) The *Taylor graphs* are nonbipartite double-covers of complete graphs, i.e., distance-regular graphs with intersection array of the form  $\{k, c_2, 1; 1, c_2, k\}$ , where  $c_2 < k-1$ .

They have diameter 3, and are  $Q$ -polynomial with respect to both  $\theta_1, \theta_d$ . These eigenvalues are given by  $\theta_1 = \alpha, \theta_d = \beta$ , where

$$\alpha + \beta = k - 2c_2 - 1, \quad \alpha\beta = -k,$$

and  $\alpha > \beta$ . See Taylor [19], and Seidel and Taylor [17] for more details.

Each local graph is strongly-regular with parameters  $(k, a_1, \lambda, \mu)$ , where  $a_1 = k - c_2 - 1, \lambda = (3a_1 - k - 1)/2$  and  $\mu = a_1/2$ . We note both  $a_1, c_2$  are even and  $k$  is odd. The nontrivial eigenvalues of the local graph are

$$r = \frac{\alpha - 1}{2}, \quad s = \frac{\beta - 1}{2}.$$

- (iv) The graph  $3.Sym(7)$  has intersection array  $\{10, 6, 4, 1; 1, 2, 6, 10\}$  and can be obtained from a sporadic Fisher group, cf. [3, pp. 397–400]. It is sometimes called the Conway-Smith graph. It is distance-transitive, an antipodal 3-fold cover, and is not  $Q$ -polynomial.

Each local graph is a *Petersen graph*, with parameters  $(10, 3, 0, 1)$  and nontrivial eigenvalues  $r = 1, s = -2$ , see [11], [3, 13.2.B].

- (v) The graph  $3.O_6^-(3)$  has intersection array  $\{45, 32, 12, 1; 1, 6, 32, 45\}$  and can be obtained from a sporadic Fisher group, cf. [3, pp. 397–400]. It is distance-transitive, an antipodal 3-fold cover, and is not  $Q$ -polynomial.

Each local graph is a *generalized quadrangle*  $GQ(4, 2)$ , with parameters  $(45, 12, 3, 3)$  and nontrivial eigenvalues  $r = 3, s = -3$ . See [3, p. 399].

- (vi) The graph  $3.O_7(3)$  has intersection array  $\{117, 80, 24, 1; 1, 12, 80, 117\}$  and can be obtained from a sporadic Fisher group, cf. [3, pp. 397–400]. It is distance-transitive, an antipodal 3-fold cover, and is not  $Q$ -polynomial.

Each local graph is strongly-regular with parameters  $(117, 36, 15, 9)$ , and nontrivial eigenvalues  $r = 9, s = -3$ . [3, 13.2.D].

- (vii) The graph  $3.Fi_{24}$  has intersection array  $\{31671, 28160, 2160, 1; 1, 1080, 28160, 31671\}$  and can be obtained from a sporadic Fisher group, cf. [3, p. 397]. It is distance-transitive, an antipodal 3-fold cover, and is not  $Q$ -polynomial.

Each local graph is strongly-regular with parameters  $(31671, 3510, 693, 351)$  and nontrivial eigenvalues  $r = 351, s = -9$ . They are related to  $Fi_{23}$ .

- (viii) The *Soicher1 graph* has intersection array  $\{56, 45, 16, 1; 1, 8, 45, 56\}$ , cf. [2], [4, 11.41], [18]. It is distance-transitive, an antipodal 3-fold cover, and is not  $Q$ -polynomial.

Each local graph is a *Gewirtz graph* with parameters  $(56, 10, 0, 2)$  and nontrivial eigenvalues  $r = 2, s = -4$ , [3, p. 372].

- (ix) The *Soicher2 graph* has intersection array  $\{416, 315, 64, 1; 1, 32, 315, 416\}$ , cf. [18] [4, 13.8A]. It is distance-transitive, an antipodal 3-fold cover, and is not  $Q$ -polynomial.

Each local graph is strongly-regular with parameters  $(416, 100, 36, 20)$  and nontrivial eigenvalues  $r = 20, s = -4$ .

