THE 2-ADIC ORDER OF SOME GENERALIZED FIBONACCI NUMBERS

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Abstract

Let $T_n = T_n(k)$ be the generalized Fibonacci sequence of order k defined by the recurrence $T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}, n \geq k$, with $T_0 = 0$ and $T_1 = T_2 = \cdots = T_{k-1} = 1$. In this paper, we fully and partially characterize the 2-adic valuations of $T_n(4)$ and $T_n(5)$, respectively. Moreover, we provide new addition formulas and congruences for the sequences $\{T_n(k)\}_{n\geq 0}$.

1. Introduction

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. The p-adic order, $\nu_p(r)$, of r is the exponent of the highest power of a prime p which divides r. The p-adic order of a Fibonacci number was completely characterized, see [4]. Much less is known about the generalized Fibonacci sequences. Let $T_n = T_n(k), n \geq 0$, denote the generalized Fibonacci sequence of order k defined by the recurrence relation

$$T_n = T_{n-1} + T_{n-2} + \cdots + T_{n-k}, n \geq k, \tag{1}$$

and the initial conditions $T_0 = 0, T_1 = T_2 = \cdots = T_{k-1} = 1$. Note that sometimes the initial conditions are given by

$$B_0 = B_1 = \cdots = B_{k-2} = 0, B_{k-1} = 1 \tag{2}$$

with $B_n = B_n(k)$ while the recurrence $B_n = B_{n-1} + B_{n-2} + \cdots + B_{n-k}, n \geq k$, is preserved. By convention, we also set $B_{-1}(k) = 0$. Clearly, $F_n = T_n(2) = B_n(2)$.

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Our goal is to present a systematic approach that helps establish the 2-adic order of $T_n(k)$, at least for some specialized sequences of the index n (we point out that the 2-adic valuation of $T_n(3)$ was fully determined in [6]). Here, we focus on $T_n(k)$ for $k = 4$ and 5. The first few terms of the sequence $\{T_n(4)\}_{n\geq 0}$ are

$$0, 1, 1, 1, 3, 6, 11, 21, 41, 79, 152, 293, 565, 1089, 2099, 4046, 7799, \ldots$$

while those of $\{T_n(5)\}_{n\geq 0}$ are

$$0, 1, 1, 1, 1, 4, 8, 15, 29, 57, 113, 222, 436, 857, 1685, 3313, 6513, \ldots.$$

Our main results are Theorems 1, 2, and Lemmas 2, 5, and 6. We also suggest several conjectures, cf. Conjectures 1 and 2.

Throughout the paper, we emphasize the experimental aspects of finding and discovering relations, e.g., recurrence relations and congruences.

Theorem 1. *For $n \geq 1$, we have*

$$\nu_2(T_n(4)) = \begin{cases} 0, & \text{if } n \not\equiv 0 \pmod 5, \\ 1, & \text{if } n \equiv 5 \pmod{10}, \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{10}. \end{cases} \tag{3}$$

With $n \geq 1$ and $s \geq 1$ odd, this yields that

$$\nu_2(T_{5\cdot s\cdot 2^n}(4)) = n + 2. \tag{4}$$

We refer the reader to [2] for the 2-adic valuation of $B_n(4)$.

We make the following conjecture for the case of $k = 5$.

Conjecture 1. *For $n \geq 1$, we have*

$$\nu_2(T_n(5)) = \begin{cases} 0, & \text{if } n \not\equiv 0 \text{ or } 5 \pmod 6, \\ 2, & \text{if } n \equiv 5 \pmod{12}, \\ 1, & \text{if } n \equiv 11 \pmod{12}, \\ \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) < 8, \\ \nu_2(n+43266), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n+2) \geq 8, \\ \nu_2(n), & \text{if } n \equiv 0 \pmod{12}. \end{cases} \tag{5}$$

Here, we prove it in the following weaker form.

