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THE EXISTENCE OF HOMOCLINIC SOLUTIONS FOR HYPERBOLIC EQUATIONS

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Abstract. Studying homoclinic solutions of equations is one of the steps to go deeper in the understanding of dynamics. As it is known to the authors there are no papers studying homoclinic solutions of hyperbolic systems. In the paper we present a new variational method general enough to treat the problem of the existence of homoclinic solutions for the following semi-linear wave equation: $x_{tt}(t,y) - x_{yy}(t,y) + g(t,y,x(t,y)) = 0$ for $0 < y < Y$, $t \in \mathbf{R}$, $x(t,0) = 0$, $x(t,Y) = 0$ for $t \in \mathbf{R}$. Our approach covers both sublinear and superlinear cases.

1. Introduction. Several recent papers have used global variational methods to establish the existence of multibump solutions of families of superquadratic finite dimensional Hamiltonian systems. Such solutions are homoclinic solutions of equations (see e.g. Sere (1992), Bessi (1993) and Coti Zelati and Rabinowitz (1991), (1992), (1994)). These implies that studying homoclinic solutions of equations is one of the steps to go deeper in the understanding of the dynamics. As it is known to the authors there are no papers studying homoclinic solutions of hyperbolic systems. The reason is that such problems are especially difficult for topological and analytical methods. Simultanously, in eighties there were developed many variation

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al methods for obtaining different existence results for hyperbolic partial differential equations (see e.g. Brezis (1983), Mawhin (1987)). Our purpose is just to work out a new variational method general enough to treat the problem of the existence of homoclinic solutions for the following semi–linear wave equation:

$$x_{tt}(t,y) - x_{yy}(t,y) + g(t,y,x(t,y)) = 0 \quad \text{for } 0 < y < Y,\ t \in \mathbf{R}, \qquad (1)$$
$$x(t,0) = 0, x(t,Y) = 0 \quad \text{for } t \in \mathbf{R}.$$

We shall further develop the ideas presented in our earlier papers — Nowakowski & Rogowski (1993), (1995a), (1995b). It is well known that to work with variational methods it is necessary to consider (1) as a kind of the Euler-Lagrange equation for some functional. One of many possibilities is to consider (1) as an infinite dimensional Hamiltonian system (see e.g. Nowakowski (1992))

$$\frac{\partial}{\partial t}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} H_p \\ -H_x \end{pmatrix} \qquad (2)$$

where the Hamiltonian $H(t,x,p)$ is defined on $\mathbf{R} \times \mathbf{H^1_0(0,Y)} \times \mathbf{L^2(0,Y)}$ by

$$H(t,x,p) = \frac{1}{2}\int_0^Y (2G(t,y,x(t,y)) + x_y(t,y)^2 + p(t,y)^2)dy,$$

where G is a primitive of g with respect to x. It is well known that Hamiltonian system (2) is the Euler-Lagrange equation for the functional

$$J_H(x,p) = \int_{\mathbf{R}} (< \dot{x}(t), p(t) > -H(t,x(t),p(t))dt,$$

where $< \cdot, \cdot >$ is a scalar product in $\mathbf{H} = \mathbf{L^2(0,Y)}$, "$\cdot$"$= \frac{d}{dt}$ in a weak sense of the space $\mathbf{H}$, $x(t) = x(t,\cdot)$, $p(t) = p(t,\cdot)$. However the functional J_H is still unconvenient to study, because the Hamiltonian H does not depend on the derivative $\dot{x}$. This is why we calculate the Lagrange functional corresponding to J_H. It is obtained by Fenchel transform of $H(t,x,\cdot)$ i.e.

