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The Distance-Regular Graphs of Valency Four

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Abstract. We show that each distance-regular graph of valency four has known parameters.

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In this note we report on a computer search that proves that each distance-regular graph of valency four has known parameters. Here we describe first the known examples, next how putative arrays were disposed of, and finally how the search could be limited to a manageable number of arrays.

The distance-regular graphs of valency 3 have been determined by Biggs et al. [6]. Bannai and Ito worked on the general project of bounding the diameter of a distance-regular graph as a function of its valency k. They succeeded in the bipartite case [3] and in case $k = 4$[4][1]. This means that finding the feasible arrays for distance-regular graphs of valency 4 was reduced to a finite amount of work, but the diameter bounds obtained were not small enough to straightforwardly settle this case. In this note we obtain some additional conditions, and thus reduce the parameter space to be searched, and describe a way to test a parameter set using (small) integer arithmetic, thus avoiding accuracy problems.

Our notation for distance-regular graphs is standard (cf. [1, 5, 8]).

1. The known distance-regular graphs of valency four

In the table below, the parameters of the known distance-regular graphs of valency four are given. (We give an ordinal number, the number of vertices v, the diameter d, the intersection array and the spectrum.)

Descriptions of these graphs. 1. Complete graph K_5. 2. $K_{3\times 2}$ (octahedron). 3. Complete bipartite graph $K_{4,4}$. 4. 3×3 grid. 5. $K_{5,5}$ minus a matching. 6. Nonincidence graph of $PG(2, 2)$. 7. Line graph of the Petersen graph. 8. 4-cube. 9. Flag graph of $PG(2, 2)$. 10. Incidence graph of $PG(2, 3)$. 11. Incidence graph of $AG(2, 4)$ minus a parallel class. 12. Odd graph O_4. 13. Flag graph of $GQ(2, 2)$. 14. Doubled Odd graph. 15. Incidence graph of $GQ(3, 3)$. 16. Flag graph of $GH(2, 2)$. 17. Incidence graph of a $GH(3, 3)$. (Here $PG(2, q)$ and $AG(2, q)$ denote the projective and affine planes of order q, $GQ(q, q)$ and $GH(q, q)$ denote a generalized quadrangle or hexagon of order q.)

In each of these cases there is a unique graph with these parameters, except possibly in the last case, since uniqueness of $GH(3, 3)$ (a generalized hexagon of order 3) is not known.

No.	v	d	Intersection array	Spectrum
1.	5	1	{4; 1}	$4^1\ (-1)^4$
2.	6	2	{4,1; 1,4}	$4^1\ 0^3\ (-2)^2$
3.	8	2	{4,3; 1,4}	$\pm 4^1\ 0^6$
4.	9	2	{4,2; 1,2}	$4^1\ 1^4\ (-2)^4$
5.	10	3	{4,3,1; 1,3,4}	$\pm(4^1\ 1^4)$
6.	14	3	{4,3,2; 1,2,4}	$\pm(4^1\ \sqrt{2}^6)$
7.	15	3	{4,2,1; 1,1,4}	$4^1\ 2^5 (-1)^4\ (-2)^5$
8.	16	4	{4,3,2,1; 1,2,3,4}	$\pm(4^1\ 2^4)\ 0^6$
9.	21	3	{4,2,2; 1,1,2}	$4^1\ (1 \pm \sqrt{2})^6\ (-2)^8$
10.	26	3	{4,3,3; 1,1,4}	$\pm(4^1\ \sqrt{3}^{12})$
11.	32	4	{4,3,3,1; 1,1,3,4}	$\pm(4^1\ 2^{12})\ 0^6$
12.	35	3	{4,3,3; 1,1,2}	$4^1\ 2^{14}\ (-1)^{14}\ (-3)^6$
13.	45	4	{4,2,2,2; 1,1,1,2}	$4^1\ 3^9\ 1^{10}\ (-1)^9\ (-2)^{16}$
14.	70	7	{4,3,3,2,2,1,1; 1,1,2,2,3,3,4}	$\pm(4^1\ 3^6\ 2^{14}\ 1^{14})$
15.	80	4	{4,3,3,3; 1,1,1,4}	$\pm(4^1\ \sqrt{6}^{24})\ 0^{30}$
16.	189	6	{4,2,2,2,2,2; 1,1,1,1,1,2}	$4^1\ (1 \pm \sqrt{6})^{21}\ (1 \pm \sqrt{2})^{27}\ 1^{28}\ (-2)^{64}$
17.	728	6	{4,3,3,3,3,3; 1,1,1,1,1,4}	$\pm(4^1\ 3^{104}\ \sqrt{3}^{168})\ 0^{182}$

Each of these graphs is distance-transitive, except for those under 15 and 16—indeed, $GQ(3,3)$ and $GH(2,2)$ are not self-dual. (The single known example of a $GH(3,3)$ is distance-transitive; any further examples will not be.)

Our main theorem is:

Theorem 1.1 *Any distance-regular graph of valency* 4 *has one of the* 17 *intersection arrays listed above* (*and hence is one of the* 16 *graphs described above*, *or is the point-line incidence graph a generalized hexagon of order* 3).

Nomura [14] already found the seven distance-regular graphs with valency four and girth three.

(The classification is very easy: If $a_1 = 3$ then we are in case 1; if $a_1 = 2$ then Γ is locally a quadrangle, and hence is the octahedron, case 2; finally, if $a_1 = 1$, then Γ is locally $2K_2$, and hence the line graph of a cubic graph. But the distance-regular line graphs are known ([8]; [13], 4.2.16) and we find cases 4 and 7, and the flag graphs of generalized polygons of order (2, 2), cases 9, 13, 16. In all cases the graph is uniquely determined by the parameters. For the uniqueness (up to duality) of $GH(2,2)$, see [9].)

Thus, below we need only consider the case $a_1 = 0$.

2. A test for feasibility

Let Γ be a distance-regular graph with v vertices, of diameter d, and with inter-section array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$. Then Γ is regular of valency $k := b_0$, and there are

$k_i := b_0 b_1 \cdots b_{i-1}/c_1 c_2 \cdots c_i$ vertices at distance i from any given vertex. Let M be the tridiagonal matrix

$$\begin{pmatrix} 0 & b_0 & & & \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & b_2 & \\ & & . & . & . \\ & & & c_d & a_d \end{pmatrix}$$

and define polynomials u_i of degree i ($0 \le i \le d$) by $u_0(x) = 1$ and $c_i u_{i-1}(x) + a_i u_i(x) + b_i u_{i+1}(x) = x u_i(x)$, i.e., $u_{i+1}(x) = ((x - a_i)u_i(x) - c_i u_{i-1}(x))/b_i$. (Here $c_0 u_{-1}(x) = 0$.) Put $f(x) = (x - a_d)u_d(x) - c_d u_{d-1}(x)$, and $F(x) = \sum_{i=0}^{d} k_i u_i(x)^2$. Then f has $d + 1$ distinct roots, the eigenvalues of Γ, and if $f(\theta) = 0$, then θ is an eigenvalue of Γ of multiplicity $f_\theta = v/F(\theta)$. (All this is completely standard—see [1, 5, 8].)

A well-known and very strong criterion for the existence of a distance-regular graph with given intersection array is the condition that the $d + 1$ multiplicities f_θ must be integral. However, actually computing the θ and v and $v/F(\theta)$ numerically yields practical difficulties: v is very large, possibly of the order of $(k - 1)^d$, and one would have to compute θ to an extreme precision in order to conclude that $v/F(\theta)$ is not integral. Therefore, we chose a different approach that allowed us to compute with small integers only.

First observe that if θ_1 and θ_2 are algebraically conjugate, then $f_{\theta_1} = f_{\theta_2}$, so that $F(\theta_1) = F(\theta_2) = c$, say. If $m(x)$ is the irreducible factor of $f(x)$ that has θ_1 as zero, we find that $m(x) \mid (F(x) - c)$.

This is a strong existence condition. Indeed, a priori one would expect $F(x)$ mod $m(x)$ to have degree one less than the degree of $m(x)$, while in fact it has degree zero, so the higher the degree of $m(x)$, the stronger this condition. In fact, we do not know of examples, apart from the polygons, where $m(x)$ has degree higher than three. Degree 3 occurs for the Biggs-Smith graph but for no other known graph of valency more than two.

