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Symmetric Versus Non-Symmetric Spin Models for Link Invariants

FRANÇOIS JAEGER†
Laboratoire Leibniz, IMAG, 38031 Grenoble Cedex, France

KAZUMASA NOMURA
College of Liberal Arts and Sciences, Tokyo Medical and Dental University, Kohnodai, Ichikawa, 272 Japan

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Abstract. We study *spin models* as introduced in [20]. Such a spin model can be defined as a square matrix satisfying certain equations, and can be used to compute an *associated link invariant*. The link invariant associated with a symmetric spin model depends only trivially on link orientation. This property also holds for *quasi-symmetric* spin models, which are obtained from symmetric spin models by certain "gauge transformations" preserving the associated link invariant. Using a recent result of [16] which asserts that every spin model belongs to some Bose-Mesner algebra with duality, we show that the transposition of a spin model can be realized by a permutation of rows. We call the order of this permutation the *index* of the spin model. We show that spin models of odd index are quasi-symmetric. Next, we give a general form for spin models of index 2 which implies that they are associated with a certain class of symmetric spin models. The symmetric Hadamard spin models of [21] belong to this class and this leads to the introduction of *non-symmetric Hadamard spin models*. These spin models give the first known example where the associated link invariant depends non-trivially on link orientation. We show that a non-symmetric Hadamard spin model belongs to a certain triply regular Bose-Mesner algebra of dimension 5 with duality, and we use this to give an explicit formula for the associated link invariant involving the Jones polynomial.

Keywords: spin model, link invariant, Bose-Mesner algebra

1. Introduction

Symmetric spin models were introduced in [18] as basic data to compute certain invariants of oriented links in 3-space; by construction, these invariants depend only trivially on the link orientation. A non-symmetric generalization of a spin model was introduced in [20]. While one could hope that the associated link invariants would depend non-trivially on the link orientation, no such examples were known until the present work. Finally, a further generalization called 4-*weight spin models* was introduced in [1].

A 4-weight spin model can be defined as a 5-tuple (X, W_1, W_2, W_3, W_4), where X is a finite non-empty set and the W_i are complex matrices with rows and columns indexed by X which satisfy certain equations. When $W_1 = W_2 = W^+$, $W_3 = W_4 = W^-$, we call the triple (X, W^+, W^-) a 2-*weight spin model* (this is exactly a "generalized spin model" as defined in [20]). The triple (X, W^+, W^-) can be defined in terms of the matrix W^+ alone, and we call this matrix a *spin model* for simplicity.

We review the basic tools used in this paper in Section 2. They include the following results.

In [15], some transformations of 4-weight spin models, called *gauge transformations*, are introduced. These gauge transformations preserve the associated link invariant (Theorem A). Two 4-weight spin models are said to be *gauge equivalent* if they can be related by gauge transformations, and this definition applies in particular to 2-weight spin models.

In [16], generalizing previous results of [13, 22], it is shown that for any 2-weight spin model (X, W^+, W^-) there exists a (commutative) Bose-Mesner algebra $\mathcal{A}$ which contains W^+, W^- and which admits a duality Ψ given by the identity $\Psi(M) = a\,{}^tW^- \circ (W^+(W^- \circ M))$, where a is the diagonal element of W^+ and $\circ$ denotes Hadamard product (Theorem B).

In Section 3 we introduce the concept of a *dual-permutation matrix*. Such matrices are defined so that in a Bose-Mesner algebra $\mathcal{A}$ with duality Ψ, $\Psi(R)$ is a dual-permutation matrix whenever R is a permutation matrix. In this situation the dual-permutation matrices in $\mathcal{A}$ form an abelian group $\mathcal{A}'_1$ under Hadamard product, which is isomorphic to the group $\mathcal{A}_1$ of permutation matrices in $\mathcal{A}$. We show that when $\mathcal{A}$ arises from a 2-weight spin model (X, W^+, W^-) as in Theorem B, the matrix $W^+ \circ W^-$ belongs to $\mathcal{A}'_1$. By the *index* of (X, W^+, W^-) we mean the order of $W^+ \circ W^-$ in the abelian group $\mathcal{A}'_1$. Dually, $|X|^{-1}\,{}^tW^+W^-$ belongs to $\mathcal{A}_1$ and its order is the index. This leads us to introduce the *quasi-symmetric spin models*, a class of 2-weight spin models which are gauge equivalent to symmetric ones. Thus the link invariant associated with a quasi-symmetric spin model depends only trivially on the link orientation. We show that spin models of odd index are quasi-symmetric. The same holds when $\mathcal{A}$ (given by Theorem B) is the Bose-Mesner algebra of some abelian group.

In Section 4, we give a convenient general form of spin models of index 2. This shows that they are closely related with a certain class of symmetric spin models of similar form. This class contains the *symmetric Hadamard spin models* constructed in [21] from Hadamard matrices.

This leads us to define *non-symmetric Hadamard spin models* in Section 5. For each such spin model we introduce a non-symmetric Bose-Mesner algebra $\mathcal{A}$ of dimension 5 which contains it; we establish that Ψ as given in Theorem B is a duality. The Bose-Mesner algebra $\mathcal{A}$ is closely related with Bose-Mesner algebras of Hadamard graphs used in the study of symmetric Hadamard spin models. Using this relationship, we show that $\mathcal{A}$ is *triply regular* (see [11]). Then, using a simple example, we show that the associated link invariant depends non-trivially on the link orientation.

Finally, we obtain a formula for the associated link invariant which is similar to the formula previously obtained in the symmetric case [14]. This formula essentially involves the Jones polynomials (see [17]) of the various “sublinks” of a link. The proof is also similar and consists of two main steps. In the first step, we show that the associated link invariant is given by a rational function of one variable u, where u is a parameter which gives the size of the spin model. In the second step, we show that this rational function coincides with the required formula for infinitely many special values of u.

We conclude in Section 6 with some open questions.

2. Preliminaries

2.1. Spin models for link invariants

For more details concerning this section the reader can refer to [12]. An (oriented) *link* is a finite collection of disjoint simple oriented closed curves (the *components* of the link) smoothly embedded in 3-space. Any such link can be represented by a *diagram*, which is a generic plane projection (there are only a finite number of multiple points, each of which is a simple crossing), together with an indication at each crossing of the corresponding spatial structure. A *link invariant* is a quantity attached to diagrams which is invariant under certain diagram deformations called Reidemeister moves (these moves generate a combinatorial equivalence of diagrams which represents a natural topological equivalence of links). *Spin models* are basic data to compute link invariants in the following way.

In general, the link invariant will take the form

$$Z(L) = a^{-T(L)} D^{-\chi(L)} \sum_{\sigma: B(L) \to X} \prod_{v \in V(L)} \langle v, \sigma \rangle \tag{1}$$

for any diagram L of a link. Here

— X is a finite non-empty set of *spins*;
— a is a non-zero complex number, called the *modulus* of the spin model, and $T(L)$, the *Tait number* of L, is the sum of signs of the crossings of L, where the sign of a crossing is defined on figure 1;
— D is some square root of $|X|$, called the *loop variable* of the spin model;
— The regions of L (connected components of $\mathbf{R}^2 - L$) are colored with two colors, black and white, in such a way that adjacent regions of L receive different colors; $B(L)$ denotes the set of black regions of L, and $\chi(L)$ denotes the Euler characteristic of the union of these black regions; when L is connected, $\chi(L)$ is just the number of black regions;
— $V(L)$ is the set of crossings of L, and for $\sigma: B(L) \to X$, $v \in V(L)$, the quantity $\langle v, \sigma \rangle$ only depends on the values of σ on the two black regions incident with v, and on the geometry of this crossing-region incidence.

This dependence takes the following two forms.

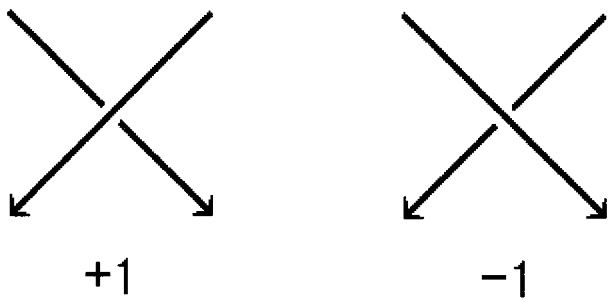

Figure 1.

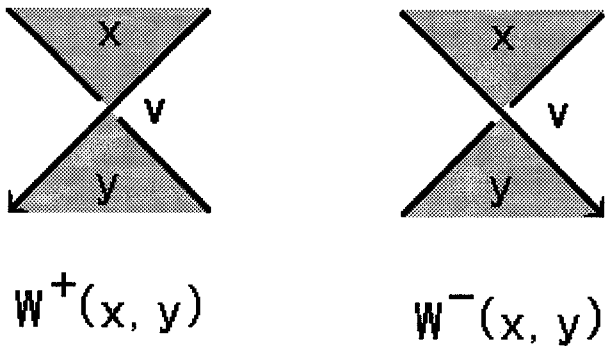

Figure 2.

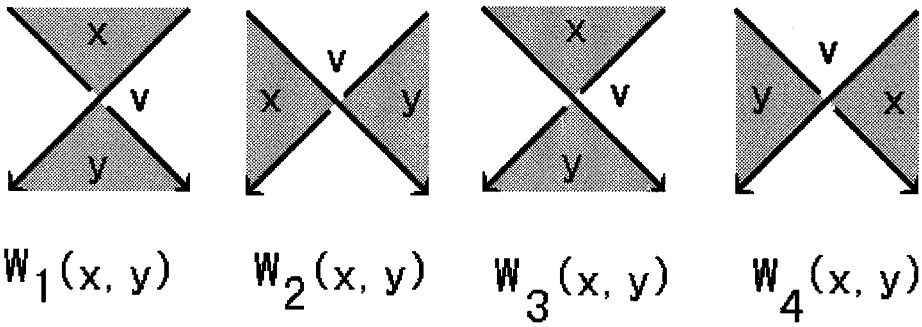

Figure 3.

In a 2-*weight spin model*, we have two matrices W^+, W^- in M_X (the set of complex matrices with rows and columns indexed by X) and $\langle v, \sigma \rangle$ is defined on figure 2 (where x, y are the values of σ on the black regions incident with v).

In a 4-*weight spin model*, we have four matrices W_1, W_2, W_3, W_4 in M_X and $\langle v, \sigma \rangle$ is defined on figure 3.

In the case of 2-weight spin models, it is shown in [20] that $Z(L)$ defined by (1) is a link invariant provided the following properties hold (for every $\alpha, \beta, \gamma \in X$):

$$W^+(\alpha, \alpha) = a, \quad W^-(\alpha, \alpha) = a^{-1}, \quad \sum_{x \in X} W^+(\alpha, x) = \sum_{x \in X} W^+(x, \alpha) = Da^{-1},$$
$$\sum_{x \in X} W^-(\alpha, x) = \sum_{x \in X} W^-(x, \alpha) = Da, \tag{2}$$

$$W^+(\alpha, \beta) W^-(\beta, \alpha) = 1, \quad \sum_{x \in X} W^+(\alpha, x) W^-(x, \beta) = |X| \delta_{\alpha,\beta} \tag{3}$$

(where δ is the Kronecker symbol),

$$\sum_{x \in X} W^+(\alpha, x) W^+(\beta, x) W^-(x, \gamma) = D W^+(\alpha, \beta) W^-(\beta, \gamma) W^-(\gamma, \alpha). \tag{4}$$

Remark (4) can be replaced by other identities, see [20].

We shall take as our definition of 2-*weight spin model* a triple $(X,\ W^+,\ W^-)$, where X is a finite non-empty set and W^+, W^- are two matrices in M_X satisfying (2), (3), (4) for some a, D in $\mathbf{C} - \{0\}$ with $D^2 = |X|$.

A 2-weight spin model (X, W^+, W^-) is said to be *symmetric* if W^+, W^- are symmetric matrices. Symmetric spin models were introduced in [18], and the non-symmetric generalization of [20] was studied later.

We observe that the link invariant associated with a symmetric 2-weight spin model depends only trivially on the link orientation, i.e. via the factor $a^{-T(L)}$ in (1). The main issue addressed in this paper is the possibility of a more complicated dependence for general 2-weight spin models.

In the case of 4-weight spin models, it is shown in [1] that $Z(L)$ defined by (1) is a link invariant if the following properties hold (for every α, β, γ in X):

$$\begin{gathered} W_1(\alpha,\alpha) = a, \quad W_3(\alpha,\alpha) = a^{-1}, \quad \sum_{x\in X} W_2(\alpha,x) = \sum_{x\in X} W_2(x,\alpha) = Da^{-1}, \\ \sum_{x\in X} W_4(\alpha,x) = \sum_{x\in X} W_4(x,\alpha) = Da, \end{gathered} \tag{5}$$

$$\begin{gathered} W_1(\alpha,\beta)W_3(\beta,\alpha) = 1, \quad \sum_{x\in X} W_1(\alpha,x)W_3(x,\beta) = |X|\delta_{\alpha,\beta}, \\ W_2(\alpha,\beta)W_4(\beta,\alpha) = 1, \quad \sum_{x\in X} W_2(\alpha,x)W_4(x,\beta) = |X|\delta_{\alpha,\beta}, \end{gathered} \tag{6}$$

$$\begin{aligned} &\sum_{x\in X} W_2(\alpha,x)W_2(\beta,\ x)W_4(x,\gamma) = DW_1(\beta,\alpha)W_3(\alpha,\gamma)W_3(\gamma,\beta) \\ &\quad = \sum_{x\in X} W_2(x,\alpha)W_2(x,\beta)W_4(\gamma,x) = DW_1(\alpha,\beta)W_3(\beta,\gamma)W_3(\gamma,\alpha). \end{aligned} \tag{7}$$

We shall take as our definition of 4-*weight spin model* a 5-tuple (X, W_1, W_2, W_3, W_4), where X is a finite non-empty set and W_i, $i = 1, \ldots, 4$ are matrices in M_X satisfying (5), (6), (7) for some a, D in $\mathbf{C} - \{0\}$ with $D^2 = |X|$.

Remark This is only one among many possible equivalent definitions, see [1].

Given a finite non-empty set X and W^+, W^- in M_X, one can show that (X, W^+, W^-) is a 2-weight spin model with loop variable D if and only if (X, W^+, W^+, W^-, W^-) is a 4-weight spin model with loop variable D (see [1]). In this case the two spin models have the same associated link invariant and can be identified.