$$L(t,x,\dot{x}) = \sup_{p \in \mathbf{X}} (< p, \dot{x} > -H(t,x,p)),$$

where $\mathbf{X} = \mathbf{H^1_0(0,Y)}$. By the form of H we calculate

$$L(t,x,\dot{x}) = -\frac{1}{2}\|\Lambda x\|^2 + \frac{1}{2}\|\dot{x}\|^2 - G(t,x),$$

where $\Lambda = \frac{d}{dy}$ is an operator in $\mathbf{H}$ with the domain $\mathbf{X}$, $\|\cdot\|$ denotes the norm in $\mathbf{H}$ and

$$G(t,x) = \int_0^Y G(t,y,x(y))dy \quad \text{for } x \in L^2(0,Y).$$

Then the Lagrange functional takes the form:

$$J(x) = \int_{\mathbf{R}} L(t, x(t), \dot{x}(t))dt.$$

We shall investigate J in the space $W_0(\mathbf{R})$, where $W_0(\mathbf{R}) = \mathbf{W}(\mathbf{R}) = \{\mathbf{x} \in \mathbf{L^2}(\mathbf{R}; \mathbf{X}),\ \dot{\mathbf{x}} \in \mathbf{L^2}(\mathbf{R}; \mathbf{H})\}$. Because of our assumptions on $\mathbf{X}$ and $\mathbf{H}$, $x \in W_0(\mathbf{R})$ is continuous as a function $x : \mathbf{R} \to \mathbf{H}$ (see e.g. Lions and Magenes (1968)) and $x(t) \to 0$ for $|t| \to +\infty$. The space $W_0(\mathbf{R})$ endowed with the norm $\|x\|_{W_0} = (\|\Lambda x\|^2_{L^2(\mathbf{R};\mathbf{H})} + \|\dot{x}\|^2_{L^2(\mathbf{R};\mathbf{H})})^{\frac{1}{2}}$ becomes a Hilbert space.

In view of the above, to investigate the homoclinic solutions of (1) it is enough to study the homoclinic solutions of (2). Since the notions of homoclinic solutions arose just for studying dynamics of homoclinic solutions of finite dimensional hamiltonian systems, thus studying homoclinic solutions to infinite dimensional Hamiltonian systems (2) i.e. such pairs of functions (x, p) satisfying (2) that $(x, p) \in W_0(\mathbf{R}) \times \mathbf{W_0}(\mathbf{R})$ i.e. $x(t)$, $p(t) \to 0$ as $|t| \to +\infty$ is a natural generalization of finite dimensional case. In that way by homoclinic solutions to (1) we shall understand homoclinic solutions to (2). Therefore we have a full analogy to the classical meaning of homoclinic solutions of the equation.

In the sequel we shall need to consider $W_0(\mathbf{R})$ as a closed subspace of a Cartesian product of two spaces $L^2(\mathbf{R}; \mathbf{H})$. Let us put $L_2 = L^2(\mathbf{R}; \mathbf{H}) \times \mathbf{L^2}(\mathbf{R}; \mathbf{H})$ so that to each $x \in W_0(\mathbf{R})$ we can associate the well defined vector Qx in L_2 given by $Qx = (\Lambda x, \dot{x}) \in L_2$. Since $\|Qx\|_{L_2} = \|x\|_{W_0}$, Q is an isometric isomorphism of $W_0(\mathbf{R})$ onto a subspace $W \subset L_2$. As $W_0(\mathbf{R})$ is complete, W is a closed subspace of of L_2. Therefore W endowed with the scalar product induced from L_2 becomes a Hilbert space. Since W is closed there exists its orthogonal completion $W^{\perp}$ in L_2 such that $L_2 = W \oplus W^{\perp}$. Denote by P the orthogonal projection from L_2 onto $W^{\perp}$ and by P^* its transpose acting from $W^{\perp}$ to L_2 as the map associating functions from $W^{\perp}$ functions in L_2. Let $h(v, w) = h_1(v) + h_2(w)$ be a lower semicontinuos convex, finite function on W. We shall compute h^* (the Fenchel conjugate to h).