Thus, if $f(x) = \prod_j m_j(x)$ is the factorization over $\mathbf{Q}$ of f into irreducible factors, then there are rational numbers c_j such that $m_j(x) \mid (F(x) - c_j)$, and hence

$$f(x) \,\Big|\, \prod_j \gcd(f(x), F(x) - c_j).$$

Unfortunately, we don't know the constants c_j, and they may be quite large. So, let us reduce mod p. Let p be a prime not dividing $b_0 b_1 \cdots b_{d-1}$. Then all denominators occurring in the coefficients of u_i and f and F are nonzero mod p, and we can reduce mod p to conclude that

$$f(x) \,\Big|\, \prod_{c=0}^{p-1} \gcd(f(x), F(x) - c)^{e_c} \pmod{p}$$

for certain exponents e_c.

It is possible to avoid all fractions, by using $w_i = b_0 b_1 \cdots b_{i-1} u_i$ and $g = b_0 \cdots b_{d-1} f$ and $G = b_0 \cdots b_{d-1} c_1 \cdots c_d F$. We find

Proposition 2.1 *Let Γ be a distance-regular graph of diameter d with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$. Let $c_0 = 0$. Define monic polynomials $w_i\,(0 \le i \le d)$, g and G by $w_0(x) = 1$, $w_{i+1}(x) = (x - a_i)w_i(x) - b_{i-1}c_i w_{i-1}(x)$ $(0 \le i \le d)$, $g(x) = w_{d+1}(x)$ and $G(x) = \sum_{i=0}^{d} b_i \cdots b_{d-1}c_{i+1} \cdots c_d w_i(x)^2$. Then for each positive integer p there are constants e_c such that*

$$g(x) \,\Bigg|\, \prod_{c=0}^{p-1} \gcd(g(x), G(x) - c)^{e_c} \pmod{p}.$$

For $p = 2$ this is useless (the condition reduces to the condition that a polygon exists), but for $p \ge 3$ it produces restrictions.

This is the condition we applied: for $p = 5, 7, 11, 13$ compute the w_i, g, G (all mod p), compute p times a gcd and remove all factors found from g, possibly repeatedly. (If a nonlinear factor is removed, additional gcds are necessary to see whether part of that factor can be removed more than once.) If after doing this a quotient of positive degree is left, no graph with this intersection array exists.

[Usually, taking $p = 5$ sufficed; in a few cases also $p = 7$, and in very few cases also $p = 11$ was required. After that only the actual examples and four other arrays, of diameters 4, 6, 6, 6, survived. Indeed, if g completely factors into linear factors, or if Γ is bipartite, and g factors completely into factors $x^2 - a$ and possibly x, then our condition will be empty for all p. This happens for three arrays: for {4, 3, 3, 2; 1, 1, 2, 4} we have $g(x) = x(x^2 - 5)(x^2 - 16)$, and for both {4, 3, 3, 2, 1, 1; 1, 1, 2, 3, 3, 4} and {4, 3, 3, 3, 1, 1; 1, 1, 1, 3, 3, 4} we have $g(x) = x(x^2 - 3)(x^2 - 7)(x^2 - 16)$. However, it is easy to rule out these arrays—for example, each has nonintegral multiplicities. In the nonbipartite case there is one additional parameter set: {4, 3, 3, 1, 1, 1; 1, 1, 1, 3, 3, 4} for a nonexistent double cover of O_4. Here $g(x) = x(x + 1)(x - 2)(x + 3)(x - 4)(x^2 - 7)$ and the multiplicities are integral—combinatorial considerations are required to rule out this case (cf. [8], Proposition 9.1.9).]

Note that we have the Christoffel-Darboux formula $G(x) = w_d(x)g'(x) - w'_d(x)g(x)$, so that we may replace $G(x)$ by $w_d(x)g'(x)$ in the above formula. (This will speed up the computations: the naive way of computing G takes order d^3 steps, but for $w_d(x)g'(x)$ only order d^2 steps are required.)

3. A divisibility condition

Let Γ be a distance-regular graph and p a prime, such that c_{r+1} is divisible by p, but c_i with $1 \le i \le r$ is not. Consider the parameters a_i, b_i, c_i and the matrices A_i as being defined over the integers mod p. Then $\langle I, A, \ldots, A_r \rangle$ is closed under multiplication, and $A_i = f_i(A)$ for some polynomial f_i of degree i $(1 \le i \le r)$. (If p divides the valency k of Γ, then the same holds for $\langle A, \ldots, A_r \rangle$.) Thus, $f(A) = 0$ for some polynomial f of degree $r + 1$, but for no nonzero polynomial of smaller degree.

Now suppose moreover that $b_m = c_{m+t+1} = 0 \pmod{p}$, and $c_{m+i}, b_{m+i} \ne 0 \pmod{p}$ for $1 \le i \le t$. Then $\langle A_{m+1}, \ldots, A_{m+t} \rangle$ is closed under multiplication by A, and if we put

$B := A_{m+1}$, then $A_{m+i} = g_{i-1}(A)B$ for some polynomial g_{i-1} of degree $i-1$ $(1 \le i \le t)$. Thus, $g(A)B = 0$ for some polynomial g of degree t, but for no nonzero polynomial of smaller degree. It follows that $g \mid f$.

This is a very useful condition. In order to apply it to the bipartite case, we first need a lemma.

Lemma 3.1 *Define polynomials p_i over any field F by $p_0 = 0$, $p_1(x) = x$, $p_{i+1}(x) = xp_i(x) - \lambda p_{i-1}(x)$ $(i \ge 1)$, where λ is a nonzero constant. Then $(p_i, p_j) = p_{(i,j)}$ (where $(-,-)$ denotes the g.c.d.). In particular, $p_i \mid p_j$ if and only if $i \mid j$.*

Proof: Modulo p_i we find that $p_{i+k} = -\lambda^k p_{i-k}$ for $0 \le k \le i$ (by induction on k). $\square$

Let us give two applications of the above divisibility condition.

Proposition 3.2 *Let Γ be a distance-regular graph such that $(c_i, a_i, b_i) \equiv (1, 0, 1) \pmod 2$ for $1 \le i \le r$ and for $d - t \le i \le d - 1$, while $b_0 \equiv c_{r+1} \equiv b_{d-t-1} \equiv c_d \equiv 0 \pmod 2$. Then $(t+1) \mid (r+1)$.*

Proof: Take $F = \mathbf{F}_2$, $\lambda = 1$. With the notation of the lemma we have (over F) $A_i = p_i(A)$ for $1 \le i \le r$, and $p_{r+1}(A) = 0$. Similarly, $A_{d-i} = p_i(A)B$ $(1 \le i \le t)$, where $B = A_d$, and $p_{t+1}(A)B = 0$. It follows that $p_{t+1} \mid p_{r+1}$, and the conclusion follows. $\square$

Proposition 3.3 *Let Γ be a distance-regular graph such that $(c_i, a_i, b_i) \equiv (1, 0, 1) \pmod 2$ for $1 \le i \le r$ and for $d - t \le i \le d - 1$, while $b_0 \equiv c_{r+1} \equiv b_{d-t-1} \equiv 0 \pmod 2$ and $c_d \equiv 1 \pmod 2$. Then $(2t+3) \mid (r+1)$.*

Proof: With $B = A_d$ we find $A_{d-i} = q_i(A)B$ for $1 \le i \le t$, and $q_{t+1}(A)B = 0$, where $q_i = p_i + p_{i-1} + \cdots + p_1 + 1$ (with notation as in the above lemma). By induction one sees that $p_{2t+1}(x) = xq_t(x)^2$. Thus, $q_{t+1} \mid p_{r+1}$ implies that (p_{2t+3}, p_{r+1}) has degree at least $t+2$, so $(2t+3, r+1) \ge t+2$, so $(2t+3) \mid (r+1)$. $\square$

Let Γ be a bipartite distance-regular graph of valency four. Then there are integers r, s, t such that $(c_i, a_i, b_i) = (0, 0, 4), (1, 0, 3), (2, 0, 2), (3, 0, 1), (4, 0, 0)$ for $i = 0$, for $1 \le i \le r$, for $r+1 \le i \le r+s$, for $r+s+1 \le i \le r+s+t$, and for $i = r+s+t+1$, respectively. The diameter d of Γ equals $d = r+s+t+1$. In this case the divisibility condition says: if $s > 0$, then $(t+1) \mid (r+1)$.

