JOURNAL OF APPLIED ANALYSIS
Vol. 3, No. 1 (1997), pp. 1–21

# OPTIMAL SYNTHESIS FOR NONOSCILLATORY CONTROLLED OBJECTS

V. BOLTYANSKI AND S. GORELIKOVA

*Received May 20, 1996 and, in revised form, October 16, 1996*

*Abstract.* In the paper we consider nonlinear, nonoscillatory controlled objects of second order. Main Theorem affirms that for these controlled objects there exist (in the controllability region) the time-optimal synthesis of Feldbaum's type. In the beginning of the paper, Felfbaum's $n$-interval Theorem is proved for linear controlled objects of $n$-th order with real eigenvalues (and without the requirement that the eigenvalues are pairwise distinct).

**1. Introduction.** First results in mathematical theory of optimal control were obtained by A. Feldbaum [3], [4]. He considered linear controlled objects of the form

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x \in R^n$ is a column-vector named the *state* of the object, $A$ is a constant $n \times n$ matrix with *real, distinct* eigenvalues, $B \in R^n$ is a constant column-vector, and $u$ is the *control* that runs over the segment $[-1, 1]$

1991 *Mathematics Subject Classification.* Primary 49J15, 49K15, 49N05.
*Key words and phrases.* Optimization, optimal synthesis, controlled object, Feldbaum's $n$-interval Theorem, nonlinear controlled objects, nonoscillatory controlled objects, controllability region, synthesis of Feldbaum's type.
This research was supported by CONACYT (Mexico).

ISSN 1425-6908 

(or, what is unessential, over a segment $[-f,\, f]$ with constant, positive $f$). For this object he solves the *time-optimal problem* which requires to find a control $u = u(t)$ transiting a given initial point $x_0$ to the origin in the shortest time.

In [3], Feldbaum investigates the system

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = u, \qquad -1 \le u \le 1, \tag{2}$$

i.e., the linear controlled object (1) of the second order with the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For this object, he proves: For every initial state $x_0 \in R^2$, there exists an optimal control transiting $x_0$ to the origin; this optimal control is uniquely defined, takes only the values $\pm 1$, and has no more than one switching (i.e., no more than two intervals of constancy).

Feldbaum's reasoning is short and elegant. We sketch it here. To brevity, denote a vector-column $x \in R^2$ with coordinates $x^1,\, x^2$ as $(x^1,\, x^2)^T$, i.e., as transposed row-vector. Let $(x^1(t),\, x^2(t))^T,\, t_0 \le t \le t_1$, be the trajectory connecting $x_0 = (a^1,\, a^2)^T$ with the origin and satisfying the conditions

$$\dot{x}^2(t) \equiv -1 \quad \text{as} \quad t_0 < t < \sigma, \qquad \dot{x}^2(t) \equiv 1 \quad \text{as} \quad \sigma < t < t_1$$

(if $\dot{x}^2(t)$ is equal at first $+1$ and then $-1$, the reasoning is similar). The graph of the function $x^2(t) = \dot{x}^1(t)$ is the union of two segments $[A_0,\, A_\sigma]$ and $[A_\sigma,\, A_1]$ with tangents $-1$ and $+1$ correspondingly. Assume that there exists an another process $y(t) = (y^1(t),\, y^2(t))^T,\, t_0 \le t \le \theta$, with the same endpoints $y(t_0) = x_0,\, y(\theta) = 0$ and the same restriction $|\dot{y}^2(t)| \le 1$ which is more quick than $x(t)$, i.e., $\theta \le t_1$. For $\theta < t \le t_1$, we put $y^1(t) = y^2(t) \equiv 0$. Then both the processes are defined on the same segment $[t_0,\, t_1]$. The graph of $y^2(t)$ cannot intersect the segment $[A_0,\, A_\sigma]$ at a point distinct from $A_0$ (otherwise the restriction $|y^2(t)| \le 1$ would be broken). Similarly, the graph of $y^2(t)$ cannot intersect the segment $[A_\sigma,\, A_1]$ at a point distinct from $A_1$. In other words, the graph of $y^2(t)$ is situated on *one side* of the graph of $x^2(t)$, i.e., the difference $x^2(t) - y^2(t)$ keeps the same sign on the whole segment $[t_0,\, t_1]$. Consequently,

$$\begin{aligned}(x^1(t_1) - y^1(t_1)) - (x^1(t_0) - y^1(t_0)) &= \int_{t_0}^{t_1} (\dot{x}^1(t) - \dot{y}^1(t))\, dt \\ &= \int_{t_0}^{t_1} (x^2(t) - y^2(t))\, dt \ne 0,\end{aligned}$$

contradicting $x^1(t_0) = y^1(t_0) = a_1,\, x^1(t_1) = y^1(t_1) = 0.$ □

We remark that for any initial point $x_0 \in R^2$ there is a *unique* phase trajectory going from $x_0$ to the origin and corresponding to a control with two intervals of constancy. By Feldbaum's reasoning, these and only these trajectories are optimal, i.e., this reasoning gives a necessary and sufficient condition of optimality.

Generalizing these arguments, Feldbaum established in 1953 the following result (for the proof, see Supplement 1 in [4]):

FELDBAUM'S $n$-INTERVAL THEOREM. *Assume that the eigenvalues of the controlled object*

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = x^3, \ ..., \ \dot{x}^{n-1} = x^n, \quad \dot{x}^n = -a_1 x^n - ... - a_n x^1 + u, \qquad |u| \leq 1 \quad (3)$$

*are real and distinct. A control transiting* $x_0 \in R^n$ *to the origin is optimal if and only if it is piecewise constant, takes only the values* $u = \pm 1$, *and has no more than* $n$ *intervals of constancy.*

We remark that under the condition that the eigenvalues are real, this result is *equivalent* to the maximum principle.

With his $n$-interval Theorem, Feldbaum solves the problem of *synthesis* [5] for time-optimal trajectories. Let $x(t)$, $-\infty < t < \infty$ be a trajectory of (3) with $u \equiv +1$. The part of this trajectory for $-\infty < t \leq \theta$ is said to be the *semitrajectory* with the endpoint $x(\theta)$ for the control $u \equiv +1$. Similarly a semitrajectory for the control $u \equiv -1$ is defined. Denote now by $S_n^{(+)}$ the semitrajectory of (3) for $u \equiv +1$ with the endpoint at the origin. Furthermore, by $S_{n-1}^{(-)}$ denote the union of all semitrajectories for $u \equiv -1$ with endpoints belonging to $S_n^{(+)}$. Next denote by $S_{n-2}^{(+)}$ the union of all semitrajectories for $u \equiv +1$ with endpoints in $S_{n-1}^{(-)}$, etc. At last if $n$ is even, $S_1^{(-)}$ is the union of all semitrajectories for $u \equiv -1$ with endpoints in $S_2^{(+)}$ and if $n$ is odd, $S_1^{(+)}$ is the union of all semitrajectories for $u \equiv +1$ with endpoints in $S_2^{(-)}$.

Thus if $n$ is even, we can start from any point $x_0$ of the last "cell" $S_1^{(-)}$ and pass along the "cells" $S_2^{(+)}$, ... $S_{n-1}^{(-)}$, $S_n^{(+)}$ until the arrival to the origin. We obtain a phase trajectory for (3) with $u = \pm 1$ that contains $n$ intervals of constancy. Similarly for odd $n$.

Certainly we have obtained in this way only *one-half* of trajectories corresponding to controls with $n$ intervals of constancy, since only the controls with the last part $u \equiv +1$ were considered. The second half of trajectories can be obtained analogously: we denote by $S_n^{(-)}$ the semitrajectory for $u \equiv -1$ with the endpoint 0, by $S_{n-1}^{(+)}$ the union of all semitrajectories for $u \equiv +1$ with endpoints in $S_n^{(-)}$, by $S_{n-2}^{(-)}$ the union of all semitrajectories for $u \equiv -1$ with endpoints in $S_{n-1}^{(+)}$, etc.

In the union of all the "cells", the synthesis of optimal trajectories is realizable and this union is the *controllability region*, i.e., the set of all points which can be transited to the origin. Thus for real and distinct eigenvalues, Feldbaum gave a necessary and sufficient condition for optimality and also a solution of the synthesis problem.