Lemma 1. *In the above setting let us assume that there exist convex, lower semicontinuos, finite functions $\bar{h}_1 : L^2(\mathbf{R}; \mathbf{H}) \to \mathbf{R}$, $\bar{h}_2 : L^2(\mathbf{R}; \mathbf{H}) \to \mathbf{R}$ such that $\bar{h}(v, w) = \bar{h}_1(v) + \bar{h}_2(w)$ restricted to W is equal to h. Then the Fenchel conjugate to h is given by the formulae*

$$h^*(v, w) = \inf_{(\xi,\eta) \in W^{\perp}} (\bar{h}_1^*(v + \xi) + \bar{h}_2^*(w + \eta)).$$

Proof. Since W is the kernel of the projection P we can write, using the indicator function χ of the set $\{0\}$: $h(v,w) = \bar{h}(v,w) + \chi_{\{0\}}(P(v,w))$. By the definition of P we easily check that 0 belongs to the interior of $P(\operatorname{Dom} h)$. Thus we can apply Corollary 4.4.12 from Aubin and Ekeland (1984) to calculate h^*. Consequently $h^*(v,w) = \inf_{(\xi,\eta)\in W^\perp}(\bar{h}_1^*(v+\xi)+\bar{h}_2^*(w+\eta))$. □

PROPOSITION 1. *If the sequence $\{x_n\} \subset W_0(\mathbf{R})$ converges weakly to x in $W_0(\mathbf{R})$, then $\{x_n\}$ converges strongly to x in $L^2(\mathbf{R};\mathbf{H})$.*

Proof. Since any weakly convergent sequence in $W_0(\mathbf{R})$ is bounded in the W_0–norm, therefore $\|\Lambda x_n\|_{L^2(\mathbf{R};\mathbf{H})}$ and $\|\dot{x}_n\|_{L^2(\mathbf{R};\mathbf{H})}$ are are uniformly bounded by some constant M. To end the proof it is enough to show that for every $\varepsilon > 0$ there exists a positive number $\delta > 0$ and a compact subset $G \subset \mathbf{R}$ such that for every n and every $h \in \mathbf{R}$ with $|h| < \delta$:

$$\int_{\mathbf{R}} \|x_n(t+h) - x_n(t)\|^2 dt < \varepsilon \quad \text{and} \quad \int_{\mathbf{R}\setminus\mathbf{G}} \|x_n(t)\|^2 dt < \varepsilon.$$

Choose any n and $|h| < \delta$. Then

$$\int_{\mathbf{R}} \|x_n(t+h) - x_n(t)\|^2 dt \le \int_{\mathbf{R}} \int_t^{t+h} \|\dot{x}_n(s)\|^2 ds dt \le 3hM.$$

Hence if we put $\delta = \frac{\varepsilon}{3M}$ then we get the first inequality. To obtain the second one we infer from the boundedness of $\{x_n\}$ in W_0 that the functions $t \mapsto \|x_n(t)\|_H$ are uniformly bounded in $\mathbf{R}$. Thus we are able to choose a compact set $G \subset \mathbf{R}$ such that $\int_{\mathbf{R}\setminus\mathbf{G}} \|x_n(t)\|^2 dt < \varepsilon$. □

Through the paper we need the following hypothesis:

(H): $G : \mathbf{R}\times[\mathbf{0},\mathbf{Y}]\times\mathbf{R} \to \mathbf{R}$ is measurable, convex and Gateaux differentiable in the third variable and is subject to the following growth conditions:

$$\frac{1}{r_1} a_1 |x|^{r_1} + b_1(t,y) \le G(t,y,x) \le \frac{1}{r} a |x|^r + b(t,y)$$
$$\text{for all } (t,y,x) \in \mathbf{R}\times[\mathbf{0},\mathbf{Y}]\times\mathbf{R},$$

where $a_1 > 0$, $a > 0$, $b, b_1 \in L^1(\mathbf{R}\times[\mathbf{0},\mathbf{Y}])$ and $r > 1$, $r_1 \le 2$, $r \ge r_1$;

The main result of the paper is the following

THEOREM 1. *Under the above hypothesis (H) there exists a weak homoclinic solution $\bar{x} \in W(\mathbf{R})$ to (1) such that*

$$J(\bar{x}) = \inf_{\dot{x}\in L^2(\mathbf{R};\mathbf{H})} \; \sup_{\Lambda x \in L^2(\mathbf{R};\mathbf{H})} J(x)$$

This theorem is a direct consequence of Corollary 2 and Theorem 4.