After writing the above we discovered that (the case $t > 1$ of) Proposition 3.2 is the contents of [16]. More generally, Nomura [15] communicates a result which is the case $\epsilon = -1$ of the following:

Proposition 3.4 *Let Γ be a distance-regular graph such that, for some prime p and integer $\epsilon = \pm 1$, we have $(c_i, a_i, b_i) \equiv (1, 0, -1) \pmod p$ for $1 \le i \le r$ and $(c_i, a_i, b_i) \equiv (\epsilon, 0, -\epsilon) \pmod p$ for $m+1 \le i \le m+t$, while $b_0 \equiv c_{r+1} \equiv b_m \equiv c_{m+t+1} \equiv 0 \pmod p$. If $t > 1$, then $(t+1) \mid (r+1)$.*

Proof: Put $\lambda = -1$. With $B = A_{m+1}$ we find $\epsilon^{i-1} A_{m+i} = (p_i/p_1)\,(A)B$ and $(p_{t+1}/p_1)\,(A)B = 0$ so that $p_{t+1}(x) \mid x p_{r+1}(x)$, and $(t+1, r+1) \geq t$. □

4. The intersection array in case $k = 4$

Given any distance-regular graph with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$, we put $k = b_0$ and $a_i = k - b_i - c_i$ as usual. Let e_{cab} denote the number of indices i for which $(c_i, a_i, b_i) = (c, a, b)$.

Lemma 4.1 *Let Γ be a distance-regular graph of valency 4. Then we have one of three cases.*
(i) *Γ is bipartite.*
(ii) *Γ is a generalized Odd graph ($a_i = 0$ for $i < d$, $a_d \neq 0$).*
(iii) *Γ has $a_i > 0$ for some $i < d$, and $e_{202} = 0$.*

Proof: The Brouwer-Lambeck inequalities state: if $a_i \neq 0$, and $i < d$, then $b_i \leq a_i + a_{i+1} b_i / a_i$, and if $i > 1$ then $c_i \leq a_i + a_{i-1} c_i / a_i$ (see [8], Proposition 5.5.4). It follows that if $(c_i, a_i, b_i) = (1, 1, 2)$, then $a_{i+1} \neq 0$, and if $(c_i, a_i, b_i) = (2, 1, 1)$, then $a_{i-1} \neq 0$. It follows that if $e_{202} > 0$, then $e_{211} = e_{112} = e_{121} = 0$, so that $a_i = 0$ for $i < d$. □

Once Case (i) has been handled, Case (ii) is easy: If Γ is a generalized Odd graph, then its bipartite double is distance-regular of diameter $2d + 1$, an antipodal 2-cover of Γ, so that Γ can be retrieved from it by folding (see [8], Proposition 4.2.11). We shall find that the only bipartite graphs of odd diameter that are antipodal 2-covers are $K_{5,5}$ minus a matching ($v = 10$) and the doubled Odd graph ($v = 70$); folding these we find K_5 ($v = 5$) and O_4 ($v = 35$).

From now on, we shall assume that we are not in Case (ii). This leaves us with two cases: the bipartite case, where we put $r = e_{103}$, $s = e_{202}$, $t = e_{301}$, and the case where $a_i > 0$ for some $i < d$, where we put $r = e_{103}$, $s_1 = e_{112}$, $s_2 = e_{121}$, $s_3 = e_{211}$, $t = e_{301}$.

Lemma 4.2 *Let Γ be a distance-regular graph of valency 4. Then*
(i) *$t \leq r$.*
(ii) *If $t > 0$, then $a_d = 0$.*
(iii) *If $s_1 > 0$, $s_2 = s_3 = 0$, then $t = 0$ and $a_d \neq 0$.*

Proof: (i) This follows since k_d is integral. (ii) This follows from [8], Proposition 5.5.7. (iii) This follows from the Brouwer-Lambeck inequalities. □

A bound on e_{112} is provided by the following two results.

Proposition 4.3 ([7]; cf. [8], 5.10.1) *Let Γ be a distance-regular graph of valency k. If $e_{1,1,k-2} \geq 3$ then $3 \mid e_{1,0,k-1}$, and if moreover $e_{1,0,k-1} > 0$ then $e_{1,1,k-2} \leq 4$.*

Proposition 4.4 [10] *Let Γ be a distance-regular graph of valency $k > 3$ with $a_1 = 0$. Then $e_{1,1,k-2} \leq 3$, and if $e_{1,1,k-2} = 3$, then $c_{r+4} > 1$, where $r = e_{1,0,k-1}$.*

In our case this means that $s_1 \leq 3$, and if $s_1 = 3$, then $3 \mid r$ then $s_2 = 0$ and $c_d > 1$.

Lemma 4.5 $s_3 \leq s_1$.

Proof: Indeed, $k_d = 4.3^{r-t} \cdot 2^{s_1-s_3}/c_d$. If $c_d = 4$, then the conclusion follows by integrality of k_d. Otherwise, the conclusion follows by integrality of $p^1_{dd} = k_d p^d_{1d}/k_1 = 3^{r-t} \cdot 2^{s_1-s_3} \cdot (4 - c_d)/c_d$. □

Lemma 4.6 *Assume $t > 0$, so that $s_2 + s_3 > 0$.*
(i) *If $s_3 > 0$, then either $t = r$ or $t \leq (r + s_1 - s_3 - 2)/2$.*
(ii) *If $s_3 = 0$, $s_2 > 0$, then either $t = r$ or $(s_1, s_2, s_3) = (0, 1, 0)$ or $t \leq \frac{1}{2}r - 1$.*

Proof: Γ has girth $2r + 3$, so if $t < r$, then no path of length at most $2t + 4$ can be a circuit. Fix a vertex x, and put $D := \Gamma_d(x)$. Let N_l be the number of paths of length l from D to D. If γ_m is the number of geodesics between two vertices at distance m, then there are precisely $\gamma_m \sum_{i=0}^{m} a_i$ paths of length $m + 1$ between any two such vertices.

(i) Suppose $s_3 > 0$. On the one hand, we find $N_{2t+3} = k_d p^d_{d,2t+3} c_1 \cdots c_{2t+3} = p^{2t+3}_{d,d} b_0 \cdots b_{2t+2}$. On the other hand, we have $N_{2t+3} = k_d \cdot 4 \cdot 3^t = k_{d-1-t} = 4 \cdot 3^r \cdot 2^{s_1-s_3}$. It follows that $2t + 3 \leq 1 + r + s_1 - s_3$.
(ii) Suppose $s_3 = 0, s_2 > 0$. We have $N_{2t+3} = 4 \cdot 3^t \cdot 2 \cdot k_d$ and $N_{2t+4} = 4 \cdot 3^t \cdot (2 + b_{d-t-2} - 1) \cdot k_d$ so that $b_{d-t-2} + 1 \geq 2 \sum_{i=0}^{2t+3} a_i$, and either $s_2 = 1$, $s_2 = 0$ or $2t + 3 \leq r + 1$. □

For the bipartite case we have two more restrictions:

Lemma 4.7 *If $s > 0$, then $(t + 1) \mid (r + 1)$.*

Proof: This is just Proposition 3.2. □

Proposition 4.8 [18] *Let Γ be a distance-regular graph of valency k and diameter d, and with intersection array $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$. Let $r := e_{1,0,k-1}$. If $(c_{r+1}, a_{r+1}, b_{r+1}) = (c_{r+2}, a_{r+2}, b_{r+2}) = (2, 0, k - 2)$, then r is even.*

Using this saves (more than) half of the work in case $s \geq 2$. However, since the total amount of work in the bipartite case turned out to be rather small anyway, we have not used this proposition. (But omitting it caused the prime $p = 13$ to be used twice.)

A bound on s (in the bipartite case) or s_2 (in the non-bipartite case) follows from Terwilliger's multiplicity bound, see Section 6 below.

5. Location of the eigenvalues

We shall need bounds on the eigenvalues of tridiagonal matrices T such as M (with positive entries on the diagonals above and below the main diagonal). Write $\theta_{\min}(T)$, $\theta_{\max}(T)$ and $\theta_2(T)$ for the smallest, the largest, and the second largest eigenvalue of T.

Perron-Frobenius tells us that if S is a matrix obtained from T by decreasing some elements, keeping the off-diagonal elements nonnegative, then $\theta_{\max}(S) < \theta_{\max}(T)$. Interlacing tells us that if S is a principal submatrix of T, then $\theta_{\min}(T) \le \theta_{\min}(S)$ and $\theta_2(S) \le \theta_2(T)$ and $\theta_{\max}(S) \le \theta_{\max}(T)$. But we can be more precise. If p_n is a series of orthogonal polynomials, then for $n > m$ there is a root of p_n between any two roots of p_m. Since the characteristic polynomials u_i of the upper left-hand corner T_i (of order i) of T form a sequence of orthogonal polynomials, there is an eigenvalue of T between any two eigenvalues of T_i.