Later on a complete proof of $n$-interval Theorem for real eigenvalues of the matrix $A$, without the assumption that they are distinct, was established (see, for example, [6]). In this paper we give another proof.

We remark that for $n = 2$, the object (1) with real eigenvalues of $A$ is *nonoscillatory.* In the second part of the article, we generalize $n$-interval theorem for *nonlinear* nonoscillatory objects of the second order. This gives a solution of the synthesis problem for a class of nonlinear objects.

**2. Linear controlled objects.** In the sequel, we assume that the vector $B \in R^n$ is not situated in any proper invariant subspace of $A$, i.e., the vectors

$$B,\ AB,\ \dots,\ A^{n-1}B \tag{4}$$

are linearly independent. In this case, the maximum principle is a necessary and sufficient condition for time-optimality (regardless $A$ has real or complex eigenvalues). More detailed, consider the conjugate equation

$$\dot{\psi} = -\psi A, \tag{5}$$

$\psi$ being a row-vector. Let $\psi(t)$ be a nontrivial solution of (5) and $u(t)$, $t_0 \le t \le t_1$, be an admissible control (i.e., $-1 \le u(t) \le 1$ for all $t$). The control $u(t)$ satisfies the *maximum condition* with respect to $\psi(t)$ if

$$\langle \psi(t),\, Bu(t)\rangle = \max_{-1\le u\le 1} \langle \psi(t),\, Bu\rangle \quad \text{almost everywhere on} \quad [t_0,\, t_1]. \tag{6}$$

The maximum principle affirms that a process $x(t)$, $u(t)$, $t_0 \le t \le t_1$, transiting a point $x_0$ to the origin, is optimal if and only if $u(t)$ satisfies the maximum condition with respect to a nontrivial solution of (5).

Let $u(t)$, $t_0 \le t \le t_1$, be an admissible control satisfying the maximum condition (6) with respect to a nontrivial solution $\psi(t)$ of the conjugate equation (5). We recall that the control $u(t)$, $t_0 \le t \le t_1$, is defined *uniquely* by $\psi(t)$, takes only the values $\pm 1$, and has a finite number of switchings. (For definiteness, here and in the sequel we assume that $u(t) = u(t+0)$ at every point $t < t_1$.) If the matrix $A$ has complex eigenvalues, the number of the intervals of constancy can be arbitrary big (as the interval $[t_0,\, t_1]$ is large enough). But for real eigenvalues (as $n$-interval theorem affirms), the number of the intervals of constancy is not grater than $n$. To establish this, we recall the notion of a quasipoiynomial and prove auxiliary propositions.

A function $f(t)$ is a *quasipolynomial* if it can be represented in the form

$$f(t) = p_1(t)e^{\lambda_1 t} + ... + p_k(t)e^{\lambda_k t}, \tag{7}$$

$p_1(t), ..., p_k(t)$ being polynomials. We consider here only the quasipolynomials with *real, distinct* exponents $\lambda_1, ..., \lambda_k$ and *real* coefficients of all the polynomials $p_1(t), ..., p_k(t)$. If $m_i$ is the power of the polynomial $p_i(t)$, $i = 1, ..., k$, the number $m_1 + ... + m_k + k$ is said to be the *weight* of the quasipolynomial (7).

**Lemma 1.** *An arbitrary nonzero quasipolynomial $f(t)$ of the weight $m$ with real exponents and real coefficients has less than $m$ roots (i.e., the equation $f(t) = 0$ has less than $m$ real, distinct solutions). If $f(t)$ has $m-1$ real, distinct roots, then $f'(t) \neq 0$ at any its root, i.e., $f(t)$ changes sign at each its root.*

*Proof.* Every nonzero quasipolynomial of the weight 1 has the form $f(t) = ce^{\lambda t}$ with $c \neq 0$. Since $\lambda$ and $c$ are real, this quasipolinomial has no roots and this is the beginning for induction.

Assume that Lemma 1 holds for the weight $m-1$ and prove it for the weight $m$. Admit on the contrary, that a nonzero quasipolynomial (7) with the weight $m$ has $m$ (or more) roots. We can assume that each of the polynomials $p_1(t), ..., p_k(t)$ is nontrivial (i.e., at least one of its coefficients is nonzero) and all the exponents $\lambda_1, ..., \lambda_k$ are distinct. The quasipolynomial

$$g(t) = f(t)e^{-\lambda_1 t} = p_1(t) + p_2(t)e^{(\lambda_2-\lambda_1)t} + ... + p_k(t)e^{(\lambda_k-\lambda_1)t}$$

also has $m$ (or more) real roots. Consequently its derivative $g'(t)$ has no less than $m-1$ real roots. This derivative has the form

$$g'(t) = p_1'(t) + q_2(t)e^{(\lambda_2-\lambda_1)t} + ... + q_k(t)e^{(\lambda_k-\lambda_1)t},$$

where every polynomial $q_i(t)$ has *the same* power $m_i$ as $p_i(t)$, $i = 2, ..., k$. Hence the weight of the quasipolynomial $g(t)$ is equal to $m-1$ (indeed, if $m_1 > 0$, then $p_1'(t)$ has the power $m_1 - 1$ and if $m_1 = 0$, then $p_1'(t) = 0$, i.e., the number of exponents in $g(t)$ is equal to $k-1$). Thus $g(t)$ is a quasipolynomial of the weight $m-1$ with $m-1$ (or more) real roots, contradicting the inductive assumption. This completes the proof of the first assertion of the Lemma.

Assume now that $f(t)$ has $m-1$ real roots $\theta_1 < ... < \theta_{m-1}$. Then $g(t)$ has the same roots and hence $g'(t)$ has a root in an *interior* point of every segment $[\theta_i, \theta_{i+1}]$, $i = 1, ..., m-2$. If moreover $f'(\theta_k) = 0$ for an index $k$, then $g'(\theta_k) = 0$, i.e., $g'(t)$ has $m-1$ real, distinct roots, contradicting what was proved above. □

**Lemma 2.** *Let* $\Psi(t)$ *be the matrix solution of the equation* $\dot{\Psi}(t) = -\Psi(t)A$ *with the initial condition* $\Psi(0) = I$, *where* $I$ *is the identity matrix. Then for* $t_0 < \theta_1 < ... < \theta_{n-1}$, *the vectors*

$$\Psi(t_0)B, \quad \Psi(\theta_1)B, \quad \dots, \quad \Psi(\theta_{n-1})B \tag{8}$$

*are linearly independent.*

*Proof.* Assume that the vectors (8) are linearly dependent, i.e., there exists a nonzero row-vector $c = (c_0, c_1, ..., c_{n-1})$ such that $c\Psi(t)B = 0$ for $t = t_0, \theta_1, ..., \theta_{n-1}$. In other words, the function $c\Psi(t)B$ has $n$ real roots $t_0, \theta_1, ..., \theta_{n-1}$. But the row-function $\psi(t) = c\Psi(t)$ is a solution of the equation (5) with the initial condition $\psi(0) = c$. Hence $c\Psi(t)B$ is a quasipolynomial of a weight no greater than $n$ with $n$ real roots. Therefore it is equal to zero *identically* (by Lemma 1), i.e., $\langle c\Psi(t),\, B\rangle \equiv 0$. Consequently its derivative also is equal to zero identically: $\langle c(-\Psi(t)A),\, B\rangle \equiv 0$, i.e., $\langle c\Psi(t),\, AB\rangle \equiv 0$. Taking the derivative once more, we obtain $\langle c\Psi(t),\, A^2B\rangle \equiv 0$ etc. Thus $\langle c\Psi(t),\, A^pB\rangle \equiv 0$ for $p = 0, 1, ..., n-1$, i.e., the row-function $\psi(t) = c\Psi(t)$ is orthogonal to every vector (4) for any $t$. Since the vectors (4) are linearly independent, we conclude $\psi(t) \equiv 0$, contradicting $\psi(0) = c \neq 0$. □

We now establish $n$-interval theorem (without the assumption that the eigenvalues are distinct).

