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MINIMAX INEQUALITY OF KY FAN TYPE IN H-SPACES WITH APPLICATIONS

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Abstract. In this paper, we prove a main result about Ky Fan minimax inequality in any compact H-space. Then, we apply this result to fixed point problem, intersection problem of sets with convex sections, minimax inequality of the Von Neumann type, and variational inequality.

1. Some elementary concepts

We recall some elementary concepts on an H-space (see [3]).

1) Let X be a topological space and $F(X)$ be a family of all nonempty finite subsets of X. Let Γ_A be a family of nonempty contractible subsets of X indexed by $A \in F(X)$ such that $\Gamma_A \subset \Gamma_{A'}$ where $A \subset A'$. The pair $(X, \{\Gamma_A\})$ is called an H-space.

2) The set $C \subset X$ is called H-convex in $(X, \{\Gamma_A\})$ if $\Gamma_A \subset C$ for any finite subset $A \subset C$.

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3) Let $(X,\{\Gamma_A\})$ be an H-space and Y a topological space. Let $\Phi : C \times Y \mapsto \mathbb{R} \cup \{\pm\infty\}$ is a functional and $\lambda \in \mathbb{R}$. For each $y \in Y, \Phi(x,y)$ is called to be *H-quasiconvex* (or *H-quasiconcave*) in X about λ, iff the set

$$\{x \in X : \Phi(x,y) < \lambda\} \quad (\text{or } \{x \in X : \Phi(x,y) > \lambda\})$$

is H-convex in $(X,\{\Gamma_A\})$.

4) Let X and Y be two topological spaces and $r \in \mathbb{R}$. A functional, $\Phi : X \times Y \mapsto \mathbb{R} \cup \{\pm\infty\}$, is called transfer lower–semicontinuous in Y iff, for each $x \in X$ and $y \in Y$, $\Phi(x,y) > r$ implies that there exists a point $x' \in X$ and an open neighborhood $N(y)$ of y such that $\Phi(x',z) > r$ for all $z \in N(y)$.

2. Main results

Lemma 1 (see [8]). *Let $(X,\{\Gamma_A\})$ be a compact H-space and let $T : X \mapsto 2^X$ be a set–valued mapping such that*

(1.1) *for each $x \in X$, $T(x)$ is a nonempty H-convex subset;*
(1.2) *for each $y \in X$, $T^{-1}(y) = \{x \in X : y \in T(y)\}$ contains an open set O_y (O_y may be empty for some y);*
(1.3) $\bigcup\{O_y : y \in X\} = X$.

Then there is a point $x_0 \in T(x_0)$.

Theorem 2. *Let $(X,\{\Gamma_A\})$ be a compact H-space, and $f, g : X \times X \mapsto \mathbb{R}$ be functional such that*

(2.1) *$f \le g$ on $X \times X$;*
(2.2) *$f(x,y)$ is transfer lower-semicontinuous about y;*
(2.3) *$g(\cdot,y)$ is H-quasiconcave.*

Then, for each $\lambda \in \mathbb{R}$, one of the following property holds:

(a) *there exists a point $x_0 \in X$ such that $g(x_0,x_0) > \lambda$;*
(b) *there exists a point $y_0 \in X$ such that $f(x,y_0) \le \lambda$ for all $x \in X$.*

Proof. For each $\lambda \in \mathbb{R}$, let

$$S(y) = \{x \in X : f(x,y) > \lambda\} \quad \text{and} \quad T(y) = \{x \in X : g(x,y) > \lambda\}$$

for each $y \in X$. Then $S(y) \subset T(y)$ by (2.1).

Suppose that there is an $x \in X$, such that $f(x,y) > \lambda$ for each $y \in X$. Then $S : X \mapsto 2^X$ is a multivalued mapping with nonempty values, so is T. By (2.3) we know that $T(y)$ is H-convex for each $y \in X$. Hence the condition (1.1) of Lemma 1 holds.