The eigenvalues distinct from k of the tridiagonal matrix M are the eigenvalues of

$$M' = \begin{pmatrix} -c_1 & b_1 & & & \\ c_1 & k-b_1-c_2 & b_2 & & \\ & \cdot & \cdot & \cdot & \\ & & c_{d-2} & k-b_{d-2}-c_{d-1} & b_{d-1} \\ & & & c_{d-1} & k-b_{d-1}-c_d \end{pmatrix}$$

(cf. [8]).

Lemma 5.1 *Let $\iota = \{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ be an intersection array, and put $r = e_{1,0,k-1}$ and $t = e_{k-1,0,1}$, where $k = b_0$. Then the second largest eigenvalue θ_2 of the array will decrease if we decrease r or t or a_d $(=k - c_d)$.*

Proof: By interlacing and Perron-Frobenius. (i) Decreasing r by one means removing the first row and column of M' and then decreasing the top left corner element. (ii) Decreasing t by one means removing the last row and column of M' possibly followed by decreasing the bottom right corner element. (iii) a_d only occurs in the diagonal element $a_d - b_{d-1}$ of M'. □

Let us apply these ideas in the case of valency 4.

Lemma 5.2 *Let Γ be a bipartite distance-regular graph of valency 4, and put $s = e_{202}$. Then $\theta_2(\Gamma) > 4\cos\frac{\pi}{s+1}$.*

Proof: Decrease both r and t to 0. Now M is twice the tridiagonal matrix of a circuit of size $2(s+1)$ and has eigenvalues $4\cos\frac{2\pi j}{2s+2} (0 \le j \le 2s+1)$. □

Similarly, we have for the nonbipartite case:

Lemma 5.3 *Let Γ be a distance-regular graph of valency 4, with $s_2 := e_{121} > 1$. Then $\theta_2(\Gamma) > 2 + 2\cos\frac{\pi}{s_2}$. Moreover, if both $s_1 > 0$ and $s_3 > 0$, then $\theta_2(\Gamma) > 2 + 2\cos\frac{\pi}{s_2+1}$.*

Proof: M' has a submatrix $2I + A$, where A is the adjacency matrix of a path of $s_2 - 1$ vertices, and hence has largest eigenvalue $2 + 2\cos\frac{\pi}{s_2}$. If both s_1 and s_3 are nonzero, then we can pick a submatrix of size $s_2 + 1$ and find $2I + C'$ where C' is a matrix that has as its eigenvalues the different eigenvalues other than 2 of a circuit of size $2(s_2 + 1)$, so that this submatrix has largest eigenvalue $2 + 2\cos\frac{2\pi}{2(s_2+1)}$. □

Lemma 5.4 *Let* Γ *be a distance-regular graph of valency* 4, *and put* $r := e_{103}$. *If* $r > 0$ *then* $\theta_2(\Gamma) > 2\sqrt{3}\cos\frac{\pi}{r}$ *and* $\theta_{\min}(\Gamma) < -2\sqrt{3}\cos\frac{\pi}{r+1}$. *Moreover, each interval* $(2\sqrt{3}\cos\frac{\pi(j+1)}{r}, 2\sqrt{3}\cos\frac{\pi j}{r+1})$ $(j = 1, \ldots, r-1)$ *contains an eigenvalue of* Γ.

Proof: The submatrix of M' formed by rows and columns 1 up to r has eigenvalues ψ_j with $2\sqrt{3}\cos\frac{\pi j}{r} < \psi_j < 2\sqrt{3}\cos\frac{\pi j}{r+1}$ $(j = 1, \ldots, r)$. □

Using Sturm sequences, we can show that in the nonbipartite case the smallest eigenvalue is not too small. (In the bipartite case the smallest eigenvalue equals $-k$, and only a bound on the second smallest eigenvalue would be interesting).

Theorem 5.5 *Let* Γ *be a distance-regular graph of diameter* $d > 1$, *and* σ *a positive real number satisfying*

(i) $\sigma^2 + a_1\sigma - \frac{1}{2}k \geq 0$, *and*
(ii) $\sigma^2 + a_i\sigma - b_{i-1}c_i \geq 0$ $(2 \leq i \leq d-1)$, *and*
(iii) $\sigma^2 + \frac{1}{2}a_d\sigma - \frac{1}{2}b_{d-1}c_d \geq 0$.

Let θ *be the smallest eigenvalue of* Γ. *Then* $\theta \geq -2\sigma$ *with equality if and only if equality holds in all inequalities* (i), (ii), (iii).

Proof: The number of eigenvalues larger than or equal to α equals the number of sign changes in the sequence $u_i(\alpha)$ $(0 \leq i \leq d+1)$ (where a sign change is either a zero entry or a pair of subsequent elements of opposite sign), so we want to show that $u_i(-2\sigma)$ has sign $(-1)^i$ for all i. The u_i are given by $u_0 = 1$, $u_1 = -2\sigma/k$, $c_i u_{i-1} + (a_i + 2\sigma)u_i + b_i u_{i+1} = 0$. Scale the u_i by putting $q_i = b_0 b_1 \cdots b_{i-1} u_i / (-\sigma)^i$. Then $q_0 = 1, q_1 = 2$ and $q_{i+1} = (2 + \frac{a_i}{\sigma})q_i - \frac{b_{i-1}c_i}{\sigma^2}q_{i-1}$. Now the number of eigenvalues smaller than or equal to -2σ equals the number of sign changes of q_i $(0 \leq i \leq d+1)$. By induction on i we show that $q_{i+1} \geq q_i \geq 2$ $(1 \leq i \leq d-1)$. For $i = 1$ this follows from (i), and for $2 \leq i \leq d-1$ from (ii). Finally, $q_{d+1} \geq 0$ then follows from (iii). □

Examples with equality are the flag graphs (of diameter m) of the generalized m-gons of order $(s, t) = (q, q)$. (These have intersection array $\{2q, q, \ldots, q; 1, \ldots, 1, 2\}$. For $q = 1$ we find the even polygons. For $m = 2$ these are the lattice graphs ($(q+1) \times (q+1)$ grid graphs). Examples exist for $m = 3, 4, 6$). All these examples have $\sigma = 1$.

Corollary 5.6 *Let* Γ *be a distance-regular graph of valency* 4, *not bipartite and not a generalized Odd graph. Then the smallest eigenvalue of* Γ *is larger than* $-2\sqrt{3}$.

Proof: This follows directly from the above theorem and Lemma 4.2 (iii). □

According to Lemma 5.4, for large r many roots lie close to $-2\sqrt{3}$, so this bound cannot be improved.

6. Terwilliger's multiplicity bound

Proposition 6.1 (cf. [19]) *Let Γ be a distance-regular graph of valency k, and T a tree in Γ such that for all vertices $u, v, w \in T$, if $d_T(u, v) = d_T(u, w)$ then also $d_\Gamma(u, v) = d_\Gamma(u, w)$. Then the multiplicity f of any eigenvalue $\theta \neq \pm k$ of Γ is at least the number of leaves in T.*

Corollary 6.2 *If $(c_1, a_1, b_1) = (c_r, a_r, b_r) = (1, 0, 3)$, then $f \geq 2 \cdot 3^{r/2}$. Moreover, if r is odd, then $f \geq 4 \cdot 3^{(r-1)/2}$.*

This lower bound on the multiplicity implies that the second largest eigenvalue θ of Γ cannot be too large, otherwise its multiplicity f would be too small.

Let us work out the details for bipartite Γ of valency 4. As before, let $r = e_{103}$, $s = e_{202}$, $t = e_{301}$, so that $d = r + s + t + 1$. Then

$$v = 1 + \underbrace{4 + \cdots + 4 \cdot 3^{r-1}}_{r \text{ terms}} + \underbrace{2 \cdot 3^r + \cdots + 2 \cdot 3^r}_{s \text{ terms}} + \underbrace{4 \cdot 3^{r-1} + \cdots + 4 \cdot 3^{r-t}}_{t \text{ terms}} + 3^{r-t}$$
$$= 1 + 2(3^r - 1) + 2s \cdot 3^r + 2 \cdot 3^{r-t}(3^t - 1) + 3^{r-t} \leq 2(s + 2)3^r - 2$$

(since $t \leq r$).

For any eigenvalue θ distinct from $\pm 2\sqrt{3}$, let us compute $u_i = u_i(\theta)$. Using $u_0 = 1$, $u_1 = \frac{1}{4}\theta$ and the three-term recurrence relation, we find

$$u_i = \alpha\lambda^i - \beta\mu^i \quad (\text{for } 0 \leq i \leq r + 1)$$

where $\alpha = (\frac{1}{4}\theta - \mu)/(\lambda - \mu)$ and $\beta = (\frac{1}{4}\theta - \lambda)/(\lambda - \mu)$, and λ, μ are the two roots of $3x^2 - \theta x + 1 = 0$. Now assume that $2\sqrt{3} < \theta < 4$. Then λ and μ are real, and we can choose them such that $\frac{1}{3} < \mu < \frac{1}{\sqrt{3}} < \lambda < 1$.