**Theorem 1.** *Assume that the eigenvalues of the matrix* $A$ *are real. Let* $x(t)$, $u(t)$, $t_0 \le t \le 1$, *be an admissible process for the controlled object* (1), $|u| \le 1$, *transiting an initial point* $x_0 = x(t_0)$ *to the origin. This process is optimal if and only if the control* $u(t)$, $t_0 \le t \le t_1$, *is piecewise constant, takes only the values* $\pm 1$, *and has no more than* $n$ *intervals of constancy* (*i.e., no more than* $n-1$ *switchings).*

*Proof.* First establish the part 'only if'. The matrix $-A$ has real eigenvalues. Hence for any solution $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ of (5), the function $b^1\psi_1(t) + ... + b^n\psi_n(t)$ is a quasipolynomial of a weight $m \le n$ with real coefficients and real exponents.

Let now $x(t)$, $u(t)$, $t_0 \le t \le 1$, be an optimal process. Then $u(t)$ satisfies the maximum condition with respect to a nontrivial solution $\psi(t)$ of the conjugate equation. The scalar product

$$\langle \psi(t),\, Bu\rangle = \sum_{i=1}^{n} \psi_i(t) b^i u = \Big(b^1\psi_1(t) + ... + b^n\psi_n(t)\Big)\, u$$

is a linear function of the variable $u \in R^1$. By (6), $u = \operatorname{sign}(b^1\psi_1(t) + \ldots + b^n\psi_n(t))$. Under 'sign' we have a quasipolynomial of a weight $\leq n$. According to Lemma 1, the number of switchings (i.e., the number of real roots of the quasipolynomial) is less than $n$.

We now prove the part 'if'. Let $x(t)$, $u(t)$, $t_0 \leq t \leq 1$, be a process transiting a point $x_0$ to the origin such that the control $u(t)$, $t_0 \leq t \leq t_1$, is piecewise constant, takes only the values $\pm 1$, and has switching points $\theta_1, ..., \theta_p \in [t_0, t_1]$, $p < n$. For definiteness, assume that $u(t) \equiv +1$ as $t_0 < t < \theta_1$, further on $u(t) \equiv -1$ as $\theta_1 < t < \theta_2$ etc. If $p < n - 1$, we choose arbitrary points $\theta_{p+1}, ..., \theta_{n-1}$ with $t_1 < \theta_{p+1} < ... < \theta_{n-1}$.

Denote by $\Gamma$ the hyperplane spanned by all the vectors (8) except for $\Psi(t_0)B$ and by $\bar{c}$ the row-vector orthogonal to this hyperplane with $\langle \bar{c}, \Psi(t_0)B \rangle = 1$ (by Lemma 2, this is possible). Then $\langle \bar{c}\Psi(\theta_i), B \rangle = 0$ for $i = 1, ..., n-1$, i.e., the quasipolinomial $f(t) = \bar{c}\Psi(t)B$ has the roots $\theta_1, ..., \theta_{n-1}$. Moreover, $f(t_0) = \bar{c}\Psi(t_0)B = \langle \bar{c}, \Psi(t_0)B \rangle = 1$. Since $f(t)$ changes its sign at every point $\theta_i$, $i = 1, ..., n-1$ (by Lemma 1), we have $f(t) > 0$ as $t < \theta_1$, further on $f(t) < 0$ as $\theta_1 < t < \theta_2$ etc. In other words, $u(t) = \operatorname{sign} f(t) = \operatorname{sign} \langle \bar{c}\Psi(t), B \rangle$, i.e., $u(t)$ satisfies the maximum condition with respect to the nontrivial solution $\psi(t) = \bar{c}\Psi(t)$ of (5). Consequently by the maximum principle, the considered process is optimal. $\square$

**3. Optimal synthesis for nonoscillatory, nonlinear controlled objects of second order.** Under a condition of *nonoscillation* (defined below), we describe here the synthesis of time-optimal trajectories for a nonlinear controlled object

$$\ddot{y} = f(y, \dot{y}, u), \qquad -1 \leq u \leq 1, \tag{9}$$

where $y$ is a scalar variable. In phase coordinates $x^1 = y$, $x^2 = \dot{y}$ the object (9) is described by the system

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = f(x^1, x^2, u), \qquad -1 \leq u \leq 1. \tag{10}$$

In the sequel, we assume that $f$ has continues derivatives with respect to $x^1$, $x^2$ and

$$f(0, 0, 1) > 0, \quad f(0, 0, -1) < 0, \tag{11}$$

$$f(x^1, x^2, u) \text{ increases with respect to } u \in [-1, 1] \text{ for any fixed } x^1, x^2. \tag{12}$$

By (11), (12), there exists a unique point $u_0 \in [-1, 1]$ such that $f(0, 0, u_0) = 0$. Thus with $u \equiv u_0$, the origin is an *equilibrium point* of the system (10), i.e., as we get the origin by an admissible control, we can stay there any time, putting $u \equiv u_0$.

Besides, we assume that the following two conditions are satisfied:

(A). No phase trajectory of the controlled object (10) goes to infinity in a finite time. For example, this condition is satisfied if there exists a positive constant $M$ such that

$$\left|\frac{\partial f}{\partial x^1}\right| \leq M, \quad \left|\frac{\partial f}{\partial x^2}\right| \leq M \quad \text{for all} \quad x^1,\, x^2,\, u.$$

(B). There exists a function $\varphi(x^1,\, x^2,\, u)$ such that $\varphi,\ \frac{\partial\varphi}{\partial x^1},\ \frac{\partial\varphi}{\partial x^2}$ are continuous and

$$x^2\,\frac{\partial\varphi}{\partial x^1} + f\,\frac{\partial\varphi}{\partial x^2} + (\varphi)^2 - \varphi\,\frac{\partial f}{\partial x^2} - \frac{\partial f}{\partial x^1} \leq 0 \quad \text{for} \quad u = \pm 1 \quad \text{and any} \quad x^1,\, x^2.$$

In particular, (B) is satisfied (putting $\varphi \equiv 0$) if

$$\frac{\partial f}{\partial x^1} \geq 0 \quad \text{for} \quad u = \pm 1 \quad \text{and any} \quad x^1,\, x^2.$$

We remark that if the linear controlled object

$$\ddot{y} = ay + b\dot{y} + u, \qquad -1 \leq u \leq 1, \tag{13}$$

has real eigenvalues, i.e., $b^2 + 4a \geq 0$, then the conditions (11), (12), (A), (B) are satisfied. Indeed to satisfy (B), we can take $\varphi \equiv \frac{b}{2}$. Thus nonlinear controlled objects satisfying (11), (12), (A), (B) generalize linear ones with real eigenvalues.

According to Theorem 1 for the linear controlled object (13), the synthesis of optimal trajectories is realizable in an open set $G \subset R^2$ and every optimal control has no more than two intervals of constancy. We show under the conditions (11), (12), (A), (B), that every nonlinear object (10) possesses the same properties. In the sequel, we consider the object (10) and assume the conditions (11), (12), (A), (B) are satisfied.

**Lemma 3.** *Every optimal control takes only the values* $u = \pm 1$ *and has no more than two intervals of constancy, i.e., no more than one switching.*

*Proof.* For the object (10), the Hamilton function takes the form

$$H(\psi,\, x,\, u) = \psi_1 x^2 + \psi_2 f(x^1,\, x^2,\, u).$$

By (12), the function $f(x^1,\, x^2,\, u)$ takes its maximal and minimal values (with respect to $u \in [-1,\, 1]$) at the points $u = 1$, $u = -1$ respectively. This means that the maximum condition

$$H(\psi(t),\, x(t),\, u(t)) = \max_{-1 \leq u \leq 1} H(\psi(t),\, x(t),\, u)$$

is equivalent to the relation

$$u = \operatorname{sign} \psi_2(t) \quad \text{as} \quad \psi_2(t) \neq 0. \tag{14}$$

Let now $x(t)$, $u(t)$, $t_0 \le t \le t_1$, be an optimal process transiting an initial point $x_0$ to the origin. Then $u(t)$ satisfies the maximum condition with respect to a nontrivial solution $\psi(t) = (\psi_1(t), \psi_2(t))$, $t_0 \le t \le t_1$, of the conjugate system

$$\begin{aligned} \dot{\psi}_1 &= -\frac{\partial H}{\partial x^1} = -\psi_2 \frac{\partial f(x(t),\, u(t))}{\partial x^1}, \\ \dot{\psi}_2 &= -\frac{\partial H}{\partial x^2} = -\psi_1 - \psi_2 \frac{\partial f(x(t),\, u(t))}{\partial x^2}. \end{aligned} \tag{15}$$

We have to prove that the function $\psi_2(t)$ has no interval of constancy and no more than one changing of sign.