For $\lambda \in \mathbf{C} - \{0\}$, it is clear from (5), (6), (7) that if (X, W_1, W_2, W_3, W_4) is a 4-weight spin model, then $(X, \lambda W_1, \lambda^{-1}W_2, \lambda^{-1}W_3, \lambda W_4)$ is also a 4-weight spin model. These two

4-weight spin models will be said to be *proportional*, and it is easy to see that they yield the same link invariant.

We shall need more general transformations of 4-weight spin models which preserve the associated link invariant. The following theorem sums up some results of [15] (see also [7]). In the statement of this theorem, X is a finite non-empty set and W_i and W_i' ($i = 1, \ldots, 4$) are matrices in M_X.

Theorem A *Let (X, W_1, W_2, W_3, W_4) be a 4-weight spin model with loop variable D.*

(i) *$(X, W_1', W_2, W_3', W_4)$ is a 4-weight spin model with loop variable D if and only if there exists an invertible diagonal matrix Δ such that $W_1' = \Delta W_1 \Delta^{-1}$, $W_3' = \Delta W_3 \Delta^{-1}$.*

(ii) *$(X, W_1, W_2', W_3, W_4')$ is a 4-weight spin model with loop variable D if and only if there exists a permutation matrix P such that $W_2^{-1} P W_2$ is also a permutation matrix and $W_2' = P W_2$, $W_4' = W_4 {}^tP$.*

(iii) *Two 4-weight spin models related as in* (i) *or* (ii) *yield the same link invariant.*

The transformation relating the two 4-weight spin models in (i) (respectively (ii)) of Theorem A is called an *odd* (respectively *even*) *gauge transformation*. Two 4-weight spin models which, up to proportionality, are related by odd or even gauge transformations will be said to be *gauge equivalent*. Thus gauge equivalent 4-weight spin models have the same associated link invariant.

2.2. Spin models and Bose-Mesner algebras

A *Bose-Mesner algebra* on a finite non-empty set X is a commutative subalgebra of M_X which contains the identity I, which is also an algebra under the Hadamard (that is, entry-wise) product $(A, B) \to A \circ B$ with identity J (the all-one matrix), and which is closed under the transposition operation $A \to {}^tA$. It can easily be shown that Bose-Mesner algebras and (commutative) association schemes are equivalent concepts (see [6] Theorem 2.6.1 which is easily extended to the non-symmetric case). We shall only work here with the concept of Bose-Mesner algebra (note that for convenience we have incorporated the commutativity property of the ordinary matrix product into our definition). The reader is referred to [4] for details on material reviewed in the rest of the section.

Every Bose-Mesner algebra $\mathcal{A}$ has a basis of Hadamard idempotents $\{A_i,\ i = 0, \ldots, d\}$ satisfying

$$A_i \neq 0, \quad A_i \circ A_j = \delta_{i,j} A_i, \tag{8}$$

$$\sum_{i=0}^{d} A_i = J. \tag{9}$$

It is easy to show that I belongs to this basis and, as usual, we take $A_0 = I$. Similarly, $\mathcal{A}$ has a basis of ordinary idempotents $\{E_i,\ i = 0, \ldots, d\}$ satisfying

$$E_i \neq 0, \quad E_i E_j = \delta_{i,j} E_i, \tag{10}$$

$$\sum_{i=0}^{d} E_i = I. \tag{11}$$

It is easy to show that $|X|^{-1}J$ belongs to this basis and as usual we take $E_0 = |X|^{-1}J$. One can also show that

$$ {}^tE_i = \bar{E}_i \ (i = 0, \dots, d). \tag{12}$$

A *duality* of $\mathcal{A}$ is a linear map $\Psi : \mathcal{A} \to \mathcal{A}$ such that

$$\Psi^2(A) = |X|\, {}^tA \quad \text{for } A \in \mathcal{A}, \tag{13}$$

$$\Psi(AB) = \Psi(A) \circ \Psi(B) \quad \text{for } A, B \in \mathcal{A}. \tag{14}$$

It follows easily that

$$\Psi(A \circ B) = |X|^{-1}\Psi(A)\Psi(B) \quad \text{for } A, B \in \mathcal{A}, \tag{15}$$

$$\Psi(I) = J, \quad \Psi(J) = |X|I, \tag{16}$$

$$ {}^t\Psi(A) = \Psi({}^tA) \quad \text{for } A \in \mathcal{A}. \tag{17}$$

The main result relating spin models to Bose-Mesner algebras is the following. Here the form of Ψ is obtained from [16], Theorem 11 by using the 2-weight spin model $(X, {}^tW^-, {}^tW^+)$ instead of (X, W^+, W^-), this being allowed by Proposition 2 of [20]. See also [13, 22].

Theorem B *Let (X, W^+, W^-) be a 2-weight spin model with modulus a. Then there is a Bose-Mesner algebra $\mathcal{A}$ on X containing W^+, W^- with duality Ψ given by*

$$\Psi(A) = a\, {}^tW^- \circ (W^+(W^- \circ A))$$

for every A in $\mathcal{A}$.

Remarks (i) We may rewrite (2), (3) as

$$\begin{aligned} &I \circ W^+ = aI, \quad I \circ W^- = a^{-1}I, \quad W^+J = JW^+ = Da^{-1}J, \\ &\quad W^-J = JW^- = DaJ, \end{aligned} \tag{2'}$$

$$W^+ \circ {}^tW^- = J, \quad W^+W^- = |X|I. \tag{3'}$$

(ii) Using (3') and (2'), one easily sees that the duality Ψ given by Theorem B satisfies $\Psi({}^tW^+) = D\, {}^tW^-$, or equivalently $\Psi(W^+) = DW^-$ by (17). In addition, $\Psi(W^-) = D\, {}^tW^+$ by (13).

3. Some general results on 2-weight spin models

3.1. *Permutation matrices and dual-permutation matrices*

A matrix R in M_X is a permutation matrix if and only if $R \circ R = R$ and $R\, {}^tR = I$. The set of permutation matrices which belong to a Bose-Mesner algebra $\mathcal{A}$ obviously forms an abelian group $\mathcal{A}_1$ under ordinary matrix product. Expressing such a matrix in the basis of Hadamard idempotents of $\mathcal{A}$ we see that all coefficients, except one equal to 1, must be zero, and hence $\mathcal{A}_1 \subseteq \{A_i, i = 0, \ldots, d\}$. It is well possible that $\mathcal{A}_1 = \{I\}$.

On the other hand, the equality $\mathcal{A}_1 = \{A_i,\ i = 0, \ldots, d\}$ occurs in the following situation. Let X be an abelian group written additively. For every i in X, define the matrix A_i in M_X by the identity $A_i(x, y) = \delta_{i,y-x}$. Then it is easy to check that $\{A_i,\ i \in X\}$ is the basis of Hadamard idempotents of a Bose-Mesner algebra on X, called the *Bose-Mesner algebra of the abelian group* X.

Let A_i be an element of order $k > 1$ in $\mathcal{A}_1$. Since a permutation represented by a matrix in $\{A_j,\ j = 1, \ldots, d\}$ has no fixed points by (8), all the cycles of the permutation represented by A_i have length k. Hence we may establish a bijection between X and $\{1, \ldots, k\} \times \{1, \ldots, \ell\}$, where $\ell = |X|/k$, so that $A_i((r, s),\ (t, u)) = 1$ iff $s = u$ and $t \equiv r + 1 \pmod k$ $(r\ , t \in \{1, \ldots, k\}, s, u \in \{1, \ldots, \ell\})$.

Let us now consider the dual concepts. A matrix F in M_X is a *dual-permutation matrix* if $|X|^{-1}F^2 = F$ and $F \circ {}^tF = J$. So if $\mathcal{A}$ is a Bose-Mesner algebra with duality Ψ and if $R \in \mathcal{A}_1$, then $\Psi(R)$ is a dual-permutation matrix. Indeed, applying Ψ to $R \circ R = R$ and using (15) we obtain $|X|^{-1}\Psi(R)^2 = \Psi(R)$; applying Ψ to $R\,{}^tR = I$ and using (14), (16), (17) we obtain $\Psi(R) \circ {}^t\Psi(R) = J$.

Proposition 1 *The following properties are equivalent for a matrix F in M_X:*
- (i) *F is a dual-permutation matrix*,
- (ii) *$|X|^{-1}F$ is a rank 1 idempotent with constant diagonal*,
- (iii) *There is an invertible diagonal matrix Δ in M_X such that $F = \Delta J \Delta^{-1}$.*

Proof: (i) $\Rightarrow$ (ii): Since $|X|^{-1}F^2 = F$, $|X|^{-1}F$ is an idempotent. The rank of this idempotent is $|X|^{-1}$ Trace$(F) = |X|^{-1}\sum_{x\in X} F(x, x)$. Since $F \circ {}^tF = J$, $F(x, x)^2 = 1$ and $F \neq 0$. It follows that $F(x, x) = 1$ for every x in X and $|X|^{-1}F$ has rank 1.

(ii) $\Rightarrow$ (iii): Since $|X|^{-1}F$ has rank 1, there exists functions f, g from X to **C** such that $|X|^{-1}F(x, y) = f(x)g(y)$ for all $x, y \in X$. The constant diagonal element $f(x)g(x)$ $(x \in X)$ of the matrix $|X|^{-1}F$ is $|X|^{-1}$ Trace$(|X|^{-1}F) = |X|^{-1}$, so $f(x) \neq 0$ for all $x \in X$ and $F(x, y) = f(x)f(y)^{-1}$ for all $x, y \in X$. Take $\Delta(x, y) = \delta_{x,y} f(x)$ for all $x, y \in X$.

(iii) $\Rightarrow$ (i): $|X|^{-1}F^2 = F$ is immediate, $F \circ {}^tF = J$ follows from $F(x, y) = \Delta(x, x)$ $\Delta(y, y)^{-1}$. □

Clearly the set of dual-permutation matrices in M_X forms an abelian group under Hadamard product (the identity element is J and the inverse of F is ${}^tF$). Hence the set of dual-permutation matrices which belong to a Bose-Mesner algebra $\mathcal{A}$ form an abelian group $\mathcal{A}'_1$ under Hadamard product. For F in $\mathcal{A}'_1$, let us express $|X|^{-1}F$ in the basis of ordinary idempotents of $\mathcal{A}$. By (ii) of Proposition 1, taking the trace we see that all coefficients, except one equal to 1, must be zero, and hence $\mathcal{A}'_1 \subseteq \{|X|E_i, i = 0, \ldots, d\}$. Again we may well have $\mathcal{A}'_1 = \{J\}$. On the other hand if $\mathcal{A}$ is the Bose-Mesner algebra of an abelian group X, then the equality $\mathcal{A}'_1 = \{|X|E_i, i = 0, \ldots, d\}$ holds, since by (11) each E_i has rank 1 since $|X| = d + 1$.

Let $|X|E_i$ be an element of order $k > 1$ in $\mathcal{A}'_1$. So k is the smallest positive integer ℓ such that $(|X|E_i(x, y))^\ell = 1$ for all $x, y \in X$. It follows that $\{|X|E_i(x, y), x, y \in X\} =$

$\{\eta^u, u \in U\}$ where $\eta = \exp(2\pi\sqrt{-1}/k)$ and $U \subseteq \{0, \ldots, k-1\}$ contains a non-zero element which is not a proper divisor of k.

By (10), $JE_j = 0$ for every element $|X|E_j \neq J$ of the subgroup of $\mathcal{A}'_1$ generated by $|X|E_i$. Expressing $|X|E_i$ in the basis of Hadamard idempotents, we see that there exist positive integers p_u ($u \in U$) such that every column of $|X|E_i$ takes the value η^u exactly p_u many times. Then $\sum_{u\in U} p_u(\eta^u)^v = 0$ ($v = 1, \ldots, k-1$). Taking $p_u = 0$ when $u \in \{0, \ldots, k\} - U$, we may write $\sum_{u=0}^{k-1} p_u\eta^{uv} = 0$ ($v = 1, \ldots, k-1$). Thus the vector $(p_0, \ldots, p_{k-1})$ is orthogonal to every vector representing a non-trivial character of $\mathbf{Z}/k\mathbf{Z}$. Hence this vector is a multiple of the trivial character. It follows that $(p_0, \ldots, p_{k-1}) = (|X|/k, \ldots, |X|/k)$.

Now let Δ be an invertible diagonal matrix such that $|X|E_i = \Delta J \Delta^{-1}$. Thus $|X|E_i(x, y) = \Delta(x, x)\Delta(y, y)^{-1}$ for every x, y in X. Let us fix $y \in X$ and assume without loss of generality that $\Delta(y, y) = 1$. Then we see that the diagonal values of Δ are the powers of η, each repeated $|X|/k$ times. In other words, we may establish a bijection between X and $\{1, \ldots, k\} \times \{1, \ldots, \ell\}$, where $\ell = |X|/k$, so that $\Delta((r, s), (r, s)) = \eta^{r-1}$ ($r \in \{1, \ldots, k\}$, $s \in \{1, \ldots, \ell\}$). Then $|X|E_i((r, s), (t, u)) = \eta^{r-t}$ ($r, t \in \{1, \ldots, k\}$, $s, u \in \{1, \ldots, \ell\}$). Note that this formula is compatible with (12).

Finally we observe that if the Bose-Mesner algebra $\mathcal{A}$ has a duality Ψ, then Ψ is a group isomorphism between $\mathcal{A}_1$ and $\mathcal{A}'_1$. Indeed we have already shown that $\Psi(\mathcal{A}_1) \subseteq \mathcal{A}'_1$. Conversely, let F belong to $\mathcal{A}'_1$. By (13) and (17), we may write $F = |X|^{-1}\Psi^2({}^tF) = \Psi(|X|^{-1}\,{}^t\Psi(F))$. Let us show that $R = |X|^{-1}\,{}^t\Psi(F)$ belongs to $\mathcal{A}_1$. First, $R \circ R = |X|^{-2}\,{}^t\Psi(F) \circ {}^t\Psi(F)$; applying Ψ to $|X|^{-1}F^2 = F$, using (14) and transposing, we get $|X|^{-1}\,{}^t\Psi(F) \circ {}^t\Psi(F) = {}^t\Psi(F)$ and hence $R \circ R = R$. Second, $R\,{}^tR = |X|^{-2}\,{}^t\Psi(F)\Psi(F)$; applying Ψ to $F \circ {}^tF = J$, using (15), (16), (17), we obtain $|X|^{-1}\Psi(F)\,{}^t\Psi(F) = |X|I$, so that $R\,{}^tR = I$. Thus we have shown that Ψ is a bijection from $\mathcal{A}_1$ to $\mathcal{A}'_1$. This bijection is a group isomorphism by (14).