Moreover for each $x \in X$, since $S(x) \neq \emptyset$, there is a point $y' \in S(x)$, i.e. $f(y',x) > \lambda$. According to (2.2) there exists an open neighborhood $N(x)$

of x and a point $y \in X$ such that $f(y,z) > \lambda$ for each $z \in N(x)$. i.e. for all $z \in N(x), y \in S(z) \subset T(z)$. Hence, $z \in T^{-1}(y)$ for all $z \in N(x)$ which implies $N(x) \subset T^{-1}(y)$. Let $N(x) = O_y$. Then $T^{-1}(y)$ contains an open set O_y, which shows that for each $x \in X$, there exists a point $y \in X$ and an open subset $O_y \subset T^{-1}(y)$ such that

$$\bigcup_{y \in X} O_y = \bigcup_{x \in X} N(x) = X$$

where O_y may be empty for some $y \in X$.

By virtue of Lemma 1, there exists a point $x_0 \in X$ such that $x_0 \in T(x_0)$, i.e. $g(x_0, x_0) > \lambda$.

This completes the proof of Theorem 2. □

Remark 1. If X be a nonempty convex subset of a Hausdorff topological vector space, then result of Theorem 2 was given in [1], but its proof of Theorem is based on the KKM-mapping principle.

Corollary 3. *Let* $(X, \{\Gamma_A\})$ *be a compact* H*-space, and let* $\Phi, \Psi : X \times X \mapsto \mathbb{R}$ *be functionals such that*

(3.1) $\Phi \leq \Psi$ *on the diagonal* $\triangle = \{(x,x) : x \in X\}$ *and* $\Phi \geq \Psi$ *on* $(X \times X) \setminus \triangle$*;*

(3.2) *the function* $\Phi(y,y) - \Phi(x,y)$ *is transfer lower–semicontinuous about* y*;*

(3.3) *the function* $\Psi(\cdot, y)$ *is* H*-quasiconvex.*

Then there exists a point $y_0 \in X$ *such that* $\Phi(y_0, y_0) \leq \Phi(x, y_0)$ *for all* $x \in X$.

Proof. Define f and g on $X \times X$ by setting

$$f(x,y) = \Phi(y,y) - \Phi(x,y) \quad \text{and} \quad g(x,y) = \Psi(y,y) - \Psi(x,y).$$

Then f and g satisfy the hypothesis of Theorem 2. Since $g(x,x) = 0$ for all $x \in X$, Theorem 2 implies that there exists a point $y_0 \in X$, such that $f(x, y_0) \leq 0$ for all $x \in X$ i.e. $\Phi(y_0, y_0) \leq \Phi(x, y_0)$ for all $x \in X$. This completes the proof. □

Remark 2. When $\Phi \equiv \Psi$, Corollary 3 was given by Ky Fan ([4] p. 118, and Corollary 1 in [7]).

3. Some applications

Theorem 4 (Ky Fan minimax inequality)**.** *Let $(X, \{\Gamma_A\})$ be a compact H-space, and $f, g : X \times X \mapsto \mathbb{R}$ be two functionals such that*

(4.1) *$f \leq g$ on $X \times X$;*
(4.2) *$f(x, y)$ is transfer lower–semicontinuous about y;*
(4.3) *$g(\cdot, y)$ is H-quasiconcave.*

Then the minimax inequality $\min_{y\in X} \sup_{x\in X} f(x, y) \leq \sup_{x\in X} g(x, x)$ holds.

Proof. Let $\lambda = \sup_{x\in X} g(x, x)$. If $\lambda = +\infty$ then Theorem is obviously true. So we may assume that $\lambda < +\infty$. Since there is not an $x_0 \in X$ such that $g(x_0, x_0) > \lambda$, Theorem 2 implies that there exists a point $y_0 \in X$ such that $f(x, y_0) \leq \lambda$ for all $x \in X$, which results in $\min_{y\in X} \sup_{x\in X} f(x, y) \leq \sup_{x\in X} g(x, x)$. □

Remark 3. Obviously, Theorem 4 implies Theorem 2. When $f \equiv g$, Theorem 4 reduces to the Ky Fan minimax inequality [6].