Assuming $\psi_2(t) \equiv 0$ on a segment $I \subset [t_0, t_1]$, we obtain from (15), $\psi_1(t) \equiv 0$ and hence $\psi(t) \equiv 0$ on $[t_0, t_1]$, contradicting $\psi(t)$ is a *nontrivial* solution of the conjugate system. Thus the equation $\psi_2(t) = 0$, $t_0 \le t \le t_1$, has a closed, 0-dimensional set of roots. Assume that this equation has more than one root. Let $\alpha < \beta$ be two *adjacent* roots, i.e., $\psi_2(\alpha) = \psi_2(\beta) = 0$ and $\psi_2(t) \ne 0$ for $\alpha < t < \beta$. Assume $\psi_2(t)$ is *negative* as $\alpha < t < \beta$ (if it is positive, the reasoning is similar). Then $\dot{\psi}_2(\alpha) \le 0$, $\dot{\psi}_2(\beta) \ge 0$. By (15),

$$\psi_1(\alpha) = -\dot{\psi}_2(\alpha) \ge 0, \quad \psi_1(\beta) = -\dot{\psi}_2(\beta) \le 0.$$

Besides, $\psi_1(\alpha) \ne 0$, $\psi_1(\beta) \ne 0$ (since the solution $\psi(t)$ is nontrivial) and hence

$$\psi_1(\alpha) > 0, \quad \psi_1(\beta) < 0.$$

The condition (14) implies $u \equiv -1$ for $\alpha < t < \beta$. Consequently by (15), $\psi(t)$ satisfies (for $\alpha < t < \beta$) the relations

$$\dot{\psi}_1 = -\psi_2 \frac{\partial f(x(t),\, -1)}{\partial x^1}, \quad \dot{\psi}_2 = -\psi_1 - \psi_2 \frac{\partial f(x(t),\, -1)}{\partial x^2}.$$

We now put

$$\eta(t) = \psi_1(t) + \psi_2(t)\, \varphi(x^1(t),\, x^2(t),\, -1),$$

where $\varphi$ is the function indicated in the condition (B). Then

$$\begin{aligned} \dot{\eta} = \dot{\psi}_1 + \dot{\psi}_2 \varphi + \psi_2 \dot{\varphi} &= -\psi_2 \frac{\partial f}{\partial x^1} - \varphi \left( \psi_1 + \psi_2 \frac{\partial f}{\partial x^2} \right) + \psi_2 \left( \frac{\partial \varphi}{\partial x^1} x^2 + \frac{\partial \varphi}{\partial x^2} f \right) \\ &= \psi_2 \left( x^2 \frac{\partial \varphi}{\partial x^1} + f \frac{\partial \varphi}{\partial x^2} + (\varphi)^2 - \varphi \frac{\partial f}{\partial x^2} - \frac{\partial f}{\partial x^1} \right) - \varphi \eta. \end{aligned}$$

Since by the condition (B), the expression in the last brackets is nonpositive and $\psi_2(t) \le 0$ for $t \in [\alpha, \beta]$, we obtain

$$\dot{\eta} \ge -\eta \varphi(x^1(t),\, x^2(t),\, -1), \qquad \alpha \le t \le \beta. \tag{16}$$

Taking into account that $\psi_2(\alpha) = \psi_2(\beta) = 0$, we find, according to definition of $\eta(t)$,

$$\eta(\alpha) = \psi_1(\alpha) > 0, \quad \eta(\beta) = \psi_1(\beta) < 0,$$

and hence there is at least one root of the equation $\eta(t) = 0$ between $\alpha$ and $\beta$. Let $\gamma \in [\alpha, \beta]$ be the root nearest to $\alpha$. Then

$$\eta(\gamma) = 0 \quad \text{and} \quad \eta(t) > 0 \quad \text{for} \quad \alpha \le t < \gamma.$$

By (16), this implies

$$\frac{\dot{\eta}}{\eta} \ge -\varphi(x^1(t), x^2(t), -1) \quad \text{for} \quad \alpha \le t < \gamma.$$

Integrating, we find

$$\ln \eta(t) - \ln \eta(\alpha) \ge -\int_\alpha^t \varphi(x^1(t), x^2(t), -1)\, dt \quad \text{for} \quad \alpha \le t < \gamma.$$

Hence

$$\eta(\alpha) \le \eta(t) e^{\int_\alpha^t \varphi(x^1(t), x^2(t), -1)\, dt} \quad \text{for} \quad \alpha \le t < \gamma.$$

As $t \to \gamma$, this implies

$$\eta(\alpha) \le \eta(\gamma) e^{\int_\alpha^\gamma \varphi(x^1(t), x^2(t), -1)\, dt},$$

contradicting $\eta(\gamma) = 0$, $\eta(\alpha) > 0$. □

We now repeat Feldbaum's synthesis construction for the nonlinear controlled object (10). Denote by $L^{(+)}$ the semitrajectory of (10) for $u \equiv +1$ with the endpoint at the origin. Furthermore, by $S^{(-)}$ denote the union of all semitrajectories for $u \equiv -1$ with endpoints belonging to $L^{(+)}$. Similarly, denote by $L^{(-)}$ the semitrajectory for $u \equiv -1$ with the endpoint at the origin and by $S^{(+)}$ the union of all semitrajectories with $u \equiv +1$ and endpoints in $L^{(-)}$. Thus we can start from any point $x_0$ of the 2-dimensional cell $S^{(-)}$ and pass along the cells $S^{(-)}$, $L^{(+)}$ until the arrival to the origin. We also can start from any point $x_0$ of the 2-dimensional cell $S^{(+)}$ and pass along the cells $S^{(+)}$, $L^{(-)}$ until the arrival to the origin. Finally, we can start from any point $x_0 \ne 0$ of the one-dimensional cell $L^{(+)}$ (or $L^{(-)}$) and move along this cell (with constant control $u = \pm 1$) until the arrival to the origin. We obtain the phase trajectories of the object (10) for $u = \pm 1$ which contain no more than two intervals of constancy.

Put $L = \{0\} \cup L^{(+)} \cup L^{(-)}$ and $G = L \cup S^{(+)} \cup S^{(-)}$. In the set $G$, we have the synthesis of trajectories for the object (10) with no more than two intervals of constancy. The curve $L$ is the *switching line* for these trajectories. In the sequel, this synthesis is said to be *Felfbaum's synthesis* for the nonlinear controlled object (10).

We are going to prove that this is the *optimal* synthesis in the set $G$ and moreover, $G$ is the *controllability region*, i.e., if $x_0 \notin G$, then it is impossible to transit $x_0$ to the origin by any admissible control.

**Lemma 4.** *Let $x(t) = (x^1(t), x^2(t))^T$ be a trajectory of* (10) *corresponding to a constant control $u = \pm 1$. Then $x(t)$ has no more than one common point with $x^1$-axis.*

*Proof.* For definiteness, assume that $x(t)$ corresponds to the control $u \equiv 1$ (for $u \equiv -1$ the reasoning is similar). We can assume that $x(t)$ is not an equilibrium point, i.e., the relation $x(t) \equiv \text{const}$ does not hold. If $x(\tau)$ belongs to $x^1$-axis, then

$$\dot{x}^1(\tau) = x^2(\tau) = 0, \quad \dot{x}^2(\tau) = f(x^1(\tau), 0, 1) \neq 0, \tag{17}$$

since $x(\tau)$ is not an equilibrium point for (10). In other words, the trajectory $x(t)$ has at the moment $\tau$ a vertical tangent and hence for $t$ close to $\tau$ the point $x(t)$ is not situated in $x^1$-axis. Thus the trajectory $x(t)$ intersects $x^1$-axis only at *isolated* moments.

Assume that $x(t)$ intersects $x^1$-axis more than one time. Let $\alpha$ and $\beta$ be two *adjacent* intersectional moments, i.e., $x^2(\alpha) = x^2(\beta) = 0$ and $x^2(t) \neq 0$ for $\alpha < t < \beta$. For definiteness, assume that $x^2(t)$ is *positive* for $\alpha < t < \beta$.