3.2. *The index of a 2-weight spin model*

Let (X, W^+, W^-) be a two-weight spin model with modulus a and let $\mathcal{A}$ be the Bose-Mesner algebra introduced in Theorem B. Exchanging α and β in (4), we obtain:

$$\sum_{x\in X} W^+(\alpha, x)W^+(\beta, x)W^-(x, \gamma) = DW^+(\beta, \alpha)W^-(\alpha, \gamma)W^-(\gamma, \beta). \tag{18}$$

Hence, comparing (4) and (18),

$$W^+(\alpha, \beta)W^-(\beta, \gamma)W^-(\gamma, \alpha) = W^+(\beta, \alpha)W^-(\alpha, \gamma)W^-(\gamma, \beta).$$

Using (3), we obtain

$$\frac{W^+(\alpha, \beta)}{W^+(\beta, \alpha)} = \frac{W^+(\alpha, \gamma)}{W^+(\gamma, \alpha)} \cdot \frac{W^+(\gamma, \beta)}{W^+(\beta, \gamma)}, \quad \text{for every } \alpha, \beta, \gamma \in X.$$

Fixing γ and defining the diagonal matrix Δ in M_X by

$$\Delta(x,x) = \frac{W^+(x,\gamma)}{W^+(\gamma,x)},$$

this becomes

$$\frac{W^+(\alpha,\beta)}{W^+(\beta,\alpha)} = \frac{\Delta(\alpha,\alpha)}{\Delta(\beta,\beta)},$$

or equivalently

$$W^+(\alpha,\beta)W^-(\alpha,\beta) = \frac{\Delta(\alpha,\alpha)}{\Delta(\beta,\beta)}.$$

Hence, by Proposition 1, $W^+ \circ W^-$ is a dual-permutation matrix. Let Ψ be the duality given by Theorem B. By Remark (ii) following Theorem B and (15), $\Psi(W^+ \circ W^-) = |X|^{-1}\Psi(W^+)\Psi(W^-) = W^- \, {}^tW^+$. Also, since $\Psi(\mathcal{A}_1) = \mathcal{A}'_1$, $\Psi(\mathcal{A}'_1) = \{\Psi^2(R),\ R \in \mathcal{A}_1\} = \{|X|\, {}^tR,\ R \in \mathcal{A}_1\} = \{|X|R,\ R \in \mathcal{A}_1\}$ by (13). Hence we have proved the following result.

Proposition 2 *$W^+ \circ W^- \in \mathcal{A}'_1$ and $|X|^{-1}\, {}^tW^+W^- \in \mathcal{A}_1$.*

We note that $\Psi(\{E_i,\ i = 0,\dots,d\}) = \{A_i,\ i = 0,\dots,d\}$. We shall choose the indices so that $\Psi(E_i) = A_i$, $i = 0,\dots,d$. We shall write $W^+ \circ W^- = |X|E_s$, $s \in \{0,\dots,d\}$, and consequently ${}^tW^+W^- = |X|A_s$.

Since Ψ is a group isomorphism from $\mathcal{A}_1$ to $\mathcal{A}'_1$, the order of the element $|X|^{-1}\, {}^tW^+W^-$ of the group $\mathcal{A}_1$ is equal to the order of the element

$$|X|^{-1}\Psi({}^tW^+W^-) = |X|^{-1}\Psi({}^tW^+) \circ \Psi(W^-) = {}^tW^- \circ {}^tW^+$$

of the group $\mathcal{A}'_1$, which is equal to the order of $W^+ \circ W^-$. This positive integer will be denoted by m and will be called the *index* of the 2-weight spin model (X, W^+, W^-). Note that a 2-weight spin model has index 1 if and only if it is symmetric, and that $m \le |\mathcal{A}_1| = |\mathcal{A}'_1| \le d+1$.

Remarks (i) For 2-weight spin models $(X_i,\ W_i^+,\ W_i^-)$, $i = 1, 2$, their tensor product $(X,\ W^+,\ W^-)$ is defined by $X = X_1 \times X_2$ and $W^\pm = W_1^\pm \otimes W_2^\pm$, where $A \otimes B$ denotes the Kronecker product: $(A \otimes B)((x_1,x_2),(y_1,y_2)) = A(x_1,y_1)B(x_2,y_2)$ for $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2$. As easily shown, $(X,\ W^+,\ W^-)$ is a 2-weight spin model. The index m of (X, W^+, W^-) is given by the least common multiple of m_1 and m_2, where m_i denotes the index of $(X_i,\ W_i^+,\ W_i^-)$, $i = 1, 2$. This fact can be shown by computing the order of $W^+ \circ W^-$ with respect to Hadamard product.

(ii) In particular, the index is invariant under taking tensor product with any symmetric 2-weight spin model.

Proposition 3 (i) *There is a partition of X into m parts $X_1, \dots, X_m$ of equal sizes such that $W^+(x, y) = \eta^{i-j} W^+(y, x)$ for all $i, j \in \{1, \dots, m\}$ and $x \in X_i$, $y \in X_j$, where $\eta = \exp(2\pi\sqrt{-1}/m)$.*

(ii) *Let $x, y \in X$ be such that $A_s(x, y) = 1$. Then $W^+(z, x) = W^+(y, z)$ for all z in X.*

(iii) *Write $W^+ = \sum_{i=0}^{d} t_i A_i$ and ${}^tA_s = A_{s'}$. Then $t_{s'} = t_0$.*

Proof: (i) Follows immediately from the analysis at the end of section 3.1 (applied with $i = s$ and $k = m$) and from the equality $W^+ \circ W^- = |X|E_s$.

(ii) Let σ be the permutation of X such that for $x, y \in X$, $A_s(x, y) = 1$ iff $x = \sigma(y)$. We want to show that for all $y, z \in X$, $W^+(z, \sigma(y)) = W^+(y, z)$. We note that

$$\begin{aligned}({}^tA_s\, {}^tW^+)(y, z) &= \sum_{u\in X} {}^tA_s(y, u)\, {}^tW^+(u, z)\\ &= \sum_{u\in X} A_s(u, y) W^+(z, u)\\ &= \sum_{u\in X} \delta_{u,\sigma(y)} W^+(z, u) = W^+(z, \sigma(y)).\end{aligned}$$

On the other hand, recall that $|X|A_s = {}^tW^+ W^-$. Using (3') (see Remark (i) following Theorem B), we get

$$|X|\, {}^tA_s\, {}^tW^+ = {}^tW^- W^+\, {}^tW^+ = W^+\, {}^tW^-\, {}^tW^+ = W^+(|X|I),$$

that is, ${}^tA_s\, {}^tW^+ = W^+$. The result follows.

(iii) Take $z = y$ in (ii): $W^+(y, x) = W^+(y, y)$ whenever $A_s(x, y) = 1$. From the equality $W^+ = \sum_{i=0}^{d} t_i A_i$, $W^+(y, x) = t_{s'}$ whenever $A_s(x, y) = 1$, and $W^+(y, y) = t_0$ for all $y \in X$. □

It is clear that the partition $X_1, \dots, X_m$ in (i) above is uniquely determined up to ordering. In particular, such a partition characterizes the index m. The significance of Proposition (iii) is that in a non-symmetric 2-weight spin model (X, W^+, W^-), the value which appears in the diagonal of W^+ also appears elsewhere in this matrix.

Part of the following result also appears in [15], Proposition 12.

Proposition 4 *Let R be an element of $\mathcal{A}_1$ and let $F = \Psi(R) \in \mathcal{A}_1'$. Then RW^+ and $F \circ W^+$ are scalar multiples of one another. Write $W'^+ = \lambda^{-1} RW^+ = \lambda F \circ W^+$ for some non-zero complex number λ, and define W'^- by the equality $W'^+ \circ {}^tW'^- = J$. Then*

(i) *(X, W'^+, W'^-) is a 2-weight spin model gauge equivalent to (X, W^+, W^-).*

(ii) *The index of (X, W'^+, W'^-) is the order of $A_s({}^tR)^2$ in $\mathcal{A}_1$.*

(iii) *(X, W'^+, W'^-) can be chosen symmetric iff A_s is a square in $\mathcal{A}_1$ or equivalently $|X|E_s$ is a square in $\mathcal{A}_1'$. In this case the link invariant associated with (X, W^+, W^-) depends only trivially on the link orientation.*

Proof: From Theorem B, $F = \Psi(R) = a\,{}^tW^- \circ (W^+(W^- \circ R))$. Since $R \in \{A_i,\ i = 0, \ldots, d\}$, there is a complex number μ such that $W^- \circ R = \mu R$. Note that $\mu \neq 0$ since W^- has non-zero entries by (3). Then $F = a\,{}^tW^- \circ (W^+(\mu R)) = \mu a\,{}^tW^- \circ (W^+R)$ and by (3'), $F \circ W^+ = \mu a W^+ R = \mu a R W^+$.

So we may write $W'^+ = \lambda^{-1} R W^+ = \lambda F \circ W^+$ for $\lambda^{-2} = \mu a \in \mathbf{C} - \{0\}$. Let Δ be an invertible diagonal matrix such that $F(x, y) = \Delta(x, x)\Delta(y, y)^{-1}$ for all $x, y \in X$ (see Proposition 1). Then $F \circ W^+ = \Delta W^+ \Delta^{-1}$. It follows from (3') that $({}^tF \circ {}^tW^-) \circ (F \circ W^+) = J$ and hence ${}^tW'^- = \lambda^{-1}({}^tF \circ {}^tW^-)$, that is $W'^- = \lambda^{-1}(F \circ W^-) = \lambda^{-1}\Delta W^- \Delta^{-1}$.

We have $F \circ W^+ = \Delta W^+ \Delta^{-1} = \lambda^{-2} R W^+$. Taking the inverses and using (3') we obtain $\Delta W^- \Delta^{-1} = \lambda^2 W^-\,{}^tR$, and hence $W'^- = \lambda W^-\,{}^tR$.

Let us consider the 4-weight spin model (X, W^+, W^+, W^-, W^-). Then (see Theorem A),

$$(X, \Delta W^+ \Delta^{-1}, W^+, \Delta W^- \Delta^{-1}, W^-) = (X, \lambda^{-1} W'^+, W^+, \lambda W'^-, W^-)$$

is a 4-weight spin model obtained from it by an odd gauge transformation. Noting that $(W^+)^{-1} R W^+ = R$ is a permutation matrix, we now perform an even gauge transformation to obtain a 4-weight spin model

$$(X, \lambda^{-1} W'^+, R W^+, \lambda W'^-, W^-\,{}^tR) = (X, \lambda^{-1} W'^+, \lambda W'^+, \lambda W'^-, \lambda^{-1} W'^-),$$

which is proportional to $(X, W'^+, W'^+, W'^-, W'^-)$. Hence (X, W'^+, W'^-) is a 2-weight spin model gauge equivalent to (X, W^+, W^-).

Finally,

$$\begin{aligned} |X|^{-1}\,{}^tW'^+ W'^- &= |X|^{-1}(\lambda^{-1}\,{}^tW^+\,{}^tR)(\lambda W^-\,{}^tR) \\ &= (|X|^{-1}\,{}^tW^+ W^-)({}^tR)^2 \\ &= A_s({}^tR)^2 \end{aligned}$$

and the index of (X, W'^+, W'^-) is the order of this element of $\mathcal{A}_1$. □

A 2-weight spin model will be said to be *quasi-symmetric* if A_s is a square in $\mathcal{A}_1$ or equivalently $|X|E_s$ is a square in $\mathcal{A}'_1$. Thus the link invariant associated with a quasi-symmetric 2-weight spin model depends only trivially on the link orientation.

Proposition 5 (i) *Every* 2-*weight spin model is gauge equivalent to a* 2-*weight spin model whose index is a power of* 2.

(ii) *A* 2-*weight spin model of odd index is quasi-symmetric. In particular*, *a* 2-*weight spin model defined on a set* X *of odd size is quasi-symmetric.*

Proof: (i) Write $m = (2p + 1)2^k$ $(p \geq 0, k \geq 0)$. Then $A_s^{2p+1} = A_s(A_s^p)^2$ has order 2^k and by Proposition 4 we obtain a 2-weight spin model of index 2^k which is gauge equivalent to (X, W^+, W^-).

(ii) In particular if m is odd, $k = 0$ and we obtain a symmetric 2-weight spin model gauge equivalent to (X, W^+, W^-). Since m divides $|X|$ by Proposition 3 (i), if $|X|$ is odd then m is also odd. □

When X is an abelian group with the Bose-Mesner algebra $\mathcal{A}$, there exists a 2-weight spin model satisfying the situation of Theorem B (see [1, 3]). The following result also appears as part of Proposition 13 of [15] with a different proof.

Proposition 6 *If $\mathcal{A}$ is isomorphic to the Bose-Mesner algebra of an abelian group, then the 2-weight spin model (X, W^+, W^-) is quasi-symmetric.*

Proof: We have (identifying X and $\{0, \ldots, d\}$) $\mathcal{A}_1 = \{A_i,\ i = 0, \ldots, d\}$ and $\mathcal{A}_1' = \{|X|E_i,\ i = 0, \ldots, d\}$. Write $A_j = \sum_{i=0}^{d} P_{ij} E_i\ \ (j = 0, \ldots, d)$. Let $A_j,\ A_k,\ A_\ell$ be three elements of $\mathcal{A}_1$ such that $A_j A_k = A_\ell$. Then, for $i \in \{0, \ldots, d\}$, $P_{i\ell} E_i = E_i A_\ell = E_i A_j A_k = P_{ij} E_i A_k = P_{ij} P_{ik} E_i$, so that $P_{i\ell} = P_{ij} P_{ik}$. Hence the map χ_i from $\mathcal{A}_1$ to $\mathbf{C}$ defined by $\chi_i(A_j) = P_{ij}$ for every A_j in $\mathcal{A}_1$ is a character of $\mathcal{A}_1$. Let $\mathcal{A}_1^*$ be the group of characters of $\mathcal{A}_1$ and let φ be the mapping from $\mathcal{A}_1'$ to $\mathcal{A}_1^*$ defined by $\varphi(|X|E_i) = \chi_i$ for $i = 0, \ldots, d$. The matrix with entries P_{ij} has no repeated rows since it is a matrix of change of basis from $\{A_i,\ i = 0, \ldots, d\}$ to $\{E_i,\ i = 0, \ldots, d\}$, and hence φ is injective. Since $|\mathcal{A}_1'| = |\mathcal{A}_1| = |\mathcal{A}_1^*|$, φ is a bijection. Moreover the bijection φ is a group isomorphism. Indeed let $|X|E_i$, $|X|E_j$, $|X|E_k$ belong to $\mathcal{A}_1'$ with $|X|E_k = |X|E_i \circ |X|E_j$. For A_ℓ in $\mathcal{A}_1$, $\chi_k(A_\ell)E_k = E_k A_\ell = |X|(E_i \circ E_j)A_\ell = |X|(E_i A_\ell \circ E_j A_\ell)$ (since A_ℓ is a permutation matrix) $= |X|(\chi_i(A_\ell)E_i \circ \chi_j(A_\ell)E_j) = \chi_i(A_\ell)\chi_j(A_\ell)E_k$. Hence $\chi_k = \chi_i \chi_j$.