It is worth to note that the growth conditions in hypothesis (H) include both sublinear and superlinear cases of (1). As it is known to the authors sublinear and superlinear cases were always treated by different methods. In this paper however, we establish the general method, developed in Sections 2 and 3, for studying both cases simultaneously — sublinear and superlinear.

2. Duality results. To obtain duality results we need a space over which a perturbation of J will be built as well as a proper duality pairing associating it with the space $W_0(\mathbf{R})$ can be defined. In our case we take just $W_0(\mathbf{R})$ with the scalar product $< x, p >_{W_0(\mathbf{R})} = < \Lambda x, \Lambda p >_{L^2(\mathbf{R};\mathbf{H})} + < \dot{x}, \dot{p} >_{L^2(\mathbf{R};\mathbf{H})}$. The choice of the space $W_0(\mathbf{R})$ is related to the results we intend to derive from the duality principle.

First we define for each $x \in W_0(\mathbf{R})$ the perturbation of J as:

$$J_x(g) = \int_{\mathbf{R}} \left(\frac{1}{2} \|\Lambda x + \Lambda g\|^2 + G(t, x(t) + \dot{g}(t)) - \frac{1}{2} \|\dot{x}(t)\|^2 \right) dt$$

for $g \in W_0(\mathbf{R})$. Of course $J_x(0) = -J(x)$. For $x \in W_0(\mathbf{R})$, $p \in W_0(\mathbf{R})$ we define a type of conjugate of J by

$$J_x^{\#}(p) = \sup_{(\Lambda g, \dot{g}) \in W} \int_{\mathbf{R}} (< p(t), \Lambda g(t) > + < \dot{p}(t) - \Lambda p(t), \dot{g}(t) > + \\ - \frac{1}{2} \|\Lambda x(t) + \Lambda g(t)\|^2 - G(t, x(t) + \dot{g}(t))) dt + \frac{1}{2} \int_{\mathbf{R}} \|\dot{x}(t)\|^2 dt.$$

Using formulae for the conjugate in the space W we compute that

$$J_x^{\#}(p) = \min_{(v,w) \in W^{\perp}} \int_{\mathbf{R}} (\frac{1}{2} \|p(t) + v(t)\|^2 + G^*(t, \dot{p}(t) - \Lambda p(t) + \\ + w(t)) + \frac{1}{2} \|\dot{x}(t)\|^2) dt. \quad (3)$$

Now it is easy to calculate

$$\inf_{\dot{x} \in L^2(\mathbf{R};\mathbf{H})} J_x^{\#}(p) = \min_{(v,w) \in W^{\perp}} \int_{\mathbf{R}} (\frac{1}{2} \|p(t) + v(t)\|^2 + G^*(t, \dot{p}(t) - \Lambda p(t) + \\ + w(t)) - \frac{1}{2} \|p(t)\|^2) dt. \quad (4)$$

The right–hand side of (4) we shall denote by $J_D(p)$ i.e.

$$J_D(p) = \min_{(v,w) \in W^{\perp}} \int_{\mathbf{R}} (\frac{1}{2} \|p(t) + v(t)\|^2 + G^*(t, \dot{p}(t) - \Lambda p(t) + \\ + w(t)) - \frac{1}{2} \|p(t)\|^2) dt \quad (5)$$

and we shall call it the *functional dual* to J. For $g \in W_0(\mathbf{R})$ we put

$$J_x^{\#\#}(g) = \sup_{(\Lambda p, \dot{p}) \in W} \left(\int_{\mathbf{R}} (< p(t), \Lambda g(t) > + < \dot{p}(t) - \Lambda p(t), \dot{g}(t) >) dt - J_x^{\#}(p) \right).$$

