Journal of Algebraic Combinatorics **10** (1999), 61–68

Fractional Multiples of Graphs and the Density of Vertex-Transitive Graphs

CLAUDE TARDIF tardif@math.uregina.ca
Département de mathématiques et de statistique, Université de Montréal, C.P. 6128 succ. Centre-ville, Montreal, Québec H3C 3J7, Canada

Received December 23, 1996; Revised February 6, 1998

Abstract. We introduce a construction called the *fractional multiple* of a graph. This construction is used to settle a question raised by E. Welzl: We show that if G and H are vertex-transitive graphs such that there exists a homomorphism from G to H but no homomorphism from H to G, then there exists a vertex-transitive graph that is homomorphically "in between" G and H.

Keywords: fractional chromatic number, graph, homomorphism

1. Introduction

The concept of graph homomorphisms gives rise to a natural quasi-order on the class of graphs: For two graphs G and H, we write $G \trianglelefteq H$ if there exists a homomorphism (i.e., an edge-preserving map) from G to H. We consider graphs without loops, and it often happens that for given graphs G and H, there exists no homomorphism from G to H. The quasi-order defined above therefore has a non-trivial structure. We write $G \triangleleft H$ if $G \trianglelefteq H$ and there exists no homomorphism from H to G. Also, we call G and H *homomorphically equivalent* if both relations $G \trianglelefteq H$ and $H \trianglelefteq G$ hold. It can be shown that the relation $\trianglelefteq$ induces a lattice order on the classes of homomorphically equivalent graphs. Welzl [10] investigated the quasi-order $\trianglelefteq$ with respect to density, and proved the following result:

Theorem 1 [10, Theorem 5.1] *Let G, H be finite graphs such that H is not bipartite and $G \triangleleft H$. Then there exists a finite graph K such that $G \triangleleft K \triangleleft H$.*

An elegant short proof of this result has recently been found independently by Nešetřil [7] and Perles (see [5]). The same type of investigation is also possible in other relational structures. In this spirit, Nešetřil and Zhu [8] characterized the dense intervals in the class of oriented paths. An earlier problem dealt with a specific subclass of the class of undirected graphs: Welzl asked in [11] if a density result also holds for the class of vertex-transitive graphs. Some particular instances of this question were answered affirmatively in [1, 11].

Partially supported by grant FCAR ER-1647 of the Ministère de l'éducation du Québec.
Current address: Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada, S4S0A2.

Albertson and Booth [1] have also shown that any vertex-transitive graph is the lower end of an interval that is void of vertex-transitive graphs. Our main result is the following:

Theorem 2 *Let G, H be finite vertex-transitive graphs such that $G \lhd H$. Then there exists a finite vertex-transitive graph K such that $G \lhd K \lhd H$.*

Our method relies on a class of graphs which we call the *fractional multiples* of a graph. These bear the same relation to some numerical parameters associated with graphs as do the complete graphs to the chromatic number, or the Kneser graphs to the fractional chromatic number. This class of graphs is presented in the next section, and Section 3 presents the related parameters. The proof of Theorem 2 is given in Section 4. All graphs considered in this paper will be finite.

2. Fractional multiples of graphs

For a graph G and integers r, s such that $1 \leq r \leq s$, the (r, s)th multiple $G^{(r,s)}$ of G is defined[1] by putting

$$V\big(G^{(r,s)}\big) = \{ f : D_f \to V(G) : D_f \subseteq \{1, \ldots, s\}, |D_f| = r\}$$
$$E\big(G^{(r,s)}\big) = \{[f, g] : [f(i), g(i)] \in E(G) \text{ for all } i \in D_f \cap D_g\}.$$

Some instances of this construction are well known. For example, $G^{(s,s)}$ is the sth categorical power of G in the sense of Miller [6] (see also (3) below); to within homomorphic equivalence, it is the same as G. At the other extreme is $G^{(1,s)}$, the Zykov join of s mutually disjoint copies of G, which from the point of view of homomorphic equivalence may be regarded as an integral multiple of G. The "fractional multiples" or "fractional joins" $G^{(r,s)}$ lie in between these two extremes. However, note that the word "fractional" represents a genuine abuse of language in the sense that even up to homomorphic equivalence, $G^{(r,s)}$ depends on both values r and s rather than solely on the ratio s/r. For instance, if $G = K_1$ and $2 \leq r < s/2$, $G^{(r,s)}$ is the well-known Kneser graph $K(r, s)$, and it is known (see [9]) that for r and s relatively prime, there exists a homomorphism from $K(\alpha r, \alpha s)$ to $K(\beta r, \beta s)$ only if β is a multiple of α.