Theorem 2. *For $n \geq 1$, we have*

$$\nu_2(T_n(5)) = \begin{cases} 0, & \text{if } n \not\equiv 0 \text{ or } 5 \pmod 6, \\ 2, & \text{if } n \equiv 5 \pmod{12}, \\ 1, & \text{if } n \equiv 11 \pmod{12}, \\ \nu_2(n+2), & \text{if } n \equiv 6 \pmod{12} \text{ and } \nu_2(n-6) \neq 3, \\ \nu_2(n), & \text{if } n \equiv 0 \pmod{12}. \end{cases} \tag{6}$$

With $n \geq 1$ and $s \geq 1$ odd, this yields that

$$\nu_2(T_{6\cdot s\cdot 2^n}(5)) = n + 1. \tag{7}$$

We also propose the following conjecture.

Conjecture 2. *For $n \geq 1$ and $k \geq 2$ integers and $s \geq 1$ odd integer, we have*

$$\nu_2(T_{s\cdot(k+1)\cdot 2^n}(k)) = n + c(k)$$

where $c(2) = 2$ and otherwise,

$$c(k) = \begin{cases} 2, & \text{if } k \equiv 0 \pmod 4; \\ 1, & \text{if } k \equiv 1 \pmod 4; \\ \nu_2(k-2)+1, & \text{if } k \equiv 2 \pmod 8; \\ 1, & \text{if } k \equiv 3 \pmod 8; \\ 3, & \text{if } k \equiv 6 \pmod 8; \\ 1, & \text{if } k \equiv 7 \pmod 8). \end{cases}$$

Remark 1. Conjecture 2 can be easily verified for $k = 2$ and 3. In fact, we proved for $n \geq 1$ that $\nu_2(T_n(2)) = \nu_2(n) + 2$ if $n \equiv 0, 6 \pmod{12}$ in [4] and $\nu_2(T_n(3)) = \nu_2(n) - 1$ if $n \equiv 0, 8 \pmod{16}$ in [6]. In this paper, we prove the conjecture for $k = 4$ and 5.

We outline a plan that can be followed in order to prove Conjecture 2. In fact, we will apply the plan in the cases of $k = 4$ and 5 in Sections 4 and 5, respectively.

Step 1. First we establish an addition formula for $T_{q+r}(k)$ in terms of $T_{q'}(k)$ and $T_{r'}(k)$ with q' and r' close to q and r, respectively; more precisely, with $q - k + 2 \leq q' \leq q + k$ and $r \leq r' \leq r + k - 1$.

Step 2. The second step is to come up with a set of induction hypotheses for $T_{s\cdot(k+1)\cdot 2^n+i}(k) \pmod{2^{n+c(k)+1}}$ for all $i : 0 \leq i \leq k-1$ and $n \geq n_0(k)$ with some functions $c(k)$ and $n_0(k)$, e.g., $T_{s\cdot(k+1)\cdot 2^n} \equiv s \cdot 2^{n+c(k)} \pmod{2^{n+c(k)+1}}, n \geq 1$, in Lemmas 5 and 6 and prove it simultaneously by using the recurrence relation for $T_{q+r}(k)$ from the first step. Note that the congruence $T_{s\cdot(k+1)\cdot 2^n+i}(k) \pmod{2^{n+c(k)+1}}$ will follow for any $i \leq -1$ and $i \geq k$ by the recurrence (1).

Step 3. In the induction proof, first we deal with the case $s = 1$ and we prove this case by induction on n. The same procedure will work for other values of s.

In conclusion, this process yields that if $m = s \cdot (k+1) \cdot 2^n$ and $s \geq 1$ is odd then $\nu_2(T_m(k)) = n + c(k)$ for $n \geq n_0(k)$.

We illustrate the actual steps in Sections 2 and 3. Section 2 is devoted to the process of obtaining recurrence relations while Section 3 contains the congruences that are the essential tools in proving Theorems 1 and 2.

The actual calculations and proofs in the cases of $k = 4$ and 5 are presented in Sections 4 and 5. They lead to identities (11) and (12) that are crucial in proving the congruences (14), (15), and (16).