- (x) The *Meixner1 graph* has intersection array  $\{176, 135, 24, 1; 1, 24, 135, 176\}$ , cf. [13] [4, 12.4A]. It is distance-transitive, an antipodal 2-fold cover, and is  $Q$ -polynomial.



Each local graph is strongly-regular with parameters  $(176, 40, 12, 8)$  and nontrivial eigenvalues  $r = 8, s = -4$ .

- (xi) The *Meixner2 graph* has intersection array  $\{176, 135, 36, 1; 1, 12, 135, 176\}$ , cf. [13] [4,12.4A]. It is distance-transitive, an antipodal 4-fold cover, and is not  $Q$ -polynomial.

Each local graph is strongly-regular with parameters  $(176, 40, 12, 8)$  and nontrivial eigenvalues  $r = 8, s = -4$ .

- (xii) The *Patterson graph* has intersection array  $\{280, 243, 144, 10; 1, 8, 90, 280\}$ , and can be constructed from the Suzuki group, see [3, 13.7]. It is primitive and distance-transitive, but not  $Q$ -polynomial.

Each local graph is a generalized quadrangle  $GQ(9,3)$  with parameters  $(280, 36, 8, 4)$  and nontrivial eigenvalues  $r = 8, s = -4$ , [3, Thm. 13.7.1].

Name	$\theta_1$	$\theta_d$	$\{\sigma_i\}$	$\{\rho_i\}$	$\varepsilon$
$J(2d, d)$	$d(d-2)$	$-d$	$\sigma_i = \frac{d-2i}{d}$	$\rho_i = \frac{(-1)^i \cdot 1 \cdot 2 \cdots i}{d(d-1) \cdots (d-i+1)}$	$\frac{d+2}{d}$
$\frac{1}{2}H(2d, 2)$	$(2d-1)(d-2)$	$-d$	$\sigma_i = \frac{d-2i}{d}$	$\rho_i = \frac{(-1)^i \cdot 1 \cdot 3 \cdots (2i-1)}{(2d-1)(2d-3) \cdots (2d-2i+1)}$	$\frac{d+1}{d-1}$
Taylor	$\alpha$	$\beta$	$\left(1, \frac{\alpha}{k}, \frac{-\alpha}{k}, -1\right)$	$\left(1, \frac{\beta}{k}, \frac{-\beta}{k}, -1\right)$	$\frac{k+1}{\alpha-\beta}$
3.Sym(7)	5	-4	$\left(1, \frac{1}{2}, 0, \frac{-1}{4}, \frac{-1}{2}\right)$	$\left(1, \frac{-2}{5}, \frac{3}{10}, \frac{-2}{5}, 1\right)$	$\frac{4}{3}$
$3.O_6^-(3)$	15	-9	$\left(1, \frac{1}{3}, 0, \frac{-1}{6}, \frac{-1}{2}\right)$	$\left(1, \frac{-1}{5}, \frac{1}{10}, \frac{-1}{5}, 1\right)$	2
$3.O_7(3)$	39	-9	$\left(1, \frac{1}{3}, 0, \frac{-1}{6}, \frac{-1}{2}\right)$	$\left(1, \frac{-1}{13}, \frac{2}{65}, \frac{-1}{13}, 1\right)$	$\frac{5}{2}$
$3.Fi_{24}$	3519	-81	$\left(1, \frac{1}{9}, 0, \frac{-1}{18}, \frac{-1}{2}\right)$	$\left(1, \frac{-1}{391}, \frac{5}{17204}, \frac{-1}{391}, 1\right)$	$\frac{44}{5}$
Soicher1	14	-16	$\left(1, \frac{1}{4}, 0, \frac{-1}{8}, \frac{-1}{2}\right)$	$\left(1, \frac{-2}{7}, \frac{1}{7}, \frac{-2}{7}, 1\right)$	2
Soicher2	104	-16	$\left(1, \frac{1}{4}, 0, \frac{-1}{8}, \frac{-1}{2}\right)$	$\left(1, \frac{-1}{26}, \frac{1}{91}, \frac{-1}{26}, 1\right)$	$\frac{7}{2}$
Meixner1	44	-16	$\left(1, \frac{1}{4}, 0, \frac{-1}{4}, -1\right)$	$\left(1, \frac{-1}{11}, \frac{1}{33}, \frac{-1}{11}, 1\right)$	3
Meixner2	44	-16	$\left(1, \frac{1}{4}, 0, \frac{-1}{12}, \frac{-1}{3}\right)$	$\left(1, \frac{-1}{11}, \frac{1}{33}, \frac{-1}{11}, 1\right)$	3
Patterson	80	-28	$\left(1, \frac{2}{7}, \frac{1}{21}, \frac{-2}{63}, \frac{-1}{9}\right)$	$\left(1, \frac{-1}{10}, \frac{1}{45}, \frac{-1}{54}, \frac{5}{27}\right)$	$\frac{8}{3}$

