VARIATIONS ON NARROW DOTS-AND-BOXES AND DOTS-AND-TRIANGLES

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Received: 7/29/15, Revised: 8/9/16, Accepted: 2/5/17, Published: 2/13/17

Abstract

We verify a conjecture of Nowakowski and Ottaway that closed $1 \times n$ Dots-and-Triangles is a first-player win when $n \neq 2$. We also prove that in both the open and closed $1 \times n$ Dots-and-Boxes games where n is even, the first player can guarantee a tie.

1. Introduction

The classic children's game of Dots-and-Boxes has been well studied in [1] and [2]. The game begins with a square array of dots. Players take turns drawing an edge that connects two neighboring dots. A player who draws the fourth edge of a box claims the box and immediately takes another turn. The game ends when all boxes have been completed, and the player who has claimed more boxes is declared the winner. Variations of the game may be played, such as Dots-and-Triangles, which is played on a triangular board shape, and the Swedish and Icelandic games, in which some edges are drawn before the game begins.

In this paper we consider $1 \times n$ ("narrow") versions of both Dots-and-Boxes and Dots-and-Triangles. The $1 \times n$ Dots-and-Boxes game consists of two rows of $n+1$ dots each; on completion of the game, n boxes will have been enclosed. The $1 \times n$ Dots-and-Triangles game consists of a row of n dots on top and $n+1$ dots on bottom; on completion of the game, $2n-1$ triangles will have been enclosed. In the *closed* narrow games, the top and side exterior edges have been drawn before the game begins. The *open* version of the game begins with no such edges. The starting configurations for all four variants are shown in Figure 1.

To simplify our analysis, we will consider the graph-theoretic dual version of Dots-and-Boxes/Triangles known as Strings-and-Coins; this dual version of the game is thoroughly presented by Berlekamp in [1]. In this dual game, each box or triangle in the original game corresponds to a vertex, or "coin." Two vertices are adjacent in the dual game if and only if their corresponding faces share an edge *that has not yet been drawn* in the original game. In addition, we think of the exterior of the game board as a single vertex called the "ground." The ground vertex is not drawn. Drawing an edge in the original game corresponds to removing an edge, or "cutting a string," in the dual game. Completing and claiming a face in the original game corresponds to isolating and taking the vertex, or "capturing the coin," in the dual game. The ground vertex cannot be taken. Every face-enclosing game such as Dots-and-Boxes can be converted to an equivalent Strings-and-Coins game, although the converse is not true. The Strings-and-Coins presentations of the games in Figure 1 are shown in Figure 2. Following Berlekamp's convention, we use a small arrow to denote an edge that goes to the ground.

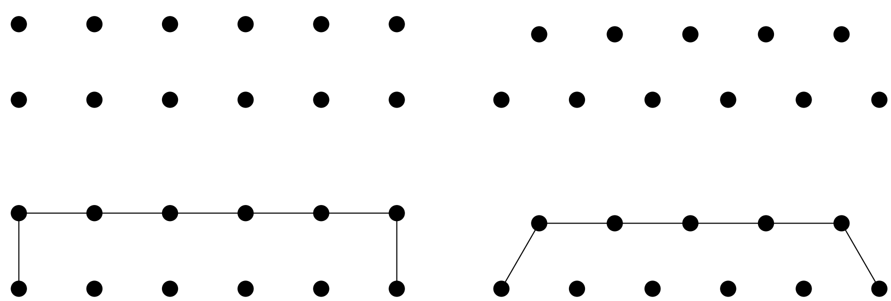

Figure 1: Starting positions for open (top) and closed (bottom) 1×5 games of Dots-and-Boxes (left) and Dots-and-Triangles (right).

We note the distinction between a player's move (which consists of cutting a single string) and a player's turn (which consists of one or more moves). A *capturable coin* is a degree one vertex. Without loss of generality, we will assume that a Strings-and-Coins player will take all coins that become capturable at some point during their turn except in the special cases shown in Figure 3, as argued on pages 42-43 of [1]. In each of the three cases shown, a player has two options. Either the player may remove edge X first and then edge Y, in which case they take two vertices and move again (if possible) in the remaining game, or the player may remove edge Y first and end their turn. A player who is faced with any of the cases shown in

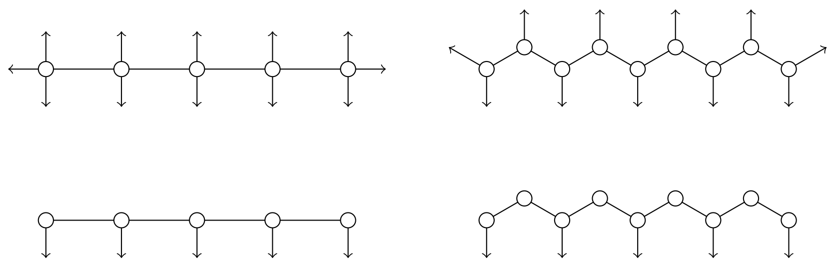

Figure 2: Strings-and-Coins equivalents to open (top) and closed (bottom) 1×5 games of Dots-and-Boxes (left) and Dots-and-Triangles (right).