For large r we find $u_r \sim \alpha\lambda^r$, and

$$2 \cdot 3^{r/2} \leq f \leq \frac{v}{k_r u_r^2} \lesssim \frac{2(s + 2)3^r}{\frac{4}{3}3^r \alpha^2 \lambda^{2r}}$$

so that

$$(\lambda^2\sqrt{3})^r \lesssim \frac{3(s + 2)}{4\alpha^2} \leq \frac{3(r + 2)}{4\alpha^2}$$

(since $s \leq r$, by Terwilliger, cf. [8], 5.2.5). Consequently, we find a bound on r provided that $\lambda^2\sqrt{3} > 1$ (i.e., $\lambda \gtrsim 0.76$), i.e., provided that $\theta > 3^{1/4} + 3^{3/4}$ (i.e., $\theta \gtrsim 3.6$).

Let us do the precise calculations. Assume that $\theta > 3^{1/4} + 3^{3/4}$ so that $\lambda^2\sqrt{3} > 1$. Since $\theta < 4$ and $\lambda + \mu = \frac{1}{3}\theta$ and $\lambda - \mu = \sqrt{\frac{1}{9}\theta^2 - \frac{4}{3}} > \frac{1}{3}$ we find $3\lambda - \mu > \frac{2}{3} + \frac{1}{3}\theta > \frac{1}{2}\theta$, so that $\alpha > 3\beta$.

Also, $\alpha > 1$ since $\lambda < \frac{1}{4}\theta$ (because $(\lambda - \frac{1}{4}\theta)(\lambda - \frac{1}{12}\theta) = \frac{1}{48}(\theta^2 - 16) < 0$).

Thus, $u_r > ((\frac{\lambda}{\mu})^r - \frac{1}{3})\mu^r$.

Since $\frac{\lambda}{\mu} = \frac{\lambda^2}{\lambda\mu} > \sqrt{3}$ we find for $r \geq 7$ that $u_r \geq 0.99\lambda^r$. Thus, for $r \geq 7$, we have

$$(\lambda^2\sqrt{3})^r < \frac{s+2}{\frac{4}{3}(0.99)^2} < 0.77(s+2) \leq 0.77(r+2).$$

From Lemma 5.2 we know that if s is large, then $\theta := \theta_2$ is large.

Suppose $s \geq 8$. Then $\theta > 4\cos\frac{\pi}{9} > 3.758$. Next, $\lambda = (\theta + \sqrt{\theta^2 - 12})/6 > 0.869$ and $\lambda^2\sqrt{3} > 1.3$ and $(\lambda^2\sqrt{3})^8 > 8 > 0.77 \cdot 10$, a contradiction. Hence $s \leq 7$.

Suppose $s = 7$. Then $\theta > 4\cos\frac{\pi}{8} > 3.695$ and $\lambda > 0.83$ and $\lambda^2\sqrt{3} > 1.193$. Now from $1.193^r < 0.77(s+2) = 6.93$ we find $r \leq 10$.

Suppose $s = 6$ and $r > 0$. Then by Lemma 5.1 we have $\theta > 3.64 > \sqrt{13}$. But if $\theta \geq \sqrt{13}$ and $s \leq 6$, then $\lambda^2\sqrt{3} > 1.02$. Now from $1.02^r < 0.77(s+2) \leq 6.16$ we find $r \leq 91$.

Thus we proved: $s \leq 7$, and if either $s \geq 6$ or $\theta \geq \sqrt{13}$, then $r \leq 91$. Moreover, we have seen already that if $s > 0$, then $(t+1) \mid (r+1)$.

A small computer search of the region $\{(r,s,t) \mid r \leq 100, s \leq 7, t \leq r$ and if $s > 0$ then $(t+1) \mid (r+1)\}$ (using the test described in Section 2) finds only the known examples.

Thus we may now assume in the bipartite case that $r > 100$ and $s \leq 5$ and $\theta < \sqrt{13}$.

Next, consider the non-bipartite case. As before, let $r = e_{103}$, $s_1 = e_{112}$, $s_2 = e_{121}$, $s_3 = e_{211}$, $t = e_{301}$, so that $d = r + s_1 + s_2 + s_3 + t + 1$. Then

$$\begin{aligned}
v = 1 &+ \underbrace{4 + \cdots + 4 \cdot 3^{r-1}}_{r \text{ terms}} + \underbrace{4 \cdot 3^r + \cdots + 2 \cdot 2^{s_1}3^r}_{s_1 \text{ terms}} + \underbrace{4 \cdot 2^{s_1}3^r + \cdots + 4 \cdot 2^{s_1}3^r}_{s_2 \text{ terms}} \\
&+ \underbrace{4 \cdot 2^{s_1-1}3^r + \cdots + 4 \cdot 2^{s_1-s_3}3^r}_{s_3 \text{ terms}} + \underbrace{4 \cdot 2^{s_1-s_3}3^{r-1} + \cdots + 4 \cdot 2^{s_1-s_3}3^{r-t}}_{t \text{ terms}} \\
&+ 4 \cdot 2^{s_1-s_3}3^{r-t}/c_d \\
&= 1 + 2(3^r - 1) + 4 \cdot 3^r(2^{s_1} - 1) + 4 \cdot 2^{s_1}3^r s_2 + 4 \cdot 2^{s_1-s_3}3^r(2^{s_3} - 1) \\
&\quad + 2 \cdot 2^{s_1-s_3}3^{r-t}(3^t - 1) + 4 \cdot 2^{s_1-s_3}3^{r-t}/c_d \\
&= 4(s_2+2)2^{s_1}3^r - 1 - 2 \cdot 3^r - 2 \cdot 2^{s_1-s_3}3^r - (2 - 4/c_d)2^{s_1-s_3}3^{r-t} \\
&\leq 4(s_2+2)2^{s_1}3^r - 1 - 2 \cdot 3^r.
\end{aligned}$$

Thus, we find here for $s_2 > 0$ that

$$2 \cdot 3^{r/2} \leq f \leq \frac{v}{k_r u_r^2} \leq \frac{4(s_2+2)2^{s_1}3^r}{\frac{4}{3}3^r u_r^2}$$

so that for $r \geq 7$ (using $u_r \geq 0.9928\lambda^r$ and $s_1 \leq 2$)

$$(\lambda^2\sqrt{3})^r \leq \frac{3(s_2+2)2^{s_1}}{2(0.9928)^2} < 6.09(s_2+2).$$

Now we want to bound s_2. In the bipartite case we could use $s \leq r$. Here we can use Ivanov's results (cf. [8], Corollary 5.9.6), and find $s_2 \leq r + s_1 + 1 \leq r + 3$.

Suppose $s_2 \geq 13$. Then (by Lemma 5.3) $\theta > 2 + 2\cos\frac{\pi}{13} > 3.94$. Next, $\lambda = (\theta + \sqrt{\theta^2 - 12})/6 > 0.969$ and $\lambda^2\sqrt{3} > 1.626$ and $(\lambda^2\sqrt{3})^r > 129 > 15 \cdot 6.09$, a contradiction. Hence $s_2 \leq 12$.

Suppose $s_2 \geq 6$. Then $\theta > 2 + 2\cos\frac{\pi}{6} = 2 + \sqrt{3}$, and $\lambda > .8534$ and $\lambda^2\sqrt{3} > 1.261$, so that $1.261^r < 85.26$, and $r < 20$.

Suppose $s_2 = 5$ or $s_2 = 4$, $s_1 > 0$, $s_3 > 0$. Then $\theta > 2 + 2\cos\frac{\pi}{5} > 3.61803$, and $\lambda > 0.777$, and $\lambda^2\sqrt{3} > 1.0456$, so that $1.0456^r < 48.72$, and $r < 88$.

If $\theta \geq \sqrt{13} > 3.605$, then $\lambda > 0.7675$ and $\lambda^2\sqrt{3} > 1.02$, $r < 182$.

A computer search of the region $r < 200$ finds only the known parameter sets. Thus, we may now assume in the nonbipartite case that $r \geq 200$, $s_2 \leq 4$, $\theta < \sqrt{13} \approx 3.60555$.