The smallest example of a fractional multiple of a graph which does not belong to any of these classes is the graph $K_2^{(2,3)}$ depicted in figure 1. It is not clear from the figure that this graph is vertex-transitive. However, the next proposition shows that the definition of fractional multiples of graphs allows one to define more automorphisms than those that appear at first glance.

Proposition 1 *The wreath product $S_s \operatorname{wr} \operatorname{Aut}(G)$ is a subgroup of $\operatorname{Aut}(G^{(r,s)})$. In particular, if G is vertex-transitive, then so is $G^{(r,s)}$.*

Proof: Let π be a permutation of $\{1, \ldots, s\}$ and $\phi_1, \ldots, \phi_s$, automorphisms of a graph G. Then the map $\psi : G^{(r,s)} \to G^{(r,s)}$ defined by putting $\psi(f) = g$, where $D_g = \pi(D_f)$ and $g(i) = \phi_i \circ f(\pi^{-1}(i))$, is an automorphism of $G^{(r,s)}$. □

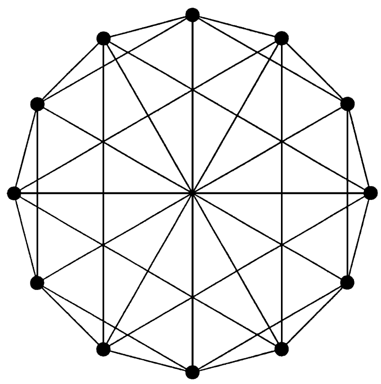

Figure 1. $K_2^{(2,3)}$.

Also, note that if $\phi : G \to H$ is a homomorphism, then for $1 \leq r \leq s$, we can define a homomorphism $\psi : G^{(r,s)} \to H^{(r,s)}$ by putting $\psi(f) = \phi \circ f$. So, $G \trianglelefteq H$ implies $G^{(r,s)} \trianglelefteq H^{(r,s)}$.

3. Generalizations of fractional colourings

For graphs G and H, let $\mathcal{H}_G(H)$ denote the family of induced subgraphs H' of H such that $H' \trianglelefteq G$, and $\alpha_G(H) = \max\{|H'| : H' \in \mathcal{H}_G(H)\}$ (where $|H'|$ denotes the cardinality of the vertex set of H'). We define the *G-fractional chromatic number* $\chi_G(H)$ of H as the solution of the following linear program:

$$\begin{aligned} \chi_G(H) = \min_{\mu} &\sum_{H' \in \mathcal{H}_G(H)} \mu(H') \\ &\text{subject to} \sum_{u \in V(H')} \mu(H') \geq 1 \quad \text{for all } u \in V(H). \end{aligned} \tag{1}$$

The minimum is taken over non-negative weight functions μ on $\mathcal{H}_G(H)$, so the linear program has one variable for each member of $\mathcal{H}_G(H)$ and one constraint for each vertex of H. The dual expression for the value of this program is the following:

$$\begin{aligned} \chi_G(H) = \max_{\nu} &\sum_{u \in V(H)} \nu(u) \\ &\text{subject to} \sum_{u \in V(H')} \nu(u) \leq 1 \quad \text{for all } H' \in \mathcal{H}_G(H). \end{aligned} \tag{2}$$

When G is the one-vertex graph K_1, $\mathcal{H}_G(H)$ is the set of all independent sets of H, $\alpha_G(H)$ is the stability number of H and $\chi_G(H)$ is the fractional chromatic number of H. Many of the known properties of the fractional chromatic number extend in a natural way to all parameters χ_G. The next lemma presents some basic results on these parameters.

Lemma 1

(i) *For any graph* H, $\chi_G(H) \geq |H|/\alpha_G(H)$.

(ii) $\chi_G(H) = 1$ *if and only if* $H \trianglelefteq G$.

(iii) *If* $K \trianglelefteq H$, *then* $\chi_G(K) \leq \chi_G(H)$.

Proof:

(i) $|H|/\alpha_G(H)$ is the value of the feasible solution to the linear program (2) obtained by putting $\nu(u) = 1/\alpha_G(H)$ for all $u \in V(H)$.