2. Obtaining a Recurrence by an Addition Formula

As a reminder, we note the addition formula, given in Lemma 4 of [6], which yields a recurrence for $T_{q+r}(3)$. For all integers q and r with $q \geq 3$ and $r \geq 0$, we have that

$$T_{q+r} = T_{q-2}T_r + (T_{q-3} + T_{q-2})T_{r+1} + T_{q-1}T_{r+2}.$$

Note that $T_{q-1} = T_{q-4} + T_{q-3} + T_{q-2}$. It is determined in Theorem 2.1 of [7] that with $T_n = T_n(4)$ and $B_n = B_n(4)$, we have

$$T_q = B_{q-2}T_1 + (B_{q-2} + B_{q-3})T_2 + (B_{q-2} + B_{q-3} + B_{q-4})T_3 + B_{q-1}T_4, \tag{8}$$

for $q \geq 5$ where $B_{q-1} = B_{q-2} + B_{q-3} + B_{q-4} + B_{q-5}$. The formula (8) can be easily generalized to

Lemma 1. *For $q \geq 5$ and $r \geq 0$ with $T_n = T_n(4)$ and $B_n = B_n(4)$, we have that*

$$T_{q+r} = B_{q-2}T_{r+1} + (B_{q-2} + B_{q-3})T_{r+2} + (B_{q-2} + B_{q-3} + B_{q-4})T_{r+3} + B_{q-1}T_{r+4}.$$

To obtain similar identities for a general k, we use the fact that one can relate the sequences $\{T_n(k)\}_{n\geq 0}$ and $\{B_n(k)\}_{n\geq 0}$. In fact, we have the following general result

Lemma 2. *Let $k \geq 2$ be an integer and set $T_n = T_n(k)$ and $B_n = B_n(k)$. For integers $q > k$ and $r \geq 0$, we have that*

$$T_q = \sum_{i=1}^{k} \left(\sum_{j=2}^{i+1} B_{q-j} \right) T_i \qquad \textit{and} \qquad T_{q+r} = \sum_{i=1}^{k} \left(\sum_{j=2}^{i+1} B_{q-j} \right) T_{r+i}. \tag{9}$$

Remark 2. We also use identity (9) in its equivalent form

$$T_{q+r} = \sum_{i=0}^{k-1} \left(\sum_{j=1}^{i+1} B_{q-j} \right) T_{r+i}. \tag{10}$$

with $q \geq k \geq 2$ and $r \geq 0$, cf. (11) and (12).

We omit the proof which can be easily done by mathematical induction on $q > k$ for every fixed $r \geq 0$.

Remark 3. Identity (9) also works for sequences $T_n(k)$ of real numbers satisfying (1) with arbitrary initial conditions.

Our next step is to determine $B_{q'}(k)$ in (9) in terms of the sequence $\{T_n(k)\}_{n\geq 0}$. We note that although $B_{n+1}(3) = T_n(3)$, usually there is a non-trivial linear relationship between the two sequences. We use the approach outlined in [1]. The result is derived in (18) and (19) as well in (22) and (23), and used by (11) and (12) in Lemmas 3 and 4, respectively.

Lemma 3. *For $T_{q+r}(4)$ with $q \geq 2$ and $r \geq 0$, we have the recurrence*

$$\begin{aligned} T_{q+r} &= \left(\frac{5}{3}T_q + \frac{1}{3}T_{q+1} + 2T_{q+2} - \frac{4}{3}T_{q+3}\right)T_r \\ &+ \left(\frac{5}{3}T_{q-1} + 2T_q + \frac{7}{3}T_{q+1} + \frac{2}{3}T_{q+2} - \frac{4}{3}T_{q+3}\right)T_{r+1} \\ &+ \left(\frac{5}{3}T_{q-2} + 2T_{q-1} + 4T_q + T_{q+1} + \frac{2}{3}T_{q+2} - \frac{4}{3}T_{q+3}\right)T_{r+2} \\ &+ \left(\frac{5}{3}T_{q+1} + \frac{1}{3}T_{q+2} + 2T_{q+3} - \frac{4}{3}T_{q+4}\right)T_{r+3}. \end{aligned} \tag{11}$$