A few more cases can be ruled out using Lemma 5.1. Indeed, if $r = 1$, $t = 0$, $(s_1, s_2, s_3) = (2, 3, 2)$ we find $\theta > 3.61$. For $r = 1$, $t = 0$, $(s_1, s_2, s_3) = (1, 4, 0)$ we find $\theta > 3.61$. For $r = t = 0$, $(s_1, s_2, s_3) = (2, 4, 0)$ we find $\theta > 3.64$. Thus, (s_1, s_2, s_3) is not $(2, 3, 2)$, $(1, 4, 0)$ or $(2, 4, 0)$.

For the middle part (s_1, s_2, s_3) the following 27 possibilities are left: $(0, s_2, 0)$ $(1 \leq s_2 \leq 4)$, $(1, s_2, 0)$, $(2, s_2, 0)$, $(1, s_2, 1)$, $(2, s_2, 1)$ $(0 \leq s_2 \leq 3)$, $(2, s_2, 2)$, $(0 \leq s_2 \leq 2)$, $(3, 0, s_3)$ $(0 \leq s_3 \leq 3)$.

So, what is left now (in both cases) is to find an upper bound on r. To this end, we follow Bannai and Ito [4]. The idea is to compute the multiplicity f_θ of an eigenvalue θ and show that it is different from the multiplicity $f_{\theta'}$ of an algebraically conjugate eigenvalue θ', thus deriving a contradiction. We first need some result that shows that conjugates θ' exist that are sufficiently distinct from θ.

7. The distribution of conjugates of a totally real algebraic number

Given an eigenvalue θ of Γ, we shall want to find a conjugate θ' of θ, not very close to θ. The following theorem shows that not all conjugates can lie in a short interval.

Theorem 7.1 [12] *Suppose θ is an algebraic integer such that it and all its conjugates are real and lie in* $[-2, 2]$. *Then* $\theta = 2\cos\frac{2\pi j}{m}$ *for certain integers j and m.*

All numbers $2\cos\frac{2\pi j}{m}$ with fixed m and $(j, m) = 1$ are conjugate. It follows that if θ and all its conjugates lie in $(-2, 2\cos\frac{2\pi}{n})$, then $\theta = 2\cos\frac{2\pi j}{m}$ with $2 < m < n$. In particular, if θ and all its conjugates lie in $(-2, 2\cos\frac{2\pi}{7})$ (where $2\cos\frac{2\pi}{7} \approx 1.2469796$), then $\theta \in \{-1, 0, 1, (-1 \pm \sqrt{5})/2\}$.

More generally, Schur [17] (p. 391) shows that, given an integer a_0 and a real interval $[p, q]$ of length less than 4, there are only finitely many polynomials $a_0x^n + \cdots + a_n$ with integral coefficients and real distinct roots, all in $[p, q]$.

A slightly better interval is provided by the following:

Theorem 7.2 *Let* $\alpha = (-3+\sqrt{7+2\sqrt{5}})/2 \approx 0.193527$. *Let* θ *be an algebraic integer such that all of its conjugates are real. If* $-\alpha < \theta < 3+\alpha$, *then* θ *has an algebraic conjugate* θ' *with* $\theta' \le -\alpha$ *or* $\theta' \ge 3+\alpha$, *unless* θ *is one of the numbers* $0, 1, 2, 3, (3 \pm \sqrt{5})/2$.

Proof: Let $p \in \mathbf{Z}[X]$ be such that $0 < |p(\theta)| < 1$. Since $\prod p(\theta')$ is integral, where θ' runs over all conjugates of θ, and is nonzero (since θ and θ' are roots of the same polynomials in $\mathbf{Z}[X]$), it has absolute value at least 1, so that for some θ' conjugate to θ we have $|p(\theta')| > 1$. Remains to find, given any β with $0 \le \beta < \alpha$ and $-\beta < \theta < 3+\beta$, a polynomial $p \in \mathbf{Z}[X]$ that satisfies $|p(x)| \le 1$ for $-\beta \le x \le 3+\beta$ and $0 < |p(\theta)| < 1$.

Put $\tau = (1+\sqrt{5})/2$. For any real ξ with $|\xi| < \tau$, the sequence $\xi^{(i)}$ defined by $\xi^{(0)} = \xi$ and $\xi^{(i+1)} = (\xi^{(i)})^2 - 1$ satisfies $|\xi^{(i)}| \le 1$ for almost all i. Starting with the function $f(X) = X^2 - 3X + 1$, which satisfies $|f(x)| < \tau$ for $-\alpha < x < 3+\alpha$, we find after finitely many steps a function $g(X) := f(X)^{(m)}$ that satisfies $|g(x)| \le 1$ for $-\beta \le x \le 3+\beta$ and $|g(\theta)| < 1$.

Remains the question whether perhaps $g(\theta) = 0$. We have $f^{(1)}(X) = X(X-1)(X-2)(X-3)$, which vanishes only on integers. If $f^{(2)}(x) = 0$, then $|f^{(1)}(x)| = 1$, and we find $x = (3 \pm \sqrt{5})/2$. If $f^{(3)}(x) = 0$ but $f^{(1)}(x) \ne 0$, then $f^{(2)}(x) = 1$, $f^{(1)}(x) = \pm\sqrt{2}$, but this only happens for x that have non-real conjugates. □

For the application to distance-regular graphs, suppose that θ is an eigenvalue close to $2\sqrt{3}$. Then θ^2 is close to 12, and has a conjugate outside $[9-\alpha, 12+\alpha]$. In other words, θ has a conjugate θ' with $|\theta'| < 2.968$ or $|\theta'| > 3.491$. Similarly, if $\sqrt{10} < \theta < \sqrt{13}$, then θ^2 has a conjugate outside $[10-\alpha, 13+\alpha]$, so there is a conjugate θ' of θ with $|\theta'| < 3.132$ or $|\theta'| > 3.632$. In this latter case we need not worry about the possibility that $\theta^2 = 10 + (3+\sqrt{5})/2$ in the nonbipartite case, because θ would have a conjugate $-\sqrt{10+(3+\sqrt{5})/2}$ and this is smaller than $-2\sqrt{3}$, contradicting Corollary 5.6.

8. Formulas for the multiplicity

Fix an eigenvalue θ of the tridiagonal matrix M. If we define right and left eigenvectors u and $v^\top$ of M by $Mu = \theta u$ and $v^\top M = \theta v^\top$ and $u_0 = v_0 = 1$, then $v_i = k_i u_i$ and θ has multiplicity $f_\theta = v / \sum k_i u_i^2 = v / \sum u_i v_i = v / \sum (v_i^2/k_i)$.

The u_i satisfy the recurrence

$$c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \theta u_i \quad (\text{and } u_0 = 1, u_{-1} = 0)$$

and the v_i satisfy the recurrence

$$b_{i-1} v_{i-1} + a_i v_i + c_{i+1} v_{i+1} = \theta v_i \quad (\text{and } v_0 = 1, v_{-1} = 0).$$

In order to avoid fractions (and problems with the interpretation of v_{d+1}), it is useful to define $w_i = b_0 \cdots b_{i-1} u_i = c_1 \cdots c_i v_i$. The w_i satisfy the recurrence

$$w_{i+1} = (\theta - a_i) w_i - b_{i-1} c_i w_{i-1} \quad \text{(and } w_0 = 1, w_{-1} = 0).$$

If we regard θ as a variable, then these recurrences define polynomials u_i, v_i, w_i of degree i in θ.

Lemma 8.1

$$\sum_{i=0}^{l} b_i \cdots b_{l-1} c_{i+1} \cdots c_l w_i(X)^2 = w'_{l+1}(X) w_l(X) - w'_l(X)(w_{l+1}(X).$$

Proof: Use induction on l. We have to show that $w'_{l+2}(X) w_{l+1}(X) - w'_{l+1}(X) w_{l+2}(X) = b_l c_{l+1} (w'_{l+1}(X) w_l(X) - w'_l(X) w_{l+1}(X)) + w_{l+1}(X)^2$, and this is clear from the recurrence relation (applied to w_{l+2}). □

Lemma 8.2

$$f_\theta = \frac{v b_0 \cdots b_{d-1} c_1 \cdots c_d}{w'_{d+1}(\theta) w_d(\theta)}.$$

Proof: From the above we find $f_\theta = v / \sum k_i u_i^2 = v b_0 \cdots b_{d-1} c_1 \cdots c_d / (w'_{d+1} w_d - w'_d w_{d+1})$ but $w_{d+1}(\theta) = 0$. □

Put $F_i = c_1 \cdots c_i (v_0 + \cdots + v_i) = \sum_{j=0}^{i} c_{j+1} \cdots c_i w_j$, then F_i satisfies the recurrence

$$F_{i+1} = (\theta - k + b_i + c_{i+1}) F_i - b_i c_i F_{i-1} \quad \text{(and } F_0 = 1, F_{-1} = 0).$$

Now $w_{d+1} = (\theta - k) F_d$ and $w_d = F_d - c_d F_{d-1}$.