Thus it will be enough to prove that χ_s is a square in $\mathcal{A}_1^*$. Taking the trace in the equality $E_s A_i = \chi_s(A_i) E_s$ we obtain $\chi_s(A_i) = \mathrm{Trace}(E_s A_i) = \sum(E_s \circ {}^t A_i)$, where $\sum$ denotes the sum of entries of a matrix.

Write $W^+ = \sum_{j=0}^{d} t_j A_j$. Then by (3'), $W^- = \sum_{j=0}^{d} t_j^{-1}\, {}^t A_j$. Let $A_i \in \mathcal{A}_1$ with $A_i^2 = I$, or equivalently ${}^t A_i = A_i$. Then the coefficient of $|X|E_s = W^+ \circ W^-$ for ${}^t A_i$ is 1, and hence $E_s \circ {}^t A_i = |X|^{-1}\, {}^t A_i$. It follows that $\chi_s(A_i) = \sum |X|^{-1}\, {}^t A_i = 1$. Let π be the group homomorphism from $\mathcal{A}_1$ to itself defined by $\pi(A_j) = A_j^2$ for all A_j in $\mathcal{A}_1$. Thus χ_s takes the value 1 on $\mathrm{Ker}\,\pi$. Let π^* be the group homomorphism from $\mathcal{A}_1^*$ to itself defined by $\pi^*(\chi) = \chi^2$ for all χ in $\mathcal{A}_1^*$. Clearly

$$\mathrm{Im}\,\pi^* \subseteq \{\chi \in \mathcal{A}_1^* \mid \chi(\mathrm{Ker}\,\pi) = \{1\}\}$$

since $\chi^2(A_i) = \chi(A_i^2) = \chi(I) = 1$ for $\chi \in \mathcal{A}_1^*$ and $A_i \in \mathrm{Ker}\,\pi$.

$\{\chi \in \mathcal{A}_1^* \mid \chi(\mathrm{Ker}\,\pi) = \{1\}\}$ is isomorphic to the group of characters of $\mathcal{A}_1/\mathrm{Ker}\,\pi$ and hence has size $|\mathcal{A}_1|/|\mathrm{Ker}\,\pi| = |\mathrm{Im}\,\pi| = |\mathrm{Im}\,\pi^*|$ (since $\mathcal{A}_1$ and $\mathcal{A}_1^*$ are isomorphic). Hence $\mathrm{Im}\,\pi^* = \{\chi \in \mathcal{A}_1^* \mid \chi(\mathrm{Ker}\,\pi) = \{1\}\}$ and $\chi_s \in \mathrm{Im}\,\pi^*$, that is, χ_s is a square in $\mathcal{A}_1^*$. □

We shall now look for non quasi-symmetric spin models. For this purpose, in view of Proposition 5, we shall study the simplest case of even index, namely the case of index 2.

4. General form of 2-weight spin models of index 2

Proposition 7 *Let* (X, W^+, W^-) *be a* 2*-weight spin model of index* 2*. Then* X *can be ordered and split into* 4 *blocks of equal sizes so that* W^+ *takes the following form*:

$$W^+ = \begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ -{}^tB & {}^tB & C & C \\ {}^tB & -{}^tB & C & C \end{pmatrix} \quad \textit{with } A, C \textit{ symmetric}.$$

Proof: We first split X into two blocks X_1, X_2 of equal sizes so that $W^+(x, y) = (-1)^{i-j} W^+(y, x)$ for all $i, j \in \{1, 2\}$ and $x \in X_i$, $y \in X_j$ (Proposition 3 (i)). We order X so that

$$|X|E_s = W^+ \circ W^- = \begin{array}{c} \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{cc} X_1 & X_2 \end{array} \\ \begin{pmatrix} J & -J \\ -J & J \end{pmatrix} \end{array}.$$

Write $W^+ = \sum_{i=0}^{d} t_i A_i$. Since $A_s^2 = I$ and hence ${}^tA_s = A_s$, if $A_s(x, y) = 1$ for $x, y \in X$, $W^+(x, y) = {}^tW^+(x, y) = t_s$ and hence $|X|E_s(x, y) = 1$, so that $x \in X_1$, $y \in X_1$ or $x \in X_2$, $y \in X_2$. Since all cycles of the permutation represented by A_s have length 2 (Section 3.1), we may split X_1 (respectively: X_2) into two blocks of equal sizes X_{11}, X_{12} (respectively: X_{21}, X_{22}) so that if (x, y) is such a cycle (i.e. $A_s(x, y) = 1$), x and y belong to different blocks. We order X so that

$$A_s = \begin{array}{c} \\ X_{11} \\ X_{12} \\ X_{21} \\ X_{22} \end{array} \begin{array}{c} \begin{array}{cccc} X_{11} & X_{12} & X_{21} & X_{22} \end{array} \\ \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix} \end{array}.$$

Now write

$$W^+ = \begin{array}{c} \\ X_{11} \\ X_{12} \\ X_{21} \\ X_{22} \end{array} \begin{array}{c} \begin{array}{cccc} X_{11} & X_{12} & X_{21} & X_{22} \end{array} \\ \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \end{array}.$$

The equality $W^+ = (|X|E_s) \circ {}^tW^+$ becomes

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} = \begin{pmatrix} {}^tA_{11} & {}^tA_{21} & -{}^tA_{31} & -{}^tA_{41} \\ {}^tA_{12} & {}^tA_{22} & -{}^tA_{32} & -{}^tA_{42} \\ -{}^tA_{13} & -{}^tA_{23} & {}^tA_{33} & {}^tA_{43} \\ -{}^tA_{14} & -{}^tA_{24} & {}^tA_{34} & {}^tA_{44} \end{pmatrix} \tag{19}$$

and the equality ${}^tW^+ = A_s W^+$ becomes

$$\begin{pmatrix} {}^tA_{11} & {}^tA_{21} & {}^tA_{31} & {}^tA_{41} \\ {}^tA_{12} & {}^tA_{22} & {}^tA_{32} & {}^tA_{42} \\ {}^tA_{13} & {}^tA_{23} & {}^tA_{33} & {}^tA_{43} \\ {}^tA_{14} & {}^tA_{24} & {}^tA_{34} & {}^tA_{44} \end{pmatrix} = \begin{pmatrix} A_{21} & A_{22} & A_{23} & A_{24} \\ A_{11} & A_{12} & A_{13} & A_{14} \\ A_{41} & A_{42} & A_{43} & A_{44} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{pmatrix}. \tag{20}$$

Combining (19) and (20) we obtain

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} {}^tA_{11} & {}^tA_{21} \\ {}^tA_{12} & {}^tA_{22} \end{pmatrix} = \begin{pmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \end{pmatrix}.$$

Hence ${}^tA_{11} = A_{11} = A_{21}$, ${}^tA_{22} = A_{22} = A_{12}$ and $A_{12} = {}^tA_{21}$. It follows that the above matrix takes the form

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

with A symmetric. Similarly,

$$\begin{pmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{pmatrix} = \begin{pmatrix} {}^tA_{33} & {}^tA_{43} \\ {}^tA_{34} & {}^tA_{44} \end{pmatrix} = \begin{pmatrix} A_{43} & A_{44} \\ A_{33} & A_{34} \end{pmatrix}$$

shows that this matrix is of the form

$$\begin{pmatrix} C & C \\ C & C \end{pmatrix}$$

with C symmetric. We have also

$$\begin{pmatrix} A_{13} & A_{14} \\ A_{23} & A_{24} \end{pmatrix} = \begin{pmatrix} -{}^tA_{31} & -{}^tA_{41} \\ -{}^tA_{32} & -{}^tA_{42} \end{pmatrix} = \begin{pmatrix} -A_{23} & -A_{24} \\ -A_{13} & -A_{14} \end{pmatrix}$$

and

$$\begin{pmatrix} A_{31} & A_{32} \\ A_{41} & A_{42} \end{pmatrix} = \begin{pmatrix} -{}^tA_{13} & -{}^tA_{23} \\ -{}^tA_{14} & -{}^tA_{24} \end{pmatrix} = \begin{pmatrix} -A_{41} & -A_{42} \\ -A_{31} & -A_{32} \end{pmatrix}.$$

The last equality yields by transposition

$$\begin{pmatrix} A_{13} & A_{14} \\ A_{23} & A_{24} \end{pmatrix} = \begin{pmatrix} {}^tA_{41} & {}^tA_{31} \\ {}^tA_{42} & {}^tA_{32} \end{pmatrix}.$$

Hence

$$\begin{aligned} \begin{pmatrix} A_{13} & A_{14} \\ A_{23} & A_{24} \end{pmatrix} &= \begin{pmatrix} -A_{23} & -A_{24} \\ -A_{13} & -A_{14} \end{pmatrix} \\ &= \begin{pmatrix} -{}^tA_{31} & -{}^tA_{41} \\ -{}^tA_{32} & -{}^tA_{42} \end{pmatrix} = \begin{pmatrix} {}^tA_{41} & {}^tA_{31} \\ {}^tA_{42} & {}^tA_{32} \end{pmatrix} \end{aligned}$$

and this matrix is of the form

$$\begin{pmatrix} B & -B \\ -B & B \end{pmatrix}.$$

The result follows. □

Remarks (i) The form of W^+ given in Proposition 7 satisfies (19), (20), so we cannot go further with this method.

(ii) As easily shown, 2-weight spin model with the form of W^+ in Proposition 7 has index 2.

In the sequel we shall use the following convenient description of 2-weight spin models. Assuming the identity $W^+(\alpha, \beta)W^-(\beta, \alpha) = 1$, which is the first part of (3), the second part of (3) can be written

$$\sum_{x \in X} \frac{W^+(\alpha, x)}{W^+(\beta, x)} = |X|\delta_{\alpha,\beta} \tag{21}$$

and (4) can be written

$$\sum_{x \in X} \frac{W^+(\alpha, x)W^+(\beta, x)}{W^+(\gamma, x)} = D\frac{W^+(\alpha, \beta)}{W^+(\alpha, \gamma)W^+(\gamma, \beta)}. \tag{22}$$

A matrix W^+ in M_X with non-zero entries satisfying (21) and (22) (with $D^2 = |X|$) for all α, β, γ in X will be called a *spin model* (with loop variable D). Then, defining W^- in M_X by the identity $W^+(\alpha, \beta)W^-(\beta, \alpha) = 1$, (3) and (4) hold. Moreover, it can be shown (see [20]) that (2) holds for some non-zero complex number a. Thus (X, W^+, W^-) is a 2-weight spin model. Finally, a matrix W^+ in M_X with non-zero entries satisfying (21) for all α, β in X will be called a *type II matrix*.

In the following result, the parameter ϵ is used to introduce a class of symmetric spin models closely related to the spin models of index 2 as described in Proposition 7.

Proposition 8 *Let $\epsilon \in \{1, -1\}$ and A, B, C be three matrices in M_Y with non-zero entries, A and C being symmetric. Let*

$$W^+ = \begin{array}{c} \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{array} \begin{array}{c} \begin{array}{cccc} Y_1 & Y_2 & Y_3 & Y_4 \end{array} \\ \begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ \epsilon\, {}^tB & -\epsilon\, {}^tB & C & C \\ -\epsilon\, {}^tB & \epsilon\, {}^tB & C & C \end{pmatrix} \end{array},$$

where the blocks Y_1, Y_2, Y_3, Y_4 are copies of Y. Then W^+ is a spin model with loop variable $2D$, where $D^2 = |Y|$, if and only if the following (i) *and* (ii) *hold.*

(i) *A, C are spin models with loop variable D and B is a type II matrix.*

(ii) *The following identities hold for all α, β, γ in Y:*

$$\sum_{y\in Y} \frac{A(\alpha, y)B(y, \beta)}{B(y, \gamma)} = D\frac{B(\alpha, \beta)}{C(\beta, \gamma)B(\alpha, \gamma)}, \tag{23}$$

$$\sum_{y\in Y} \frac{C(\alpha, y)B(\beta, y)}{B(\gamma, y)} = D\frac{B(\beta, \alpha)}{A(\beta, \gamma)B(\gamma, \alpha)}, \tag{24}$$

$$\sum_{y\in Y} \frac{B(y, \beta)B(y, \gamma)}{A(\alpha, y)} = \epsilon D\frac{C(\beta, \gamma)}{B(\alpha, \beta)B(\alpha, \gamma)}, \tag{25}$$

$$\sum_{y\in Y} \frac{B(\beta, y)B(\gamma, y)}{C(\alpha, y)} = \epsilon D\frac{A(\beta, \gamma)}{B(\beta, \alpha)B(\gamma, \alpha)}. \tag{26}$$

Proof: The rows and columns of W^+ are indexed by the union X of the disjoint blocks Y_i, $i = 1, \ldots, 4$. We must consider several cases which will be described by statements of the form $\alpha \in Y_i$, $\beta \in Y_j$ and (for (22)) $\gamma \in Y_k$. To reduce the number of these cases we introduce the following transformations.

Transformation (T1): Exchange of Y_1 and Y_2. This transforms the study of W^+ into the study of the following reordering of W^+:

$$\begin{array}{c} \\ Y_2 \\ Y_1 \\ Y_3 \\ Y_4 \end{array} \begin{array}{c} \begin{array}{cccc} Y_2 & Y_1 & Y_3 & Y_4 \end{array} \\ \begin{pmatrix} A & A & -B & B \\ A & A & B & -B \\ -\epsilon\, {}^tB & \epsilon\, {}^tB & C & C \\ \epsilon\, {}^tB & -\epsilon\, {}^tB & C & C \end{pmatrix} \end{array},$$

We observe that this matrix can also be obtained from W^+ by replacing B by $-B$.

Transformation (T2): Exchange of Y_3 and Y_4. The corresponding reordering of W^+ gives the same matrix as for (T1).