We now put

$$\eta(t) = \dot{x}^2(t) - \varphi(x^1(t), x^2(t), 1)\,\dot{x}^1(t) = f(x^1(t), x^2(t), 1) - \varphi(x^1(t), x^2(t), 1)\,x^2(t),$$

where $\varphi$ is the function indicated in the condition (B). Then

$$\dot{\eta} = -x^2(t)\left(\frac{\partial\varphi}{\partial x^1}x^2 + \frac{\partial\varphi}{\partial x^2}f + (\varphi)^2 - \varphi\frac{\partial f}{\partial x^2} - \frac{\partial f}{\partial x^1}\right) + \eta\left(\frac{\partial f}{\partial x^2} - \varphi\right).$$

Taking into account the condition (B) and the relation $x^2(t) \geq 0$ for $\alpha \leq t \leq \beta$, we obtain

$$\dot{\eta} \geq \eta\left(\frac{\partial f}{\partial x^2} - \varphi\right) \quad \text{for} \quad \alpha \leq t \leq \beta.$$

Besides,

$$\eta(\alpha) = \Big(f - \varphi x^2\Big)_{t=\alpha} = f(x^1(\alpha), x^2(\alpha), 1) = \dot{x}^2(\alpha),$$

$$\eta(\beta) = \Big(f - \varphi x^2\Big)_{t=\beta} = f(x^1(\beta), x^2(\beta), 1) = \dot{x}^2(\beta).$$

Since $\dot{x}^2(\alpha) \neq 0$, $\dot{x}^2(\beta) \neq 0$ (cf. (17)) and $x^2(t)$ is positive for $\alpha < t < \beta$, we conclude that $\dot{x}^2(\alpha) > 0$, $\dot{x}^2(\beta) < 0$, i.e., $\eta(\alpha) > 0$, $\eta(\beta) < 0$. Hence there is at least one root of the equation $\eta(t) = 0$ between $\alpha$ and $\beta$.

Now we come to a contradiction by just the same way as in the proof of Lemma 3. □

**Lemma 5.** *The switching line* $L$ *is situated in the union of* 2*-nd and* 4*-th quadrants and is projected in a one-to-one manner into* $x^1$*-axis.*

*Proof.* At the origin, the semitrajectory $L^{(+)}$ has the phase velocity $\dot{x}^1(0) = 0$, $\dot{x}^2(0) = f(0, 0, 1) > 0$, i.e., the phase point moves upwards. This means that $L^{(+)}$ approaches to the origin from the lower half-plane. Moreover, having a common point with $x^1$-axis at the origin, the semitrajectory $L^{(+)}$ cannot have other common points with $x^1$-axis (by Lemma 4). Consequently this semitrajectory is completely contained in the open lower half-plane. Now (10) implies $\dot{x}^1 = x^2 < 0$ on the semitrajectory $L^{(+)}$, i.e., the phase point moves along $L^{(+)}$ *to the left*. It follows that the semitrajectory is situated in the interior of the 4-th quadrant and is projected in a one-to-one manner into positive $x^1$-semi axis.

Similarly, the semitrajectory $L^{(-)}$ is situated in the interior of the 2-nd quadrant and is projected in a one-to-one manner into negative $x^1$-semi axis. Hence the switching line $L$ is projected in a one-to-one manner into $x^1$-axis. At the origin, the switching line has vertical tangent. □

**Lemma 6.** *Let* $X$ *be a semitrajectory of* (10) *corresponding to the control* $u \equiv -1$ *and ending at a point* $a \in L^{(+)}$*. Then* $X$ *does not have any another common point with* $L$ *except* $a$*. For semitrajectories of* (10) *corresponding to* $u \equiv 1$ *and ending at points of* $L^{(-)}$*, the similar assertion holds.*

*Proof.* Since $X$ and $L^{(-)}$ satisfy the same system

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = f(x^1, x^2, -1),$$

they have no common points. Thus we have to prove that $X$ and $L^{(+)}$ have no common points except $a$. At the point $a = (a^1, a^2)^T$, the semitrajectories $X$, $L^{(+)}$ have the phase velocities $\big(a^2, f(a^1, a^2, -1)\big)^T$ and $\big(a^2, f(a^1, a^2, +1)\big)^T$, respectively. By (12), $f(a^1, a^2, +1) > f(a^1, a^2, -1)$ and hence at the point $a$, the semitrajectory $X$ approaches to $L^{(+)}$ *from above*.

Assume that there is a point $b \in X \cap L^{(+)}$ distinct from $a$. Then $X$ intersects $L$ *from above* at $b$ and goes *to the left* under the arc of $L^{(+)}$ with endpoints $b$ and $0$ (since $\dot{x}^1 = x^2 < 0$ under this arc). Hence after passing through $b$, the semitrajectory $X$ intersects negative $x^2$-semi axis. Then (to get the point $a$) $X$ has to intersect negative $x^1$-semi axis (for going to the right) and to intersect $x^1$-axis once more (since $a$ is situated in the *lower* half-plane). But by Lemma 4, $X$ cannot intersect $x^1$-axis twice. □

**Lemma 7.** *The cells $S^{(+)}$ and $S^{(-)}$ have no common points outside of $L$.*

*Proof.* Assume, on the contrary, that there is a point $p \in S^{(+)} \cap S^{(-)}$ which is not belonging to $L$. Since $p \in S^{(-)}$, there is a semitrajectory $X'$ through $p$ corresponding to $u \equiv -1$ and ending at a point $a' \in L^{(+)}$. Similarly since $p \in S^{(+)}$, there is a semitrajectory $X''$ through $p$ corresponding to $u \equiv +1$ and ending at a point $a'' \in L^{(-)}$.

The point $a' \in L^{(+)}$ is situated in 4-th quadrant and hence $X'$ either is situated completely in 4-th quadrant or comes into 4-th quadrant, intersecting positive $x^1$-semi axis. In both the cases, the point $p \in X'$ does not belong to negative $x^1$-semi axis. Considering $X''$, we conclude similarly that $p$ does not belong to negative $x^1$-semi axis.

For definiteness, assume that $p$ is situated *above* $x^1$-axis (if below, the reasoning is similar). Then $X'$ goes firstly into the upper half-plane (and passes through $p$) and then enters into the 4-th quadrant, intersecting positive $x^1$-semi axis at a point $q$. Consider the arc of $X''$ with the endpoints $p$, $a''$ and the arc of $L^{(-)}$ with the endpoints $a''$, 0. The union of these arcs denote by $P^{(-)}$. By $P^{(+)}$ denote the arc of $X'$ with the endpoints $p$, $q$. Both the arcs $P^{(-)}$, $P^{(+)}$ are situated in the upper half-plane and hence $\dot{x}^1 = x^2 > 0$ on them, i.e., phase points describing these arcs move along them *to the right*. Consequently both the arcs are projected in one-to-one manner into $x^1$-axis.

The part $Q$ of $P^{(-)}$ with the endpoints $p$, $a''$ corresponds to $u \equiv +1$ and hence (by (12)) the phase trajectories corresponding to $u \equiv -1$ intersect $Q$ from above. In particular, $P^{(+)}$ goes out of the point $p$, being situated *below* $Q$. Going further to the right, $P^{(+)}$ cannot intersect $Q$ (since the trajectories with $u \equiv -1$ intersect $Q$ from above) and also cannot intersect $L^{(-)}$ (since $P^{(+)}$ and $L^{(-)}$ satisfy (10) with the same control $u \equiv -1$). But this contradicts the fact that $P^{(+)}$ arrives to the point $q$ situated in *positive* $x^1$-semi axis. □

For every point $x \in G$, denote by $\omega(x)$ the transit time along the trajectory of Feldbaum's synthesis starting at $x$ and arriving to the origin (*Bellman's function*). We remark that, by Lemma 7, for any $x \in G$, the trajectory of Feldbaum's synthesis going from $x$ to the origin is defined uniquely and hence $\omega(x)$ is well-defined on $G$.

**Lemma 8.** *The set* $G \subset R^2$ *is open and Bellman's function* $\omega(x)$ *defined in* $G$ *is continuous. For every* $T > 0$, *the set* $\Sigma_T = \{x \in G : \omega(x) \le T\}$ *is compact.*

*Proof.* Consider the trajectory $\Gamma$ of the object (10) corresponding to the control $u \equiv 1$ and passing through the origin. The part of $\Gamma$ in the lower half–plane coincides with $L^{(+)}$. The part of $\Gamma$ in the upper halfplane is situated in the first quadrant (since $\dot{x}^1 = x^2 > 0$ in the upper halfplane). Hence the halftrajectory $L^{(-)}$ approaches to $\Gamma$ from the left.