(ii) Follows easily from (i).

(iii) If $K \trianglelefteq H$, then any homomorphism from K to H can be used to pull back (by composition) any feasible weight function on $\mathcal{H}_G(H)$ to a feasible weight function on $\mathcal{H}_G(K)$ with the same value. $\square$

In [4], Hahn et al. characterize the fractional chromatic number in terms of graph homomorphisms. The next result shows that a similar characterization holds for any parameter χ_G.

Proposition 2 *For any graphs G and H,*

$$\chi_G(H) = \max\{|K|/\alpha_G(K) : K \trianglelefteq H\}.$$

Proof: By Lemma 1, we have

$$\chi_G(H) \geq \max\{|K|/\alpha_G(K) : K \trianglelefteq H\}.$$

Let ν be rational nonnegative weight function on $V(H)$ such that $\nu(u) = s_u/r$ for each $u \in V(H)$ and $\sum_{u \in V(H)} \nu(u) = s/r$. Define a graph K as follows: For each $u \in V(H)$, $V(K)$ contains an independent set S_u of size s_u, and if $[u, v] \in E(H)$, then $[u', v'] \in E(K)$ for all $u' \in S_u$ and $v' \in S_v$. Clearly, $K \trianglelefteq H$, and it is easily seen that an induced subgraph K' of K belongs to $\mathcal{H}_G(K)$ if and only if $V(K') \subseteq \bigcup_{u \in V(H')} S_u$ for some $H' \in \mathcal{H}_G(H)$. So, if ν is a feasible solution to the linear program (2), then $\alpha_G(K) \leq r$, and

$$\frac{s}{r} = \frac{|K|}{r} \leq \frac{|K|}{\alpha_G(K)}.$$

Thus,

$$\chi_G(H) \leq \max\{|K|/\alpha_G(K) : K \trianglelefteq H\}.$$

$\square$

This alternative definition of χ_G allows a slight generalization of the No-Homomorphism lemma of Albertson and Collins [2] that was also obtained by Bondy and Hell [3].

Proposition 3

(i) *Let H be a vertex-transitive graph. Then* $\chi_G(H) = |H|/\alpha_G(H)$.

(ii) [3, *Proposition* 4] *Let H, K be graphs such that H is vertex-transitive and $K \trianglelefteq H$. Then* $|K|/\alpha_G(K) \le |H|/\alpha_G(H)$.

(iii) *Let H, K be graphs such that H is vertex-transitive, $K \trianglelefteq H$ and $|K|/\alpha_G(K) = |H|/\alpha_G(H)$. Then $|\phi^{-1}(H')| = \alpha_G(K)$ for any homomorphism $\phi : K \to H$ and $H' \in \mathcal{H}_G(H)$ such that $|H'| = \alpha_G(H)$.*

Proof:

(i) Let $\mathcal{H}_G^*(H)$ denote the set of members of $\mathcal{H}_G(H)$ of size $\alpha_G(H)$. Then any vertex of H belongs to the same number, say m, of members of $\mathcal{H}_G^*(H)$. Furthermore, we have $m \cdot |H| = \alpha_G(H) \cdot |\mathcal{H}_G^*(H)|$. Putting $\nu(u) = 1/\alpha_G(H)$ for all $u \in V(H)$ yields a feasible solution to the linear program (2) with value $|H|/\alpha_G(H)$, and putting $\mu(H') = 1/m$ for all $H' \in \mathcal{H}_G^*(H)$ provides a solution to the linear program (1) with value $|\mathcal{H}_G^*(H)|/m$. Since these values are equal, they coincide with the optimal value $\chi_G(H)$.

(ii) This follows from (i) and Proposition 2.

(iii) Clearly, $|\phi^{-1}(H')| \le \alpha_G(K)$ for any $H' \in \mathcal{H}_G^*(H)$. Also, for $u \in V(K)$, $\phi(u)$ is in precisely m members of $\mathcal{H}_G^*(H)$, so we have

$$m \cdot |K| = \sum_{H' \in \mathcal{H}_G^*(H)} |\phi^{-1}(H')| \le |\mathcal{H}_G^*(H)| \cdot \alpha_G(K).$$

However, since $|K|/\alpha_G(K) = |H|/\alpha_G(H) = |\mathcal{H}_G^*(H)|/m$, the inequality cannot be strict, and since the summation was majorized termwise, we must have $|\phi^{-1}(H')| = \alpha_G(K)$ for all $H' \in \mathcal{H}_G^*(H)$. □

We are mainly interested in the role played by the fractional multiples $G^{(r,s)}$ in the computation of χ_G. This is summarized in the next result.