Lemma 4. *For $T_{q+r}(5)$ with $q \geq 3$ and $r \geq 0$, we have the recurrence*

$$\begin{aligned} T_{q+r} &= \left(\frac{35T_q}{46} + \frac{11T_{q+1}}{23} + \frac{15T_{q+2}}{46} + \frac{18T_{q+3}}{23} - \frac{27T_{q+4}}{46}\right)T_r \\ &+ \left(\frac{35T_{q-1}}{46} + \frac{57T_q}{46} + \frac{37T_{q+1}}{46} + \frac{51T_{q+2}}{46} + \frac{9T_{q+3}}{46} - \frac{27T_{q+4}}{46}\right)T_{r+1} \\ &+ \left(\frac{35T_{q-2}}{46} + \frac{57T_{q-1}}{46} + \frac{36T_q}{23} + \frac{73T_{q+1}}{46} + \frac{12T_{q+2}}{23} \right. \\ &+ \left. \frac{9T_{q+3}}{46} - \frac{27T_{q+4}}{46}\right)T_{r+2} \\ &+ \left(\frac{35T_{q-3}}{46} + \frac{57T_{q-2}}{46} + \frac{36T_{q-1}}{23} + \frac{54T_q}{23} + T_{q+1} \right. \\ &+ \left. \frac{12T_{q+2}}{23} + \frac{9T_{q+3}}{46} - \frac{27T_{q+4}}{46}\right)T_{r+3} \\ &+ \left(\frac{35T_{q+1}}{46} + \frac{11T_{q+2}}{23} + \frac{15T_{q+3}}{46} + \frac{18T_{q+4}}{23} - \frac{27T_{q+5}}{46}\right)T_{r+4}. \end{aligned} \tag{12}$$

3. Congruences

We note that for $k = 3$ the congruences in (4) of Lemma 6 in [6] are equivalent to the following statement. For $s \geq 1$, $n \geq 3$, and $T_m = T_m(3)$, we have the congruences

$$\begin{aligned} T_{s\cdot 2^n} &\equiv s \cdot 2^{n-1} && \pmod{2^n}, \\ T_{s\cdot 2^n+1} &\equiv 1 && \pmod{2^n}, \\ T_{s\cdot 2^n+2} &\equiv 1 + s \cdot 2^{n-1} && \pmod{2^n}. \end{aligned} \tag{13}$$

Now we establish similar congruences for $k = 4$.

Lemma 5. *For $s \geq 1$, $n \geq 2$, and $T_m = T_m(4)$, we have that*

$$\begin{aligned} T_{5\cdot s\cdot 2^n} &\equiv s \cdot 2^{n+2} && \pmod{2^{n+3}}, \\ T_{5\cdot s\cdot 2^n+1} &\equiv 1 + s \cdot 2^{n+1} && \pmod{2^{n+3}}, \\ T_{5\cdot s\cdot 2^n+2} &\equiv 1 + s \cdot 2^{n+1} + s \cdot 2^{n+2} && \pmod{2^{n+3}}, \\ T_{5\cdot s\cdot 2^n+3} &\equiv 1 && \pmod{2^{n+3}}, \end{aligned} \tag{14}$$

while for $n = 1$, we have that

$$\begin{aligned} T_{10\cdot s} &\equiv 8s && \pmod{16}, \\ T_{10\cdot s+1} &\equiv 1 + 4s && \pmod{16}, \\ T_{10\cdot s+2} &\equiv 1 + 4s && \pmod{16}, \\ T_{10\cdot s+3} &\equiv 1 && \pmod{16}, \end{aligned} \tag{15}$$

which yields that $\nu_2(T_{5\cdot s\cdot 2^n}(4)) = n + 2$ if $n \geq 1$ and $s \geq 1$ odd.

Proof of Lemma 5. We closely follow the steps of the proof of Lemma 6 of [6]. First, we deal with the basis case $s = 1$. We have to prove (14) for $n \geq 2$. We use induction on n. Clearly, the congruences hold for $n = 2$. We suppose that they are true for $n \geq 2$, and then we use (11) for $T_{5\cdot 2^{n+1}+i} = T_{(5\cdot 2^n)+(5\cdot 2^n+i)}, 0 \leq i \leq 3$, to obtain the required congruences for $T_{5\cdot 2^{n+1}+i}$. Next, by the induction hypothesis, we suppose that the congruences (14) hold for $s \geq 1$. Then, we use exactly the same procedure and (11) as before for $T_{5\cdot(s+1)\cdot 2^n+i} = T_{(5\cdot s\cdot 2^n)+(5\cdot 2^n+i)}$. In a similar fashion, we use induction on $s \geq 1$ to prove the congruences (15), corresponding to the case with $n = 1$. We omit the details. □