Let $p, q \in \Gamma$ be points in lower and upper half–planes correspondingly and $Q$ be a disk centered at $q$ and contained in the upper half–plane. By Theorem on continuity of solutions of ordinary differential equation with respect to initial condition, any trajectory of (10) with $u \equiv 1$ starting at a point $p'$ close enough to $p$ has common points with $Q$. Hence if $p'$ (close enough to $p$) is situated *under* $L^{(+)}$, then the trajectory $\gamma$ of (10) with $u \equiv 1$ starting from $p'$ passes to the left from the origin and then enters into the first quadrant. This means that $\gamma$ intersects $L^{(-)}$ at a point $q'$. Thus if we move along $\gamma$ from $p'$ till $q'$ and then along $L^{(-)}$, we can get the origin.

This reasoning shows that all the points close enough to $L^{(+)}$ and situated *under* $L^{(+)}$ belong to $G$, and also all the points close enough to the origin and situated *to the left* from $L$ belong to $G$. Similarly, all the points close enough to $L^{(-)}$ and situated *above* $L^{(-)}$ belong to $G$, and also all the points close enough to the origin and situated *to the right* from $L$ belong to $G$. Consequently the origin is an *interior* point of $G$.

Furthermore, near every point of $L^{(+)}$ distinct from the origin, the semi-trajectories of (10) with $u \equiv -1$ approach $L^{(+)}$ from *above*. Hence all the points close enough to $L^{(+)}$ and situated *above* $L^{(+)}$ belong to $G$. Comparing with aforesaid, we conclude that every point of $L^{(+)}$ is an *interior* point of $G$. Similarly for $L^{(-)}$. Thus all the points of the switching line $L$ are interior ones for $G$.

Let finally, $x_0 \in G \setminus L$, say, $x_0 \in S^{(-)}$. Starting from $x_0$ with $u \equiv -1$, we reach $L^{(-)}$ at a nonzero angle $\alpha$ and then get the origin with $u \equiv 1$. Since $\alpha \ne 0$, moving from any point close enough to $x_0$ with $u \equiv -1$, we also reach $L^{(+)}$ and then can get the origin along $L^{(+)}$. This means that all the points close enough to $x_0$ are contained in $G$, i.e., $x_0$ is an interior point of $G$.

We have established that $G$ is open. At the same time, this reasoning shows that the function $\omega(x)$ is continuous.

Let now $T$ be a positive number. Denote by $X_T^{(+)}(0)$ the arc of the trajectory with $u \equiv 1$ ending at the origin and corresponding to the time

$T$. In other words, we consider the process $x(t)$, $u(t) \equiv 1$, $0 \le t \le T$, with the endpoint $x(T) = 0$ and denote by $X_T^{(+)}(0)$ the set of all phase points of these processes. Similarly, denote by $X_T^{(-)}(0)$ the arc of the trajectory with $u \equiv -1$ ending at the origin and corresponding to the time $T$. Furthermore for every point $x_0 \in X_T^{(+)}(0)$, denote by $X_T^{(-)}(x_0)$ the arc of the trajectory with $u \equiv -1$ ending at $x_0$ and corresponding to the time $T$. Similarly, for every point $x_0 \in X_T^{(-)}(0)$, denote by $X_T^{(+)}(x_0)$ the arc of the trajectory with $u \equiv 1$ ending at $x_0$ and corresponding to the time $T$. The union $M^T$ of all these arcs is compact. Indeed, by the condition (A), every arc $X_T^{(+)}(x_0)$ is a compact set depending continuously on $x_0$; similarly for $X_T^{(-)}(x_0)$.

It remains to remark that the set $\Sigma_T = \{x \in G : \omega(x) \le T\}$ is *contained* in $M_T$ and is closed in $M_T$ (since $\omega(x)$ is continuous on the set $M_T \subset G$ and the inequality $\omega(x) \le T$ defines its *closed* subset). □

**Lemma 9.** *Any process of the Feldbaum's synthesis for nonlinear controlled object* (10) *satisfies the maximum condition with respect to a nontrivial solution of the conjugate system.*

*Proof.* For definiteness, consider a process $u(t)$, $x(t)$, $t_0 \le t \le t_1$, transiting $x_0$ to the origin and satisfying $u(t) \equiv 1$ as $t_0 \le t \le \tau$, $u(t) \equiv -1$ as $\tau < t \le t_1$, where $t_0 < \tau < t_1$. Denote by $\psi(t) = (\psi_1(t), \psi_2(t))$ the solution of the conjugate system (15) corresponding to the considered process with $\psi_1(\tau) = 1$, $\psi_2(\tau) = 0$. Then by (15), $\dot{\psi}_2(\tau) = -\psi_1(\tau) = -1$. Consequently (since $\psi_2(t)$ can change sign no more than *one* time, cf. the proof of Lemma 3),

$$\psi_2(t) > 0 \quad \text{as} \quad t_0 < t < \tau \quad \text{and} \quad \psi_2(t) < 0 \quad \text{as} \quad \tau < t < t_1.$$

This means that the process $u(t)$, $x(t)$, $t_0 \le t \le t_1$, satisfies the maximum condition with respect to $\psi(t)$. □

To formulate further lemmas, we remark that for every $x_0 \in G \setminus L$, the trajectory of Feldbaum's synthesis going from $x_0$ to the origin meets the switching line $L$ at a *nonzero* angle. Consequently $\omega(x)$ is a *smooth* function on $G \setminus L$, i.e., the derivatives $\frac{\partial \omega(x)}{\partial x^1}$, $\frac{\partial \omega(x)}{\partial x^2}$ exist and are continuous in $G \setminus L$. In other words, in $G \setminus L$ the function $\omega(x)$ has a nonzero, continuous *gradient*

$$\operatorname{grad} \omega(x_0) = \left( \frac{\partial \omega(x_0)}{\partial x^1}, \frac{\partial \omega(x_0)}{\partial x^2} \right).$$

If now we start from a point $x \in G \setminus L$ under action of a control $u(t)$, then change of the function $\omega(x)$ along the corresponding trajectory $x(t)$ is described by the derivative

$$\frac{d\omega(x(t))}{dt} = \langle \operatorname{grad} \omega(x),\, \dot{x}(t) \rangle = \frac{\partial \omega(x)}{\partial x^1} x^2 + \frac{\partial \omega(x)}{\partial x^2} f(x^1, x^2, u).$$

Along the trajectories of Feldbaum's synthesis this derivative is equal to $-1$ (by definition of the function $\omega(x)$). It is naturally to expect (assuming that Feldbaum's synthesis is in fact optimal) that for any other trajectory, the module of this derivative is *less* than 1, i.e., the following *Bellman's equation* holds:

$$-\frac{\partial \omega(x)}{\partial x^1} x^2 - \frac{\partial \omega(x)}{\partial x^2} f(x^1, x^2, v(x)) = \max_{-1 \le u \le 1} \Big( -\frac{\partial \omega(x)}{\partial x^1} x^2 - \frac{\partial \omega(x)}{\partial x^2} f(x^1, x^2, u) \Big) = 1, \quad (18)$$

where $v(x)$ is the control for Feldbaum's synthesis, i.e., $v(x) = 1$ as $x \in S^{(+)}$ and $v(x) = -1$ as $x \in S^{(-)}$.

**Lemma 10.** *For nonlinear controlled object* (10), *Bellman's equation holds in* $G \setminus L$.

*Proof.* Let $x_0 \in G \setminus L$, and $u_0(t), x_0(t),\ t_0 \le t \le t_1,$ be the process belonging to Feldbaum's synthesis and transiting $x_0$ to the origin.