We see, taking into accout (3), that for $x \in W_0(\mathbf{R})$

$$J_x^{\#\#}(0) = -J(x).$$

Using the "min–max" theorem (Brezis (1973)) we are able to compute (see (4) and (5))

$$\sup_{\dot{x} \in L^2(\mathbf{R};\mathbf{H})} \inf_{\Lambda x \in L^2(\mathbf{R};\mathbf{H})} J_x^{\#\#}(0) = \sup_{\dot{x}} \inf_{\Lambda x} \sup_{(\Lambda p, \dot{p}) \in W} -J_x^{\#}(p) =$$
$$\sup_{(\Lambda p, \dot{p}) \in W} \sup_{\dot{x}} \inf_{\Lambda x} -J_x^{\#}(p) = \sup_{p \in W_0(\mathbf{R})} -J_D(p).$$

Therefore, we come to the following duality principle.

THEOREM 2. *Functionals J and J_D are subject to the following relation*

$$\inf_{\dot{x} \in L^2(\mathbf{R};\mathbf{H})} \sup_{\Lambda x \in L^2(\mathbf{R};\mathbf{H})} J(x) = \inf_{p \in W_0(\mathbf{R})} J_D(p).$$

The next result formulates a variational principle for "min–max" arguments. However, as it is much more easily to prove that the infimum of $J_D(p)$ over $W_0(\mathbf{R})$ is attained than "min–max" for $J(x)$, therefore we shall investigate the dual functional J_D. To this effect define the perturbation of J_D as:

$$J_{Dp}(\dot{g}) = -\int_{\mathbf{R}} \Big(\frac{1}{2} \|p(t) + v_p(t)\|^2 + G^*(t, \dot{p}(t) - \Lambda p(t) + w_p(t)) + \\ - \frac{1}{2} \|p(t) + \dot{g}(t)\|^2 \Big) dt. \quad (6)$$

where $(v_p, w_p) \in W^{\perp}$ is a pair for which a minimum in (4) is attained.

THEOREM 3. *Let $\bar{p} \in W_0(\mathbf{R})$ be such that $J_D(\bar{p}) = \inf_{p \in W_0} J_D(p) > -\infty$ and let the set $\partial J_{D\bar{p}}(0)$ be nonempty. Then there exists $\dot{\bar{x}} \in L^2(\mathbf{R};\mathbf{H})$, $-\dot{\bar{\mathbf{x}}} \in \partial \mathbf{J}_{\mathbf{D}\bar{\mathbf{p}}}(\mathbf{0})$ and $\bar{x}$ corresponding to $\dot{\bar{x}}$ belongs to $W_0(\mathbf{R})$ and satisfies*

$$J(\bar{x}) = \inf_{\dot{x}} \sup_{\Lambda x} J(x).$$

Furthermore

$$J_{D\bar{p}}(0) + J_{\bar{x}}^{\#}(\bar{p}) = 0, \qquad J(\bar{x}) - J_{\bar{x}}^{\#}(\bar{p}) = 0. \quad (7)$$

Proof. By Theorem 2 to prove the first assertion it suffices to show that $J_D(\bar{p}) \geq \sup_{\Lambda\tilde{x}} J(\tilde{x}) = J(\bar{x})$, where $\tilde{x}(t)$ is a function in $W_0(\mathbf{R})$ determined by $\dot{\tilde{x}}$, and $-\dot{\tilde{x}} \in \partial J_{D\bar{p}}(0)$. By the definition of $\partial J_{D\bar{p}}(0)$ we see that

$$\frac{1}{2}\int_{\mathbf{R}} \|\bar{p}(t) + \dot{g}(t)\|^2 dt \geq \frac{1}{2}\int_{\mathbf{R}} \|\bar{p}(t)\|^2 dt + \int_{\mathbf{R}} < \dot{g}(t), -\dot{\tilde{x}}(t) > dt$$

$$\text{for all } g \in W_0(\mathbf{R}). \quad (8)$$

This implies, that for all $\dot{g} \in L^2(\mathbf{R};\mathbf{H})$ such that $g \in W_0(\mathbf{R})$ the following inequality holds

$$J_{D\bar{p}}(g) \geq -J_D(\bar{p}) + \int_{\mathbf{R}} < \dot{g}(t), -\dot{\tilde{x}}(t) > dt. \quad (9)$$