Theorem 5. *Let $(X, \{\Gamma_A\})$ be a compact H-space and $A, B \subset X \times X$ such that*

(5.1) *$A \subset B$;*
(5.2) *for each fixed $x \in X$, the set $\{y \in X : (x, y) \in A\}$ is open in X;*
(5.3) *for each fixed $y \in X$, the set $\{x \in X : (x, y) \in B\}$ is H-convex.*

Then one of the following properties holds:

(a) *there exists a point $x_0 \in X$ such that $(x_0, x_0) \in B$;*
(b) *there exists a point $y_0 \in X$ such that $\{x \in X : (x, y_0) \in A\} = \emptyset$.*

Proof. Let f and g be two characteristic functions defined on A and B respectively. Since $A \subset B$, we have $f \leq g$ on $X \times X$. For each fixed $x \in X$ and any $t \in \mathbb{R}$, we have

$$\{y \in X : f(x, y) > t\} = \begin{cases} X, & \text{if} \quad t < 0; \\ \{y \in X : (x, y) \in A\}, & \text{if} \quad 0 \leq t < 1; \\ \emptyset, & \text{if} \quad 1 \leq t. \end{cases}$$

Thus, $f(x, y)$ is transfer lower–semicontinuous about y for each fixed $x \in X$. For each fixed $y \in X$ and any $t \in \mathbb{R}$, we have

$$\{x \in X : g(x, y) > t\} = \begin{cases} X, & \text{if} \quad t < 0; \\ \{x \in X : (x, y) \in B\}, & \text{if} \quad 0 \leq t < 1; \\ \emptyset, & \text{if} \quad 1 \leq t. \end{cases}$$

Thus, $g(x, y)$ is H-quasiconcave on X for each fixed $y \in X$. Let $0 \leq \lambda < 1$. By Theorem 2, there exists a point $x_0 \in X$ such that $g(x_0, x_0) > \lambda$, or there exists a point $y_0 \in X$ such that $f(x, y_0) \leq \lambda$ for all $x \in X$. This implies that there exists a point $x_0 \in X$ such that $(x_0, x_0) \in B$ or there exists a point $y_0 \in X$ such that $(x, y_0) \notin A$ for all $x \in X$, that is, $\{x \in X : (x, y_0) \in A\} = \emptyset$. The conclusion of Theorem be proved. □

Remark 4. Obviously, Theorem 5 implies Theorem 2. Let $\lambda \in \mathbb{R}$ and define

$$A=\{(x, y) \in X \times X : f(x, y) > \lambda\} \text{ and } B=\{(x, y) \in X \times X : g(x, y) > \lambda\}.$$

When $A = B$, Theorem 5 is due to Ky Fan [7] and can be formulated in terms of the complement M of A and the complement N of B as follows.

Corollary 6. *Let $(X, \{\Gamma_A\})$ be a compact H-space and $M, N \subset X \times X$ such that*

(6.1) *$N \subset M$;*
(6.2) *the set $\{y \in X : (x, y) \in M\}$ is closed in X for each fixed $x \in X$;*
(6.3) *the set $\{x \in X : (x, y) \notin N\}$ is H-convex (possibly empty) for each fixed $y \in X$.*

Then one of the following properties holds:

(a) *there exists a point $x_0 \in X$ such that $(x_0, x_0) \notin N$;*
(b) *there exists a point $y_0 \in X$ such that $X \times \{y_0\} \subset M$.*

Proof. In terms of the complement M of A and the complement N of B in Theorem 5, we obtain the conclusion immediately. □

Theorem 7 (Fixed point version)**.** *Let $(X, \{\Gamma_A\})$ be a compact H-space and let $F, G : X \mapsto 2^X$ be two set–valued maps such that*

(7.1) *$F(x) \subset G(x)$ for each $x \in X$;*
(7.2) *$G^{-1}(y)$ is H-convex for each $y \in X$;*
(7.3) *$F(x)$ is open in X for each $x \in X$.*

Then one of the following properties holds:

(a) *there exists a point $w \in X$ such that $w \in G(w)$;*
(b) *there exists a $y_0 \in X$ such that $F^{-1}(y_0) = \emptyset$.*