Consider the level curve of the function $\omega(x)$ through $x_0$ and let $x_\varepsilon$ be the point of this level curve at the distance $\varepsilon$ from $x_0$. Then $x_\varepsilon = x_0 + \varepsilon v + o(\varepsilon)$, where $v$ is the unit tangent vector of the level curve. Denote by $u_\varepsilon(t), x_\varepsilon(t)$ the process belonging to Feldbaum's synthesis and transiting $x_\varepsilon$ to the origin. Since the points $x_0, x_\varepsilon$ are situated on the same level curve of $\omega(x)$, the transit time for $x_\varepsilon$ is *the same* as for $x_0$, i.e., we may assume that the process $u_\varepsilon(t), x_\varepsilon(t)$, is defined on the same time-segment $t_0 \le t \le t_1$. The control $u_\varepsilon(t)$ has the form

$$u_\varepsilon(t) \equiv 1 \quad \text{as} \quad t_0 \le t < t_\varepsilon, \qquad u_\varepsilon(t) \equiv -1 \quad \text{as} \quad t_\varepsilon \le t \le t_1,$$

where $t_\varepsilon$ is the switching moment. Since the trajectory $x_0(t)$ meets the switching line $L$ at a nonzero angle, we have $t_\varepsilon = \tau + \varepsilon k + o(\varepsilon)$, where $k$ is a constant not depending on $\varepsilon$. Let, for definiteness, $k > 0$ (if $k < 0$, the reasoning is similar). Then $x_0(t) \equiv x_\varepsilon(t)$ for $t_\varepsilon \le t \le t_1$, since on this time interval both the trajectories correspond to the same control $u \equiv -1$ and $x_0(t_1) = x_\varepsilon(t_1) = 0$. Furthermore,

$$x_0(t_\varepsilon) = x_0(\tau) + \varepsilon\, k w_0 + o(\varepsilon), \quad x_\varepsilon(t_\varepsilon) = x_\varepsilon(\tau) + \varepsilon\, k w_\varepsilon + o(\varepsilon), \qquad (19)$$

where $w_0$, $w_\varepsilon$ are the phase velocities at the point $x_0(\tau)$ for $u=-1$, $u=1$ respectively, i.e., $w_0 = (x_0^2(\tau),\, f(x_0^1(\tau),\, x_0^2(\tau),\, -1))^T$, $w_\varepsilon = (x_0^2(\tau),\, f(x_0^1(\tau),\, x_0^2(\tau),\, 1))^T$. Since $x_0(t_\varepsilon) = x_\varepsilon(t_\varepsilon)$, we conclude from (19)

$$x_0(\tau) - x_\varepsilon(\tau) = -\varepsilon k(w_0 - w_\varepsilon) + o(\varepsilon). \tag{20}$$

By Lemma 9, the process $u_0(t)$, $x_0(t)$ satisfies the maximum condition with respect to a nontrivial solution $\psi^{(0)}(t)$ of the conjugate system. In particular, for $t_0 \le t \le \tau$, we have

$$\langle \psi^{(0)}(t),\, \Big(x_0^2(t),\, f(x_0^1(t),\, x_0^2(t),\, 1)\Big)^T\rangle \ge \langle \psi^{(0)}(t),\, \Big(x_0^2(t),\, f(x_0^1(t),\, x_0^2(t),\, -1)\Big)^T\rangle,$$

and for $\tau \le t \le t_1$ the inequality

$$\langle \psi^{(0)}(t),\, \Big(x_0^2(t),\, f(x_0^1(t),\, x_0^2(t),\, -1)\Big)^T\rangle \ge \langle \psi^{(0)}(t),\, \Big(x_0^2(t),\, f(x_0^1(t),\, x_0^2(t),\, 1)\Big)^T\rangle$$

holds. Consequently $\langle \psi^{(0)}(\tau),\, w_0\rangle = \langle \psi^{(0)}(\tau),\, w_\varepsilon\rangle$, i.e.,

$$\langle \psi^{(0)}(\tau),\, w_0 - w_\varepsilon\rangle = 0. \tag{21}$$

We now consider the trajectories $x_0(t)$, $x_\varepsilon(t)$ for $t_0 \le t \le \tau$. On this time interval, both the trajectories correspond to the same control $u \equiv 1$. Hence on this time interval,

$$x_0(t) - x_\varepsilon(t) = \varepsilon\, \delta x(t) + o(\varepsilon), \tag{22}$$

where $\delta x(t) = \big(\delta x^1(t),\, \delta x^2(t)\big)^T$ is the solution of the system of variational equations

$$\delta\dot{x}^1 = \delta x^2, \quad \delta\dot{x}^2 = \frac{\partial f(x^1,\, x^2,\, 1)}{\partial x^1}\delta x^1 + \frac{\partial f(x^1,\, x^2,\, 1)}{\partial x^2}\delta x^2 \tag{23}$$

with the initial condition $\delta x(\tau) = -k(w_0 - w_\varepsilon)$ (cf. (20)). Moreover, on considered time interval the relation $\langle \psi^{(0)}(t),\, \delta x(t)\rangle = \mathrm{const}$ holds, since $\delta x(t)$ satisfies the system (23) and $\psi^{(0)}(t)$ satisfies the conjugate system (15) with $u \equiv 1$. In particular,

$$\langle \psi^{(0)}(t_0),\, \delta x(t_0)\rangle = \langle \psi^{(0)}(\tau),\, \delta x(\tau)\rangle = \langle \psi^{(0)}(\tau),\, -k(w_0 - w_\varepsilon)\rangle = 0 \tag{24}$$

(cf. (21)).

Finally by (22), we have $\varepsilon\, \delta x(t_0) = x_0(t_0) - x_\varepsilon(t_0) + o(\varepsilon) = x_0 - x_\varepsilon + o(\varepsilon) = -\varepsilon v + o(\varepsilon)$, i.e., $\delta x(t_0) = -v$. Now the relation (24) means that $\langle \psi^{(0)}(t_0),\, v\rangle = 0$, i.e., the vector $\psi(t_0)$ is *orthogonal* to the level curve of the function $\omega(x)$ at the point $x_0$. The vector $\operatorname{grad}\omega(x_0)$ also is orthogonal to this level curve at $x_0$. Hence

$$\psi(t_0) = \lambda \operatorname{grad}\omega(x_0) \tag{25}$$

with a suitable real number $\lambda$.

By the maximum condition,

$$H = \langle \psi^{(0)}(t),\, \Big(x_0^2(t),\, f(x_0^1(t),\, x_0^2(t),\, 1)\Big)^T\rangle =$$

$$\max_{-1\le u\le 1} \langle \psi^{(0)}(t),\, \Big(x_0^2(t),\, f(x_0^1(t),\, x_0^2(t),\, u)\Big)^T\rangle \ge 0, \quad t_0 \le t \le \tau.$$

In particular, this relation holds for $t = t_0$, i.e.,

$$\langle \psi^{(0)}(t_0),\, W\rangle = \max_{-1\le u\le 1} \langle \psi^{(0)}(t_0),\, \Big(x_0^2,\, f(x_0^1,\, x_0^2,\, u)\Big)^T\rangle \ge 0,$$

where $W = \big(x_0^2,\, f(x_0^1,\, x_0^2,\, 1)\big)^T$ is the phase velocity along the trajectory $x_0(t)$ at the point $x_0$. At the same time, $\langle \operatorname{grad}\omega(x_0),\, W\rangle < 0$, since the vector $\operatorname{grad}\omega(x_0)$ indicates the direction of *maximal increasing* for the function $\omega(x)$, whereas along the trajectory $x_0(t)$, i.e., in the direction of the vector $W$, the function $\omega(x)$ *decreases.* This means that $\lambda < 0$ in the relation (25). Thus

$$\langle -\operatorname{grad}\omega(x_0),\, W\rangle = \frac{1}{|\lambda|}\langle \psi^{(0)}(t_0),\, W\rangle =$$

$$\frac{1}{|\lambda|}\max_{-1\le u\le 1} \langle \psi^{(0)}(t_0),\, \Big(x_0^2,\, f(x_0^1,\, x_0^2,\, u)\Big)^T\rangle =$$

$$\max_{-1\le u\le 1} \langle -\operatorname{grad}\omega(x_0),\, \Big(x_0^2,\, f(x_0^1,\, x_0^2,\, u)\Big)^T\rangle.$$

In other words, at the point $x = x_0$ the equality

$$-\frac{\partial\omega(x)}{\partial x^1}x^2 - \frac{\partial\omega(x)}{\partial x^2}f(x^1, x^2, 1) =$$

$$\max_{-1\le u\le 1}\Big(-\frac{\partial\omega(x)}{\partial x^1}x^2 - \frac{\partial\omega(x)}{\partial x^2}f(x^1, x^2, u)\Big) = 1,$$

holds, where $1$ is written on the right, since the left-hand side is equal to $-\frac{d\omega(x_0(t))}{dt} = 1$. Thus Bellman's equation (18) holds at any point $x_0 \in S^{(+)} \setminus L$.