After simple transformation and taking into account the definition of $J_x^{\#}(p)$ and $J_D(p)$ we obtain from (9):

$$J_{\hat{x}}^{\#}(\bar{p}) = \sup_{\dot{g}} \left(\int_{\mathbf{R}} < \dot{g}(t), -\dot{\tilde{x}}(t) > dt - J_{D\bar{p}}(\dot{g}) \right) \leq J_D(\bar{p}) < +\infty, \quad (10)$$

where $\hat{x}$ is a primitive of $\dot{\tilde{x}}$. Since $\dot{\tilde{x}}$ satisfies (8) therefore $\dot{\tilde{x}}(t) = -\bar{p}(t)$ and so the primitive of $\dot{\tilde{x}}$ must belong to $W_0(\mathbf{R})$. Thus $\hat{x} = \tilde{x} \in W_0(\mathbf{R})$. From (10) we infer that

$$-\sup_{\Lambda\tilde{x}} J(\tilde{x}) = \inf_{\Lambda\tilde{x}} \sup_{p \in W_0(\mathbf{R})} -J_{\tilde{x}}^{\#}(p) \geq -J_D(\bar{p}), \quad \text{i.e. } \sup_{\Lambda\tilde{x}} J(\tilde{x}) \leq J_D(\bar{p}).$$

By assumption (H) the supremum over $\Lambda\tilde{x}$ is attained i.e. there exist $\bar{x} \in W_0(\mathbf{R})$ with $\frac{d}{dt}\bar{x}(t) = \dot{\tilde{x}}$, such that $\sup_{\Lambda\tilde{x}} J(\tilde{x}) = J(\bar{x})$.

The second assertion is a simple consequence of two facts: $J_{D\bar{p}}(0) = -J_D(\bar{p})$ and $-\dot{\tilde{x}} \in \partial J_{D\bar{p}}(0)$, which ends the proof. □

From equations (7) we are able to derive a dual to the Euler–Lagrange equation (2).

Corollary 1. *Let $\bar{p} \in W_0(\mathbf{R})$ satisfy: $\inf_{p \in W_0} J_D(p) = J_D(\bar{p})$ and let $J_{D\bar{p}}(0)$ be finite. Then there exists $\bar{x} \in W_0(\mathbf{R})$ such that the pair $(\bar{x}, \bar{p})$ (together with suitable $v_{\bar{p}}, w_{\bar{p}}$ realizing minimum in (4)) satisfies relations:*

$$\dot{\bar{x}}(t) = -\bar{p}(t), \quad (11)$$

$$\Lambda\bar{x}(t) = -\bar{p}(t) - v_{\bar{p}}(t), \quad (12)$$

$$\dot{\bar{p}}(t) - \Lambda\bar{p}(t) + w_{\bar{p}}(t) = g(t, \bar{x}(t)) \quad (13)$$

$$J_D(\bar{p}) = \inf_{p \in W_0(\mathbf{R})} J_D(p) = \inf_{\dot{x}} \sup_{\Lambda x} J(x) = J(\bar{x}). \quad (14)$$

Proof. From the form of $J_{D\bar{p}}(\dot{g})$ we see that $\dot{g} \to J_{D\bar{p}}(\dot{g})$ is convex, lower semicontinuous and finite in $L_2(\mathbf{R};\mathbf{H})$, and therefore continuous in that space. Hence $\partial J_{D\bar{p}}(0)$ is nonempty and so the existence of $\bar{x} \in W_0(\mathbf{R})$ is now clear by Theorem 3. Equations (11)–(13) are a direct consequence of (7). Relation (14) is a consequence of Theorem 2. □