Transformation (T3): Replacement of (Y_1, Y_2, Y_3, Y_4) by (Y_3, Y_4, Y_1, Y_2). The corresponding reordering of W^+ is

$$\begin{array}{c} \\ Y_3 \\ Y_4 \\ Y_1 \\ Y_2 \end{array}\begin{array}{c} \begin{array}{cccc} Y_3 & Y_4 & Y_1 & Y_2 \end{array} \\ \begin{pmatrix} C & C & \epsilon\, {}^tB & -\epsilon\, {}^tB \\ C & C & -\epsilon\, {}^tB & \epsilon\, {}^tB \\ B & -B & A & A \\ -B & B & A & A \end{pmatrix} \end{array}.$$

This matrix can also be obtained from W^+ by exchanging A and C and replacing B by $\epsilon\, {}^tB$.

It is easy to check that the conditions (i) and (ii) of Proposition 8 are invariant under the replacement of B by $-B$, and are also invariant under the simultanous exchange of A and C and replacement of B by $\epsilon\, {}^tB$. The only non-trivial fact is that if B is a type II matrix, ${}^tB$ is also a type II matrix. To see this, define $B^- \in M_Y$ by the identity $B^-(y, z) = B(z, y)^{-1}$. Then (21) with B instead of W^+ and Y instead of X can be written $BB^- = |Y|I$, which gives the result by transposition.

Thus we may freely use the transformations (T1), (T2), (T3) to reduce the number of cases.

A) Let us consider first (21). We may assume $\alpha \neq \beta$.

First case: α and β belong to the same block Y_i. We may assume $i = 1$. Then (21) becomes

$$2\sum_{y\in Y} \frac{A(\alpha, y)}{A(\beta, y)} + 2\sum_{y\in Y} \frac{B(\alpha, y)}{B(\beta, y)} = 0,$$

where now α, β are considered as elements of Y.

Second case: $\alpha \in Y_i$, $\beta \in Y_j$ with $i \neq j$. We may assume $i = 1$ and $j \in \{2, 3\}$. For $i = 1$, $j = 2$, (21) becomes

$$2\sum_{y\in Y} \frac{A(\alpha, y)}{A(\beta, y)} - 2\sum_{y\in Y} \frac{B(\alpha, y)}{B(\beta, y)} = 0.$$

For $i = 1$, $j = 3$, (21) holds.

The pair of identities which we have obtained reduces to the identities (21) for A and B, that is to the fact that A and B are type II matrices.

B) Let us consider (22), where D must now be replaced by $2D$.

First case: α, β, γ belong to the same block Y_i. We may assume that $i = 1$. Then (22) becomes:

$$2\sum_{y\in Y}\frac{A(\alpha,y)A(\beta,y)}{A(\gamma,y)} = 2D\frac{A(\alpha,\beta)}{A(\alpha,\gamma)A(\gamma,\beta)},$$

which is the identity (22) for the matrix A. Thus the condition which we obtain is that the type II matrix A is actually a spin model with loop variable D.

Second case: α, β belong to the same block Y_i and γ belongs to another block Y_j. We may assume $i = 1$ and $j \in \{2, 3\}$. For $i = 1$, $j = 2$, (22) becomes the identity (22) for A. For $i = 1$, $j = 3$, (22) becomes:

$$2\sum_{y\in Y}\frac{B(\alpha,y)B(\beta,y)}{C(\gamma,y)} = 2D\frac{A(\alpha,\beta)}{B(\alpha,\gamma)\epsilon\, {}^tB(\gamma,\beta)},$$

i.e.

$$\sum_{y\in Y}\frac{B(\alpha,y)B(\beta,y)}{C(\gamma,y)} = \epsilon D\frac{A(\alpha,\beta)}{B(\alpha,\gamma)B(\beta,\gamma)},$$

which is (26) up to a change of variables.

Third case: α, γ belong to the same block Y_i and β belongs to another block Y_j. We may assume $i = 1$ and $j \in \{2, 3\}$. For $i = 1$, $j = 2$, (22) becomes the identity (22) for A. For $i = 1$, $j = 3$, (22) becomes the identity (24) up to a change of variables.

Fourth case: β, γ belong to the same block Y_i and α belongs to another block Y_j. We may assume $i = 1$ and $j \in \{2, 3\}$. For $i = 1$, $j = 2$, (22) becomes the identity (22) for A. For $i = 1$, $j = 3$, (22) becomes the identity (24) since A is symmetric.

Fifth case: $\alpha \in Y_i$, $\beta \in Y_j$, $\gamma \in Y_k$ with i, j, k distinct. We may assume $i = 1$ and $j \in \{2, 3\}$. For $i = 1$, $j = 2$, we may assume $k = 3$. Then (22) becomes the identity (26) up to a change of variables. For $i = 1$, $j = 3$, we have two cases: $(i, j, k) = (1, 3, 2)$ and $(i, j, k) = (1, 3, 4)$. When $(i, j, k) = (1, 3, 2)$, (22) becomes the identity (24) up to a change of variables. When $(i, j, k) = (1, 3, 4)$, (22) becomes the identity (23) since C is symmetric. This completes the proof. □

Remark Makoto Matsumoto (private communication) has shown that (23), (24) are equivalent and (25), (26) are also equivalent.

We now establish a correspondence between the symmetric ($\epsilon = 1$) and non-symmetric ($\epsilon = -1$) spin models appearing in Proposition 8.

Proposition 9 *Let A, B, C be three matrices in M_Y, A and C being symmetric, and assume $\eta^4 = -1$, $D^2 = |Y|$. Then*

$$\begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ {}^tB & -{}^tB & C & C \\ -{}^tB & {}^tB & C & C \end{pmatrix}$$

is a (symmetric) spin model with loop variable $2D$ if and only if

$$\begin{pmatrix} A & A & \eta B & -\eta B \\ A & A & -\eta B & \eta B \\ -\eta\, {}^tB & \eta\, {}^tB & C & C \\ \eta\, {}^tB & -\eta\, {}^tB & C & C \end{pmatrix}$$

is a (non-symmetric) spin model with loop variable $2D$.

Proof: It is easy to check that the conditions (i), (ii) of Proposition 8 are invariant under the simultanous multiplication of B by η and change of sign of ϵ. □

We shall now study an interesting example of the correspondence given by Proposition 9.

5. Symmetric and non-symmetric Hadamard spin models

5.1. Definition of the Hadamard spin models

In [21], the second author constructed symmetric spin models associated with Hadamard matrices. To construct these spin models, we first need another, simpler spin model.

Let Y be a set of size $n \geq 2$, let $D^2 = n$, and let $u \in \mathbf{C}$ satisfy the equation $-u^2 - u^{-2} = D$. Then the matrix

$$A = -u^3 I + u^{-1}(J - I)$$

in M_Y is a symmetric spin model, called a *Potts model*, see [9, 18].

Now let H be a Hadamard matrix in M_Y, i.e. a matrix with entries $+1$ or -1 such that $H\,{}^tH = nI$. We shall assume in the sequel that $n \geq 4$, and hence n is a multiple of 4. Let ω be a fourth root of 1. Let

$$U^+ = \begin{pmatrix} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \omega\, {}^tH & -\omega\, {}^tH & A & A \\ -\omega\, {}^tH & \omega\, {}^tH & A & A \end{pmatrix}.$$

Then U^+ is a symmetric spin model with loop variable $2D$, which we shall call a *symmetric Hadamard spin model*. For the sake of completeness, we give a proof using Proposition 8 (with $B = \omega H$, $C = A$ and $\epsilon = 1$).

Condition (i) is satisfied since every Hadamard matrix H is a type II matrix: (21) for H can be written $\sum_{y\in Y} H(\alpha, y)H(\beta, y) = |Y|\delta_{\alpha,\beta}$, which is equivalent to the equation $H\,{}^tH = nI$.

The equations (23)–(26) of condition (ii) become:

$$\sum_{y\in Y} A(\alpha, y)H(y, \beta)H(y, \gamma) = D\frac{H(\alpha, \beta)H(\alpha, \gamma)}{A(\beta, \gamma)}, \tag{23'}$$

$$\sum_{y\in Y} A(\alpha, y)H(\beta, y)H(\gamma, y) = D\frac{H(\beta, \alpha)H(\gamma, \alpha)}{A(\beta, \gamma)}, \tag{24'}$$

$$\sum_{y\in Y} \frac{H(y, \beta)H(y, \gamma)}{A(\alpha, y)} = DA(\beta, \gamma)H(\alpha, \beta)H(\alpha, \gamma), \tag{25'}$$

$$\sum_{y\in Y} \frac{H(\beta, y)H(\gamma, y)}{A(\alpha, y)} = DA(\beta, \gamma)H(\beta, \alpha)H(\gamma, \alpha). \tag{26'}$$

We observe that (24') is equivalent to (23') with H replaced by ${}^tH$ and similarly (26') is equivalent to (25') with H replaced by ${}^tH$. So it is enough to prove (23') and (25') for any Hadamard matrix H in M_Y.

Let $A^- = -u^{-3}I + u(J - I)$. Then A^- is obtained from A by the exchange of u and u^{-1}, and hence is a Potts model as defined above. Moreover $A^-(y, z) = A(y, z)^{-1}$ for all y, z in Y. Since (25') is equivalent to (23') with A replaced by A^-, it is enough to prove (23') for any Hadamard matrix H and Potts model A. (23') can be rewritten as

$$-u^3H(\alpha, \beta)H(\alpha, \gamma) + u^{-1} \sum_{y\in Y-\{\alpha\}} H(y, \beta)H(y, \gamma) = D\frac{H(\alpha, \beta)H(\alpha, \gamma)}{A(\beta, \gamma)}.$$

When $\beta = \gamma$ this becomes

$$-u^3 + u^{-1}(n - 1) = \frac{D}{-u^3},$$

i.e.

$$-u^3 + u^{-1}(u^4 + u^{-4} + 1) = u^{-3}(u^2 + u^{-2}).$$

When $\beta \neq \gamma$ the LHS is equal to

$$\begin{aligned} &-(u^3 + u^{-1})H(\alpha, \beta)H(\alpha, \gamma) + u^{-1} \sum_{y\in Y} H(y, \beta)H(y, \gamma) \\ &\quad = -(u^3 + u^{-1})H(\alpha, \beta)H(\alpha, \gamma) \end{aligned}$$

since H is a Hadamard matrix, and the RHS is $DH(\alpha, \beta)H(\alpha, \gamma)(u^{-1})^{-1}$. This completes the proof.

Assume $\eta^4 = -1$. Let

$$W^+ = \begin{pmatrix} A & A & \eta H & -\eta H \\ A & A & -\eta H & \eta H \\ -\eta\, {}^t H & \eta\, {}^t H & A & A \\ \eta\, {}^t H & -\eta\, {}^t H & A & A \end{pmatrix}.$$

Then it follows from Proposition 9 that W^+ is a non-symmetric spin model which we shall call a *non-symmetric Hadamard spin model*.

5.2. *Two related Bose-Mesner algebras*

Consider the following matrices (where the blocks belong to M_Y):

$$B_0 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & \frac{J+H}{2} & \frac{J-H}{2} \\ 0 & 0 & \frac{J-H}{2} & \frac{J+H}{2} \\ \frac{J+{}^tH}{2} & \frac{J-{}^tH}{2} & 0 & 0 \\ \frac{J-{}^tH}{2} & \frac{J+{}^tH}{2} & 0 & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} J-I & J-I & 0 & 0 \\ J-I & J-I & 0 & 0 \\ 0 & 0 & J-I & J-I \\ 0 & 0 & J-I & J-I \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & \frac{J-H}{2} & \frac{J+H}{2} \\ 0 & 0 & \frac{J+H}{2} & \frac{J-H}{2} \\ \frac{J-{}^tH}{2} & \frac{J+{}^tH}{2} & 0 & 0 \\ \frac{J+{}^tH}{2} & \frac{J-{}^tH}{2} & 0 & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix}.$$

Then the symmetric Hadamard spin model U^+ of the previous section is

$$U^+ = -u^3 B_0 + \omega B_1 + u^{-1} B_2 - \omega B_3 - u^3 B_4.$$

Let X be the union of four copies of Y which indexes the rows and columns of these matrices. Then there is a distance-regular graph Γ on the vertex-set X with intersection array

$$\left\{ n,\ n-1,\ \frac{n}{2},\ 1;\quad 1,\ \frac{n}{2},\ n-1,\ n \right\}$$

(called a *Hadamard graph*, see [6]) such that for x, x' in X, $B_i(x, x') = 1$ iff the distance between x, x' in Γ is i. It follows that $\{B_i,\ i = 0, \ldots, 4\}$ is the basis of Hadamard idempotents of a Bose-Mesner algebra $\mathcal{B}$. Properties of this Bose-Mesner algebra are extensively used to construct symmetric Hadamard spin models in [21] and to determine the associated link invariant in [11, 14]. We now introduce another Bose-Mesner algebra to study non-symmetric Hadamard spin models.

Let us consider the following matrices in M_X:

$$A_0 = B_0, \quad A_2 = B_2, \quad A_4 = B_4,$$

$$A_1 = \begin{pmatrix} 0 & 0 & \frac{J+H}{2} & \frac{J-H}{2} \\ 0 & 0 & \frac{J-H}{2} & \frac{J+H}{2} \\ \frac{J-{}^tH}{2} & \frac{J+{}^tH}{2} & 0 & 0 \\ \frac{J+{}^tH}{2} & \frac{J-{}^tH}{2} & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & \frac{J-H}{2} & \frac{J+H}{2} \\ 0 & 0 & \frac{J+H}{2} & \frac{J-H}{2} \\ \frac{J+{}^tH}{2} & \frac{J-{}^tH}{2} & 0 & 0 \\ \frac{J-{}^tH}{2} & \frac{J+{}^tH}{2} & 0 & 0 \end{pmatrix}.$$

Let $\mathcal{A}$ be the linear span of the matrices A_i, $i = 0, \ldots, 4$.