Figure 3 is said to have a *double-dealing opportunity.* A player double-deals when they remove edge Y, thus declining to take two vertices that would have been taken had they removed edge X.

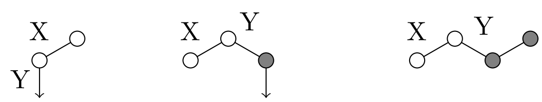

Figure 3: Double-dealing opportunities. Each shaded vertex may be incident to other edges not shown in the figure.

Remark 1. The first player to be presented with a double-dealing opportunity at a point in the game when that player has a nonnegative net score can guarantee at least a tie [1, 3].

A move that gives the opponent a double-dealing opportunity is called *loony.* A chain of length k is a collection of $k+1$ edges such that the removal of any single edge allows the opponent to claim all k vertices during their next turn; removing this edge is called *opening the chain.* A chain of length $k \geq 3$ is *long*; otherwise the chain is *short.* Every move in a long chain is loony, but in every short chain there is at least one non-loony move. Thus when a player opens a chain of length k, they immediately cede k coins to their opponent if $k \leq 2$ or $k-2$ coins and a double-dealing opportunity if $k \geq 3$.

For the remainder of the paper, we will use the convention that Alice is Player 1 and Bob is Player 2 *in the original game.* We let G denote a position in a Strings-and-Coins game *in progress* after any capturable coins (except possibly those of the type shown in Figure 3) have been taken.

The next lemma is used in the proof of the main result. Although the proof of the lemma is a simple application of the man-in-the-middle technique, we include it here for completeness.

Lemma 1. *Let $G + G$ denote two copies of G. The player who plays second in $G + G$ can guarantee at least a tie.*

Proof. We proceed by induction on the number of edges m in G. Since we assume that G has no capturable coins except possibly those of the type shown in Figure 3, the base case is $m = 2$; it is easily verified that Player 2 can guarantee at least a tie in this case. Consider the game $G + G$ when $m > 2$. If Player 1's first move is loony, then Player 2 can guarantee at least a tie by Remark 1. If Player 1's first move is not loony, then Player 2 collects any coins and plays the corresponding move in G's copy. In doing so, Player 2 cedes a number of coins equal to the number of coins they just captured. At the beginning of Player 1's second turn, the net score is zero and the game is of the form $G' + G'$, where G' has fewer edges than G. Thus Player 2 can guarantee at least a tie by the inductive hypothesis. □

As an immediate consequence, we have the following.

Corollary 1. *If $n \geq 4$ is even, then Player 1 can guarantee at least a tie in $1 \times n$ open or closed Dots-and-Boxes.*

Proof. Alice begins by taking the middle edge. Since $n \geq 4$ this does not give Bob any points, and reduces the game to the form $G + G$. Bob then plays first in $G + G$, so Alice can guarantee a tie by Lemma 1. □

2. Proof of Main Result

We turn our attention now to the game of closed $1 \times n$ Dots-and-Triangles. Alice's "take the middle edge" strategy will still be the correct first move, but the argument is not as straightforward as above. If n is odd and Alice takes the middle edge (i.e., the edge connecting the middle coin to the ground), the remaining game is not of the form $G + G$, and if n is even and Alice takes one of the two middlemost edges, she cedes the middle coin to Bob, in which case her net score is no longer nonnegative. Thus we require two additional lemmas. Note that the first is true of any Strings-and-Coins game, while the second applies to narrow $1 \times n$ closed games only.

Lemma 2. *Let $G \oplus G$ denote the game obtained from $G + G$ by inserting a long chain between a vertex of G and its corresponding vertex in the copy of G. The player who plays second in $G \oplus G$ can guarantee at least a tie.*

Proof. Proceed by induction on m, where m is the number of edges in G (not including the edges in the long chain). If Player 1's first move is loony, then Player 2 can guarantee a tie by Remark 1. If Player 1's first move is not loony, then Player

2 can play the corresponding move in the copy of G and the inductive hypothesis applies. □

Lemma 3. *In a game of closed* $1 \times n$ *Dots-and-Triangles where* $n \geq 2$*, eventually one player must open a long chain.*

Proof. Proceed by induction on n. If $n = 2$, every move opens a long chain. Now suppose that in a $1 \times n$ game where $n > 2$, the first move is to remove an edge which is incident to a degree 3 vertex. If it is a ground edge, then a long chain is created and it must eventually be opened. If it is a non-ground edge, then the remaining game consists of two components; at least one of these will be a $1 \times n'$ game where $2 \leq n' < n$.

Suppose instead that the first edge to be removed is not incident to a vertex of degree 3. If it is a non-ground edge, then after the next player collects all capturable coins, a $1 \times (n-1)$ game remains. If it is a ground edge, then either the next player collects all capturable coins or double-deals. In either case, after the next player's turn one of the components in the remaining game is a $1 \times (n-1)$ game. □

This leads us to our main result, the proof of a conjecture of Nowakowski and Ottaway (see p. 470 of [4]).