Similarly, for every point $x_0 \in S^{(-)} \setminus L$, Bellman's equation (18) holds too with $v(x_0) = -1$ instead of $v(x_0) = 1$. □

**Lemma 11.** *Let $x(t)$, $u(t)$, $t_0 \le t \le t_1$, be an admissible process transiting a point $x_0$ to the origin, where the trajectory $x(t)$ is situated in the set $G$. Then the estimate $t_1 - t_0 \ge \omega(x_0)$ holds.*

*Proof.* Let $\varepsilon$ be an arbitrary positive number. Since the function $\omega(x)$ is continuous and $\omega(0) = 0$, there is a neighborhood $U \subset G$ of the origin such that $\omega(x) < \varepsilon$ for any $x \in U$.

Let now $p < 1$ be a positive number. Denote by $x_p(t)$, $t_0 \le t \le t_1$, the phase trajectory with the initial point $x_p(t_0) = x_0$ corresponding to the control $u_p(t) = p\,u(t)$. If $p$ is close enough to 1, then the endpoint $x_p(t_1)$ is arbitrary close to the origin. We fix a number $p < 1$ such that $x_p(t_1) \in U$ i.e., $\omega(x_p(t_1)) < \varepsilon$.

Assume that at a moment $t'$, the trajectory $x_p(t)$ intersects the switching line $L$, i.e., $x_p(t') \in L$. Since $|u_p(t)| = p\,|u(t)| < 1$, the phase velocity $\dot{x}_p(t)$ (at the moment $t'$ and close moments) does not touch the line $L$. Hence $t'$ is an *isolated* time moment at which $x_p(t) \in L$. This means that there are only finitely many moments $t \in [t_0,\, t_1]$ with $x_p(t) \in L$, i.e., $x_p(t) \in G \setminus L$ almost everywhere in $[t_0,\, t_1]$. Consequently by Bellman's equation,

$$-\frac{d\omega(x_p(t))}{dt} = -\frac{\partial\omega(x_p(t))}{\partial x^1} x_p^{\,2}(t) - \frac{\partial\omega(x_p(t))}{\partial x^2} f(x_p^{\,1}(t), x_p^{\,2}(t), u_p(t)) \le 1$$

almost everywhere in $[t_0,\, t_1]$. Integrating this inequality (and taking into consideration that $x_p(t)$ ia absolutely continuous), we obtain

$$-\omega(x_p(t_1)) + \omega(x_p(t_0)) \le t_1 - t_0,$$

i.e., $-\varepsilon + \omega(x_0) \le t_1 - t_0$. By arbitrariness of $\varepsilon > 0$, this means that $\omega(x_0) \le t_1 - t_0$. □

Finally, for considered nonoscillatory nonlinear object we establish that Feldbaum's synthesis is in fact the *optimal* synthesis.

**Theorem 2.** *Assume that a nonlinear controlled object* (10) *satisfies the conditions* (11), (12), (A), (B). *A process* $x(t)$, $u(t)$, $t_0 \le t \le t_1$, *transiting a point* $x_0$ *to the origin is time-optimal if and only if the control* $u(t)$ *is piecewise constant, takes only the values* $\pm 1$, *and has no more than two intervals of constancy. The set* $G = L \cup S^{(+)} \cup S^{(-)}$ *is open and coincides with the controllability region. The synthesis of optimal controls is realized in* $G$ *by the following synthesis function* $v(x)$ :

$$v(x) = -1 \quad as \quad x \in S^{(-)} \cup L^{(-)}, \qquad v(x) = +1 \quad as \quad x \in S^{(+)} \cup L^{(+)}.$$

*In other words, a trajectory* $x(t)$ *is optimal if and only if it satisfies the system*

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = f(x^1,\, x^2,\, v(x)).$$

*Proof.* First we establish that $G$ coincides with the controllability region. Assume, on the contrary, that there is an admissible process $u(t)$, $x(t)$, $t_0 \le t \le t_1$, transiting a point $x_0 \notin G$ to the origin. Since $x_0 = x(t_0) \notin$

$G$, $x(t_1) = 0 \in G$, the trajectory $x(t)$ intersects the boundary $\mathrm{bd}\, G$ of the open set $G$. Let $t'$ be the *last* moment of intersection of the trajectory $x(t)$ with $\mathrm{bd}\, G$ (the moment $t'$ exists, since $\mathrm{bd}\, G$ is a *closed* set). The point $x(t')$ does not belong to $G$, since $G$ is an open set and $x(t')$ is its boundary point. At the same time, $x(t) \in G$ for any $t > t'$. Denote the number $t_1 - t'$ by $T$ and consider the set $\Sigma_T$ as in Lemma 8. Then $x(t') \notin \Sigma_T$ (since $\Sigma_T \subset G$, $x(t') \notin G$). By compactness of $\Sigma_T$, we can choose a moment $t^* > t'$ such that $x(t^*) \notin \Sigma_T$. The process $u(t)$, $x(t)$, $t^* \leq t \leq t_1$, transits the point $x(t^*)$ to the origin (inside the set $G$). Hence by Lemma 11, $\omega(x(t^*)) \leq t_1 - t^*$, i.e., $\omega(x(t^*)) < T$. But this means that $x(t^*) \in \Sigma_T$, contradicting the aforesaid. Thus $G$ coincides with the controllability region.

Finally, let $\bar{u}(t)$, $\bar{x}(t)$, $\bar{t}_0 \leq t \leq \bar{t}_1$, be a process belonging to Feldbaum's synthesis, i.e., this process transits the point $\bar{x}_0 = \bar{x}(t_0)$ to the origin and the control $\bar{u}(t)$ takes the values $\pm 1$ and has no more than two intervals of constancy. By definition of the function $\omega(x)$, we have $\omega(\bar{x}_0) = \bar{t}_1 - \bar{t}_0$. If now $u(t), x(t)$, $t_0 \leq t \leq t_1$, is an another process transiting $\bar{x}_0$ to the origin, then the trajectory $x(t)$ is contained in $G$ (since $G$ is the controllability region), and by Lemma 11, $t_1 - t_0 \geq \omega(\bar{x}_0)$, i.e., $t_1 - t_0 \geq \bar{t}_1 - \bar{t}_0$. This means that the process $\bar{u}(t)$, $\bar{x}(t)$, $\bar{t}_0 \leq t \leq \bar{t}_1$, transits $\bar{x}_0$ to the origin in the *shortest* time, i.e., it is time-optimal. Thus all the processes of Feldbaum's synthesis are optimal. Conversely, every optimal process is contained in Feldbaum's synthesis (Lemma 3). □

Remark. This result is contained in [1]. But in [1] the proof is very complicated (using *regular synthesis* introduced in [2]). Moreover, in [1] there are some skips in the proofs of lemmas. Thus the proof offered above is more correct, short, and preferred.

It would be interesting to generalize Theorem 2 for nonlinear controlled objects in $R^n$, introducing a class of "nonoscillatory" objects.

## References


[1] Boltyanski, V.G., *Mathematical Methods of Optimal Control*, Holt, Rinehart and Winston, Inc., New York — Montreal, Que., — London (1971).

[2] Boltyanski, V.G., *Sufficient conditions for optimality and the justification of the Dynamic Programming Method*, J. SIAM Control, Vol 4, No. 2 (1966), 326–361.

[3] Feldbaum, A.A., *Optimal processes in systems of automatic control*, Avtomatika i Telemechanika, Vol. 14, No. 6 (1953), 712–728 (in Russian).

[4] Feldbaum, A.A., *On synthesis of optimal systems of automatic control*, Transactions of the 2-nd National Conference on the Theory of automatic control, Vol. 2 (1955), Izdat. AN SSSR, Moscow — Leningrad (in Russian).

[5] FELDBAUM, A.A., *On synthesis of optimal systems with the help of phase space*, Avtomatika i Telemechanika, Vol. 16, No. 2 (1955), 129–149 (in Russian).
[6] PERVOZVANSKI, A.A., *Theory of automatic control*, Izdat. Nauka, Moscow (1988) (in Russian).

CENTRO DE INVESTIGACION
EN MATEMATICAS, A.P. 402
36000 GUANAJUATO, GTO
MEXICO