Example 1. We illustrate the above proof in the case of $k = 4$, $n \geq 2$, $s \geq 1$, and $i = 0$. With the setting $r = 5\cdot s\cdot 2^n$ and $q = 5\cdot 2^n$, we obtain by (11) that $T_{5\cdot 2^n(s+1)} = \Big(\frac{5}{3}T_{5\cdot 2^n} + \frac{1}{3}T_{5\cdot 2^n+1} + 2T_{5\cdot 2^n+2} - \frac{4}{3}T_{5\cdot 2^n+3}\Big)T_{5\cdot 2^n s} + \Big(2T_{5\cdot 2^n} + \frac{5}{3}T_{5\cdot 2^n-1} + \frac{7}{3}T_{5\cdot 2^n+1} + \frac{2}{3}T_{5\cdot 2^n+2} - \frac{4}{3}T_{5\cdot 2^n+3}\Big)T_{5\cdot 2^n s+1} + \Big(4T_{5\cdot 2^n} + \frac{5}{3}T_{5\cdot 2^n-2} + 2T_{5\cdot 2^n-1} + T_{5\cdot 2^n+1} + \frac{2}{3}T_{5\cdot 2^n+2} -$

$\frac{4}{3}T_{5\cdot 2^n+3}\Big)T_{5\cdot 2^n s+2} + \Big(\frac{5}{3}T_{5\cdot 2^n+1} + \frac{1}{3}T_{5\cdot 2^n+2} + 2T_{5\cdot 2^n+3} - \frac{4}{3}T_{5\cdot 2^n+4}\Big)T_{5\cdot 2^n s+3}$, which results in $\frac{1}{3}2^{n+2}s - \frac{1}{3}2^{n+3}s + \frac{1}{3}2^{n+4}s + \frac{5}{3}2^{2n+3}s + 2^{2n+4}s + \frac{1}{3}2^{2n+6}s + \frac{1}{3}2^{2n+7}s - \frac{1}{3}2^{2n+8}s + \frac{1}{3}2^{2n+9}s + \frac{2^{n+2}}{3} + \frac{2^{n+3}}{3} \pmod{2^{n+3}}$ by the induction hypothesis. We get $\frac{1}{3}\cdot 2^{n+2}\cdot(s+1) \equiv 2^{n+2}\cdot(s+1) \pmod{2^{n+3}}$ by replacing any term including a factor with a "high" power of 2 with 0. More precisely, any term including $2^{c\cdot n+d}$ with $d \geq 3$ or $c > 1$ combined with $d \geq 1$ is dropped. It implies that the statement $T_{5\cdot(s+1)\cdot 2^n} \equiv (s+1)\cdot 2^{n+2} \pmod{2^{n+3}}$ in (14) is also true.

Note that the substitutions and simplifications above can be easily preformed by using *Mathematica*.

In the case of $k = 5$ we proceed similarly.

Lemma 6. *For $s \geq 1$, $n \geq 1$, and $T_m = T_m(5)$, we have that*

$$\begin{aligned} T_{6\cdot s\cdot 2^n} &\equiv s\cdot 2^{n+1} && \pmod{2^{n+2}},\\ T_{6\cdot s\cdot 2^n+1} &\equiv 1 && \pmod{2^{n+2}},\\ T_{6\cdot s\cdot 2^n+2} &\equiv 1 + s\cdot 2^{n+1} && \pmod{2^{n+2}},\\ T_{6\cdot s\cdot 2^n+3} &\equiv 1 && \pmod{2^{n+2}},\\ T_{6\cdot s\cdot 2^n+4} &\equiv 1 && \pmod{2^{n+2}}, \end{aligned} \tag{16}$$

which yields that $\nu_2(T_{6\cdot s\cdot 2^n}(5)) = n+1$ if $n \geq 1$ and $s \geq 1$ odd.

The proof essentially duplicates the steps of the proof of Lemma 5 and we leave the details to the reader.