Proposition 10 *$\mathcal{A}$ is a Bose-Mesner algebra.*

Proof: First note that $\sum_{i=0}^{4} A_i = J$ and $A_i \circ A_j = \delta_{i,j} A_i$ ($i, j \in \{0, \ldots, 4\}$). Hence $\mathcal{A}$ is an algebra with identity J for the Hadamard product. Moreover,

$$A_0, A_2, A_4 \text{ are symmetric and } A_3 = {}^tA_1. \tag{27}$$

Hence $\mathcal{A}$ is closed under transposition. Note that $I = A_0 \in \mathcal{A}$. Thus it remains to show that $\mathcal{A}$ is a commutative algebra under ordinary matrix product. For this it is enough to check the following equalities:

$$
\begin{aligned}
A_1^2 &= \frac{n}{2}(A_2 + 2A_4),\\
A_1A_2 &= A_2A_1 = (n-1)(A_1 + A_3),\\
A_2^2 &= (2n-2)(A_0 + A_4) + (2n-4)A_2,\\
A_1A_3 &= A_3A_1 = \frac{n}{2}(A_2 + 2A_0),\\
A_2A_3 &= A_3A_2 = (n-1)(A_1 + A_3),\\
A_3^2 &= \frac{n}{2}(A_2 + 2A_4),\\
A_1A_4 &= A_4A_1 = A_3,\\
A_2A_4 &= A_4A_2 = A_2,\\
A_3A_4 &= A_4A_3 = A_1,\\
A_4^2 &= A_0.
\end{aligned}
\tag{28}
$$

Actually computation is needed only for the first three lines of (28). The last four lines are easily derived by considering the permutation of rows (respectively: columns) corresponding to multiplication by A_4 on the left (respectively: right). Then left and right multiplication of the first line by A_4 yield the fourth line. The fifth line is obtained from the second line by transposition, and the sixth line is obtained similarly from the first line. The matrix computations for the second and third line are easy and will be left to the reader. Finally let us consider the first line.

Let

$$
S = \begin{pmatrix} \frac{J+H}{2} & \frac{J-H}{2} \\ \frac{J-H}{2} & \frac{J+H}{2} \end{pmatrix},
$$

$$
T = \begin{pmatrix} \frac{J-{}^tH}{2} & \frac{J+{}^tH}{2} \\ \frac{J+{}^tH}{2} & \frac{J-{}^tH}{2} \end{pmatrix},
$$

so that

$$
A_1 = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}.
$$

Then

$$
A_1^2 = \begin{pmatrix} ST & 0 \\ 0 & TS \end{pmatrix}.
$$

Now

$$
ST = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}
$$

with

$$V_1 = \left(\frac{J+H}{2}\right)\left(\frac{J-{}^tH}{2}\right) + \left(\frac{J-H}{2}\right)\left(\frac{J+{}^tH}{2}\right),$$

$$V_2 = \left(\frac{J+H}{2}\right)\left(\frac{J+{}^tH}{2}\right) + \left(\frac{J-H}{2}\right)\left(\frac{J-{}^tH}{2}\right),$$

$$V_3 = \left(\frac{J-H}{2}\right)\left(\frac{J-{}^tH}{2}\right) + \left(\frac{J+H}{2}\right)\left(\frac{J+{}^tH}{2}\right) = V_2,$$

$$V_4 = \left(\frac{J-H}{2}\right)\left(\frac{J+{}^tH}{2}\right) + \left(\frac{J+H}{2}\right)\left(\frac{J-{}^tH}{2}\right) = V_1.$$

Expanding V_1 we obtain

$$\begin{aligned} V_1 &= \frac{1}{4}(J^2 + HJ - J\,{}^tH - H\,{}^tH + J^2 - HJ + J\,{}^tH - H\,{}^tH) \\ &= \frac{1}{2}(J^2 - H\,{}^tH) = \frac{1}{2}(nJ - nI) = \frac{n}{2}(J - I). \end{aligned}$$

Similarly

$$\begin{aligned} V_2 &= \frac{1}{4}(J^2 + HJ + J\,{}^tH + H\,{}^tH + J^2 - HJ - J\,{}^tH + H\,{}^tH) \\ &= \frac{1}{2}(J^2 + H\,{}^tH) = \frac{1}{2}(nJ + nI) = \frac{n}{2}(J + I). \end{aligned}$$

Hence

$$ST = \frac{n}{2}\begin{pmatrix} J-I & J+I \\ J+I & J-I \end{pmatrix}.$$

We observe that S and T can be obtained from each other by exchanging H and $-{}^tH$. Since ST does not depend on the choice of the Hadamard matrix H, $TS = ST$. Hence

$$\begin{aligned} A_1^2 &= \frac{n}{2}\begin{pmatrix} J-I & J+I & 0 & 0 \\ J+I & J-I & 0 & 0 \\ 0 & 0 & J-I & J+I \\ 0 & 0 & J+I & J-I \end{pmatrix} \\ &= \frac{n}{2}(A_2 + 2A_4), \end{aligned}$$

as required. □

Note that

$$W^+ = -u^3(A_0 + A_4) + u^{-1}A_2 + \eta(A_1 - A_3), \tag{29}$$

so that W^+ belongs to the Bose-Mesner algebra $\mathcal{A}$. We define:

$$\begin{aligned} W^- &= -u^{-3}(A_0 + A_4) + uA_2 + \eta^{-1}({}^tA_1 - {}^tA_3) \\ &= -u^{-3}(A_0 + A_4) + uA_2 - \eta^{-1}(A_1 - A_3), \end{aligned} \tag{30}$$

so that (X, W^+, W^-) is a 2-weight spin model (with loop variable $2D$).

We note (using (28)) that $A_4W^+ = {}^tW^+$. Moreover

$$W^+ \circ W^- = A_0 + A_4 + A_2 - A_1 - A_3$$

(this also follows of course from our construction of W^+).

The following result together with Proposition 10 gives an explicit form of Theorem B for non-symmetric Hadamard spin models.

Proposition 11 *$\mathcal{A}$ has a duality Ψ given by*

$$\Psi(M) = -u^3\, {}^tW^- \circ (W^+(W^- \circ M)) \text{ for every } M \text{ in } \mathcal{A}.$$

The matrix of Ψ in the basis $\{A_i,\ i = 0, \ldots, 4\}$ is

$$P = \begin{pmatrix} 1 & n & 2n-2 & n & 1 \\ 1 & -\eta^{-2}D & 0 & \eta^{-2}D & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & \eta^{-2}D & 0 & -\eta^{-2}D & -1 \\ 1 & -n & 2n-2 & -n & 1 \end{pmatrix}.$$

Proof: We show that the matrix of Ψ in the basis $\{A_i,\ i = 0, \ldots, 4\}$ is the above matrix P. Observe that it is enough to compute $\Psi(A_i)$, $i = 0, \ldots, 4$. These are routine computations using $A_i \circ A_j = \delta_{i,j}$, (28), (29) and (30). By way of example, we compute $\Psi(A_1)$.

$$\begin{aligned} \Psi(A_1) &= -u^3\, {}^tW^- \circ (W^+(W^- \circ A_1)). \\ W^- \circ A_1 &= -\eta^{-1}A_1. \\ W^+(W^- \circ A_1) &= -\eta^{-1}W^+A_1 \\ &= -\eta^{-1}\big(-u^3A_0A_1 + \eta A_1^2 + u^{-1}A_2A_1 - \eta A_3A_1 - u^3A_4A_1\big). \end{aligned}$$

Using (28) we obtain

$$\begin{aligned}&W^+(W^- \circ A_1)\\ &= \eta^{-1}u^3A_1 - \frac{n}{2}(A_2+2A_4) - \eta^{-1}u^{-1}(n-1)(A_1+A_3) + \frac{n}{2}(A_2+2A_0) + \eta^{-1}u^3A_3\\ &= nA_0 + \eta^{-1}(u^3 - u^{-1}(n-1))A_1 + \eta^{-1}(u^3 - u^{-1}(n-1))A_3 - nA_4.\end{aligned}$$

Now

$$-u^{3\,t}W^- = A_0 - \eta^{-1}u^3A_1 - u^4A_2 + \eta^{-1}u^3A_3 + A_4.$$

So finally

$$\Psi(A_1) = n(A_0 - A_4) + \eta^{-1}(u^3 - u^{-1}(n-1))\eta^{-1}u^3(A_3 - A_1).$$

The coefficient of $A_3 - A_1$ is

$$\eta^{-2}(u^3 - u^{-1}(u^4 + u^{-4} + 1))u^3 = \eta^{-2}(-u^{-2} - u^2) = \eta^{-2}D.$$

Hence

$$\Psi(A_1) = nA_0 - \eta^{-2}DA_1 + \eta^{-2}DA_3 - nA_4.$$

We now show that Ψ is a duality.

Checking (13), i.e. $\Psi^2(M) = 4n\,{}^tM$ for every M in $\mathcal{A}$, amounts to checking that $P^2 = 4nR$, where R is the matrix of the transposition operator in the basis $\{A_i,\ i = 0,\dots,4\}$. This is an easy computation which is left to the reader.

To verify (14), we shall check that $\Psi(A_iA_j) = \Psi(A_i) \circ \Psi(A_j)$ for $i, j \in \{0,\dots,4\}$. Since $\Psi(I) = J$ this is true if $i = 0$ or $j = 0$, so we assume $i, j \in \{1,\dots,4\}$. The following tables give the expressions of the $\Psi(A_i) \circ \Psi(A_j)$ in the basis $\{A_i,\ i = 0,\dots,4\}$. They are easily obtained from the matrix P by computing entrywise products of its column vectors.

	$\Psi(A_1)\circ\Psi(A_1)$	$\Psi(A_1)\circ\Psi(A_2)$	$\Psi(A_2)\circ\Psi(A_2)$	$\Psi(A_1)\circ\Psi(A_3)$	$\Psi(A_2)\circ\Psi(A_3)$
A_0	n^2	$n(2n-2)$	$(2n-2)^2$	n^2	$n(2n-2)$
A_1	$-n$	0	0	n	0
A_2	0	0	4	0	0
A_3	$-n$	0	0	n	0
A_4	n^2	$-n(2n-2)$	$(2n-2)^2$	n^2	$-n(2n-2)$

	$\Psi(A_3)\circ\Psi(A_3)$	$\Psi(A_1)\circ\Psi(A_4)$	$\Psi(A_2)\circ\Psi(A_4)$	$\Psi(A_3)\circ\Psi(A_4)$	$\Psi(A_4)\circ\Psi(A_4)$
A_0	n^2	n	$2n-2$	n	1
A_1	$-n$	$\eta^{-2}D$	0	$-\eta^{-2}D$	1
A_2	0	0	-2	0	1
A_3	$-n$	$-\eta^{-2}D$	0	$\eta^{-2}D$	1
A_4	n^2	$-n$	$2n-2$	$-n$	1

It is now easy to compare these values of the $\Psi(A_i) \circ \Psi(A_j)$ with the values of the $\Psi(A_i A_j)$ described by linear combinations of columns of P given by (28). We leave this verification to the reader. □

Remark Transposition of the $\Psi(A_i)$ is realized by the exchange of the second and fourth row of P. Since $\eta^{-2} = \pm\sqrt{-1}$, this amounts to complex conjugation of P, in agreement with (12).

A Bose-Mesner algebra $\mathcal{A}$ on X with basis of Hadamard idempotents $\{A_i,\ i = 0, \ldots, d\}$ is called *triply regular* if the following property holds [11]: for every i, j, k in $\{0, \ldots, d\}$, the number

$$|\{x \in X \mid A_i(x, \alpha) = 1,\ A_j(x, \beta) = 1,\ A_k(x, \gamma) = 1\}|,$$

where $\alpha, \beta, \gamma \in X$, only depends on the indexes u, v, w in $\{0, \ldots, d\}$ such that $A_u(\beta, \gamma) = 1$, $A_v(\gamma, \alpha) = 1$, $A_w(\alpha, \beta) = 1$. Then this number is denoted by $K(ijk \mid uvw)$ and called a *triple intersection number*.

The Bose-Mesner algebra $\mathcal{B}$ associated with a Hadamard graph and introduced at the beginning of this section is triply regular, as shown in [21]. To avoid confusion we shall denote its triple intersection numbers by $K_s(ijk \mid uvw)$, where s stands for "symmetric". From now on, $\mathcal{A}$ is again the Bose-Mesner algebra associated with a non-symmetric Hadamard spin model.

Proposition 12 *$\mathcal{A}$ is triply regular. Moreover its triple intersection numbers are triple intersection numbers of $\mathcal{B}$.*

Proof: Let us split again X into two blocks X_1 and X_2 of equal sizes, X_1 corresponding to the first half of X with respect to the ordering we have chosen for X when writing down the matrices B_i and A_i $(i = 0, \ldots, 4)$.

We observe that $A_1(\alpha, \beta) = B_3(\alpha, \beta)$ if $\beta \in X_1$ and $A_1(\alpha, \beta) = B_1(\alpha, \beta)$ if $\beta \in X_2$. Similarly, $A_3(\alpha, \beta) = B_1(\alpha, \beta)$ if $\beta \in X_1$ and $A_3(\alpha, \beta) = B_3(\alpha, \beta)$ if $\beta \in X_2$. Denote by $i \mapsto i'$ the permutation of $\{0, 1, 2, 3, 4\}$ which exchanges 1 and 3. Then $A_i(\alpha, \beta) = B_{i'}(\alpha, \beta)$ if $\beta \in X_1$, and $A_i(\alpha, \beta) = B_i(\alpha, \beta)$ if $\beta \in X_2$.

A triple $(u, v, w) \in \{0, \ldots, d\}^3$ will be said to be *feasible* if there exist α, β, γ in X such that $A_u(\beta, \gamma) = 1$, $A_v(\gamma, \alpha) = 1$, $A_w(\alpha, \beta) = 1$. Note that the triple intersection numbers $K(ijk \mid uvw)$ are defined only for feasible triples (u, v, w). Clearly (u, v, w) is feasible iff ${}^tA_w$ appears in the expression of $A_u A_v$ in the basis of Hadamard idempotents. Then (28) shows that if (u, v, w) is feasible, there is an even number of odd indexes among u, v, w ("parity rule").

We shall need the following property of the triple intersection numbers of $\mathcal{B}$: for all i, j, k, u, v, w in $\{0, \ldots, 4\}$, $K_s(i'j'k' \mid u'v'w') = K_s(ijk \mid uvw)$. This is because B_3 is the adjacency matrix of a Hadamard graph Γ' on the vertex-set X such that for x, x' in X, $B_1(x, x') = 1$ iff the distance of x, x' in Γ' is 3 (exchange H and $-H$ in the definition of the B_i), and the $K_s(ijk \mid uvw)$ do not depend on the choice of Hadamard graph on the vertex-set X (see [21]).

Now let α, β, γ be three elements of X such that $A_u(\beta, \gamma) = 1$, $A_v(\gamma, \alpha) = 1$, $A_w(\alpha, \beta) = 1$, and let us compute the number

$$N = |\{x \in X \mid A_i(x, \alpha) = 1, A_j(x, \beta) = 1, A_k(x, \gamma) = 1\}|.$$

Thanks to the parity rule, we have only to consider the following two cases.

Case 1: u, v, w are all even.