From the above corollary we infer at once

Corollary 2. *By the same assumptions as in Corollary 1 there exists a pair* $(\bar{x}, \bar{p}) \in W_0(\mathbf{R}) \times \mathbf{W_0}(\mathbf{R})$ *being a weak solution to (1) and satisfying (14).*

Proof. Since the right-hand side of (11) has the weak derivative in t, there exist also $\frac{d}{dt}\dot{\bar{x}}$ in the weak sense. As $(v_{\bar{p}}, w_{\bar{p}}) \in W^{\perp}$ we infer from (12) that for $x \in W_0(\mathbf{R})$

$$\langle \Lambda\bar{x}, \Lambda x\rangle_{L^2(\mathbf{R};\mathbf{H})} = \langle -\bar{p} - v_{\bar{p}}, \Lambda x\rangle_{L^2(\mathbf{R};\mathbf{H})} = \langle \Lambda\bar{p}, x\rangle_{L^2(\mathbf{R};\mathbf{H})} - \langle v_{\bar{p}}, \Lambda x\rangle_{L^2(\mathbf{R};\mathbf{H})}$$

and from (11) and (13) for the same x:

$$\int_{\mathbf{R}} (< -\dot{\bar{x}}(t), \dot{x}(t) > + < \Lambda\bar{x}(t), \Lambda x(t) > + < g(t, \bar{x}(t)), x(t) >)dt = 0.$$

The last means that $\bar{x}$ is a weak solution to (1). □

3. The existence of a minimum for the dual functional $\mathbf{J_D}$. The last problem which we must solve to obtain the existence of solutions to (1) or (2) is to prove, in view of Corollary 2 , the existence of $\bar{p}$ together with the pair $(v_{\bar{p}}, w_{\bar{p}})$, satisfying:

$$J_D(\bar{p}) = \min_{p \in W_0(\mathbf{R})} \int_{\mathbf{R}} (\frac{1}{2}\|p(t) + v_p(t)\|^2 + G^*(t, \dot{p}(t) - \\ - \Lambda p(t) + w_p(t)) - \frac{1}{2}\|p(t)\|^2)dt. \quad (15)$$

To obtain this we use hypothesis (H) and Proposition 1.

Theorem 4. *Under hypothesis (H) there exist* $\bar{p} \in W_0(\mathbf{R})$ *along with* $(v_{\bar{p}}, w_{\bar{p}})$ *such that (15) holds.*

Proof. First note that for each fixed $p \in W_0(\mathbf{R})$ minimum over $(v, w) \in W^{\perp}$ is attained by some (v_p, w_p) as the space $W^{\perp}$ is closed and convex and the integral in (15) is a lower semicontinuous and convex functional of (v, w) and tends to $+\infty$ as $\|(v, w)\| \to +\infty$ in $W^{\perp}$. Since for each $p \in W_0(\mathbf{R})$

the pair $(p, \dot{p} - \Lambda p)$ does not belong to $W^{\perp}$ therefore $(v_p, w_p) \neq (p, \dot{p} - \Lambda p)$. Next we show that the functional J_D is bounded below. Really

$$J_D(p) \geq \int_{\mathbf{R}} (< p(t), v_p(t) > + \frac{1}{2}\|v_p(t)\|^2) dt + \\ + \int_{\mathbf{R}} \frac{1}{r'} a^{1-r'} \|\dot{p}(t) - \Lambda p(t) + w_p(t))\|^{r'} dt + \int_{\mathbf{R}} b(t) dt, \quad (16)$$

where $\frac{1}{r} + \frac{1}{r}' = 1$. Thus $J_D(p)$ is bounded below and satisfies the above growth conditions. From (16) we infer that for minimizing sequence $\{p_n\} \subset W_0(\mathbf{R})$, the sequence $\{\|v_{p_n}\|_{L^2}\}$ is also bounded and hence $\{p_n\}$ is bounded in $L^2(\mathbf{R}; \mathbf{H})$ (because v_p minimizes $\int_{\mathbf{R}} \|p(t) + v(t)\|^2 dt$). From (16) we also conclude that the sequence $\{\dot{p}_n - \Lambda p_n + w_{p_n}\}$ is bounded in $L^{r'}$. Therefore for each $\dot{p} \in L^2(\mathbf{R}; \mathbf{H})$ the sequence