Lemma 8.3 *If $\theta \neq k$, then*

$$f_\theta = \frac{v b_0 \cdots b_{d-1} c_1 \cdots c_{d-1}}{(k - \theta) F'_d(\theta) F_{d-1}(\theta)}.$$

Proof: From the above, since $F_d(\theta) = 0$. □

The following theorem, due to Bannai and Ito [4], expresses the dependence of the multiplicity of an eigenvalue θ on $r = e_{1,0,k-1}$. We see that if θ stays away from $\pm 2\sqrt{k-1}$ the multiplicity behaves like Cr^{-1}, while close to $\pm 2\sqrt{k-1}$ the multiplicity is much smaller. A bound on r is obtained by showing that there are conjugate eigenvalues, one close to $\pm 2\sqrt{k-1}$, the other not.

Theorem 8.4 *Let Γ be a distance-regular graph with v vertices, and with valency k, and let θ be an eigenvalue of Γ distinct from $\pm k$ and from $\pm 2\sqrt{k-1}$. Put $r := e_{1,0,k-1}$ and $t := e_{k-1,0,1}$. Put $\delta := 1 - a_d$ if $t > 0$ and $\delta := 0$ if $t = 0$. If $a_1 = 0$, then the multiplicity f_θ of θ is given by*

$$f_\theta = \frac{1}{2} vk \frac{4(k-1)-\theta^2}{k^2-\theta^2} \frac{1}{M_\theta}$$

where

$$M_\theta = r + t \frac{\lambda+\delta}{\lambda+1} \frac{\mu+\delta}{\mu+1} \frac{P\bar{P} - Q\bar{Q}}{R\bar{R}} + \frac{\lambda-\mu}{2} \frac{R\bar{D} - \bar{R}D}{R\bar{R}}$$

where λ, μ are the two roots of $X^2 - \theta X + k - 1 = 0$ (so that $\lambda + \mu = \theta$ and $\lambda\mu = k-1$ and $(\lambda+1)(\mu+1)(\lambda-\mu) \neq 0$) and $P, \bar{P}, Q, \bar{Q}$ are defined by

$$\begin{pmatrix} \bar{P} & \bar{Q} \\ Q & P \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 1 \\ \mu^{-1} & 1 \end{pmatrix} T_{r+1} T_{r+2} \cdots T_{d-t} \begin{pmatrix} -\mu & -\lambda \\ 1 & 1 \end{pmatrix}$$

where

$$T_i = \begin{pmatrix} 0 & -b_{i-1}c_{i-1} \\ 1 & \theta - k + b_{i-1} + c_i \end{pmatrix}$$

(so that $P\bar{P} - Q\bar{Q} = (4(k-1)-\theta^2) b_{r+1}c_{r+1} \cdots b_{d-t-1}c_{d-t-1}$) and $R, \bar{R}$ are defined by

$$R = \frac{\mu+\delta}{\lambda+1} P - \frac{\lambda+\delta}{\lambda+1} Q\sigma^t \quad \text{and} \quad \bar{R} = \frac{\lambda+\delta}{\mu+1} \bar{P} - \frac{\mu+\delta}{\mu+1} \bar{Q}\sigma^{-t}$$

with $\sigma = \lambda/\mu$, and $D, \bar{D}$ are defined by

$$D = \left(\frac{\mu+\delta}{\lambda+1} P\right)' - \left(\frac{\lambda+\delta}{\lambda+1} Q\right)' \sigma^t \quad \text{and} \quad \bar{D} = \left(\frac{\lambda+\delta}{\mu+1} \bar{P}\right)' - \left(\frac{\mu+\delta}{\mu+1} \bar{Q}\right)' \sigma^{-t}.$$

Here $(\cdots)'$ denotes differentiation with respect to θ (so that $\lambda' = \frac{\lambda}{\lambda-\mu}$, $\mu' = \frac{-\mu}{\lambda-\mu}$, $\sigma' = \frac{2\sigma}{\lambda-\mu}$). Note that in case $\theta^2 < 4(k-1)$ the roots λ, μ are conjugate complex numbers, and the bars above denote complex conjugation. In general, the bars denote interchange of λ and μ. For an eigenvalue θ of Γ, we have $\bar{R}\sigma^{r+t} + R = 0$.

Proof: Apply Lemma 8.3 and compute. See [4]. □

This theorem is essentially the special case $a = 0$ of Theorem 2 of [4]. (We could have written the general case, but have no need for that here.) But note that Bannai and Ito take

$\delta = 1 - a_d$, which is correct only if $t > 0$ (that is, the second sentence of their proof is false). The restriction $a_1 = 0$ (that is, $r > 0$) is needed because otherwise $P\bar{P} - Q\bar{Q} = R = 0$ and the expressions become indefinite.

9. Estimates

Now let us estimate f_θ for the case $|\theta| < 2\sqrt{3}$. Continue the notation of the foregoing theorem. Put $\theta = 2\sqrt{3}\cos\phi$, so that $\lambda = \sqrt{3}e^{i\phi}$ and $\mu = \sqrt{3}e^{-i\phi}$ and $\sigma = e^{2i\phi}$. Put $S = (12 - \theta^2)b_{r+1}c_{r+1}\cdots b_{d-t-1}c_{d-t-1}$ so that $P\bar{P} - Q\bar{Q} = S$. From $|P - Q| \geq |P| - |Q|$ we find

$$|R| \geq \left|\frac{\lambda + \delta}{\lambda + 1}\right| \frac{S}{|P| + |Q|}.$$

Finally, using $\lambda\mu = 3$ and $(\lambda + 1)(\mu + 1) = 4 + \theta$,

$$|\lambda - \mu| \cdot |D| \leq \left|\frac{\lambda + \delta}{\lambda + 1}\right| \cdot |\lambda - \mu|(|P'| + |Q'|) + \left|\frac{6 + \mu + \delta\lambda}{4 + \theta}\right| |P| + \left|\frac{(\delta - 1)\lambda}{4 + \theta}\right| |Q|$$

and

$$M_\theta \leq r + t\frac{(|P| + |Q|)^2}{S} + |\lambda - \mu|\frac{|D|}{|R|}.$$

Since $|P|^2 - |Q|^2 = S > 0$, we have $|Q| < |P|$. If $t = 0$, then $\delta = 0$ and $2|P|^2/S$ gets coefficient $(|6 + \lambda| + |\lambda|)/|3 + \lambda| \leq 3 + \sqrt{3}$. If $t > 0$, then $\delta = 1$ and $2|P|^2/S$ gets coefficient $(6 + \theta)/(4 + \theta) \leq 3 + \sqrt{3}$. So, in both cases we have the estimate

$$M_\theta \leq r + t\frac{4|P|^2}{S} + \frac{2|P|}{S}|\lambda - \mu|(|P'| + |Q'|) + \frac{2(3 + \sqrt{3})|P|^2}{S}.$$

For the choices for (s_1, s_2, s_3) listed below, we find for $r \geq 100$, using Sturm, that Γ has precisely two eigenvalues larger than $2\sqrt{3}$ namely the valency 4 and an eigenvalue θ bounded below as listed. Estimating as in Section 6 we find a lower bound on v/f_θ that is exponential in r:

$$\frac{v}{f_\theta} > \frac{4}{3}\alpha^2\left(1 - \frac{\beta}{\alpha}\left(\frac{\mu}{\lambda}\right)^{100}\right)^2 (3\lambda^2)^r := a \cdot c^r.$$

On the other hand, by Theorem 7.2, θ has a conjugate θ_1 with $|\theta_1| < \sqrt{10}$, and the above estimate yields (using $v/f_{\theta_1} < \frac{3}{2}M_{\theta_1}$ and $t \leq r$) an upper bound on v/f_{θ_1} that is linear in r. For sufficiently large r this will yield a contradiction.

In the table below we list the estimates used to obtain a contradiction (for $r \geq r_{\min}$). We write $\mathbf{s} := (s_1, s_2, s_3)$ and $\kappa := \lambda - \mu$ to save some space.