Let us assume first that α belongs to X_1. Then, since v and w are even, β and γ also belong to X_1. Hence

$$N = |\{x \in X \mid B_{i'}(x, \alpha) = 1, B_{j'}(x, \beta) = 1, B_{k'}(x, \gamma) = 1\}|.$$

Also $B_{u'}(\beta, \gamma) = 1$, $B_{v'}(\gamma, \alpha) = 1$, $B_{w'}(\alpha, \beta) = 1$. It follows that

$$N = K_s(i'j'k' \mid u'v'w') = K_s(ijk \mid uvw).$$

Assume now that $\alpha \in X_2$. Then $\beta \in X_2$, $\gamma \in X_2$. Hence

$$N = |\{x \in X \mid B_i(x, \alpha) = 1,\ B_j(x, \beta) = 1,\ B_k(x, \gamma) = 1\}|.$$

Also $B_u(\beta, \gamma) = 1$, $B_v(\gamma, \alpha) = 1$, $B_w(\alpha, \beta) = 1$. It follows that

$$N = K_s(ijk \mid uvw).$$

Thus we may define $K(ijk \mid uvw)$ which is equal to $K_s(ijk \mid uvw)$.

Case 2: exactly one of the indexes u, v, w is even.

We may assume without loss of generality that u is even and v, w are odd. Let us assume first that α belongs to X_1. Then β, γ belong to X_2. Hence

$$N = |\{x \in X \mid B_{i'}(x, \alpha) = 1,\ B_j(x, \beta) = 1,\ B_k(x, \gamma) = 1\}|.$$

Also $B_u(\beta, \gamma) = 1$, $B_{v'}(\gamma, \alpha) = 1$, $B_w(\alpha, \beta) = 1$. It follows that

$$N = K_s(i'jk \mid uv'w).$$

Assume now that $\alpha \in X_2$. Then $\beta \in X_1$, $\gamma \in X_1$. Hence

$$N = |\{x \in X \mid B_i(x, \alpha) = 1,\ B_{j'}(x, \beta) = 1,\ B_{k'}(x, \gamma) = 1\}|.$$

Also $B_{u'}(\beta, \gamma) = 1$, $B_v(\gamma, \alpha) = 1$, $B_{w'}(\alpha, \beta) = 1$. It follows that

$$N = K_s(ij'k' \mid u'vw') = K_s(i'jk \mid uv'w).$$

Thus we may define $K(ijk \mid uvw)$ which is equal to $K_s(i'jk \mid uv'w)$. □

5.3. Behaviour of the associated link invariant with respect to orientation

Could a non-symmetric Hadamard spin model be quasi-symmetric? The answer is no, as shown by the following result.

Proposition 13 *The link invariant Z associated with a non-symmetric Hadamard spin model depends non-trivially on the link orientation.*

Proof: A simple example will do. Consider the links given by the following diagrams L_1 and L_2, which differ only by their orientation (figure 4).

We have chosen a black and white coloring of the regions; x, y denote the values taken by a mapping from the set of black regions to X.

We shall show that $(-u^3)^{T(L_1)}Z(L_1) \neq (-u^3)^{T(L_2)}Z(L_2)$. Since $\chi(L_1) = \chi(L_2)$, this amounts to

$$\sum_{x,y\in X} W^-(y,x)^2 \neq \sum_{x,y\in X} W^-(x,y)W^-(y,x)$$

(see (1) and figure 2).

Note that $\sum_{x,y\in X} W^-(y,x)^2 = \sum W^- \circ W^-$ and $\sum_{x,y\in X} W^-(x,y)W^-(y,x) = \sum W^- \circ {}^tW^-$, where $\sum$ denotes the sum of entries of a matrix.

Recall that

$$W^- = -u^{-3}A_0 - \eta^{-1}A_1 + uA_2 + \eta^{-1}A_3 - u^{-3}A_4.$$

Hence

$$W^- \circ W^- = u^{-6}A_0 + \eta^{-2}A_1 + u^2A_2 + \eta^{-2}A_3 + u^{-6}A_4.$$

We have also

$${}^tW^- = -u^{-3}A_0 + \eta^{-1}A_1 + uA_2 - \eta^{-1}A_3 - u^{-3}A_4.$$

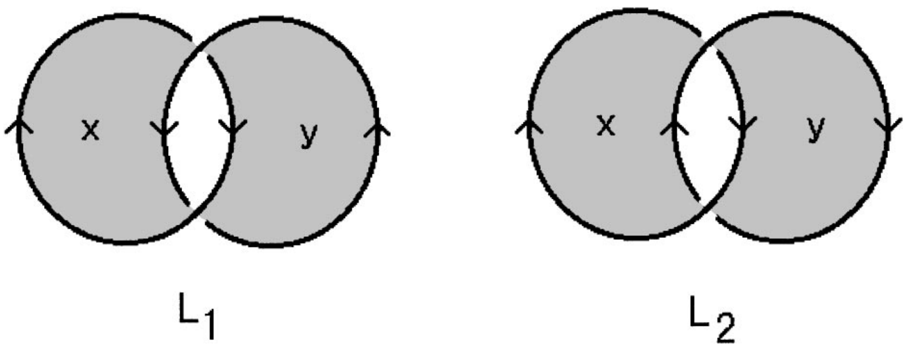

Figure 4.

Hence

$$W^- \circ {}^tW^- = u^{-6}A_0 - \eta^{-2}A_1 + u^2A_2 - \eta^{-2}A_3 + u^{-6}A_4.$$

Now $W^- \circ W^- - W^- \circ {}^tW^- = 2\eta^{-2}(A_1 + A_3)$ and

$$\sum W^- \circ W^- - \sum W^- \circ {}^tW^- = 2\eta^{-2}\sum(A_1 + A_3).$$

It easily follows from Section 5.2 that $\sum(A_1 + A_3) = 8n^2 \neq 0$. □

5.4. An explicit formula for the associated link invariant

In [14], using results of [11], an explicit formula was obtained for the link invariant associated with a symmetric Hadamard spin model. In this section we follow the same approach for the link invariant associated with a non-symmetric Hadamard spin model, denoted by Z.

The following result is similar to Proposition 13 of [14], and its proof will be essentially the same.

Proposition 14 *With every diagram L is associated a one variable rational function Q_L such that $Z(L) = Q_L(u)$.*

Proof: Let G be a directed graph with vertex-set $V(G)$ and edge-set $E(G)$. We denote by $i(e)$ (respectively: $t(e)$) the initial (respectively: terminal) end of the edge e. Let w be a mapping from $E(G)$ to M_X. Then we write

$$Z(G, w) = \sum_{\sigma:V(G)\to X} \prod_{e\in E(G)} w(e)(\sigma(i(e)), \sigma(t(e))), \tag{31}$$

an empty product being equal to 1.

Let L be a diagram with a black and white coloring of the regions such that adjacent regions have opposite color. By (1),

$$Z(L) = (-u^3)^{-T(L)}(2(-u^2 - u^{-2}))^{-\chi(L)} \sum_{\sigma:B(L)\to X} \prod_{v\in V(L)} \langle v, \sigma\rangle,$$

where $\langle v, \sigma\rangle$ is defined on figure 2. We must show that

$$\sum_{\sigma:B(L)\to X} \prod_{v\in V(L)} \langle v, \sigma\rangle$$

is given by a rational function of u.

We may construct a directed planar graph G_L and a mapping w_L from $E(G_L)$ to $\{W^+, W^-\} \subseteq \mathcal{A} \subseteq M_X$ ($\mathcal{A}$ denotes the Bose-Mesner algebra associated with the non-symmetric Hadamard spin model) such that

$$\sum_{\sigma:B(L)\to X} \prod_{v\in V(L)} \langle v, \sigma\rangle = Z(G_L, w_L).$$

Just take the set of black regions of L as the set of vertices of G_L, and for each crossing of L join the vertices of G_L corresponding to the black regions incident with this crossing by an oriented edge (the orientation is defined by that of the upper part of the link). Then, to define w_L, assign W^+ or W^- to each edge according to the prescription of figure 2.

The map $Z_G : w \mapsto Z(G, w)$ given by (31) is multilinear in the components $w(e)$, $e \in E(G)$, of w. We introduce for every directed plane graph G a vector space $\mathcal{A}_G$ which is a tensor product of copies of $\mathcal{A}$, one copy for each edge of G. Each mapping w from $E(G) = \{e_1, \ldots, e_k\}$ to $\mathcal{A}$ is represented by the element $w(e_1) \otimes \cdots \otimes w(e_k)$ of $\mathcal{A}_G$. Then Z_G can be identified with a linear form on $\mathcal{A}_G$.

The vector space $\mathcal{A}_G$ has a natural basis $B = \{A_{i_1} \otimes \cdots \otimes A_{i_k} \mid i_1, \ldots, i_k \in \{0, \ldots, 4\}\}$. Let L be a diagram. If we express w_L in the basis B, we see that the coefficients are given by rational functions of u, since this is true for the coefficients of W^+ and W^- in the basis $\{A_i,\ i = 0, \ldots, 4\}$. Hence $Z(G_L, w_L) = Z_{G_L}(w_L)$ is a linear combination, with coefficients given by rational functions of u, of terms of the form $Z_{G_L}(b)$, $b \in B$. Thus it will be enough to prove that for every directed plane graph G and $b \in B$, $Z_G(b)$ is given by a rational function of u. If G has connected components $G_1, \ldots, G_p$ and $b_1, \ldots, b_p$ denote the restrictions of b to these connected components, then $Z_G(b) = \prod_{i=1}^{p} Z_{G_i}(b_i)$. Hence we may assume that G is connected.

A slightly modified form of a result by Epifanov [8] asserts that there is a sequence $G_0, G_1, \ldots, G_k = G$ of directed plane graphs such that G_0 is the trivial graph with one vertex and no edge and for $i = 1, \ldots, k$, G_{i-1} is obtained from G_i by an elementary local transformation of one of the following types: reversal of the orientation of an edge, deletion of a loop, contraction of a pendant edge, deletion of an edge parallel to another one (the ordered pairs of ends of the two edges are the same), contraction of an edge in series with another one (its terminal end has in-degree and out-degree 1), and *star-triangle transformations*, that is, replacement of a triangle by a "star" (three edges incident to a common vertex) or the converse operation (for more details on star-triangle transformations, see section 5.4 of [11]). It is shown in [11] that, provided that $\mathcal{A}$ is *exactly triply regular*, for $i = 1, \ldots, k$, $Z_{G_i} = Z_{G_{i-1}} \rho_i$ for some easily described linear map ρ_i from $\mathcal{A}_{G_i}$ to $\mathcal{A}_{G_{i-1}}$. Since $\mathcal{A}$ is triply regular by Proposition 12 and admits a duality Ψ, $\mathcal{A}$ is indeed exactly triply regular by Proposition 9(iii) of [11]. Each ρ_i consists in the action of one of certain linear maps $\tau : \mathcal{A} \to \mathcal{A}$, $\theta : \mathcal{A} \to \mathbf{C}$, $\theta^* : \mathcal{A} \to \mathbf{C}$, $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $\mu^* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $\kappa : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, $\kappa^* : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ on appropriate factors of the tensor product $\mathcal{A}_{G_i}$. Note that $Z_G = Z_{G_0} \rho_1 \cdots \rho_k$. The map Z_{G_0} from $\mathcal{A}_{G_0} \simeq \mathbf{C}$ to $\mathbf{C}$ consists in scalar multiplication by $|X| = 4n = 4(u^4 + u^{-4} + 2)$. We want to show that the matrix of Z_G with respect to the basis B of $\mathcal{A}_G$ and the basis $\{1\}$ of $\mathbf{C}$ has entries given by rational functions of u. It will be enough to show that the matrices of τ, θ, θ^*, μ, μ^*, κ, κ^* with respect to bases appropriately chosen among $\{1\}$ for $\mathbf{C}$, $\{A_i, i = 0, \ldots, 4\}$ for $\mathcal{A}$, $\{A_i \otimes A_j, i, j \in \{0, \ldots, 4\}\}$ for $\mathcal{A} \otimes \mathcal{A}$ and $\{A_i \otimes A_j \otimes A_k, i, j, k \in \{0, \ldots, 4\}\}$ for $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ have entries given by rational functions of u.

The map $\tau : \mathcal{A} \to \mathcal{A}$ is the transposition map (it corresponds to the reversal of the orientation of an edge). Its matrix with respect to the basis $\{A_i, i = 0, \ldots, 4\}$ of $\mathcal{A}$ is a permutation matrix.

The map $\theta : \mathcal{A} \to \mathbf{C}$ gives for each matrix M in $\mathcal{A}$ its (constant) diagonal element $\theta(M)$ (θ corresponds to the deletion of a loop). Its matrix with respect to the basis $\{A_i, i = 0, \ldots, 4\}$ of $\mathcal{A}$ and the basis $\{1\}$ of $\mathbf{C}$ has entries equal to 0 or 1.

The map $\theta^* : \mathcal{A} \to \mathbf{C}$ gives for each matrix M in $\mathcal{A}$ its (constant) row sum $\theta^*(M)$ (θ^* corresponds to the contraction of a pendant edge). It easily follows from Section 5.2 that $\theta^*(A_0) = \theta^*(A_4) = 1, \theta^*(A_1) = \theta^*(A_3) = n, \theta^*(A_2) = 2n - 2$, where $n = u^4 + u^{-4} + 2$. Hence the entries of the matrix of θ^* with respect to the basis $\{A_i,\ i = 0, \ldots, 4\}$ of $\mathcal{A}$ and the basis $\{1\}$ of $\mathbf{C}$ are given by rational functions of u.

The map $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is defined by the identity $\mu(M \otimes M') = MM'$ (μ corresponds to the contraction of an edge in series with another one). Then (28) shows that the entries of the matrix of μ with respect to the basis $\{A_i \otimes A_j, i, j \in \{0, \ldots, 4\}\}$ of $\mathcal{A} \otimes \mathcal{A}$ and the basis $\{A_i, i = 0, \ldots, 4\}$ of $\mathcal{A}$ are given by polynomials of degree at most 1 in $n = u^4 + u^{-4} + 2$.

The map $\mu^* : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is defined by the identity $\mu^*(M \otimes M') = M \circ M'$ (μ^* corresponds to the deletion of an edge parallel to another one). By (8), the matrix of μ^* with respect to the basis $\{A_i \otimes A_j, i, j = 0, \ldots, 4\}$ of $\mathcal{A} \otimes \mathcal{A}$ and the basis $\{A_i, i = 0, \ldots, 4\}$ of $\mathcal{A}$ has entries equal to 0 or 1.