4. The Case of $k = 4$

Before we present the proof of Lemma 3, we explore an approach given in [1]. In fact, we use it with some modifications and with $n \geq 0$ and $m \geq 4$. We start with the matrix

$$\begin{pmatrix} T_n & T_{n+1} & T_{n+2} & T_{n+3} & T_{m+n}\\ T_{n+1} & T_{n+2} & T_{n+3} & T_{n+4} & T_{m+n+1}\\ T_{n+2} & T_{n+3} & T_{n+4} & T_{n+5} & T_{m+n+2}\\ T_{n+3} & T_{n+4} & T_{n+5} & T_{n+6} & T_{m+n+3} \end{pmatrix}. \tag{17}$$

After experimenting with different values of m and row reducing the matrix in (17), we successfully obtain the recurrence relation $T_{m+n} = B_{m-1}T_n + (B_{m-2}+B_{m-1})T_{n+1} + (B_{m-3}+B_{m-2}+B_{m-1})T_{n+2} + B_mT_{n+3}$ suggesting (9) of Lemma 2 in its equivalent form (10) for $k = 4$ with $m \geq 4$ and $n \geq 0$.

In a similar fashion, we establish the

Proof of Lemma 3. We consider the matrix

$$\begin{pmatrix} T_n & T_{n+1} & T_{n+2} & T_{n+3} & B_{m+n} \\ T_{n+1} & T_{n+2} & T_{n+3} & T_{n+4} & B_{m+n+1} \\ T_{n+2} & T_{n+3} & T_{n+4} & T_{n+5} & B_{m+n+2} \\ T_{n+3} & T_{n+4} & T_{n+5} & T_{n+6} & B_{m+n+3} \end{pmatrix}. \tag{18}$$

After setting $m = -1$ and using different values of $n \geq 1$, we observe that the row reduction always results in

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -\frac{4}{3} \end{pmatrix}, \tag{19}$$

which yields that

$$B_{n-1} = \left(\frac{5T_n}{3} + \frac{T_{n+1}}{3} + 2T_{n+2} - \frac{4T_{n+3}}{3} \right) \tag{20}$$

for $n \geq 1$, which confirms (11).

Note that once (20) is established, an easy induction proof justifies this identity. Indeed, with $n = 1, 2, 3, 4$ we get that $0 = \frac{5}{3}\cdot 1 + \frac{1}{3}\cdot 1 + 2\cdot 1 - \frac{4}{3}\cdot 3 = \frac{5}{3}\cdot 1 + \frac{1}{3}\cdot 1 + 2\cdot 3 - \frac{4}{3}\cdot 6 = \frac{5}{3}\cdot 1 + \frac{1}{3}\cdot 3 + 2\cdot 6 - \frac{4}{3}\cdot 11$ and $1 = \frac{5}{3}\cdot 3 + \frac{1}{3}\cdot 6 + 2\cdot 11 - \frac{4}{3}\cdot 21$. The induction step is trivial by (1) and (2). □

A natural approach to obtain the proof of Theorem 1 is to utilize the periodicity of the underlying sequences. In some cases we can apply multisection techniques, cf. [5], to find the complete or some partial characterization of the p-adic order of the sequences. Here we combine these methods with the applications of sets of congruences for $\{T_{s\cdot(k+1)\cdot 2^n+i}\}_{i=0}^{k-1}$ with $s \geq 1$ and $n \geq n_0(k)$ integers.

Now we can complete the proof of Theorem 1.

Proof of Theorem 1. The proof for the case $n \not\equiv 0 \pmod 5$ is trivial by taking $T_n(4) \pmod 2$ and induction on n. In fact, the sequence $\{T_n(4)\}_{n\geq 0}$ is periodic with period $\{0, 1, 1, 1, 1\}$ modulo 2.

If $n \equiv 5 \pmod{10}$ then by 5-section of the generating function $\sum_{m=0}^{\infty} T_m(4)x^m$ (cf. [5]) we get that

$$\sum_{m=0}^{\infty} T_{5m}(4)x^{5m} = \frac{2x^5(3 - 2x^5 - x^{10})}{1 - 26x^5 - 16x^{10} - 6x^{15} - x^{20}},$$

which easily yields that $\nu_2(T_n(4)) = 1$. Indeed, the denominator of the 5-sected generating function suggests the recurrence

$$T_{5m+10} = 26T_{5m+5} + 16T_{5m} + 6T_{5m-5} + T_{5m-10}, m \geq 2, \tag{21}$$

for $T_r = T_r(4)$ with r divisible by 5. We observe that $\nu_2(T_5) = 1$, $\nu_2(T_{10}) = 3$, $\nu_2(T_{15}) = 1$, and $\nu_2(T_{20}) = 4$, which yield that $\nu_2(T_{5m}) \geq 1$ for $m \geq 0$ by the initial values and (21). Now $\nu_2(T_{5m+10}) = \nu_2(T_{5m-10}) = 1$ with $m \geq 3$ odd also follows by recurrence (21).