$\mathbf{s}$	θ	$a \cdot c^r$	$\lvert P\rvert$	$\lvert\kappa P'\rvert$	$\lvert\kappa Q'\rvert$	$\frac{3}{2} M_{\theta_1}$	$r_{\min}$
(0, 4, 0)	3.593	$2 \cdot 1.7^r$	320	760	380	$308000\,(r+5)$	100
(2, 3, 1)	3.591	$2 \cdot 1.7^r$	2500	8000	4000	$2350000\,(r+5)$	100
(1, 3, 1)	3.548	$3 \cdot 1.55^r$	700	2100	860	$368000\,(r+5)$	100
(2, 3, 0)	3.537	$4 \cdot 1.5^r$	740	2200	1060	$411000\,(r+5)$	100
(2, 2, 2)	3.502	$6 \cdot 1.34^r$	1750	6000	3200	$575000\,(r+6)$	100
(1, 3, 0)	3.484	$10 \cdot 1.238^r$	210	560	230	$66200\,(r+5)$	100
(2, 2, 1)	3.4703	$29 \cdot 1.127^r$	480	1600	700	$86500\,(r+5)$	110

Thus, assuming $r > 110$, we ruled out 7 of the 27 possible triples (s_1, s_2, s_3). Left are the 20 triples 010, 020, 030, 100, 101, 110, 111, 120, 121, 200, 201, 202, 210, 211, 212, 220, 300, 301, 302, 303.

The second largest eigenvalue for an array increases with t, and increases with a_d. The above 7 cases had a second largest eigenvalue above $2\sqrt{3}$ in the worst case $t = a_d = 0$. (And the same is true for 303, but a contradiction is only obtained for large r.) In a few other cases we find a sufficiently large θ_2 assuming a lower bound on t. The conclusion is that either $r < r_{\min}$ (and our computer search will handle the case), or $t < t_{\min}$ (and we will have a sharp bound on M_θ later).

$\mathbf{s}$	$t_{\min}$	θ	$a \cdot c^r$	$\lvert P\rvert$	$\lvert\kappa P'\rvert$	$\lvert\kappa Q'\rvert$	$\frac{3}{2} M_{\theta_1}$	$r_{\min}$
(0, 3, 0)	5	3.49	$8 \cdot 1.27^r$	64	150	54	$12300\,(r+5)$	100
(3, 0, 3)	2	3.475	$17 \cdot 1.17^r$	800	3000	1800	$30100\,(r+5)$	100
(2, 2, 0)	9	3.473	$21 \cdot 1.154^r$	150	420	190	$16900\,(r+5)$	105
(1, 2, 1)	18	3.466	$86 \cdot 1.068^r$	140	400	150	$14800\,(r+5)$	160
(3, 0, 2)	13	3.466	$86 \cdot 1.068^r$	215	750	420	$4400\,(r+6)$	140
(2, 1, 2)	18	3.466	$86 \cdot 1.068^r$	350	1200	550	$23000\,(r+5)$	165

If no eigenvalue much larger than $2\sqrt{3}$ is available, we can use one very close to $-2\sqrt{3}$ and find a bound on its multiplicity that decreases cubically with r, and again find a contradiction for large r.

Theorem 9.1 ([2], Proposition 6) *If we define S_m by $S_m = \sum_{i=0}^{m} k_i u_i^2$, and put $\theta = 2\sqrt{k-1}\cos\phi$, where ϕ is imaginary if $\theta > 2\sqrt{k-1}$, then S_r is given by*

$$S_r(\theta) = 1 + \frac{2r}{k} + \frac{(k-2)^2}{2k(k-1)} \frac{r}{\sin^2\phi} - \frac{\sin r\phi}{2k(k-1)\sin^3\phi}$$
$$\times((k-1)^2\cos(r+3)\phi - 2(k-1)\cos(r+1)\phi + \cos(r-1)\phi).$$

In our case $k = 4$ this means

$$S_r(\theta) = 1 + r\left(\frac{1}{2} + \frac{1}{6\sin^2\phi}\right) - \frac{\sin r\phi}{24\sin^3\phi} \times (9\cos(r+3)\phi - 6\cos(r+1)\phi + \cos(r-1)\phi).$$

Now choose θ to be the smallest eigenvalue of Γ. We saw earlier that in the nonbipartite case we have $-2\sqrt{3} < \theta < -2\sqrt{3}\cos\frac{\pi}{r+1}$. Thus, $\theta = 2\sqrt{3}\cos\phi$ with $0 < \pi - \phi < \frac{\pi}{r+1}$. Since $S_r(\theta)$ decreases on the interval $0 \le \phi \le 2\pi/r$ (for any fixed $r \ge 11$), and increases on the interval $\pi - 2\pi/r \le \phi \le \pi$ (for any fixed $r \ge 10$), we find

$$v/f_\theta > S_r(\theta) > \frac{r^3}{6\pi^2}.$$

The eigenvalue θ has a conjugate θ' with $|\theta'| < 3.132 < \sqrt{10}$ (note that we already know that $\theta' < \sqrt{13}$), and this conjugate is used with the above estimate on $M_{\theta'}$ in the six cases listed above (where we already have an upper bound on t).

s	$\|P\|$	$\|\kappa P'\|$	$\|\kappa Q'\|$	$t_{\max}$	$\frac{3}{2}M_{\theta'}$	$r_{\min}$
(0, 3, 0)	64	150	54	4	$\frac{3}{2}(r + 65536)$	185
(1, 2, 1)	140	400	150	17	$\frac{3}{2}(r + 215600)$	270
(2, 1, 2)	331	1200	550	17	$\frac{3}{2}(r + 301300)$	300
(2, 2, 0)	150	420	190	8	$\frac{3}{2}(r + 147000)$	240
(3, 0, 2)	215	750	420	12	$\frac{3}{2}(r + 52100)$	170
(3, 0, 3)	800	3000	1800	1	$\frac{3}{2}(r + 140000)$	240

In the remaining 14 cases we known (by use of Sturm) that there are no eigenvalues $\theta' \neq \pm 4$ with $|\theta'| > 2\sqrt{3}$, and we have the stronger conclusion that θ has a conjugate θ' with $|\theta'| < 2.968$.

s	$\|P\|$	$\|\kappa P'\|$	$\|\kappa Q'\|$	$\frac{7}{6}M_{\theta'}$	$r_{\min}$
(0, 1, 0)	5	8	2	$41\,(r + 3)$	100
(0, 2, 0)	12.1	30	10	$229\,(r + 5)$	120
(1, 0, 0)	7	10	4	$40\,(r + 4)$	100
(1, 0, 1)	8	21	9	$27\,(r + 5)$	100
(1, 1, 0)	8	22	6	$51\,(r + 5)$	100
(1, 1, 1)	28	80	28	$307\,(r + 5)$	140
(1, 2, 0)	39	110	37	$1185\,(r + 5)$	270
(2, 0, 0)	12	23	15	$58\,(r + 5)$	100
(2, 0, 1)	14	55	22	$40\,(r + 6)$	100
(2, 0, 2)	50	185	100	$245\,(r + 6)$	125
(2, 1, 0)	28	80	30	$307\,(r + 5)$	140
(2, 1, 1)	90	290	120	$1577\,(r + 6)$	310
(3, 0, 0)	42	122	74	$345\,(r + 5)$	150
(3, 0, 1)	50	185	82	$245\,(r + 6)$	125

Altogether, we used in the nonbipartite case $r \geq 182$ in the Terwilliger bound, and $r \geq 310$ here, so checking $r < 310$ suffices to settle the nonbipartite case.

In the bipartite case, Lemma 5.4 guarantees the existence of an eigenvalue θ with $-2\sqrt{3} < \theta < -2\sqrt{3}\cos\frac{2\pi}{r}$, and we find

$$S_r(\theta) > \frac{r^3}{24\pi^2}.$$

Here we use the existence of a conjugate θ_1 with $|\theta_1| < \sqrt{10}$ (note that we already know that $|\theta'| < \sqrt{13}$ for all eigenvalues $\theta' \neq \pm 4$), and the same arguments as before settle these cases.

s	$\|P\|$	$\|\kappa P'\|$	$\|\kappa Q'\|$	$\frac{3}{2}M_{\theta_1}$	$r_{\min}$
0	4.6	4	0	$43\,(r+3)$	105
1	7.5	12	6	$30\,(r+4)$	100
2	19	35	22	$47\,(r+5)$	115
3	35	100	65	$40\,(r+5)$	100
4	75	260	170	$45\,(r+6)$	110
5	150	650	450	$45\,(r+7)$	110

We did a computer search up to $r = 500$ and found only the known arrays (and four others, as described in Section 2). These computations took about two months on a 275 MHz DEC Alpha running Linux. (The programs were written before many of the refinements discussed above had been discovered. Probably one week would suffice now.)

This completes the proof of our main theorem.

Note

1. The second author very recently succeeded in handling the cases $k = 5, 6, 7$.

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