The map $\kappa : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ (which corresponds to the replacement of a star by a triangle) is defined by Eq. (52) of [11], which takes the form

$$\kappa(E_i \otimes E_j \otimes E_k) = \sum_{uvw \in F(\mathcal{A})} c(ijk \mid uvw) A_u \otimes A_v \otimes A_w,$$

where $F(\mathcal{A})$ is the set of feasible triples of $\mathcal{A}$ and the $c(ijk/uvw)$ are certain complex coefficients. Hence there is a corresponding equation of the form

$$\kappa(A_i \otimes A_j \otimes A_k) = \sum_{uvw \in F(\mathcal{A})} c'(ijk \mid uvw) A_u \otimes A_v \otimes A_w.$$

It then follows from Section 5.3 of [11] that

$$c'(ikj \mid uvw) = K(ijk \mid uvw)$$

for all i, j, k, u, v, w in $\{0, \ldots, 4\}$. By Proposition 12, these numbers are triple intersection numbers of $\mathcal{B}$; they have been computed in [21] and are given by polynomials of degree at most 1 in $n = u^4 + u^{-4} + 2$. Hence the entries of the matrix of κ with respect to the basis $\{A_i \otimes A_j \otimes A_k, i, j, k \in \{0, \ldots, 4\}\}$ of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ are given by rational functions of u.

Finally the map $\kappa^* : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ (which corresponds to the replacement of a triangle by a star) is defined by Eq. (53) of [11]. We shall need several steps.

(i) $\mathcal{A}$ is generated by W^+ and I under Hadamard product.

It is enough to show that if we write $W^+ = \sum_{i=0}^{4} t_i A_i$, the t_i with $i \neq 0$ are distinct. So we check that η, u^{-1}, $-\eta$, $-u^3$ are distinct. If $u^{-1} \in \{\eta, -\eta\}$ or if $u^{-1} = -u^3$, then $u^{-4} = -1$ and hence $n = u^4 + u^{-4} + 2 = 0$, a contradiction. If $-u^3 \in \{\eta, -\eta\}$, then $u^{12} = -1$ and hence

$$D^3 = (u^4 + u^{-4} + 2)(-u^2 - u^{-2}) = -(u^6 + u^{-6} + 3(u^2 + u^{-2})) = 3D,$$

a contradiction.

(ii) $\mathcal{A}$ has a unique duality Ψ such that $\Psi(W^+) = DW^-$.
The existence of Ψ follows from Theorem B and Remark (ii) following Theorem B. The uniqueness follows from (i) above.

(iii) $\kappa^* = (4n)^{-4}(\Psi \otimes \Psi \otimes \Psi)\kappa(\Psi \otimes \Psi \otimes \Psi)$.

By Proposition 12 and 18 of [11] and by (i) and (ii) above, we obtain

$$(\Psi \otimes \Psi \otimes \Psi)\kappa = 4n\kappa^*(\tau \otimes \tau \otimes \tau)(\Psi \otimes \Psi \otimes \Psi).$$

Right multiplying by $\Psi \otimes \Psi \otimes \Psi$ we get

$$(\Psi \otimes \Psi \otimes \Psi)\kappa(\Psi \otimes \Psi \otimes \Psi) = 4n\kappa^*(\tau\Psi^2 \otimes \tau\Psi^2 \otimes \tau\Psi^2).$$

Then the required equality (iii) follows from (13). Since the entries of the matrix P of Ψ with respect to the basis $\{A_i, i = 0, \ldots, 4\}$ (see Proposition 11) are given by rational functions of u, we know that the entries of the matrices of κ and $\Psi \otimes \Psi \otimes \Psi$ with respect to the basis $\{A_i \otimes A_j \otimes A_k, i, j, k \in \{0, \ldots, 4\}\}$ of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ are given by rational functions of u. By equation (iii), the same holds for κ^*. □

The following definitions were introduced in Section 6.1 of [14] with a slightly different presentation. Consider a link with set of components K represented by a diagram L. For any set C of crossings of L we denote by $s(C)$ the sum of signs of crossings in C (see figure 1). For any subset S of K, we denote by L_S the diagram obtained from L by keeping only the part of L which represents components in S; L_S is called a *subdiagram* of L. We shall allow the empty diagram $\emptyset$ and define $L_\emptyset = \emptyset$. For any $S \subseteq K$, let $C(S; K)$ denote the set of crossings involving a component from S and a component from $K \backslash S$. Now consider two link invariants f, g which take their values in a commutative ring Ω and an invertible element λ of Ω. The *λ-composition of f, g* is the link invariant denoted by $(f, g)_\lambda$ and defined as follows:

$$(f, g)_\lambda(L) = \sum_{S \subseteq K} \lambda^{s(C(S;K))} f(L_S) g(L_{K \backslash S}).$$

The *Jones polynomial* is a link invariant introduced in [17]. Up to a change of variable and normalization, it can be defined as follows (see [19]). Consider a diagram L without its orientation and the following operations of *smoothing* of a crossing depicted on figure 5 (which gives the *signs* of the smoothings):

A *state* of L is a diagram without crossings obtained by smoothing every crossing of L. Let $S(L)$ be the set of states of L. For σ in $S(L)$, we denote by $k(\sigma)$ the sum of signs of smoothings which have created σ from L, and by $c(\sigma)$ the number of loops of σ. The

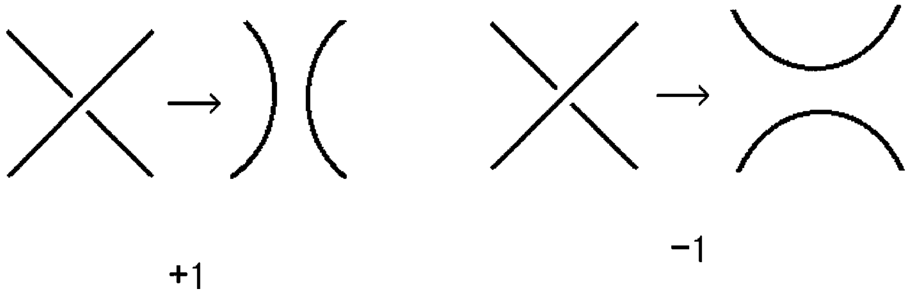

Figure 5.

bracket polynomial of L, denoted by $\langle L\rangle$, is the Laurent polynomial in the variable u defined by:

$$\langle L\rangle = \sum_{\sigma\in S(L)} u^{k(\sigma)}(-u^2 - u^{-2})^{c(\sigma)}.$$

We set $\langle\emptyset\rangle = 1$. Then $V(L) = (-u^3)^{-T(L)}\langle L\rangle$ (with $V(\emptyset) = 1$) defines a link invariant which is the Jones polynomial (up to a change of variables and normalization). It is shown in [18] (see also [9, 10]) that the link invariant associated with a Potts model as defined at the beginning of Section 5.1 is given by the Jones polynomial as defined above.

Proposition 15 *The link invariant associated with a non-symmetric Hadamard spin model is given by the* $(-\eta^{-1}u^{-3})$*-composition of two Jones polynomials.*

Proof: Proposition 11 of [14] for $Y = \mathbf{Z}_2$ ($=\mathbf{Z}/2\mathbf{Z}$) states essentially the following (see the Remark following this result). Let $((\mathbf{Z}_2)^{2k}, W_1, W_2, W_3, W_4)$ ($k \geq 1$) be a 4-weight spin model with loop variable 2^k, modulus μ and associated link invariant f, such that the four matrices W_i, $i = 0, \ldots, 4$, belong to the Bose-Mesner algebra of the group $(\mathbf{Z}_2)^{2k}$. Let λ be a non-zero complex number. Then there exists a symmetric Hadamard matrix H of size 2^{2k} such that the matrices W_i' ($i = 1, \ldots, 4$) given by

$$W_i' = \begin{pmatrix} W_i & W_i & \lambda^\epsilon H & -\lambda^\epsilon H \\ W_i & W_i & -\lambda^\epsilon H & \lambda^\epsilon H \\ \lambda^\epsilon H & -\lambda^\epsilon H & W_i & W_i \\ -\lambda^\epsilon H & \lambda^\epsilon H & W_i & W_i \end{pmatrix},$$

where $\epsilon = 1$ if $i = 1, 4$ and $\epsilon = -1$ if $i = 2, 3$, define a 4-weight spin model $((\mathbf{Z}_2)^{2k+2}, W_1', W_2', W_3', W_4')$ with loop variable 2^{k+1}, modulus μ and associated link invariant $(f, f)_{(\lambda\mu^{-1})}$.

We apply this result to the case where $W_1 = W_2 = A$ is a Potts model (see Section 5.1). That is, $A = -u^3 I + u^{-1}(J - I)$, where $-u^2 - u^{-2} = 2^k$. Then we must take $W_3 = W_4 =$

$A^- = -u^{-3}I + u(J - I)$. Moreover we take $\lambda = \eta^{-1}$. We then obtain the following matrices:

$$W_1' = \begin{pmatrix} A & A & \eta^{-1}H & -\eta^{-1}H \\ A & A & -\eta^{-1}H & \eta^{-1}H \\ \eta^{-1}H & -\eta^{-1}H & A & A \\ -\eta^{-1}H & \eta^{-1}H & A & A \end{pmatrix},$$

$$W_2' = \begin{pmatrix} A & A & \eta H & -\eta H \\ A & A & -\eta H & \eta H \\ \eta H & -\eta H & A & A \\ -\eta H & \eta H & A & A \end{pmatrix}.$$

W_3', W_4' satisfy the equations $W_1'W_3' = 2^{2k+2}I$, $W_2'W_4' = 2^{2k+2}I$ which can be used to define them.

Consider the following invertible diagonal matrix (the blocks have size 2^k)

$$\Delta = \begin{pmatrix} \eta I & 0 & 0 & 0 \\ 0 & \eta I & 0 & 0 \\ 0 & 0 & \eta^{-1}I & 0 \\ 0 & 0 & 0 & \eta^{-1}I \end{pmatrix}.$$

It is easy to check that (using $\eta^{-3} = -\eta$ and ${}^tH = H$)

$$\Delta W_1' \Delta^{-1} = \begin{pmatrix} A & A & \eta H & -\eta H \\ A & A & -\eta H & \eta H \\ -\eta\, {}^tH & \eta\, {}^tH & A & A \\ \eta\, {}^tH & -\eta\, {}^tH & A & A \end{pmatrix}$$

which is a non-symmetric Hadamard spin model W^+. Since $W^+W^- = 2^{2k+2}I$, it must be the case that $\Delta W_3' \Delta^{-1} = W^-$.

Consider now the permutation matrices (with blocks of size 2^{2k})

$$R = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

It is easy to check that $RW_2' = W_2'S = W^+$. Note that $W_2'^{-1}RW_2' = S$ is a permutation matrix and that $W_4'\,{}^tR = W^-$. Hence, by Theorem A, the 4-weight spin model

$$((\mathbf{Z}_2)^{2k+2}, W^+, W^+, W^-, W^-) = \left((\mathbf{Z}_2)^{2k+2}, \Delta W_1' \Delta^{-1}, RW_2', \Delta W_3' \Delta^{-1}, W_4'\,{}^tR\right)$$

is gauge equivalent to $((\mathbf{Z}_2)^{2k+2}, W_1', W_2', W_3', W_4')$. Since the link invariant associated with $((\mathbf{Z}_2)^{2k}, A, A, A^-, A^-)$ is the Jones polynomial V, and $\mu = -u^3$, the link invariant associated with $((\mathbf{Z}_2)^{2k+2}, W^+, W^+, W^-, W^-)$ is $(V, V)_{(-\eta^{-1}u^{-3})}$. This implies that, for every diagram L,

$$Q_L(u) = \big[(V, V)_{(-\eta^{-1}u^{-3})}(L)\big](u)$$

(see Proposition 14) whenever u is a complex number such that $-u^2 - u^{-2}$ is of the form 2^k, $k \geq 1$. Since the rational functions Q_L and $(V, V)_{(-\eta^{-1}u^{-3})}(L)$ are equal for infinitely many values of the variable u, they are equal. □

Thus the link invariant associated with a non-symmetric Hadamard spin model is given, for every link with set of components K and diagram L, by:

$$\begin{aligned} Z(L) &= \sum_{S\subseteq K} (-\eta^{-1}u^{-3})^{s(C(S;K))} V(L_S) V(L_{K\setminus S}) \\ &= \sum_{S\subseteq K} (-u^{-3})^{s(C(S;K))+T(L_S)+T(L_{K\setminus S})} \times (\eta^{-1})^{s(C(S;K))} \langle L_S\rangle\langle L_{K\setminus S}\rangle \\ &= (-u^3)^{-T(L)} \sum_{S\subseteq K} \eta^{-s(C(S;K))} \langle L_S\rangle\langle L_{K\setminus S}\rangle. \end{aligned}$$

We illustrate this formula on the examples of figure 4, Section 5.3. We denote by L_0 the link diagram with no crossings and one loop. Then for $i = 1, 2$,

$$Z(L_i) = (-u^3)^{-T(L_i)}\big(\langle L_i\rangle\langle\emptyset\rangle + \langle\emptyset\rangle\langle L_i\rangle + 2\eta^{-T(L_i)}\langle L_0\rangle^2\big).$$

Clearly $\langle L_0\rangle = -u^2 - u^{-2}$ and an easy computation gives

$$\langle L_1\rangle = \langle L_2\rangle = u^6 + u^2 + u^{-2} + u^{-6}.$$

It follows that

$$Z(L_1) = 2(-u^3)^{-2}(u^6 + u^2 + u^{-2} + u^{-6} + \eta^{-2}(u^4 + 2 + u^{-4}))$$

and

$$Z(L_2) = 2(-u^3)^2(u^6 + u^2 + u^{-2} + u^{-6} + \eta^2(u^4 + 2 + u^{-4})).$$

One can easily check these results using the the method of Section 5.3.

6. Concluding remarks

We believe that the notion of index could be a useful tool for studying non-symmetric spin models. It would be of interest to obtain new results on the structure of spin models of index

$m \neq 2$. One might expect to obtain new non-symmetric spin models in the case where m is a power of 2. The case of $m = p$ (odd prime) would also be of interest.

For index 2, it would also be nice to be able to answer the following questions. Keeping the notations of Proposition 8, can one find a spin model W^+ of index 2 with $A \neq C$, or with B not a Hadamard matrix? Note that by Proposition 9 this would also yield new symmetric spin models. Can one find an expression for the associated link invariant involving the link invariants associated with A and C, similar to the expression we have obtained in the case of non-symmetric Hadamard spin models?

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