Proof. Let $A = \{(x, y) : y \in F(x)\}$ and $B = \{(x, y) : y \in G(x)\}$ be two subsets of $X \times X$. We can easily see that Theorem 7 is equivalent to Theorem 5. □

Now, given a Cartesian product $X = \prod\{X_i : 1 \le i \le n\}$ of topological space, let

$$X^i = \prod\{X_j : 1 \le j \le n, j \ne i\} \text{ and } x^i = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X^i.$$

For $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in X$, let

$$(y_i, x^i) = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n).$$

Thus, Theorems of Ky Fan (see [5]–[7]) concerning sets with convex sections can be extended in a natural way.

Theorem 8. *Let $(X_i, \{\Gamma_{A_i}\})$ $(i = 1, 2, \dots, n)$ be n compact H-spaces, and $f_1, \dots, f_n$; $g_1, \dots, g_n$, be real–valued functions on the product space $(X, \{\Gamma_A\})$ where $A = \prod\{A_i : 1 \le i \le n\}$ such that*

(8.1) *$f_i \le g_i$ on X for $i = 1, 2, \dots, n$;*
(8.2) *for each $i = 1, 2, \dots, n$ and for each fixed $x_i \in X_i$, $f_i(x_i, x^i)$ is transfer lower–semicontinuous on X^i;*
(8.3) *for each $i = 1, 2, \dots, n$ and for each fixed $x^i \in X^i$, $g_i(x_i, x^i)$ is H-quasiconcave on X_i.*

Let $t_1, \dots, t_n$ be n real numbers. For each point $x^i \in X^i$, there exists a point $x_i \in X_i$ such that $f_i(x_i, x^i) > t_i$. There exists a point $u \in X$ such that $g_i(u) > t_i$ for $i = 1, 2, \dots, n$.

Proof. For any two points $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ of X, we define

$$\Phi(x, y) = \min_{1 \le i \le n} \{f_i(x_i, y^i) - t_i\} \quad \text{and} \quad \Psi(x, y) = \min_{1 \le i \le n} \{g_i(x_i, y^i) - t_i\}.$$

Then:

1) $\Phi \le \Psi$ on $X \times X$ by (8.1).

2) $\Phi(x, y)$ is transfer lower–semicontionuous on X for each fixed $x \in X$. (If $\Phi(x, y) > t$, where t is real number, then for all $i = 1, 2, \dots, n$ we have $f_i(x_i, y^i) > t + t_i$. By (8.2), that there exists an $\bar{x_i} \in X_i$ and an open neighborhood $N(y^i)$ of y^i such that $f_i(\bar{x_i}, z^i) > t + t_i$ $(i = 1, 2, \dots, n)$. It follows that there exists a point $(\bar{x_i}, y^i) \in X$ and an open neighborhood $N(y_i, y^i)$ such that for $i = 1, 2, \dots, n$, $f_i(\bar{x_i}, z^i) - t_i > t$. Hence, $\Phi(\bar{x}, z) = \min_{1 \le i \le n}\{f_i(\bar{x_i}, z^i) - t_i\} > t$ for all $z \in N(y_i, y^i)$.)

3) $\Psi(x, y)$ is H-quasiconcave on X for each fixed $y \in X$ by (8.3). (For each $t \in \mathbb{R}$ the set

$$\begin{aligned}&\{x \in X : \Psi(x,y) \geq t, \text{ fixed } y \in X\}\\&= \{x \in X : \min_{1\leq i\leq n}\{g_i(x_i, y^i) - t_i\} \geq t, \ 1 \leq i \leq n\}\\&= \{x \in X : \min_{1\leq i\leq n}\{g_i(x_i, y^i) \geq t + t_i\}, \ 1 \leq i \leq n\}\\&= \bigcap_{i=1}^{n}\{x \in X : g_i(x_i, y^i) \geq t + t_i\}\end{aligned}$$

which is H-convex.)