Theorem 1. *A game of closed* $1 \times n$ *Dots-and-Triangles where* $n \neq 2$ *is a first-player win.*

Proof. The $n = 1$ case is trivial. First suppose that $n \geq 3$ is odd. Alice begins the game by taking the middle (ground) edge (see Figure 4). Thus Bob plays first in a game of the form $G \oplus G$. By Lemma 2, Alice can guarantee at least a tie. But the number of vertices is odd, so Alice must win.

Now suppose $n \geq 4$ is even. Alice's first move is to take one of the two middlemost (non-ground) edges. Bob immediately collects a coin and then plays first in a game of the form $G + G$. Although Alice plays second in $G + G$. she has a net score of -1, so she must not give Bob a double-dealing opportunity. Thus she proceeds as follows. If Bob makes a non-loony move, Alice copies the move, keeping her net score the same. If Bob makes a loony move in a short chain, Alice collects the coins and makes a non-loony move in the corresponding copy of the short chain. By Lemma 3, one player must eventually open a long chain. By appropriately copying Bob's moves, Alice ensures that Bob will be the first to do so. Alice then collects at least one coin, bringing her net score up to at least zero, and she has a double-dealing opportunity. By Remark 1, Alice can guarantee at least a tie, but since the number of vertices is odd, Alice must win.

□

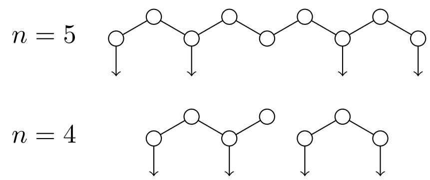

Figure 4: The game at the end of Alice's first turn

3. Further Work

There are several open questions regarding the $1 \times n$ Dots-and-Boxes/Triangles games. Of immediate interest are the narrow Dots-and-Boxes game where n is odd, and the open narrow Dots-and-Triangles game, as our results do not apply to these cases. The reason is that when n is odd and Alice attempts to use the man-in-the-middle technique, Bob has a move that is "unmirrorable;" see Figure 5. In this case both e, the edge to be removed by Bob, and e', the edge to be removed by Alice, are part of the same chain before Bob removes e. If Alice removes e', then she must move again, and now Bob may employ the man-in-the-middle strategy. In fact, if Alice attempts to mirror until Bob takes an unmirrorable edge, then Bob can win by taking ground edges which are incident to degree 3 vertices in sequence until the game is in the position shown in Figure 6, at which point Alice must open a long chain and Bob can win the game. It seems that something more complicated than the man-in-the-middle strategy is called for in the odd versions of narrow Dots-and-Boxes.

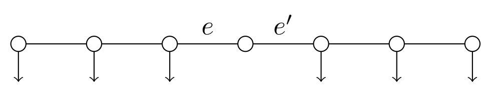

Figure 5: Alice has removed the center edge on her first turn. If Bob removes edge e and Alice responds by removing its corresponding edge e', then Alice must move again before ending her turn.

Through a computer search, we have considered the closed $1 \times n$ Dots-and-Boxes game up to $n = 21$. The results are shown in Table 1. Our results suggest that

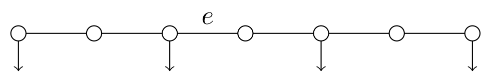

Figure 6: If Bob takes edge e, Alice will be forced to open a long chain.

under optimal play (i.e., playing so as to maximize final net score), the first player can win with a final net score of 1 when n is odd and $n \geq 9$, and that the first player can end with a final net score of 0 when n is even and $n \geq 4$. This suggests that Corollary 1 cannot be improved upon, and that the closed $1 \times n$ Dots-and-Boxes game is a first-player win when n is odd. We have also considered games of closed $1 \times n$ Dots-and-Triangles up to $n = 15$. The mirroring strategy employed in the proof above guarantees a win but does not necessarily maximize Player 1's final net score. Of particular interest are games where n is a multiple of 3. It seems that Player 1 can win with a final net score of 5 when $n = 3, 6, 9, 12$, but this pattern does not continue, as the final net score is just 1 in the $n = 15$ case. A strategy that is known to maximize the final net score would be of interest here.

Dots-and-Boxes		Dots-and-Triangles	
n	score	n	score
1	1	1	1
2	-2	2	-3
3	3	3	5
4	0	4	1
5	1	5	1
6	0	6	5
7	3	7	3
8	0	8	1
9	1	9	5
10	0	10	1
11	1	11	1
12	0	12	5
13	1	13	1
14	0	14	1
15	1	15	1
16	0		
17	1		
18	0		
19	1		
20	0		
21	1		

Table 1: Maximum final net score for Player 1 in closed versions of the games.

Acknowledgement. The authors wish to thank the referee for helpful comments which improved this paper.

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