We note that we can extend (15) by recurrence (1) to obtain $T_{10\cdot s+4} \equiv 3 \pmod{16}$ and $T_{10\cdot s+5} \equiv 6 + 8s \pmod{16}$, and the latter congruence also results in $\nu_2(T_n) = 1$ with $n \equiv 5 \pmod{10}$.

In the remaining case 10 divides n, and Lemma 5 concludes the proof. □

5. The Case of $k = 5$

Now we turn to the

Proof of Lemma 4. Similarly to (18) in the case of $k = 4$, we now consider

$$\begin{pmatrix} T_n & T_{n+1} & T_{n+2} & T_{n+3} & T_{n+4} & B_{m+n} \\ T_{n+1} & T_{n+2} & T_{n+3} & T_{n+4} & T_{n+5} & B_{m+n+1} \\ T_{n+2} & T_{n+3} & T_{n+4} & T_{n+5} & T_{n+6} & B_{m+n+2} \\ T_{n+3} & T_{n+4} & T_{n+5} & T_{n+6} & T_{n+7} & B_{m+n+3} \\ T_{n+4} & T_{n+5} & T_{n+6} & T_{n+7} & T_{n+8} & B_{m+n+4} \end{pmatrix}. \tag{22}$$

After setting $m = -1$ and using different values of $n \geq 1$, row reduction leads us to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{35}{46} \\ 0 & 1 & 0 & 0 & 0 & \frac{11}{23} \\ 0 & 0 & 1 & 0 & 0 & \frac{15}{46} \\ 0 & 0 & 0 & 1 & 0 & \frac{18}{23} \\ 0 & 0 & 0 & 0 & 1 & -\frac{27}{46} \end{pmatrix} \tag{23}$$

which results in $B_{n-1} = \frac{35T_n}{46} + \frac{11T_{n+1}}{23} + \frac{15T_{n+2}}{46} + \frac{18T_{n+3}}{23} - \frac{27T_{n+4}}{46}$ for $n \geq 1$, which is in agreement with (12). Its proof follows easily by induction as it was explained in the proof of Lemma 3 for $k = 4$. □

We are now ready to present the proof of Theorem 2.

Proof of Theorem 2. As above, the proof for the case $n \not\equiv 0$ and $5 \pmod{6}$ is trivial by taking $T_n(5) \pmod{2}$ and induction on n since the sequence $\{T_n(5)\}_{n\geq 0}$ is periodic with period $\{0, 1, 1, 1, 1, 0\}$ modulo 2.

If $n \equiv 6 \pmod{12}$ then with $n = 6 \cdot s \cdot 2^m + 6$, $s \geq 1$ odd and $m \geq 1$, we get that $T_{6\cdot s\cdot 2^m+5} \equiv 4 \pmod{2^{m+2}}$ and $T_{6\cdot s\cdot 2^m+6} \equiv 8 + s \cdot 2^{m+1} \pmod{2^{m+2}}$ by extending (16) via (1). It implies that $\nu_2(T_{6\cdot s\cdot 2^m+6}) = \nu_2(n + 2)$ as long as either $m \geq 3$ or $m = 1$, in which cases the 2-adic order is either 3 or 2, respectively. In a similar fashion, it follows that $T_{6\cdot s\cdot 2^m+11} \equiv 222 \pmod{2^{m+2}}$. Thus, with $t \geq 1$ integer, we also have that $T_{12t+5} \equiv 4 \pmod 8$ and $T_{12t+11} \equiv 222 \pmod 8$, which yield that $\nu_2(T_{12t+5}) = 2$ and $\nu_2(T_{12t+11}) = 1$.

Otherwise 12 divides n, and Lemma 6 concludes the proof. □

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