Proposition 4 *For any graphs G and H,*

$$\chi_G(H) = \min\left\{\frac{s}{r} : H \trianglelefteq G^{(r,s)}\right\}$$

Proof: For $i \in \{1, \dots, s\}$, we have

$$A_i = \left\{ f \in V\big(G^{(r,s)}\big) : i \in D_f \right\} \in \mathcal{H}_G\big(G^{(r,s)}\big).$$

Putting $\mu(A_i) = 1/r$ for $i = 1, \dots, s$ then shows that $\chi_G(G^{(r,s)}) \le s/r$. So, if $H \trianglelefteq G^{(r,s)}$, then by Lemma 1, $\chi_G(H) \le s/r$. Thus,

$$\chi_G(H) \le \min\left\{\frac{s}{r} : H \trianglelefteq G^{(r,s)}\right\}.$$

Let μ be a rational nonnegative weight function on $\mathcal{H}_G(H)$ such that $\mu(H') = s_{H'}/r$ for each $H' \in \mathcal{H}_G(H)$ and $\sum_{H' \in \mathcal{H}_G(H)} \mu(u) = s/r$. If μ satisfies the constraints of the linear program (1), we can define a map $\phi : H \to G^{(r,s)}$ as follows: For each $H' \in \mathcal{H}_G(H)$, fix a homomorphism $\phi_{H'} : H' \to G$, and assign to H' an $s_{H'}$-set $S_{H'} \subseteq \{1, \ldots, s\}$ such that different members of $\mathcal{H}_G(H)$ are assigned disjoint sets. Then for each $u \in V(H)$ we can select an r-set $S_u \subseteq \bigcup_{u \in V(H')} S_{H'}$. The map ϕ is then defined by putting $\phi(u) = f$, where $D_f = S_u$ and $f(i) = \phi_{H'}(u)$ for $i \in S_u \cap S_{H'}$. It is easily seen that ϕ is a homomorphism. Thus,

$$\chi_G(H) \geq \min\left\{\frac{s}{r} : H \trianglelefteq G^{(r,s)}\right\}.$$

$\square$

We have not been able to determine the range of the function χ_G for arbitrary G. For instance, when $G = K_1$, χ_G is the usual fractional chromatic number which can be 1 or any rational value greater than or equal to 2. It is conceivable that if G is not homomorphically equivalent to K_1, χ_G can assume any rational value greater than or equal to 1, but we have no proof of this.

4. Proof of Theorem 2

The G-fractional chromatic number $\chi_G(H)$ of a graph H can be thought of as a measure of how far H deviates from admitting a homomorphism into G (cf. Lemma 1). With this in mind, we begin this section with a numerical counterpart to the density statement of Theorem 1.

Lemma 2 *Let G, H be two graphs such that $G \lhd H$. Then for each $\epsilon > 0$, there exists a graph K such that $G \lhd K \lhd H$ and $\chi_G(K) \leq 1 + \epsilon$.*

Proof: By a repeated application of Theorem 1, we can find a sequence $\{H_n\}_{n \geq 1}$ of graphs such that $G \lhd H_1 \lhd H$ and $G \lhd H_{n+1} \lhd H_n$ for all $n \geq 1$. Further, at each stage, we can suppose that H_n is chosen with smallest cardinality. This means that for each $u \in V(H_n)$ we have $H_n - u \trianglelefteq G$, in which case u is called a *critical* vertex of H_n, or $G \not\trianglelefteq H_n - u$, in which case u is called an *essential* vertex of H_n. Note that a vertex can be critical and essential at the same time. Let c_n and e_n denote, respectively, the number of critical and essential vertices of H_n. We then have $e_n + c_n \geq |H_n|$. However, $e_n \leq |G|$ for each n, and the size of $V(H_n)$ must become arbitrarily large with n, since there is only a finite number of isomorphism classes of graphs with a given number of vertices. So, $\lim_{n\to\infty} c_n = \infty$. Also, $\chi_G(H_n) \leq c_n/(c_n - 1)$ since we get a feasible solution of the linear program (1) defining χ_G by putting $\mu(H_n - u) = 1/(c_n - 1)$ for each critical vertex u of H_n. Thus, $\lim_{n\to\infty} \chi_G(H_n) = 1$. $\square$