The last part of the hypothesis asserts that for each $y \in X$, where $y^i \in X^i$, there exists an $x_i \in X_i$, such that $f_i(x_i, y^i) > t_i$ for each i $(1 \leq i \leq n)$. Hence,

$$\Phi(x,y) = \min_{1\leq i\leq n}\{f_i(x_i, y^i) - t_i\} > 0.$$

According to Theorem 2, there exists a point $u \in X$ such that $\Psi(u,u) > 0$. Thus, $g_i(u) > t_i$ for $i = 1, 2, \ldots, n$. The theorem is proved. ☐

Theorem 9. *Let $(X_i, \{\Gamma_{A_i}\})$ be n $(n > 1)$ compact H-spaces, and let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be subsets of the product space $X = \prod\{X_i : 1 \leq i \leq n\}$. Suppose that*

(9.1) *$A_i \subset B_i$ for $i = 1, 2, \ldots, n$;*
(9.2) *for each $i = 1, 2, \ldots, n$ and for each $x_i \in X_i$, the set $A_i(x_i) = \{x^i \in X^i : (x_i, x^i) \in A_i\}$ is open in X^i;*
(9.3) *the set $B_i(x^i) = \{x_i \in X_i : (x_i, x^i) \in B_i\}$ is H-convex for each i $(i = 1, 2, \ldots, n)$, and for each point $x^i \in X^i$ the set $A_i(x^i) = \{x_i \in X_i : (x_i, x^i) \in A_i\}$ is nonempty.*

Then $\bigcap_{i=1}^{n} B_i \neq \emptyset$.

Proof. For each i $(i = 1, 2, \ldots, n)$, let f_i and g_i be characteristic functions of sets A_i and B_i respectively. i.e.

$$f_i(x) = \begin{cases} 1, & \text{if } x \in A_i; \\ 0, & \text{if } x \in X \setminus A_i. \end{cases}$$

$$g_i(x) = \begin{cases} 1, & \text{if } x \in B_i; \\ 0, & \text{if } x \in X \setminus B_i. \end{cases}$$

Since $A_i \subset B_i$ $(i = 1, 2, \ldots, n)$, then $f_i \leq g_i$ on X. For each fixed $x_i \in X_i$ $(i = 1, 2, \ldots, n)$ and any $t \in \mathbb{R}$, we have

$$\{x^i \in X^i : f_i(x_i, x^i) > t\} = \begin{cases} X^i, & \text{if } t < 0; \\ A_i(x_i), & \text{if } 0 \leq t < 1; \\ \emptyset, & \text{if } 1 \leq t. \end{cases}$$

which implies that the condition (8.2) is satisfied. Similarly,

$$\{x_i \in X_i : g_i(x_i, x^i) > t\} = \begin{cases} X_i, & \text{if} \quad t < 0; \\ B_i(x^i), & \text{if} \quad 0 \leq t < 1; \\ \emptyset, & \text{if} \quad 1 \leq t. \end{cases}$$

which implies that the condition (8.3) is satisfied.

When $0 \leq t < 1$, since $A_i(x^i) \neq \emptyset$ $(i = 1, 2, \dots, n)$, it follows that the final hypothesis of Theorem 8 is satisfied. Therefore, according to Theorem 8, there exists a point $u \in X$ such that $g_i(u) > t$ for each i $(i = 1, 2, \dots, n)$, i.e. there exists a point $u \in B_i$ for each i $(i = 1, 2, \dots, n)$. Hence, we have $\bigcap_{i=1}^n B_i \neq \emptyset$. This completes the proof. □

Remark 5. If each i $(i = 1, 2, \dots, n)$, we define $A_i = \{u \in X : f_i(u) > t_i\}$ and $B_i = \{u \in X : g_i(u) > t_i\}$, then we can easily see that the Theorem 9 implies Theorem 8.