**Appendix A: 1-homogeneous partitions of the known examples of the AT4 family and the Patterson Graph**

In [21] a tight non bipartite antipodal distance-regular graph with diameter four was parameterized by the eigen values  $r$  and  $-s$  of the local graphs and the size of its antipodal

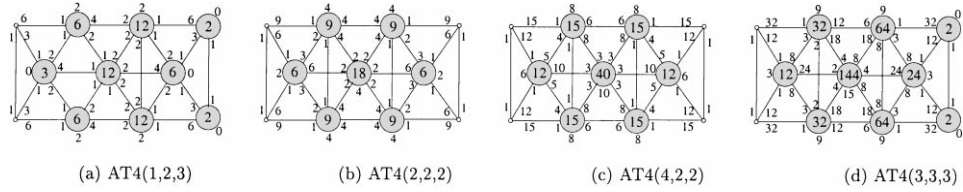


Figure A.1. 1-homogeneous partition of (a) the Conway-Smith graph (b) the Johnson graph  $J(8, 4)$ , (c) the halved cube  $\frac{1}{2}H(8, 2)$ , and (d) the  $3.O_6^-(3)$ .

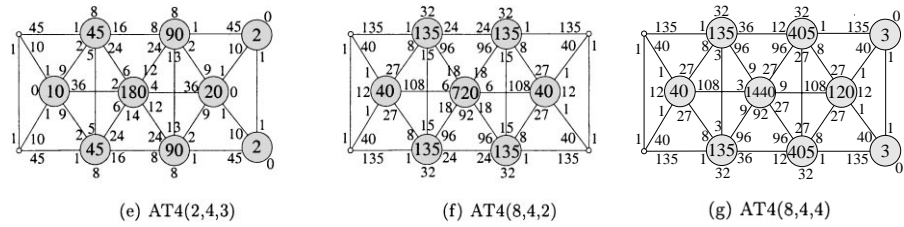


Figure A.2. 1-homogeneous partition of (e) the Soicher1 graph, (f) the Meixner1 graph, (g) the Meixner2 graph.

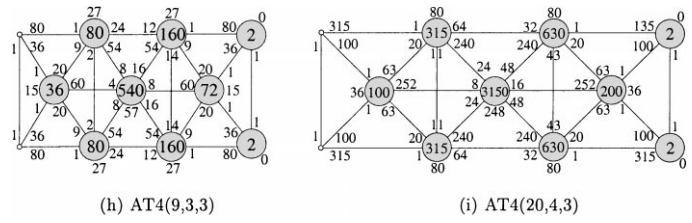


Figure A.3. 1-homogeneous partition of (h) the  $3.O_7(3)$ , (i) the Soicher2 graph.

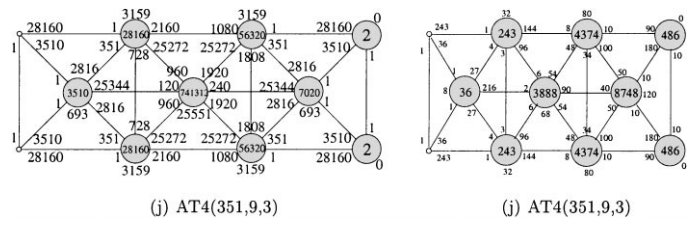


Figure A.4. 1-homogeneous partition of (j) the  $3.Fi_{24}^-$  graph and (i) the Patterson graph.

classes. The graph was called an **antipodal tight graph** of diameter four and with Parameters  $(r, s, t)$  and denoted by  $AT4(r, s, t)$ .

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