Recall that the usual *categorical product* $G \times H$ of two graphs G and H is defined by putting

$$V(G \times H) = V(G) \times V(H),$$
$$E(G \times H) = \{[(u_1, v_1), (u_2, v_2)] : [u_1, u_2] \in E(G), [v_1, v_2] \in E(H)\}. \tag{3}$$

Clearly, the categorical product of two graphs is vertex-transitive whenever both factors are. Both projections of $G \times H$ on its factors are homomorphisms. Also, for graphs G, H_1, H_2, we have $G \trianglelefteq H_1 \times H_2$ if and only if $G \trianglelefteq H_1$ and $G \trianglelefteq H_2$.

Proof of Theorem 2: Let G and H be vertex-transitive graphs such that $G \triangleleft H$. By Lemma 1, $\chi_G(H) > 1$, so by Lemma 2 there exists a finite graph K such that $G \triangleleft K \triangleleft H$ and $\chi_G(K) < \chi_G(H)$. By Proposition 4, there exist integers r, s such that $K \trianglelefteq G^{(r,s)}$ and $\chi_G(K) = s/r$. We then have $G \trianglelefteq H \times G^{(r,s)} \trianglelefteq H$, where $H \times G^{(r,s)}$ is vertex-transitive, and it remains to show that there is no homomorphism from H to $H \times G^{(r,s)}$ or from $H \times G^{(r,s)}$ to G. It is easily seen that both possibilities would contradict the choice of K, since $H \trianglelefteq H \times G^{(r,s)} \trianglelefteq G^{(r,s)}$ implies $\chi_G(H) \leq s/r = \chi_G(K)$, and $H \times G^{(r,s)} \trianglelefteq G$ implies $K \trianglelefteq H \times G^{(r,s)} \trianglelefteq G$. So, $G \triangleleft H \times G^{(r,s)} \triangleleft H$. $\square$

Yoav Kirsch recently told us that Theorem 2 was also obtained by Micha Perles, using a different construction.

Acknowledgments

The problem of density came to the author's attention through the lectures of G. Hahn during the Séminaire de Mathématiques Supérieures/NATO ASI on "Graph Symmetry" (Montréal, July 1996). The author also thanks B. Larose for valuable discussions on this subject.

Note

1. Strictly speaking we should write $(r, s)G$ instead of $G^{(r,s)}$, but the latter is considerably less cumbersome.

References

1. M.O. Albertson and V. Booth, "Homomorphisms of symmetric graphs," *Proceedings of the 17th Southeastern International Conference on Combinatorics, Graph Theory, and Computing, Boca Raton, FL., 1986, Congr. Numer.* **53** (1986), 79–86.
2. M.O. Albertson and K.L. Collins, "Homomorphisms of 3-chromatic graphs," *Discrete Math.* **54** (1985), 127–132.
3. J.A. Bondy and P. Hell, "A note on the star chromatic number," *J. Graph Theory* **14** (1990), 479–482.
4. G. Hahn, P. Hell, and S. Poljak, "On the ultimate independence ratio of a graph," *European J. Combin.* **16** (1995), 253–261.
5. G. Hahn and C. Tardif, "Graph homomorphisms: Structure and symmetry," *Graph symmetry (Montreal, PQ, 1996)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, 1997, Vol. 497, pp. 107–166.

6. D.J. Miller, "The categorical product of graphs," *Canad. J. Math.* **20** (1968), 1511–1521.
7. J. Nešetřil, "Structure of graph homomorphisms I," to appear in *Proceedings of Matrahaza Meeting*, V.T. Sos (Ed.), 1996.
8. J. Nešetřil and X. Zhu, "Path homomorphisms," *Math. Proc. Cambridge Philos. Soc.* **120** (1996), 207–220.
9. S. Stahl, "*n*-tuple colorings and associated graphs," *J. Combinatorial Theory Ser. B* **20** (1976), 185–203.
10. E. Welzl, "Color-families are dense," *J. Theoret. Comput. Sci.* **17** (1982), 29–41.
11. E. Welzl, "Symmetric graphs and interpretations," *J. Combinatorial Theory Ser. B* **37** (1984), 235–244.