Theorem 10 (Ben-Ei-Mechaiekh–Degurie–Granas)**.** *Let $(X, \{\Gamma_A\})$ and $(Y, \{\Gamma_B\})$ be two compact H-spaces, and f, s, t, g be four real–valued functions on $X \times Y$ satisfying*

(10.1) $f \leq s \leq t \leq g$ *on* $X \times Y$*;*
(10.2) $f(x, y)$ *is lower–semicontinuous on* Y *for each fixed* $x \in X$;
(10.3) $s(x, y)$ *is* H*-quasiconcave on* X *for each fixed* $y \in Y$;
(10.4) $t(x, y)$ *is* H*-quasiconvex on* Y *for each fixed* $x \in X$;
(10.5) $g(x, y)$ *is upper–semicontinuous on* X *for each fixed* $y \in Y$.

Then for each $\lambda \in \mathbb{R}$*, one of the following property holds:*

(a) *there exists a point* $y_0 \in Y$ *such that* $f(x, y_0) \leq \lambda$; *for all* $x \in X$*;*
(b) *there exists a point* $x_0 \in X$ *such that* $g(x_0, y) \geq \lambda$ *for all* $y \in Y$.

Proof. Let $A_1 = \{(x, y) \in X \times Y : f(x, y) > \lambda\}$; $A_2 = \{(x, y) \in X \times Y : g(x, y) < \lambda\}$; $B_1 = \{(x, y) \in X \times Y : s(x, y) > \lambda\}$; $B_2 = \{(x, y) \in X \times Y : t(x, y) < \lambda\}$, where $\lambda \in \mathbb{R}$.

Suppose that the assertion of theorem is false. Then for each fixed $y \in Y$, $A_1(y) = \{x \in X : f(x, y) > \lambda\} \neq \emptyset$ and for each fixed $x \in X$, $A_2(x) = \{y \in Y : g(x, y) > \lambda\} \neq \emptyset$. It is easy to see that other conditions of Theorem 9 from the hypotheses of Theorem 10. By virtue of Theorem 9, we have $B_1 \cap B_2 \neq \emptyset$. Let $(x_0, y_0) \in B_1 \cap B_2$. Then

$$\lambda < s(x_0, y_0) \leq t(x_0, y_0) < \lambda$$

which is a contradiction. The proof is completed. □

Corollary 11. *Let* $(X, \{\Gamma_A\})$ *and* $(Y, \{\Gamma_B\})$ *be two compact* H*-spaces, and let* f, s, t, g *be four real–valued functions on* $X \times Y$ *satisfying*

(11.1) $f \leq s \leq t \leq g$ *on* $X \times Y$*;*
(11.2) $f(x, y)$ *is lower–semicontinuous on* Y *for each fixed* $x \in X$*;*
(11.3) $s(x, y)$ *is* H*-quasiconcave on* X *for each fixed* $y \in Y$*;*
(11.4) $t(x, y)$ *is* H*-quasiconvex on* Y *for each fixed* $x \in X$*;*
(11.5) $g(x, y)$ *is upper–semicontinuous on* X *for each fixed* $y \in Y$*.*

Then the minimax inequality holds as follows

$$\inf_{y\in Y} \sup_{x\in X} f(x, y) \leq \sup_{x\in X} \inf_{y\in Y} g(x, y).$$

Proof. This is the immediate conclusion of Theorem 10. □

Remark 6. If X and Y are two nonempty compact convex subsets in a Hausdorff topological vector space, Corollary 11 is due to Ben-Ei-Mechaiekh, Deguire and Granas [2].

Corollary 12 (Minimax principle of Von Neumann)**.** *Let*$(X, \{\Gamma_A\})$, $(Y, \{\Gamma_B\})$ *be two compact* H*-spaces and let* $f : X \times Y \mapsto \mathbb{R}$ *be a real–valued functionon satisfying*

(12.1) $f(x, y)$ *is* H*-quasiconvex and lower–semicontinuous for each fixed* $x \in X$*;*
(12.2) $f(x, y)$ *is* H*-quasiconcave and upper–semicontinuous for each fixed* $y \in Y$*.*

Then the minimax inequality holds as follows

$$\min_{y\in Y} \max_{x\in X} f(x, y) = \max_{x\in X} \min_{y\in Y} f(x, y).$$

Proof. Taking $f = s = t = g$ in Corollary 11, we can easily see the minimax equality holds. □

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