$$< v_{p_n}, \dot{p} >_{L^2} + < \dot{p}_n - \Lambda p_n + w_{p_n}, \dot{p} >_{L^2} = \\ = < \dot{p} - \Lambda p, v_{p_n} >_{L^2} + < \dot{p}_n - \Lambda p_n, \dot{p} >_{L^2}, \quad n = 1, 2, \ldots$$

is bounded (we recall that $(\Lambda p, \dot{p}) \in W$ and $(v_{p_n}, w_{p_n}) \in W^{\perp}$). Since the set of all $\dot{p}$ with $p \in W_0(\mathbf{R})$ is dense in $L^2(\mathbf{R}; \mathbf{H})$, thus the last means that the sequence $\{\dot{p}_n - \Lambda p_n\}$ is bounded in $L^2(\mathbf{R}; \mathbf{H})$. This allows us to choose a subsequence of $\{p_n\}$, which we denote again by $\{p_n\}$, such that $\{\dot{p}_n - \Lambda p_n\}$ is weakly convergent to some $q \in L^2(\mathbf{R}; \mathbf{H})$. By the definition of Λ it maps $\mathbf{X}$ onto $\mathbf{H}$ and so Λ maps also $L^2(\mathbf{R}; \mathbf{X})$ onto $L^2(\mathbf{R}; \mathbf{H})$. Moreover we know that $\mathbf{X}$ is dense in $\mathbf{H}$. These imply that $\{\dot{p}_n\}$ is weakly convergent in $L^2(\mathbf{R}; \mathbf{H})$. Really, for all $h \in L^2(\mathbf{R}; \mathbf{X})$

$$< \dot{p}_n, h >_{L^2} = < \dot{p}_n - \Lambda p_n, h >_{L^2} + < p_n, \Lambda h >_{L^2}, \quad \text{for } n = 1, 2, \ldots .$$

As $\{< \dot{p}_n - \Lambda p_n, h >_{L^2}\}$ and $\{< p_n, \Lambda h >_{L^2}\}$ are convergent, therefore $\{< \dot{p}_n, h >_{L^2}\}$ is convergent. Similarly we get that $\{< \Lambda p_n, h >\}$ is convergent. This means that $\{\dot{p}_n\}$ and $\{\Lambda p_n\}$ are convergent weakly to some q_1 and q_2 respectively. Of course $q = q_1 + q_2$. In the same way we are able to show that $q_2 = \Lambda \bar{p}$ and next $q_1 = \dot{\bar{p}}$ for some $\bar{p} \in W_0(\mathbf{R})$. Hence we obtain that $\{p_n\} \subset W_0(\mathbf{R})$ is weakly convergent in $W_0(\mathbf{R})$ to $\bar{p} \in W_0(\mathbf{R})$. By Proposition $\{p_n\}$ is then strongly in $L^2(\mathbf{R}; \mathbf{H})$ convergent to $\bar{p}$. We can also choose some subsequence of $\{p_n\}$ (we do not change notations) such that $\{v_{p_n}\}$ and $\{w_{p_n}\}$ are weakly convergent in $L^2(\mathbf{R}; \mathbf{H})$ to $v_{\bar{p}}$ and $w_{\bar{p}}$ respectively. Since the functional

$$p \to \int_{\mathbf{R}} (\frac{1}{2}\|p(t) + v_p(t)\|^2 + G^*(t, \dot{p}(t) - \Lambda p(t) + w_p(t))) dt$$

is weakly lower semicontinuous we get that

$$\liminf J_D(p_n) \geq J_D(\bar{p})$$

with the above $(v_{\bar{p}}, w_{\bar{p}})$, which ends the proof. □

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